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Behavioral Algebraization of Logics

Abstract. We introduce and study a new approach to the theory of abstract algebraic logic (AAL) that explores the use of many-sorted behavioral logic in the role traditionally played by unsorted equational logic. Our aim is to extend the range of applicability of AAL towards providing a meaningful algebraic counterpart also to logics with a many-sorted language, and possibly including non-truth-functional connectives. The proposed behavioral approach covers logics which are not algebraizable according to the standard approach, while also bringing a new algebraic perspective to logics which are algebraizable using the standard tools of AAL. Furthermore, we pave the way towards a robust behavioral theory of AAL, namely by providing a behavioral version of the Leibniz operator which allows us to generalize the traditional Leibniz hierarchy, as well as several well-known characterization results. A number of meaningful examples will be used to illustrate the novelties and advantages of the approach.

Keywords: Abstract algebraic logic, many-sorted behavioral logic, non-truth-functionality.

1. Introduction

The general theory of *abstract algebraic logic* (AAL, from now on) was first introduced in [3] with the aim of extending the so-called *Lindenbaum-Tarski method*, as used for instance to establish the relationship between classical propositional logic and Boolean algebras, to the systematic study of the connection between a given logic and a suitable equational theory. This connection enables one to use the powerful tools of universal algebra to study the metalogical properties of the logic being algebraized, namely with respect to its axiomatizability, definability aspects, the deduction theorem, or interpolation properties [11, 13]. Despite of its success, the scope of applicability of the standard tools of AAL is relatively limited. Logics with a many-sorted language, even if well behaved, are good examples of logics that fall out of their scope. It goes without saying that rich logics, with many-sorted languages, are essential to specify and reason about complex systems, as also argued and justified by the theory of combined logics [40]. However, even in the case of propositional based (single-sorted) logics, many interesting examples simply fall out of the scope of the standard tools of AAL. With the proliferation of logical systems, with applications ranging from computer science, to mathematics and philosophy, the examples of non-algebraizable logics that, therefore, lack from a meaningful and insightful algebraic counterpart are expected to become more and more common. This is the case, for example, of certain non-truth-functional logics [2]. The notion of a (non-)truth-functional logic is relatively imprecise, and may sometimes be confused with self-extensionality or the Fregean property. We do not wish to dwell here on the exact best meaning of the notion. Instead, it will suit our purposes to dub non-truth-functional those logics that can be seen as extensions (by adding connectives and rules) of algebraizable logics, but in which some of the new connectives do not satisfy the congruence property with respect to the equivalence of their algebraizable fragment.

Although AAL can associate a class of algebras to every logic, the connection between a non-algebraizable logic and the corresponding class of algebras is, of course, not as strong as if it were algebraizable and may not be very interesting. This phenomenon is well known and may happen for several reasons, and in different degrees, depending on whether the Leibniz operator will lack the properties of injectivity, monotonicity, or commutation with inverse substitutions. The particular issue of non-injectivity, staying within the realm of protoalgebraic and equivalential logics, has been carefully studied in [11, 13, 12]. Moreover, in [20] the authors restrict the models of the protoalgebraic logic at hand by considering just the matrices with a so-called Leibniz filter, therefore obtaining a weakly algebraizable logic. Although this is a very interesting approach, the resulting logic is, of course, different from the original one. Our aim in this paper is precisely to propose and study an extension of the tools of AAL that may encompass some of these less orthodox logics while still associating to them meaningful and insightful algebraic counterparts. Therefore, contrarily to what is done in [20], we do not want, at all, to change the logic we start from. Our strategy is rather to change the algebraic perspective. This is achieved by considering behavioral many-sorted logic, rather than plain old unsorted equational logic, as the main working tool.

The motivation for our use of the term *behavioral* emerges from computer science, namely from the algebraic approach to the specification and verification of complex (namely, object oriented) systems, where abstract data types and object classes are defined by the properties of their associated operations. Algebras are considered as abstract machines where the programs are to be run. Such systems constitute a challenge for traditional algebraic methods, since they very often provide mechanisms to encapsulate internal data in order to make the updating of programs easier and, the internal data protected. Consequently, the data should naturally be split into two categories: *visible data* which can be directly accessed, and *hidden data* that can only be accessed indirectly by analyzing the meaning (output) of programs with visible output, called *experiments*. The role of experiments is to access the relevant information encapsulated in a state. Since one cannot access the hidden data, it is not possible to reason directly about the equality of two hidden values. Hence, equational logic needs to be replaced by behavioral equational logic (sometimes called hidden equational logic) based on the notion of behavioral equivalence. Two values are said to behaviorally equivalent if they cannot be distinguished by the set of available experiments (as introduced by Reichel in [34]). This restriction induces the notion of Γ-behavioral equivalence (cf. [21]), where Γ is a subset of the set of original operations. It can be shown that the Γ-behavioral equivalence is the largest Γ-congruence whose visible part is the identity relation. Notably, the possibility of having a restricted set of experiments also accommodates the existence of non-congruent operations [37]. This feature of behavioral logic will play a fundamental role in our development, since it allows us to cope also with non-truth-functionality. Note that such a setting arises quite naturally if the language of a logic is built from a many-sorted signature with a designated sort ϕ for formulas. The corresponding free many-sorted algebra will have a set of terms of each sort, but only those terms of sort ϕ will correspond to formulas of the logic. Therefore, in the logic itself, one can only observe the behavior of terms of other sorts by their indirect influence on the truth-values of the formulas where they appear.

In more concrete terms, we will introduce and study an extension of the standard theory of AAL obtained by using many-sorted behavioral logic in the role traditionally played by unsorted equational logic. Necessarily, we will not only show that our approach is a generalization of the standard approach, but we shall also present a number of meaningful examples emphasizing the importance of this generalization. Namely, we will see that logics that were not algebraizable in the standard sense may admit a neat and meaningful behavioral algebraization, such as the paraconsistent logic $\mathcal{C}_1$ of da Costa [16] (see Section 3.3.4). With our approach we can isolate and algebraizable part of a logic and moreover, based on the algebraic counterpart of this fragment, build up a meaningful and insightful algebraic counterpart for the whole logic. This can be useful even in the cases where the extension of an algebraizable logic produces an algebraizable logic. In those cases, our approach may provide a different perspective which can lead to a better insight on the logic itself and its algebraic properties. This will be illustrated by a behavioral analysis of Nelson's logic N of constructive negation [31] (see Section 3.3.6). On the other hand, we will also show that behavioral algebraization is not trivial, by providing meaningful necessary conditions. Furthermore, using these behavioral techniques, we also propose behavioral versions of several of the key notions and results of AAL, including the notions of *equivalential* and *protoalgebraic* logic, as well as some of their alternative *characterization results*, namely those involving the *Leibniz operator*. The behavioral version of the Leibniz operator that we propose is particularly appealing since it relies precisely on equality under all available experiments. A comparison of the resulting behavioral hierarchy with the standard Leibniz hierarchy will also be provided.

In Section 2 we present the necessary preliminary notions and notation, including many-sorted behavioral logic. In Section 3 we introduce the novel notion of behaviorally algebraizable logic along with some necessary conditions for behavioral algebraization. The class of algebras canonically associated with a behaviorally algebraizable logic is also introduced. Section 3 ends with some examples supporting the ideas discussed above. Then, in Section 4, we generalize a number of standard notions and results of AAL to the behavioral setting, most notably those involving the Leibniz operator. Finally, Section 5 draws some conclusions, and points to several topics for further research.

2. Basic notions

In this section we will introduce some basic notions and fix some notation for the remainder of the paper.

2.1. Many-sorted logics

First of all let us fix the notion of logic we will work with. We will adopt a Tarskian notion of logic [42].

Definition 1. (Logic)
A *logic* is a pair $\mathcal{L} = \langle L, \vdash \rangle$, where L is a set of *formulas* and $\vdash \subseteq \mathcal{P}(L) \times L$ is a *consequence relation* satisfying, for every $T_1 \cup T_2 \cup \{\varphi\} \subseteq L$: if $\varphi \in T_1$ then $T_1 \vdash \varphi$ (***reflexivity***); if $T_1 \vdash \varphi$ for all $\varphi \in T_2$, and $T_2 \vdash \psi$ then $T_1 \vdash \psi$ (***cut***); and if $T_1 \vdash \varphi$ and $T_1 \subseteq T_2$ then $T_2 \vdash \varphi$ (***weakening***).

We will consider only these three conditions. However, Tarski considered an additional property (see [44]): if $T_1 \vdash \varphi$ then $T' \vdash \varphi$ for some finite $T' \subseteq T_1$ (***finitariness***). In the sequel if $T_1, T_2 \subseteq L$, we will write $T_1 \vdash T_2$ whenever $T_1 \vdash \varphi$ for all $\varphi \in T_2$. We say that φ and ψ are *interderivable*, which is denoted by $\varphi \dashv\vdash \psi$, if $\varphi \vdash \psi$ and $\psi \vdash \varphi$. Given $T_1, T_2 \subseteq L$ we say that T_1 and T_2 are *interderivable*, if $T_1 \vdash T_2$ and $T_2 \vdash T_1$. The *theorems* of $\mathcal{L}$ are the formulas φ such that $\emptyset \vdash \varphi$. A *theory* of $\mathcal{L}$, or a $\mathcal{L}$-*theory*, is a set T

of formulas such that if $T \vdash \varphi$ then $\varphi \in T$. Given a set T, we can consider the set $T^{\vdash}$, the smallest theory containing T. The set of all $\mathcal{L}$-theories is denoted by $Th_{\mathcal{L}}$. Clearly $\langle Th_{\mathcal{L}}, \subseteq \rangle$ forms a complete partial order.

The standard tools of AAL study the logics whose language can be built from a (propositional) unsorted signature and further satisfy the usual structurality condition. In our many-sorted setting we will focus our attention on a wider class of logics: those whose language can be built from a richer many-sorted signature and further satisfy a structurality condition. A *many-sorted signature* is a pair $\Sigma = \langle S, F \rangle$ where S is a set (of *sorts*) and $F = \{F_{ws}\}_{w \in S^*, s \in S}$ is an indexed family of sets (of *operations*). For simplicity, we write $f : s_1 \dots s_n \to s \in F$ for an element f of $F_{s_1 \dots s_n s}$. As usual, we denote by $T_\Sigma(X) = \{T_{\Sigma,s}(X)\}_{s \in S}$ the S-indexed family of carrier sets of the free Σ-algebra $\mathbf{T_\Sigma(X)}$ with generators taken from a sorted family $X = (X_s)_{s \in S}$ of variable sets. We will denote by $x{:}s$ the fact that $x \in X_s$. Often, we will need terms $t \in T_\Sigma(x_1 : s_1, \dots, x_n : s_n)$ over a finite set of variables. For simplicity, we will denote such a term by $t(x_1{:}s_1, \dots, x_n{:}s_n)$. Moreover, if T is a set whose elements are all terms of this form, we will write $T(x_1{:}s_1, \dots, x_n{:}s_n)$.

A *substitution* $\sigma = \{\sigma_s : X_s \to T_{\Sigma,s}(X)\}_{s \in S}$ is an indexed family of functions. As usual, $\sigma(t)$ denotes the term obtained by uniformly applying σ to each variable in t. Given $t(x_1 : s_1, \dots, x_n : s_n)$ and terms $t_1 \in T_{\Sigma,s_1}(X), \dots, t_n \in T_{\Sigma,s_n}(X)$, we will write $t(t_1, \dots, t_n)$ to denote the term $\sigma(t)$ where σ is a substitution such that $\sigma_{s_1}(x_1) = t_1, \dots, \sigma_{s_n}(x_n) = t_n$. Extending everything to sets, given $T(x_1{:}s_1, \dots, x_n{:}s_n)$ and $U \subseteq T_{\Sigma,s_1}(X) \times \dots \times T_{\Sigma,s_n}(X)$, we will use $T[U] = \bigcup_{\langle t_1, \dots, t_n \rangle \in U} T(t_1, \dots, t_n)$. A *derived operation of type* $s_1 \dots s_n \to s$ over Σ is a term in $T_{\Sigma,s}(x_1 : s_1, \dots, x_n : s_n)$ for some n. We denote by $Der_{\Sigma, s_1 \dots s_n s}$ the set of all derived operations of type $s_1 \dots s_n \to s$ over Σ. A *(general many-sorted) subsignature* of Σ is a many-sorted signature $\Gamma = \langle S, F' \rangle$ such that, for each $w \in S^*$, $F'_w \subseteq Der_{\Sigma,w}$.

From now on we will assume fixed a signature $\Sigma = \langle S, F \rangle$ with a distinguished sort ϕ (the syntactic sort of formulas) and a S-sorted set X of variables. Moreover, for the sake of notation, we will assume that $X_\phi = \{\xi_i \mid i \in \mathbb{N}\}$ and will simply write ξ_k instead of $\xi_k{:}\phi$. Whenever Γ is a subsignature of Σ, we say that Σ is Γ-*standard* if, for every $s \in S$, there exists a ground Γ-term of sort s, that is, a Γ-term of sort s without variables. We define the induced set of *formulas* $L_\Sigma(X)$ to be the carrier set of sort ϕ of the free algebra $\mathbf{T_\Sigma(X)}$ with generators X.

We are now ready to introduce the class of logics that we will study in our many-sorted behavioral approach.

Definition 2. (Many-sorted logic)
A *(structural) many-sorted logic* is a tuple $\mathcal{L} = \langle \Sigma, \vdash \rangle$ where Σ is a many-sorted signature and $\vdash \subseteq \mathcal{P}(L_\Sigma(X)) \times L_\Sigma(X)$, such that $\langle L_\Sigma(X), \vdash \rangle$ is a logic that satisfies, for every $T \cup \{\varphi\} \subseteq L_\Sigma(X)$ and every substitution σ: if $T \vdash \varphi$ then $\sigma[T] \vdash \sigma(\varphi)$ (***structurality***).

An important remark to make here is that propositional-like logics appear as a particular case of many-sorted logics. They can be obtained by taking ϕ to be the only sort, that is, considering a signature $\Sigma = \langle S, F \rangle$ such that $S = \{\phi\}$. So, we conclude that, at least from the point of view of scope, our many-sorted tools generalize the standard tools of algebraization. In fact, as we will see later, for the particular case of propositional logics, our non-behavioral definitions and results coincide with the standard ones.

2.2. Many-sorted behavioral logic

Recall that our aim is to build a framework that is general enough to capture not only many-sorted logics but also other non-algebraizable logics, such as non-truth-functional logics. As we said before, behavioral reasoning in many-sorted equational logic will play a key role. It is not our intention to present the theory of many-sorted behavioral reasoning in full detail, but rather to focus on the definitions and tools from behavioral logic that are necessary for our exposition. Further details on this subject can be found, for example, in [36].

We will use $t_1 \approx t_2$ to represent an equation $\langle t_1, t_2 \rangle$ between terms $t_1, t_2 \in T_{\Sigma,s}(X)$ of the same sort (we use the symbol $\approx$ to avoid confusion with the usual symbol $=$ for (metalevel) equality, as used in the definitions). In this case we say that $t_1 \approx t_2$ is an s-equation. The S-sorted set of all Σ-equations will be written as Eq_Σ. We will denote quasi-equations by $(t_1 \approx u_1)\ \&\ldots\&\ (t_n \approx u_n) \rightarrow (t \approx u)$. A set Θ of equations with variables in $\{x_1 : s_1, \ldots, x_n : s_n\}$ will be dubbed $\Theta(x_1 : s_1, \ldots, x_n : s_n)$. Given a set Θ of equations and $s \in S$, we denote by Θ_s the set of all s-equations in Θ. The main distinction between many-sorted equational logic and many-sorted behavioral logic is that in the latter the set of sorts is explicitly split in two: the *visible* sorts and the *hidden* sorts. More precisely, a *hidden many-sorted signature* is a tuple $\langle \langle S, F \rangle, V \rangle$ where $\langle S, F \rangle$ is a many-sorted signature and $V \subseteq S$ is the set of visible sorts. The subset $H = S \setminus V$ will be dubbed the set of hidden sorts. When there is no risk of confusion we will denote a hidden many-sorted signature $\langle \Sigma, V \rangle$ by Σ. A *hidden subsignature* of a hidden signature $\langle \Sigma, V \rangle$ is a hidden signature $\langle \Gamma, V \rangle$ such that Γ is a many-sorted subsignature of Σ. In the remainder of this section we will consider fixed a hidden signature $\langle \Sigma, V \rangle$.

Let us now focus on the fundamental notion of *context*. Given the intuitive nature of visible and hidden sorts, the role of experiments is to evaluate the hidden data. We have argued that, in some cases, not all operations can be used to build the experiments. This leads to following definition.

DEFINITION 3. (Γ-context)
Given a hidden subsignature Γ of Σ, a Γ-*context for sort* s is a term $t(x{:}s, x_1{:}s_1, \ldots, x_m{:}s_m) \in T_\Gamma(X)$, with a distinguished variable x of sort s and parametric variables $x_1, \ldots, x_m$ of sorts $s_1, \ldots, s_m$ respectively. The set of all Γ-contexts for sort s will be denoted by $\mathcal{C}^\Gamma_\Sigma[x{:}s]$ (note that $x \in \mathcal{C}^\Gamma_\Sigma[x{:}s]$).

The Γ-contexts whose sort is visible will be dubbed Γ-*experiments*. The set of Γ-experiments for sort $s \in S$ will be denoted by $\mathcal{E}^\Gamma_\Sigma[x{:}s]$. When it is important to refer the sort of the contexts or experiments then we will follow the notation as used for terms: $\mathcal{C}^\Gamma_{\Sigma,s'}[x : s]$ denotes the set of Γ-contexts of sort s' for sort s, while $\mathcal{E}^\Gamma_{\Sigma,s'}[x{:}s]$ denotes the set of Γ-experiments of sort s' for sort s. When Γ is clear from the context we just write context instead of Γ-context. Given $c \in \mathcal{C}^\Gamma_{\Sigma,s'}[x : s]$ and $t \in T_{\Sigma,s}(X)$, we denote by $c[t]$ the term obtained from c by replacing every occurrence of x by t. Note that the interesting contexts and experiments are those for hidden sorts, that is, those with $s \in H$. Contexts of visible sort are allowed more for the sake of symmetry, to make the presentation smoother.

Given a Σ-algebra $\mathbf{A}$, a term $t(x_1 : s_1, \ldots, x_n : s_n)$ and $\langle a_1, \ldots, a_n \rangle \in A_{s_1} \times \ldots \times A_{s_n}$, then we denote by $t_{\mathbf{A}}(a_1, \ldots, a_n)$ the value that t takes in $\mathbf{A}$ when the variables $x_1, \ldots, x_n$ are interpreted by $a_1, \ldots, a_n$, respectively. More algebraically, $t_{\mathbf{A}}(a_1, \ldots, a_n) = h(t)$, where $h \in Hom(\mathbf{T_\Sigma(X)}, \mathbf{A})$ is any assignment such that $h_{s_i}(x_i) = a_i$ for all $i \leq n$.

DEFINITION 4. (Γ-behavioral equivalence)
Assume given a Σ-algebra $\mathbf{A}$, a hidden subsignature Γ of Σ and a sort $s \in S$. Then $a, b \in A_s$ *are* Γ-*behaviorally equivalent*, in symbols $a \equiv_\Gamma b$, if for every $\epsilon(x{:}s, x_1{:}s_1, \ldots, x_n{:}s_n) \in \mathcal{E}^\Gamma_\Sigma[x{:}s]$ and for all $\langle a_1, \ldots, a_n \rangle \in A_{s_1} \times \ldots \times A_{s_n}$, we have that $\epsilon_{\mathbf{A}}(a, a_1, \ldots, a_n) = \epsilon_{\mathbf{A}}(b, a_1, \ldots, a_n)$.

Now that we have defined behavioral equivalence, we can talk about behavioral satisfaction of an equation by a Σ-algebra. Let $\mathbf{A}$ be a Σ-algebra, h an assignment over $\mathbf{A}$ and $t_1 \approx t_2$ an equation of sort $s \in S$. We say that $\mathbf{A}$ and h Γ-*behaviorally satisfy* the equation $t_1 \approx t_2$, in symbols $\mathbf{A}, h \mid\mid\!\!\vdash t_1 \approx t_2$ if $h(t_1) \equiv_\Gamma h(t_2)$. We say that $\mathbf{A}$ behaviorally satisfies $t_1 \approx t_2$, in symbols

$\mathbf{A} \mid\mid\!\vdash t_1 \approx t_2$, if $\mathbf{A}, h \mid\mid\!\vdash t_1 \approx t_2$ for every assignment h over $\mathbf{A}$. Given a class K of Σ-algebras, the *behavioral consequence over* Σ *associated with* K *and* Γ, $\models_{\Sigma}^{K,\Gamma} \subseteq \mathcal{P}(Eq_{\Sigma}) \times Eq_{\Sigma}$, is such that $\Theta \models_{\Sigma}^{K,\Gamma} t_1 \approx t_2$ if for every $\mathbf{A} \in K$ and assignment h over $\mathbf{A}$ we have that $\mathbf{A}, h \mid\mid\!\vdash t_1 \approx t_2$ whenever $\mathbf{A}, h \mid\mid\!\vdash u_1 \approx u_2$ for every $u_1 \approx u_2 \in \Theta$.

Let $\models_{\Sigma}^{K,\Gamma}$ be the behavioral consequence associated with the class K of Σ-algebras as defined above. Let Θ be a set of ϕ-equations with ξ distinguished variable occurring in Θ. Then Θ is said a *compatible set of equations* if the following holds: $\xi_1 \approx \xi_2, \Theta(\xi_1) \models_{\Sigma}^{K,\Gamma} \Theta(\xi_2)$. We will denote by $Comp_{\Sigma}^{K,\Gamma}(Y)$ the set of all compatible sets of equations for the consequence relation $\models_{\Sigma}^{K,\Gamma}$, whose variables are contained in $Y \subseteq X$.

3. Generalizing algebraization

In this section we propose our behavioral version of the standard notion of algebraizable logic. In our approach, the role of unsorted equational logic in the standard theory of algebraization will be played by many-sorted behavioral logic. We will not present here the standard concepts and results of AAL. For that sake, we refer to [3, 11, 19]. Along with our proposal, we will present some necessary conditions for a logic to be behaviorally algebraizable. These will be important to show that our generalized notion is not as broad that it becomes trivial. We then turn our attention towards understanding the classes of algebras that our behavioral theory naturally associates to a given many-sorted logic. Results regarding uniqueness and axiomatization of these classes of algebras are obtained. Some of these results are already set in the direction of developing a full-fledged theory of behavioral tools in AAL, a path which we will further exploit in Section 4. We conclude the section with the analysis of a number of meaningful examples, which will help to illustrate the capabilities of our approach. In particular, we will see that two logics, $\mathcal{C}_1$ and $S5$, that are well known not to be algebraizable according to the standard definition, are in fact behaviorally algebraizable in a meaningful way. We will also study the example of Nelson's logic N, to see that our approach can be useful even for studying logics which are algebraizable in the standard sense.

3.1. Behavioral algebraization

Recall that our aim is to build a framework general enough to capture some logics that fall out of the scope of the standard tools of AAL. With respect to many-sortedness, some work has already been presented in [7]. Our aim here is to go further ahead and to capture also logics that are not algebraizable in the standard sense (although they still seem to be sufficiently well behaved to be studied in algebraic terms). This is the case, for example, of certain *non-truth-functional* logics. The notion of a (non-)truth-functional semantics for a logic as been considered by several authors, see for instance [2]. Still, the notion of a (non-)truth-functional logic is relatively imprecise, and may sometimes be confused with self-extensionality or the Fregean property. We do not wish to dwell here on the exact best meaning of the notion of a (non-)truth-functional logic. For what matters us, we will simply assume that non-truth-functional logics are those logics, algebraizable or not, that can be seen as extending some given algebraizable logic by some new rules and some new connectives that do not satisfy the congruence property with respect to the equivalence of their initial algebraizable fragment. Many-sorted behavioral logic seems to be the correct tool for this enterprise since, besides providing a rich many-sorted framework, it allows the isolation of the fragment of the language that corresponds to the initial algebraizable part of the logic. In its more general form, as introduced for instance in [21], behavioral equivalence is an equivalence relation that is only required to be compatible with respect to the operations in a given subsignature of the original signature.

Consider given a many-sorted language generated from a many-sorted signature $\Sigma = \langle S, F\rangle$. We know that we have a distinguished sort ϕ of formulas. In the many-sorted approach to AAL presented in [7] the theory was developed by replacing the role of unsorted equational logic by many-sorted behavioral logic over the same signature and taking the sort ϕ as the unique visible sort. Despite the success of this generalization to cope with many-sorted logics, a lot of non-algebraizable logics could still not be captured. This is due to the fact that, since the sort ϕ is considered visible, we have equational reasoning about formulas, which forces every connective to be compatible with behavioral congruence. To allow for non-congruent connectives, the sort ϕ must be a hidden sort too, so that one is forced to reason behaviorally about formulas as well. This can be achieved by considering behavioral logic over an extended signature. For that purpose, given a many-sorted signature $\Sigma = \langle S, F\rangle$ we define an extended signature $\Sigma^o = \langle S^o, F^o\rangle$ such that $S^o = S \biguplus \{v\}$, where v is to be considered the sort of *observations* of formulas. The indexed set of operations $F^o = \{{F^o}_{ws}\}_{w\in(S^o)^*,s\in S^o}$ is such that ${F^o}_{ws} = F_{ws}$ if $ws \in S^*$ and $F^o_{\phi v} = \{o\}$ and $F^o_{ws} = \emptyset$ otherwise. Intuitively, we are just extending the signature with a new sort v for the observations that we can perform on formulas using operation o. The extended hidden signature obtained from Σ, that we will also denote by Σ^o, can then be defined as $\langle \Sigma^o, \{v\}\rangle$. The choice of v as the name for the new sort is clear. It is intended to represent the only visible sort of the extended hidden signature.

In what follows, given a signature $\Sigma = \langle S, F\rangle$, a subsignature Γ of Σ and a class K of Σ^o-algebras, we will use $Bhv_\Sigma^{K,\Gamma}$ to refer to the logic $\langle Eq_{\Sigma^o}, \models_\Sigma^{K,\Gamma}\rangle$, where $\models_\Sigma^{K,\Gamma}$ is the behavioral consequence relation over Σ^o associated with K and Γ. Note that in this case, for each $s \in S$, $\mathcal{E}_\Sigma^\Gamma[x:s] = \{o(c) : c \in \mathcal{C}_{\Sigma,\phi}^\Gamma[x:s]\}$ is the set of possible experiments of sort s. From $Bhv_\Sigma^{K,\Gamma}$ we can define a logic $BEqn_\Sigma^{K,\Gamma} = \langle Eq_\Sigma, \vDash_{\Sigma,bhv}^{K,\Gamma}\rangle$ where $\vDash_{\Sigma,bhv}^{K,\Gamma}$ is just the restriction of $\models_\Sigma^{K,\Gamma}$ to Σ. The set of theories of $BEqn_\Sigma^{K,\Gamma}$ will be denoted by $Th_\Sigma^{K,\Gamma}$. We will use often a property of this behavioral consequence: given $t_1 \approx t_2 \in Eq_{\Sigma,s}(X)$ and $c \in \mathcal{C}_\Sigma^\Gamma[x:s]$, we have that $t_1 \approx t_2 \vDash_{\Sigma,bhv}^{K,\Gamma} c[t_1] \approx c[t_2]$. With this construction we obtain an important ingredient of our approach: a logic for reasoning behaviorally about equations over the original signature Σ.

First of all, it is important to note that the choice of $\Gamma = \Sigma$ covers those examples where there is no need to assume any non-congruent connective. As we will show later, if we take $\Gamma = \Sigma$ we get ordinary many-sorted algebraization. For details about many-sorted algebraization we point to [7]. Furthermore, when ϕ is the only sort we get in the realm of the standard tools of AAL. We now introduce the main notion of Γ-behaviorally algebraizable logic.

Definition 5. (Γ-behaviorally algebraizable logic)
Let $\mathcal{L} = \langle \Sigma, \vdash\rangle$ be a many-sorted logic and Γ a subsignature of Σ. Then, $\mathcal{L}$ is Γ-*behaviorally algebraizable* if there exists a class K of Σ^o-algebras, a set $\Theta(\xi) \subseteq Comp_\Sigma^{K,\Gamma}(\{\xi\})$ of ϕ-equations and a set $\Delta(\xi_1, \xi_2) \subseteq T_{\Gamma,\phi}(\{\xi_1, \xi_2\})$ of formulas such that, for every $T \cup \{t\} \subseteq L_\Sigma(X)$ and for every set $\Phi \cup \{t_1 \approx t_2\}$ of ϕ-equations,

i) $T \vdash t$ iff $\Theta[T] \vDash_{\Sigma,bhv}^{K,\Gamma} \Theta(t)$;

ii) $\Phi \vDash_{\Sigma,bhv}^{K,\Gamma} t_1 \approx t_2$ iff $\Delta[\Phi] \vdash \Delta(t_1, t_2)$;

iii) $\xi \dashv\vdash \Delta[\Theta(\xi)]$;

iv) $\xi_1 \approx \xi_2 \dashv\vDash_{\Sigma,bhv}^{K,\Gamma} \Theta[\Delta(\xi_1, \xi_2)]$;

As expected, conditions i) and iv) jointly imply ii) and iii), and vice-versa. Following the usual terminology and notation of AAL, Θ will be called the set of Γ-*defining equations*, Δ the set of Γ-*equivalence formulas*, and K a Γ-*behaviorally equivalent algebraic semantics* for $\mathcal{L}$. If the set of defining equations and of equivalence formulas are finite we will say that $\mathcal{L}$ is *finitely* Γ*-behaviorally algebraizable.*

Note that these definitions are all parameterized by the choice of a subsignature Γ of Σ. A many-sorted logic $\mathcal{L}$ is said to be behaviorally algebraizable if there exists a subsignature Γ of Σ such that $\mathcal{L}$ is Γ-behaviorally algebraizable.

Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a Γ-behaviorally algebraizable logic and let $\Delta(\xi_1, \xi_2)$ be its set of equivalence formulas. We define the set $CC_\Delta[x:\phi] \subseteq \mathcal{C}_\Sigma^\Sigma[x:\phi]$ by $c \in CC_\Delta[x:\phi]$ iff for every $\varphi, \psi \in L_\Sigma(X)$, we have that $\Delta(\varphi, \psi) \vdash \Delta(c[\varphi], c[\psi])$, and call it the *set of congruent contexts for* Δ.

PROPOSITION 6. *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic and Γ a subsignature of Σ. Suppose that $\mathcal{L}$ is Γ-behaviorally algebraizable logic with $\Theta(\xi)$ a set of defining equations, $\Delta(\xi_1, \xi_2)$ a set of equivalence formulas and K a Γ-behaviorally equivalent algebraic semantics. Then $\mathcal{C}_{\Gamma,\phi}[x:\phi] \subseteq CC_\Delta[x:\phi]$. Moreover, every $c \in CC_\Delta[x:\phi]$ is congruent with respect to $\vDash_{\Sigma,bhv}^{K,\Gamma}$.*

PROOF. As we already remarked, for every $c \in \mathcal{C}_{\Gamma,\phi}[x:\phi]$, we have that $\xi_1 \approx \xi_2 \vDash_{\Sigma,bhv}^{K,\Gamma} c[\xi_1] \approx c[\xi_2]$. By the properties of the equivalence set we have that $\Delta(\xi_1, \xi_2) \vdash \Delta(c[\xi_1], c[\xi_2])$. So, $\mathcal{C}_{\Gamma,\phi}[x:\phi] \subseteq CC_\Delta[x:\phi]$.

Now let $c \in CC_\Delta[x:\phi]$. So, we have $\Delta(\xi_1, \xi_2) \vdash \Delta(c[\xi_1], c[\xi_2])$. Using properties *i)* and *iv)* of the set of defining equations we can conclude that $\xi_1 \approx \xi_2 \vDash_{\Sigma,bhv}^{K,\Gamma} c[\xi_1] \approx c[\xi_2]$. So, c is congruent with respect to $\vDash_{\Sigma,bhv}^{K,\Gamma}$. ■

It is well know for behavioral logic [36] that, when a context is behaviorally congruent, we can always add it to the set of admissible contexts without changing the behavioral consequence. Hence, although we can have $\mathcal{C}_{\Gamma,\phi} \subset CC_\Delta$, the behavioral consequence is the same as if we had chosen the whole CC_Δ as the set of contexts.

It is natural to ask what are the limits of this new definition. We will see later that this notion extends the standard one. Still, this is at least as important as knowing whether the notion is so broad that everything becomes behaviorally algebraizable with an appropriate choice of Γ. We end this section by studying some necessary conditions for a logic to be behaviorally algebraizable. This will help us to show that the limits of applicability of the notion are very reasonable but not as broad as it might seem. In [32] Prucnal and Wrónski introduced the standard notion of equivalential logic. Equivalence systems generalize the well known phenomenon of classical propositional calculus where the equivalence of formulas can be expressed by the equivalence symbol $\leftrightarrow$. We extend this notion to the behavioral setting.

Definition 7. (Γ-behaviorally equivalential logic)
Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic and Γ a subsignature of Σ. Then, $\mathcal{L}$ is Γ-behaviorally equivalential if there exists a set $\Delta(\xi_1, \xi_2) \subseteq T_{\Gamma,\phi}(\{\xi_1, \xi_2\})$ of formulas such that for every $\varphi, \psi, \delta, \varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_n \in L_\Sigma(X)$:

(R) $\vdash \Delta(\varphi, \varphi)$;

(S) $\Delta(\varphi, \psi) \vdash \Delta(\psi, \varphi)$;

(T) $\Delta(\varphi, \psi), \Delta(\psi, \delta) \vdash \Delta(\varphi, \delta)$;

(MP) $\Delta(\varphi, \psi), \varphi \vdash \psi$;

(RP$_\Gamma$) $\Delta(\varphi_1, \psi_1), \ldots, \Delta(\varphi_n, \psi_n) \vdash \Delta(c[\varphi_1, \ldots, \varphi_n], c[\psi_1, \ldots, \psi_n])$ for every $c : \phi^n \to \phi \in Der_{\Gamma,\phi}$.

In this case, Δ is called a Γ-*behavioral equivalence set* for $\mathcal{L}$. Note that the main difference between this behavioral version of equivalentiality and the standard notion is that in the former the set Δ is no longer assumed to define a congruence, that is, an equivalence relation that is compatible with all operations. Instead, it is only assumed to preserve the operations of the subsignature Γ. Again, a many-sorted logic $\mathcal{L}$ is said to be behaviorally equivalential if there exists a subsignature Γ of Σ such that $\mathcal{L}$ is Γ-behaviorally equivalential.

In the following proposition we present the first necessary condition for behavioral algebraization. The result extends a well known standard result of AAL.

Proposition 8. *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic and Γ a subsignature of Σ. If $\mathcal{L}$ is Γ-behaviorally algebraizable then it is Γ-behaviorally equivalential.*

Proof. Suppose $\mathcal{L}$ is behaviorally algebraizable with $\Theta(\xi)$, $\Delta(\xi_1, \xi_2)$, Γ and K. Using the properties of $\Theta(\xi)$ and $\Delta(\xi_1, \xi_2)$ it is easy to prove that $\Delta(\xi_1, \xi_2)$ satisfies (R), (S) and (T). For (MP), note that, since $\mathcal{L}$ is algebraizable, $\Delta(\varphi, \psi), \varphi \vdash \psi$ is equivalent to $\varphi \approx \psi, \Theta(\varphi) \vDash^{K,\Gamma}_{\Sigma,bhv} \Theta(\psi)$. But the last condition follows from the fact that $\Theta(\xi) \in Comp^{K,\Gamma}_\Sigma(\{\xi\})$. Condition (RP$_\Gamma$) follows easily from the the fact that, given $t_1 \approx t_2 \in Eq_{\Sigma,s}(X)$ and $c \in \mathcal{C}^\Gamma_\Sigma[x : s]$, we have that $t_1 \approx t_2 \vDash^{K,\Gamma}_{\Sigma,bhv} c[t_1] \approx c[t_2]$. ■

From the notion of behaviorally equivalential logic we can isolate a much simpler necessary condition for behavioral algebraization. A many-sorted logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ *has an equivalence set (of formulas)* if there exists a set

$\Delta \subseteq L_\Sigma(\xi_1, \xi_2)$ of formulas that satisfies conditions (R), (S), (T) and (MP). We have that, by Proposition 8, this is a necessary condition for a logic to be behaviorally algebraizable. In some sense, to be behaviorally algebraizable, a logic must be expressive enough to enable the definition of an equivalence by means of a set of formulas in two variables. This is a natural requirement since a logic that does not have an equivalence set cannot represent within itself any kind of behavioral equivalence. So, it must fail to be behaviorally algebraizable. One such example is the inf-sup fragment of classical propositional logic, where no equivalence set can be defined. This logic is a well known example of a non-protoalgebraic logic.

We can give another necessary condition for a logic to be behaviorally algebraizable. Although it is a weaker condition, it is an important one since it is related precisely to protoalgebraicity. Recall that a characterization of the standard notion of protoalgebraic logic can be given by the existence of a set $\Delta(\xi_1, \xi_2, \underline{z}) \subseteq L_\Sigma(X)$ of formulas with two distinguished variables of sort ϕ and possibly parametric variables $\underline{z}$ satisfying $\vdash \Delta(\xi, \xi, \underline{z})$ (reflexivity) and $\xi_1, \Delta(\xi_1, \xi_2, \underline{z}) \vdash \xi_2$ (detachment). Below, we use the extension of the standard definition of protoalgebraic logic to the many-sorted setting introduced in [26].

Corollary 9. *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic. If $\mathcal{L}$ is behaviorally algebraizable then it is also protoalgebraic in the standard sense.*

We can then conclude that our generalized notion of algebraizable logic is not too broad since a behaviorally algebraizable logic necessarily belongs to what is considered the largest class of logics amenable to the tools of AAL.

3.2. Behaviorally equivalent algebraic semantics

We can now study the classes of algebras that are going to be canonically associated with a given behaviorally algebraizable logic. Issues like uniqueness and axiomatization of the algebraic counterpart are discussed. We end the section by showing that under suitable conditions it is possible to define operations in the new sort v that can be seen to represent the congruent operations of sort ϕ, thus promoting to some extent the behavioral reasoning to plain-old equational reasoning on the visible sort.

Recall that according to the usual definitions, a logic is algebraizable, or equivalential, in an essentially unique way. This property derives from the fact that the equivalence set of an algebraizable logic $\mathcal{L}$ represents within $\mathcal{L}$ the relation of equality in the algebraic models of $\mathcal{L}$. The distinctive feature of the equality relation is that it is a congruence relation, that is, an equivalence relation preserved by *all* primitive operations. Hence, it should be clear that we cannot expect this kind of uniqueness to hold in our behavioral framework. In fact, there is no guarantee that a logic cannot be behaviorally algebraizable with different (and possibly non-comparable) equivalence sets, giving rise to different behavioral algebraizations. Since uniqueness may fail, we study the relationship between existing equivalence sets within the same logic.

Consider fixed a many-sorted logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$. Given an equivalence set $\Delta(\xi_1, \xi_2)$, we can define a binary relation C_Δ over $L_\Sigma(X)$ as follows: $\langle \varphi, \psi \rangle \in C_\Delta$ iff $\vdash \Delta(\varphi, \psi)$. Due to the properties in the definition of equivalence set, it is an easy task to verify that C_Δ is indeed an equivalence relation over $\mathbf{T_\Sigma(X)}$. Let $Eqv_\mathcal{L} \subseteq Eqv_{\mathbf{T_\Sigma(X)}}$ be defined by $Eqv_\mathcal{L} = \{C_\Delta : \Delta(\xi_1, \xi_2) \text{ is an equivalence set of } \mathcal{L}\}$. Intuitively, $Eqv_\mathcal{L}$ can be seen as the set of equivalences over $T_\Sigma(X)$ that can be defined by a set of formulas with two variables over the deductive consequence of $\mathcal{L}$. Clearly, inclusion defines a partial order on $Eqv_\mathcal{L}$. We can see that $C_{\Delta_1} \subseteq C_{\Delta_2}$ iff $\Delta_2(\xi_1, \xi_2) \vdash \Delta_1(\xi_1, \xi_2)$ and that, in particular, $C_{\Delta_1} = C_{\Delta_2}$ iff $\Delta_1(\xi_1, \xi_2) \dashv\vdash \Delta_2(\xi_1, \xi_2)$. More than a partially ordered set, we can prove that $\langle Eqv_\mathcal{L}, \subseteq \rangle$ forms a complete lattice. It is an easy task to verify that the infimum of a set $\{C_{\Delta_i} : i \in I\}$ is C_Δ where $\Delta = \bigcup_{i \in I} \Delta_i$.

We have hinted before on why we cannot aim at full uniqueness in terms of the behavioral algebraization process, the main reason being that the algebraization process is parameterized by the choice of the subsignature Γ. Nevertheless, it is interesting to note that once Γ is fixed we can prove a uniqueness result with the same flavor as the standard one proved in [3].

THEOREM 10. *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic and Γ a subsignature of Σ. Suppose that $\mathcal{L}$ is Γ-behaviorally algebraizable logic and let K and K' be two Γ-behaviorally equivalent algebraic semantics for $\mathcal{L}$ such that $\Delta(\xi_1, \xi_2)$ and $\Theta(\xi)$ are equivalence formulas and defining equations for K, and similarly $\Delta'(\xi_1, \xi_2)$ and $\Theta'(\xi)$ for K'. Then we have that*

i) $\vDash^{K,\Gamma}_{\Sigma,bhv} = \vDash^{K',\Gamma}_{\Sigma,bhv}$;

ii) $\Delta(\xi_1, \xi_2) \dashv\vdash \Delta'(\xi_1, \xi_2)$;

iii) $\Theta(\xi) \,=\!\mid\!\vDash^{K,\Gamma}_{\Sigma,bhv} \Theta'(\xi)$.

PROOF. The proof follows as the proof given by Blok and Pigozzi in [3]. ■

Theorem 10 allows us to conclude that, as in the standard case, given a Γ-behaviorally algebraizable logic $\mathcal{L}$ we can consider the largest Γ-behaviorally equivalent algebraic semantics that we will denote by $K^{\Gamma}_{\mathcal{L}}$. However, in our approach, $K^{\Gamma}_{\mathcal{L}}$ is not the class of algebras that should be canonically associated with $\mathcal{L}$. Indeed, as we will see, it is a subclass of $K^{\Gamma}_{\mathcal{L}}$ that will allow us to generalize the standard results of AAL.

Consider now the particular case where a many-sorted logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ is finitary and finitely Γ-behaviorally algebraizable for some subsignature Γ of Σ. An immediate consequence of the above theorem is that, if K and K' are two Γ-behaviorally equivalent algebraic semantics for $\mathcal{L}$, then K and K' must Γ-behaviorally satisfy the same quasi-equations. Generalizing the definition given in [35], we introduce here the notion of Γ-hidden quasivariety. Remember that Σ^o is a hidden many-sorted signature with v its unique visible sort. A class Q of Σ^o-algebras is a Γ-*hidden quasivariety over* Σ^o if it is Γ-behaviorally definable by quasi-equations, in the sense that there exists a set Φ of quasi-equations such that Q contains exactly the Σ^o-algebras that Γ-behaviorally satisfy the quasi-equations of Φ. It is now clear that, if K and K' are two Γ-behaviorally equivalent algebraic semantics for $\mathcal{L}$, then K and K' generate the same Γ-hidden quasivariety and that this Γ-hidden quasivariety is also an Γ-behaviorally equivalent algebraic semantics for $\mathcal{L}$. So, we can talk about *the* equivalent Γ-hidden quasivariety semantics of a finitary and finitely Γ-behaviorally algebraizable logic. It is interesting to note that, similarly to what Blok and Pigozzi [3] did for finitary and finitely algebraizable propositional logics, we can construct a basis for the quasi-identities of the unique equivalent Γ-hidden quasivariety semantics given an axiomatization of $\mathcal{L}$.

THEOREM 11. *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a finitary many-sorted logic presented by a Hilbert-style deductive system composed of a set Ax of axioms and a set Ir of inference rules and consider Γ a subsignature of Σ. Assume that $\mathcal{L}$ is finitely Γ-behaviorally algebraizable with defining equation $\Theta(\xi)$ and equivalence formulas $\Delta(\xi_1, \xi_2)$. Then the unique equivalent Γ-hidden quasivariety semantics for $\mathcal{L}$ is Γ-behaviorally axiomatized by the following identities and quasi-identities:*

- *i)* $\Theta(\varphi)$, *for every theorem* φ *of* $\mathcal{L}$;
- *ii)* $\Theta(\Delta(\xi, \xi))$;
- *iii)* $\Theta(\psi_1) \wedge \ldots \wedge \Theta(\psi_n) \to \Theta(\varphi)$ *for every* $\langle \psi_1, \ldots, \psi_n, \varphi \rangle \in Ir$;
- *iv)* $\Theta(\Delta(\xi_1, \xi_2)) \to \xi_1 \approx \xi_2$.

PROOF. The proof is a straightforward generalization of the standard one [3], changing equational with behavioral satisfiability. ∎

It might seem unnatural that, although the language of a many-sorted logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ is over Σ, we associate to $\mathcal{L}$ a class of algebras over the extended signature Σ^o. This is actually a key point of our approach since it allows us to have behavioral reasoning over the whole Σ and in particular over the formulas. Note that since $\mathcal{L}$ is over Σ we do not have full control over the new sort v, in the sense that, given a Σ^o-algebra $\mathbf{A}$, the set A_v might contain more information than what is needed for defining the consequence $\vDash^{K,\Gamma}_{\Sigma,bhv}$. Thus, in some sense, the largest Γ-behaviorally equivalent algebraic semantics of a Γ-behaviorally algebraizable logic $\mathcal{L}$ contains more algebras than the ones we would like to canonically associate with $\mathcal{L}$. We will see how we can extract from the largest Γ-behaviorally equivalent algebraic semantics the class of algebras we are interested in canonically associating with a logic.

First of all recall that in the construction of an extended signature Σ^o from a many-sorted signature Σ, we just added a new sort v and an operation $o : \phi \to v$. No operation was defined on the new sort v. We will now see that, given a Σ^o-algebra, $\mathbf{A}$ it is possible, under some mild conditions, to define connectives in the visible sort v.

Let $\mathbf{A}$ be a Σ^o-algebra such that $o_{\mathbf{A}}$ is surjective and let $f : \phi^n \to \phi \in Der_{\Sigma,\phi}$. Assume that $\mathbf{A}$ satisfies the quasi-equation

$$o(\xi^1_1) \approx o(\xi^2_1) \& \ldots \& o(\xi^1_n) \approx o(\xi^2_n) \to o(f(\xi^1_1, \ldots, \xi^1_n)) \approx o(f(\xi^2_1, \ldots, \xi^2_n)).$$

This quasi-equation expresses the fact that $f_{\mathbf{A}}$ behaves well with respect to o. Then, we can define a n-ary operation $f^v : v^n \to v$ over $\mathbf{A}$ such that, for every $a_1, \ldots, a_n \in A_\phi$, $f^v_{\mathbf{A}}(o_{\mathbf{A}}(a_1), \ldots, o_{\mathbf{A}}(a_n)) = o_{\mathbf{A}}(f_{\mathbf{A}}(a_1, \ldots, a_n))$. It is easy to see that this operation is well defined since we are assuming that $o_{\mathbf{A}}$ is surjective and $\mathbf{A}$ satisfies the above visible quasi-equation. In this case we will say that $f : \phi^n \to \phi$ is a congruent connective on $\mathbf{A}$.

Let Σ be a many-sorted signature and Γ a subsignature of Σ. Given an Σ^o-algebra $\mathbf{A}$ we can consider the relation $\theta^\Gamma_{\mathbf{A}} = (\equiv_{\Gamma})_\phi$ over A_ϕ. Define the Σ^o-algebra $\mathbf{A}^*$ such that $\mathbf{A}^*_{|\Sigma} = \mathbf{A}_{|\Sigma}$, and $A_v = \{[a]_{\theta^\Gamma_{\mathbf{A}}} : a \in A_\phi\}$ and $o_{\mathbf{A}^*}(a) = [a]_{\theta^\Gamma_{\mathbf{A}}}$. Given a class K of Σ^o-algebras we can do this construction for every algebra in K obtaining $K^* = \{\mathbf{A}^* : \mathbf{A} \in K\}$.

Proposition 12. *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic and Γ a subsignature of Σ. Suppose that $\mathcal{L}$ is Γ-behaviorally algebraizable and K is a Γ-behaviorally equivalent algebraic semantics for $\mathcal{L}$. Then $\vDash^{K,\Gamma}_{\Sigma,bhv} = \vDash^{K^*,\Gamma}_{\Sigma,bhv}$ and, as a consequence, K^* is also a Γ-behaviorally equivalent algebraic semantics.*

Proof. The result follows easily from the observation that, given a Σ^o-algebra $\mathbf{A}$, h an assignment over $\mathbf{A}$ and $t_1, t_2 \in T_{\Sigma,s}(X)$, we have that $\mathbf{A}, h \Vdash t_1 \approx t_2$ iff $\mathbf{A}^*, h \Vdash t_1 \approx t_2$. ■

Now let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic and Γ a subsignature of Σ. Suppose that $\mathcal{L}$ is Γ-behaviorally algebraizable and let $K_{\mathcal{L}}^{\Gamma}$ be the largest Γ-behaviorally equivalent algebraic semantics. We can apply the above $*$-construction to $K_{\mathcal{L}}^{\Gamma}$ and obtain a class $(K_{\mathcal{L}}^{\Gamma})^*$ of Σ^o-algebras. By definition of $K_{\mathcal{L}}^{\Gamma}$ and by Proposition 12 we have that $(K_{\mathcal{L}}^{\Gamma})^*$ is a subclass of $K_{\mathcal{L}}^{\Gamma}$. The class $(K_{\mathcal{L}}^{\Gamma})^*$ is the class of Σ^o-algebras we will canonically associate to $\mathcal{L}$. The following lemma asserts that the connectives of Γ are all congruent for every algebra belonging to $(K_{\mathcal{L}}^{\Gamma})^*$.

LEMMA 13. *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic and Γ a subsignature of Σ. Suppose $\mathcal{L}$ is Γ-behaviorally algebraizable. Then, every operation $f : \phi^n \to \phi \in \Gamma$ is congruent in every member of $(K_{\mathcal{L}}^{\Gamma})^*$.*

PROOF. Recall that, for every $c \in \mathcal{C}_{\Sigma,\phi}^{\Gamma}[\xi]$, we have that $\xi_1 \approx \xi_2 \vdash_{\Sigma,bhv}^{K,\Gamma} c[t_1] \approx c[t_2]$. The result now follows from a easy induction and the fact that, for every $\mathbf{A} \in (K_{\mathcal{L}}^{\Gamma})^*$, h an assignment over $\mathbf{A}$ and $t_1, t_2 \in T_{\Sigma,\phi}(X)$, we have that $\mathbf{A}, h \;|||\!\!- t_1 \approx t_2$ iff $\mathbf{A}, h \Vdash o(t_1) \approx o(t_2)$. ■

The above lemma implies that we can define, for every algebra $\mathbf{A}$ in $(K_{\mathcal{L}}^{\Gamma})^*$ and for every operation $f : \phi^n \to \phi$ in Γ, its visible counterpart $f_{\mathbf{A}}^o : A_v^n \to A_v$ on $\mathbf{A}$. Thus, for every Σ^o-algebra in $(K_{\mathcal{L}}^{\Gamma})^*$, we can consider, without loss of generality, that it is endowed with the operations on the sort v that arise in this fashion from congruent operations on the sort ϕ.

We end this section with a remark. Although the traditional tools of AAL can always associate a class $Alg(\mathcal{L})$ to a given propositional (unsorted) logic $\mathcal{L}$, the relation between $Alg(\mathcal{L})$ and $\mathcal{L}$ is not always strong, nor interesting. Of course, when $\mathcal{L}$ is algebraizable the connection between $Alg(\mathcal{L})$ and $\mathcal{L}$ is very strong. So, when a logic $\mathcal{L}$ is not algebraizable but it is behaviorally algebraizable with respect to a class K of algebras, then K has a stronger connection with $\mathcal{L}$ than $Alg(\mathcal{L})$ has.

Still, we remark that, if a logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ is algebraizable, in the usual way, with respect to a class K of algebras and if, for some subsignature Γ of Σ, $\mathcal{L}$ is also Γ-behaviorally algebraizable with respect to a class K' of algebras, then K and K' have a strong connection. Intuitively, K' can be understood as K seen from a different perspective. Moreover, this different perspective can help to shed some new light on the algebraic counterpart of $\mathcal{L}$. This is precisely the case of Nelson's logic N, as we will see later on in Subsection 3.3.6. We have decided not to include the full general result, due to its length, but also to its simplicity. The proof is in the style of a reflection between the classes K and K' by providing suitable back and forth translations of algebras. Details can be found in [22].

3.3. Examples

We now present some examples to support our approach. In the first example, we show that our behavioral approach can indeed be seen as extending both the standard and the many-sorted approaches to AAL. Then, we just briefly analyze first-order logic FOL, stressing that our approach can be useful to shed light on the essential distinction between terms and formulas. We then exemplify the power of our approach by showing that it can be used, directly, to study the algebraization of k-deductive systems. We also have a look at the logics $\mathcal{C}_1$ and $S5$, whose non-algebraizability in the standard sense is well known. We show that they are both behaviorally algebraizable and provide a meaningful algebraic counterpart to each of them. Finally, we study the example of Nelson's constructive logic N with strong negation and prove that, although the logic is algebraizable according to the standard definition, its behavioral algebraization helps to give an extra insight on the distinctive role Heyting algebras play in the standard algebraic counterpart of N, the class of N-lattices.

3.3.1. Standard algebraization

In this example we prove that a logic algebraizable according to the standard notion is also behaviorally algebraizable. Recall that the standard objects of study of AAL are the structural propositional logics, that correspond in our setting to single-sorted logics.

Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a structural propositional logic. As we already observed, Σ will be considered as a single-sorted signature with ϕ the unique sort. Let $\mathbf{A}$ be a Σ-algebra and consider $\mathbf{A}^2$ a Σ^o-algebra obtained from $\mathbf{A}$ such that $A^2_v = A^2_\phi = A_\phi$ and $o_{\mathbf{A}^2}(a) = a$ for every $a \in A_\phi$ and $f_{\mathbf{A}^2}(a_1, \ldots, a_n) = f_{\mathbf{A}}(a_1, \ldots, a_n)$ for every connective f over Σ and $a_1, \ldots, a_n \in A_\phi$. Intuitively, by taking the visible sort of $\mathbf{A}^2$ to be a copy of $\mathbf{A}$ and $o_{\mathbf{A}^2}$ to be the identity function, we are aiming at a collapsing between behavioral satisfaction in $\mathbf{A}^2$ and equational satisfaction in $\mathbf{A}$. Let K be a class of Σ-algebras and consider the class $K^2 = \{\mathbf{A}^2 : \mathbf{A} \in K\}$ of Σ^o-algebras.

PROPOSITION 14. *Given a propositional signature Σ and a class K of Σ-algebras, then* $\vDash_K = \vDash^{K^2,\Sigma}_{\Sigma,bhv}$.

PROOF. The result follows from the easy to prove fact that, given a propositional signature Σ, a Σ-algebra A, an equation $t_1 \approx t_2$ and an assignment $h : T_{\Sigma^o} \to A^2$, we have that $A, h_{|\Sigma} \Vdash t_1 \approx t_2$ iff $A^2, h \Vvdash^{\Sigma}_{\Sigma} t_1 \approx t_2$. ■

It is now easy to prove that, if $\mathcal{L}$ is algebraizable according to the standard notion, with K an equivalent algebraic semantics, then $\mathcal{L}$ is Σ-behaviorally algebraizable with K^2 a Σ-behaviorally equivalent algebraic semantics.

Many-sorted algebraization, as introduced in [7], is just a particular case of the behavioral approach, again obtained by taking $\Gamma = \Sigma$.

3.3.2. First-order logic

The language of first-order logic (FOL) is naturally two-sorted. There is a clear syntactical distinction between terms and formulas. Having this in mind it should be natural to consider a many-sorted framework to study the algebraizability of first-order logic. We point to [7] for a many-sorted approach to the algebraization of FOL and recall that, as explained in Example 3.3.1, our behavioral approach is a generalization of this many-sorted approach to algebraizability. The idea is simply to use our many-sorted framework to handle first-order logic as a two-sorted logic, with a sort for terms and a sort for formulas. This perspective seems to be much more convenient, and we no longer need to view atomic FOL formulas as propositional variables, as in the standard algebraization of FOL [3]. Working out the example, whose details can be found in [22], we manage to algebraize FOL having as an equivalent algebraic semantics the class of two-sorted cylindric algebras, whose restriction to the sort ϕ is a plain old cylindric algebra, but which corresponds to a regular first-order interpretation structure on the sort of terms. More than showing the details, the objective of this example is to stress the potentiality of our approach in the algebraic treatment of extensions of FOL, namely admitting more sorts and the existence of non-congruent operations.

3.3.3. k-deductive systems

The higher dimensional systems, called k-deductive systems, constitute a natural generalization of deductive systems that encompass several other logical systems, namely equational and inequational logics. They were introduced by Blok and Pigozzi in [4] (see also [13]) to provide a context to deal with logics which are assertional and equational. The algebraic theory of these higher dimensional systems, as in the deductive system setting, is supported by properties of the Leibniz congruence. In this example we show that our approach is general and expressive enough to capture the framework of k-deductive systems directly, as a particular case. Our aim is to prove that a k-deductive system can be seen as a two-sorted logic and, moreover, that if it is algebraizable according to the standard notion then it is also behaviorally algebraizable. Therefore, we just need to work in a many-sorted setting without extending the signature and taking Γ as the whole signature. Example 3.3.1 shows that this is equivalent to working with an extended signature, and moreover we gain in simplicity of notation.

Consider given a propositional signature P. In our many-sorted framework, we show how we can present a k-deductive system over P as a two-sorted logic. From P we can consider the two-sorted signature $\Sigma_P^k = \langle \{t, \phi\}, F\rangle$ such that $F_{t^k\phi} = \{p\}$ (k-formulas), $F_{\phi^n\phi} = \{c : c \in P_n \text{ and } n \in \mathbb{N}\}$ (k-connectives) and $F_{\phi t} = \{p_i : 1 \leq i \leq k\}$ (projections).

Given a k-deductive system $S = \langle P, \vdash_S\rangle$ we can consider a many-sorted logic $\mathcal{L}_S = \langle \Sigma_P^k, \vdash\rangle$ obtained from S in the follow way:

$$\Phi \vdash p(\varphi_1, \dots, \varphi_k) \text{ iff } \{\langle \psi_1, \dots, \psi_k\rangle : p(\psi_1, \dots, \psi_k) \in \Phi\} \vdash_S \langle \varphi_1, \dots, \varphi_k\rangle.$$

Given a P-algebra $\mathbf{A}$ we can consider an induced Σ_P^k-algebra $\mathbf{A}^*$ such that $(A^*)_t = A$, $(A^*)_\phi = A^k$, $p_{\mathbf{A}^*}(a_1, \dots, a_k) = \langle a_1, \dots, a_k\rangle$ and $(p_i)_{\mathbf{A}^*}(\langle a_1, \dots, a_k\rangle) = a_i$, for every $1 \leq i \leq k$.

Now given a class K of P-algebras, we can apply this construction to the algebras of K and obtain the class $K^* = \{\mathbf{A}^* : \mathbf{A} \in K\}$ of Σ_P^k-algebras.

We now show how we can use our framework to reason about the algebraization of a k-deductive system. The representation of the algebraizability of k-deductive systems in terms of our many-sorted framework does not seem, at first sight, straightforward due to the fact that equational consequence is defined over the propositional formulas, in the case of k-deductive systems, while in our approach it is defined over tuples of propositional formulas. The following lemma asserts that, nevertheless, the expressive power is the same in both approaches. We omit the proof since it is an easy exercise.

LEMMA 15. *Let $\mathbf{A}$ be a P-algebra. Then we have that*

$$\mathbf{A}^* \Vdash p(\varphi_1, \dots, \varphi_k) \approx p(\psi_1, \dots, \psi_k) \quad \textit{iff} \quad \mathbf{A} \Vdash \varphi_i \approx \psi_i \textit{ for every } 1 \leq i \leq k.$$

In particular we have $\mathbf{A} \Vdash \varphi \approx \psi$ *iff* $\mathbf{A}^* \Vdash p(\varphi, \dots, \varphi) \approx p(\psi, \dots, \psi)$.

Before we prove the main result we need to fix some notation. First of all, note that a ϕ-equation without ϕ-variables always has the form $p(t_1, \dots, t_k) \approx p(\psi_1, \dots, \psi_k)$. Given a set Φ of ϕ-equations without ϕ-variables, we can consider, for each $1 \leq i \leq k$, the set

$$\Phi_i = \{\varphi_i \approx \psi_i : p(\varphi_1, \dots, \varphi_i, \dots, \varphi_k) \approx p(\psi_1, \dots, \psi_i, \dots, \psi_k) \in \Phi\}.$$

PROPOSITION 16. *A k-deductive system $S = \langle P, \vdash_S\rangle$ is algebraizable with* *equivalent algebraic semantics K iff $\mathcal{L}_S$ is Σ_P^k-behaviorally algebraizable with Σ_P^k-behaviorally equivalent algebraic semantics K^*.*

PROOF. Suppose first that S is algebraizable and let K be an equivalent algebraic semantics. Then, there exists a set $\Theta(x_1 : t, \ldots, x_k : t) = \Lambda_1 \approx \Lambda_2$ of k-equations and a set $\Delta(x_1 : t, x_2 : t)$ of k-formulas such that $T \vdash_S \langle \varphi_1, \ldots, \varphi_k \rangle$ iff $\Theta[T] \vDash_K \Theta(\varphi_1, \ldots, \varphi_k)$ and $\varphi_1 \approx \varphi_2 \dashv\vDash_K \Theta[\Delta(\varphi_1, \varphi_2)]$.

Now take $\Lambda_3(\xi) = \{p(\lambda(p_1(\xi), \ldots, p_k(\xi)), \ldots, \lambda(p_1(\xi), \ldots, p_k(\xi))) : \lambda \in \Lambda_1\}$ and $\Lambda_4(\xi) = \{p(\lambda(p_1(\xi), \ldots, p_k(\xi)), \ldots, \lambda(p_1(\xi), \ldots, p_k(\xi))) : \lambda \in \Lambda_2\}$.

Consider $\Theta^*(\xi) = \Lambda_3(\xi) \approx \Lambda_4(\xi)$ and $\Delta^*(\xi_1, \xi_2) = \Delta(p_1(\xi_1), p_1(\xi_2)) \cup \ldots \cup \Delta(p_k(\xi_1), p_k(\xi_2))$. It is easy to check that $\mathcal{L}_S$ is algebraizable with $\Theta^*(\xi)$, $\Delta^*(\xi_1, \xi_2)$ and K^*.

Suppose now that $\mathcal{L}_S$ is algebraizable with K^* an equivalent algebraic semantics. Then there exists a set $\Theta^*(\xi)$ of ϕ-equations and a set $\Delta^*(\xi_1, \xi_2)$ of formulas such that $T \vdash_S p(\varphi_1, \ldots, \varphi_k)$ iff $\Theta^*[T] \vDash_{K^*} \Theta^*(p(\varphi_1, \ldots, \varphi_k))$ and $p(\varphi_1, \ldots, \varphi_k) \approx p(\psi_1, \ldots, \psi_k) \dashv\vDash_{K^*} \Theta^*[\Delta^*(p(\varphi_1, \ldots, \varphi_k), p(\psi_1, \ldots, \psi_k))]$.

Now take $\Theta(x_1, \ldots, x_k) = \Theta_1^*(p(x_1, \ldots, x_k)) \cup \ldots \cup \Theta_k^*(p(x_1, \ldots, x_k))$ and $\Delta(x_1, x_2) = \Delta^*(p(x_1, \ldots, x_1), p(x_2, \ldots, x_2))$. It is now straightforward to prove that S is algebraizable with K an equivalent algebraic semantics. ■

3.3.4. da Costa's paraconsistent logic $\mathcal{C}_1$

We now analyze the behavioral algebraization of the paraconsistent logic $\mathcal{C}_1$ of da Costa [16, 14]. This is one of the motivating examples of our approach and it was inspired by the work in [6]. It was proved, first by Mortensen [30], and later by Lewin, Mikenberg and Schwarze [25], that $\mathcal{C}_1$ is not algebraizable according to the standard notion. So, we can say that $\mathcal{C}_1$ is an example of a logic whose non-algebraizability is well studied.

Nevertheless, it is rather strange that a relatively well behaved logic fails to have an interesting algebraic counterpart. The class $Alg(\mathcal{C}_1)$ standardly associated with $\mathcal{C}_1$ in AAL is not very interesting. Recall that the only congruence in $L_{\Sigma_{\mathcal{C}_1}}(X)$ compatible with the set of theorems of $\mathcal{C}_1$ is the trivial congruence, as proved by Mortensen [30]. Therefore, no work has been devoted to this class of algebras in the literature.

First of all let us introduce the logic $\mathcal{C}_1 = \langle \Sigma_{\mathcal{C}_1}, \vdash_{\mathcal{C}_1} \rangle$. The single-sorted signature of $\mathcal{C}_1$, $\Sigma_{\mathcal{C}_1} = \langle \{\phi\}, F \rangle$, is such that $F_\phi = \{\mathbf{t}, \mathbf{f}\}$, $F_{\phi\phi} = \{\neg\}$, $F_{\phi\phi\phi} = \{\wedge, \vee, \Rightarrow\}$ and $F_{ws} = \emptyset$ otherwise. We can define an unary derived connective $\sim$ over $\Sigma_{\mathcal{C}_1}$ such that $\sim \xi = (\xi^\circ \wedge (\neg \xi))$, where φ° is just an abbreviation of $\neg(\varphi \wedge (\neg \varphi))$. This derived connective is intended to correspond to classical negation. The fact that we can define classical negation within

$\mathcal{C}_1$ is indeed an essential feature of its forthcoming behavioral algebraization. The consequence relation of $\mathcal{C}_1$ can be defined in a Hilbert-style way from the following axioms:

- $\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_1)$;
- $(\xi_1 \wedge \xi_2) \Rightarrow \xi_1$;
- $(\xi_1 \wedge \xi_2) \Rightarrow \xi_2$;
- $\xi_1 \Rightarrow (\xi_2 \Rightarrow (\xi_1 \wedge \xi_2))$;
- $\xi_1 \Rightarrow (\xi_1 \vee \xi_2)$;
- $\xi_2 \Rightarrow (\xi_1 \vee \xi_2)$;
- $\neg\neg\xi_1 \Rightarrow \xi_1$;
- $\xi_1 \vee \neg\xi_1$;
- $\xi_1^\circ \Rightarrow (\xi_1 \Rightarrow (\neg\xi_1 \Rightarrow \xi_2))$;
- $(\xi_1^\circ \wedge \xi_2^\circ) \Rightarrow (\xi_1 \wedge \xi_2)^\circ$;
- $(\xi_1^\circ \wedge \xi_2^\circ) \Rightarrow (\xi_1 \vee \xi_2)^\circ$;
- $(\xi_1^\circ \wedge \xi_2^\circ) \Rightarrow (\xi_1 \Rightarrow \xi_2)^\circ$;
- $\mathbf{t} \Leftrightarrow (\xi_1 \Rightarrow \xi_1)$;
- $\mathbf{f} \Leftrightarrow (\xi_1^\circ \wedge (\xi_1 \wedge \neg\xi_1))$;
- $(\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_3)) \Rightarrow ((\xi_1 \Rightarrow \xi_2) \Rightarrow (\xi_1 \Rightarrow \xi_3))$;
- $(\xi_1 \Rightarrow \xi_3) \Rightarrow ((\xi_2 \Rightarrow \xi_3) \Rightarrow ((\xi_1 \vee \xi_2) \Rightarrow \xi_3))$;

and the rule of inference:

- $\frac{\xi_1 \quad \xi_1 \Rightarrow \xi_2}{\xi_2}$

Although it is defined as a logic weaker than Classical Propositional Logic (CPL), it happens that the defined connective $\sim$ indeed corresponds to classical negation. Therefore, the fragment $\{\sim, \wedge, \vee, \Rightarrow, \mathbf{t}, \mathbf{f}\}$ corresponds to CPL. So, despite of its innocent aspect, $\mathcal{C}_1$ is a non-truth-functional logic, namely it lacks congruence for its paraconsistent negation connective with respect to the equivalence $\Leftrightarrow$ that algebraizes the CPL fragment. In general, it may happen that $\vdash_{\mathcal{C}_1} (\varphi \Leftrightarrow \psi)$ but $\nvdash_{\mathcal{C}_1} (\neg\varphi \Leftrightarrow \neg\psi)$.

One of the objectives of this example is to behaviorally algebraizable $\mathcal{C}_1$ in order to provide it with a meaningful algebraic counterpart. Moreover, we will see that the class of algebras that we canonically associate with $\mathcal{C}_1$ coincides with a class of algebras that already appeared in the literature. This class is introduced in [6] as an algebraic semantics for $\mathcal{C}_1$, as an outgrowth of related work in the area of combining of logics.

As we have pointed out several times before, behavioral algebraization depends on the choice of the subsignature Γ. Since $\mathcal{C}_1$ can be seen as an extension of CPL by a paraconsistent negation, the key idea of this example is to leave paraconsistent negation out of the chosen subsignature, while still including the classical negation. The signature Γ represents, therefore, the algebraizable fragment of $\mathcal{C}_1$. So, consider the subsignature $\Gamma = \langle \{\phi\}, F^\Gamma \rangle$ of $\Sigma_{\mathcal{C}_1}$ such that $F^\Gamma_{\phi\phi} = \{\sim\}$ and $F^\Gamma_{ws} = F_{ws}$ for every $ws \neq \phi\phi$. Note that, since paraconsistent negation $\neg$ is used in the definition of classical negation $\sim$, the subsignature Γ is not just the reduct of $\Sigma_{\mathcal{C}_1}$ obtained by excluding $\neg$.

Let $K_{\mathcal{C}_1}$ be a class of Σ^o-algebras that Γ-behaviorally satisfy the following set of hidden equations:

i) $\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_1) \approx \mathbf{t}$;

ii) $(\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_3)) \Rightarrow ((\xi_1 \Rightarrow \xi_2) \Rightarrow (\xi_1 \Rightarrow \xi_3)) \approx \mathbf{t}$;

iii) $(\xi_1 \wedge \xi_2) \Rightarrow \xi_1 \approx \mathbf{t}$;

iv) $(\xi_1 \wedge \xi_2) \Rightarrow \xi_2 \approx \mathbf{t}$;

v) $\xi_1 \Rightarrow (\xi_2 \Rightarrow (\xi_1 \wedge \xi_2)) \approx \mathbf{t}$;

vi) $\xi_1 \Rightarrow (\xi_1 \vee \xi_2) \approx \mathbf{t}$;

vii) $\xi_2 \Rightarrow (\xi_1 \vee \xi_2) \approx \mathbf{t}$;

viii) $(\xi_1 \Rightarrow \xi_3) \Rightarrow ((\xi_2 \Rightarrow \xi_3) \Rightarrow ((\xi_1 \vee \xi_2) \Rightarrow \xi_3)) \approx \mathbf{t}$;

ix) $\neg\neg\xi_1 \Rightarrow \xi_1 \approx \mathbf{t}$;

x) $\xi_1 \vee \neg\xi_1 \approx \mathbf{t}$;

xi) $(\sim \xi_1) \Rightarrow (\neg\xi_1) \approx \mathbf{t}$;

xii) $\xi_1^\circ \wedge (\xi_1 \wedge \neg\xi_1) \approx \mathbf{f}$;

xiii) $(\xi_1^\circ \wedge \xi_2^\circ) \Rightarrow (\xi_1 \wedge \xi_2)^\circ \approx \mathbf{t}$;

xiv) $(\xi_1^\circ \wedge \xi_2^\circ) \Rightarrow (\xi_1 \vee \xi_2)^\circ \approx \mathbf{t}$;

xv) $(\xi_1^\circ \wedge \xi_2^\circ) \Rightarrow (\xi_1 \Rightarrow \xi_2)^\circ \approx \mathbf{t}$;

and Γ-behaviorally satisfies the following hidden quasi-equations:

i) $(\xi_1 \approx \mathbf{t}) \;\; \& \;\; ((\xi_1 \Rightarrow \xi_2) \approx \mathbf{t}) \;\; \rightarrow \;\; (\xi_2 \approx \mathbf{t})$;

ii) $((\xi_1 \Rightarrow \xi_2) \approx \mathbf{t}) \;\; \& \;\; ((\xi_2 \Rightarrow \xi_1) \approx \mathbf{t}) \;\; \rightarrow \;\; (\xi_1 \approx \xi_2)$.

We are interested here in the class $K^*_{\mathcal{C}_1} = \{\mathbf{A}^* : \mathbf{A} \in K_{\mathcal{C}_1}\}$. Note that, by Lemma 13, we have that $K^*_{\mathcal{C}_1}$ satisfies the following visible quasi-equations:

i) $(o(\xi_1) \approx o(\xi_2)) \rightarrow (o(\sim \xi_1) \approx o(\sim \xi_2))$;

ii) $(o(\xi_1) \approx o(\xi_2)) \& (o(\xi_3) \approx o(\xi_4)) \rightarrow (o(\xi_1 \vee \xi_3) \approx o(\xi_2 \vee \xi_4))$;

iii) $(o(\xi_1) \approx o(\xi_2)) \& (o(\xi_3) \approx o(\xi_4)) \rightarrow (o(\xi_1 \wedge \xi_3) \approx o(\xi_2 \wedge \xi_4))$;

iv) $(o(\xi_1) \approx o(\xi_2)) \& (o(\xi_3) \approx o(\xi_4)) \rightarrow (o(\xi_1 \Rightarrow \xi_3) \approx o(\xi_2 \Rightarrow \xi_4))$.

Since $K^*_{\mathcal{C}_1}$ satisfies the above quasi-equations (i)-(iv), we can define $\sim^v$: $v \to v$, $\vee^v : vv \to v$, $\wedge^v : vv \to v$ and $\Rightarrow^v : vv \to v$ over every member $\mathbf{A} \in K^*_{\mathcal{C}_1}$, respectively as $\sim^v_{\mathbf{A}} (o_{\mathbf{A}}(a)) = o_{\mathbf{A}}(\sim_{\mathbf{A}} a)$, $o_{\mathbf{A}}(a) \vee^v o_{\mathbf{A}}(b) = o_{\mathbf{A}}(a \vee_{\mathbf{A}} b)$, $o_{\mathbf{A}}(a) \wedge^v o_{\mathbf{A}}(b) = o_{\mathbf{A}}(a \wedge_{\mathbf{A}} b)$ and $o_{\mathbf{A}}(a) \Rightarrow^v o_{\mathbf{A}}(b) = o_{\mathbf{A}}(a \Rightarrow_{\mathbf{A}} b)$. For ease of notation consider the following abbreviations: $o(\mathbf{f}) = \bot, o(\mathbf{t}) = \top, \sim^v = -, \wedge^v = \sqcap, \vee^v = \sqcup$ and $\Rightarrow^v = \sqsupset$.

Due to the careful choice of the subsignature Γ, and since $K^*_{\mathcal{C}_1}$ satisfies the above quasi-equations (i)-(iv), we can obtain the following useful lemma.

Lemma 17. *Given $\mathbf{A} \in K^*_{\mathcal{C}_1}$, an equation $\varphi \approx \psi$ and h an assignment then*

$$\mathbf{A}, h \Vdash\!\vdash_\Gamma \varphi \approx \psi \text{ iff } \mathbf{A}, h \Vdash o(\varphi) \approx o(\psi).$$

Proof. The fact that $\mathbf{A}, h \Vdash\!\vdash_\Gamma \varphi \approx \psi$ implies $\mathbf{A}, h \Vdash o(\varphi) \approx o(\psi)$ follows from $\xi \in C^\Gamma_{\Sigma_{\mathcal{C}_1},\phi}[\xi]$. The other direction follows from an easy induction on the structure of contexts, recalling that $\mathbf{A}$ satisfies the quasi-equations $i)$-$iv)$. ■

The class $K^*_{\mathcal{C}_1}$ was proposed in [6] as a possible algebraic counterpart of $\mathcal{C}_1$, but the connection between $\mathcal{C}_1$ and $K^*_{\mathcal{C}_1}$ was never established at the light of the theory of algebraization. In fact, the authors introduced this class of algebras over a richer signature that contained, a priori, the visible connectives $\sqcup, \sqcap, \top, \bot, \sim$ and assumed that the visible part of every algebra in this class is a Boolean algebra. It is interesting to note that, although we define here the class $K^*_{\mathcal{C}_1}$ over a poorest signature, we are able to define the same visible connectives as abbreviations and further prove the following result.

Proposition 18. *For every algebra $\mathbf{A} \in K^*_{\mathcal{C}_1}$, $\langle A_v, \sqcup_{\mathbf{A}}, \sqcap_{\mathbf{A}}, \top_{\mathbf{A}}, \bot_{\mathbf{A}}, -_{\mathbf{A}} \rangle$ is a Boolean algebra.*

Proof. This result is a consequence of Lemma 17, the fact that $\mathcal{C}_1$ satisfies the usual axioms for positive Boolean connectives and the fact that $\sim$ defines classical negation within $\mathcal{C}_1$. ■

We are now in conditions to prove that $\mathcal{C}_1$ is behaviorally algebraizable with respect to the subsignature Γ of $\Sigma_{\mathcal{C}_1}$ introduced above.

Theorem 19. *$\mathcal{C}_1$ is Γ-behaviorally algebraizable, with $K^*_{\mathcal{C}_1}$ a Γ-behaviorally equivalent algebraic semantics with $\Theta(\xi) = \{\xi \approx \mathbf{t}\}$ a set of defining equations and $\Delta(\xi_1, \xi_2) = \{\xi_1 \Rightarrow \xi_2,\ \xi_2 \Rightarrow \xi_1\}$ a set of equivalence formulas.*

Proof. First of all, note that $\Theta(\xi) \subseteq Eq_{\Gamma,\phi}(\xi)$ and $\Delta(\xi_1, \xi_2) \subseteq T_{\Gamma,\phi(\xi_1,\xi_2)}$. Now we have to prove that for every $T \cup \{\varphi\} \subseteq L_\Sigma(X)$:

i) $T \vdash_{\mathcal{C}_1} \varphi$ iff $\{\gamma \approx \mathbf{t} : \gamma \in T\} \models^{K_{\mathcal{C}_1},\Gamma}_{\Sigma,bhv} \varphi \approx \mathbf{t}$;

ii) $\xi_1 \approx \xi_2 \dashv\models^{K_{\mathcal{C}_1},\Gamma}_{\Sigma,bhv} (\xi_1 \equiv \xi_2) \approx \mathbf{t}$;

Recall that in the visible sorts, behavioral logic coincides with equational logic. This fact, together with Lemma 17, guarantees that condition i) above can be equivalently rewritten, for every $T \cup \{\varphi\} \subseteq L_\Sigma(X)$, as:

i') $T \vdash_{\mathcal{C}_1} \varphi$ iff $\{o(\gamma) \approx \top : \gamma \in T\} \models_{K^*_{\mathcal{C}_1}} o(\varphi) \approx \top$.

Note that we have now equational consequence instead of behavioral consequence. The fact that this condition holds was already proved in [6].

Turning our attention to condition ii), and using Lemma 17, all we have to prove is that $o(\xi_1) \approx o(\xi_2) \models_{K_{\mathcal{C}_1}} o(\xi_1 \equiv \xi_2) \approx \top$ and that $o(\xi_1 \equiv \xi_2) \approx \top \models_{K_{\mathcal{C}_1}} o(\xi_1) \approx o(\xi_2)$. Both conditions follow from the fact that, for every Σ^o-algebra $\mathbf{A} \in K^*_{\mathcal{C}_1}$, $\langle A_v, \sqcup_{\mathbf{A}}, \sqcap_{\mathbf{A}}, \top_{\mathbf{A}}, \bot_{\mathbf{A}}, -_{\mathbf{A}} \rangle$ is a Boolean algebra. ■

3.3.5. Lewis's modal logic $S5$

Various logics have appeared in the literature whose theorems coincide with those of Lewis's original system for $S5$. Here, we study a Carnap style presentation of $S5$ which is well known not to be algebraizable according to the standard definition [3]. Recall that $S5$ can be seen as an extension of CPL with the modality $\Box$. So, although $S5$ is not algebraizable, we can identify an algebraizable fragment of it, CPL. Therefore, using our approach we can build up an algebraic semantics for $S5$ based on Boolean algebras, the algebraic counterpart of CPL.

Let us start by introducing the logic $S5 = \langle \Sigma_{S5}, \vdash_{S5} \rangle$ and then we show that this logic is behaviorally algebraizable. The single-sorted signature $\Sigma_{S5} = \langle \{\phi\}, F \rangle$ is such that $F_{\phi\phi} = \{\neg, \Box\}$, $F_{\phi\phi\phi} = \{\wedge, \vee, \Rightarrow\}$ and $F_{ws} = \emptyset$ otherwise. The possibility modality is obtained as usual as an abbreviation $\Diamond = \neg\Box\neg$.

The consequence relation is obtained, in a Hilbert-style way, from the following axioms:

- $\Box\varphi$ for every φ classical tautology;
- $\Box\xi \Rightarrow \xi$;
- $\Box(\Box(\xi_1 \Rightarrow \xi_2) \Rightarrow (\Box\xi_1 \Rightarrow \Box\xi_2))$;
- $\Box(\Box\xi \Rightarrow \xi)$;
- $\Box(\Diamond\xi \Rightarrow \Box\Diamond\xi)$;

and the inference rule:

- $\dfrac{\xi_1 \quad \xi_1 \Rightarrow \xi_2}{\xi_2}$.

Consider now the subsignature $\Gamma = \langle\{\phi\}, F^\Gamma\rangle$ of Σ_{S5} such that $F^\Gamma_{\phi\phi} = \{\neg\}$ and $F^\Gamma_{ws} = F_{ws}$ for every $ws \neq \phi\phi$. Note that $\Box$ is outside of Γ. We can now prove that $S5$ is Γ-behaviorally algebraizable. For the sake of simplicity we use Theorem 40 that will be only presented in section 4.2. This theorem gives a sufficient and easy to check condition for a logic to be Γ-behaviorally algebraizable. It states that to prove that a given logic $\mathcal{L} = \langle\Sigma, \vdash\rangle$ is Γ-behaviorally algebraizable it suffices to show that $\mathcal{L}$ is Γ-behaviorally equivalential and the Γ-behavioral equivalence $\Delta(\xi_1, \xi_2)$ satisfies also the so-called (G)-rule: $\xi_1, \xi_2 \vdash \Delta(\xi_1, \xi_2)$.

Theorem 20. *$S5$ is Γ-behaviorally algebraizable.*

Proof. Let $\Delta(\xi_1, \xi_2) = \{\xi_1 \Rightarrow \xi_2,\ \xi_2 \Rightarrow \xi_1\}$ be a set of formulas. Using well known properties of $S5$ it can be easily proved that Δ is a Γ-behavioral equivalence. The fact that $S5$ satisfies the (G)-rule is also well known. ■

To study the algebraic counterpart our approach associates with $S5$ we will use Theorem 11. This theorem gives an axiomatization of the largest Γ-behaviorally equivalent algebraic semantics. Consider the class K_{S5} of Σ^o_{S5}-algebras that Γ-behaviorally satisfy the following hidden equations:

- $\Box\varphi \approx \mathbf{t}$ for every φ classical tautology;
- $(\Box\xi \Rightarrow \xi) \approx \mathbf{t}$;
- $(\Box(\Box(\xi_1 \Rightarrow \xi_2) \Rightarrow (\Box\xi_1 \Rightarrow \Box\xi_2))) \approx \mathbf{t}$;
- $(\Box(\Box\xi \Rightarrow \xi)) \approx \mathbf{t}$;
- $(\Box(\Diamond\xi \Rightarrow \Box\Diamond\xi)) \approx \mathbf{t}$;

and Γ-behaviorally satisfy the hidden quasi-equations:

- $(\xi_1 \approx \mathbf{t})\ \ \&\ \ ((\xi_1 \Rightarrow \xi_2) \approx \mathbf{t})\ \rightarrow\ (\xi_2 \approx \mathbf{t})$;
- $((\xi_1 \Rightarrow \xi_2) \approx \mathbf{t})\ \ \&\ \ ((\xi_2 \Rightarrow \xi_1) \approx \mathbf{t})\ \rightarrow\ (\xi_1 \approx \xi_2)$.

Where $\mathbf{t}$ is an abbreviation of $(\xi \Rightarrow \xi)$.

We are interested here in the class $K^*_{S5} = \{\mathbf{A}^* : \mathbf{A} \in K_{S5}\}$. Note that, by Lemma 13, we have that K^*_{S5} satisfies the following visible quasi-equations:

i) $(o(\xi_1) \approx o(\xi_2)) \rightarrow (o(\neg\xi_1) \approx o(\neg\xi_2))$;

ii) $(o(\xi_1) \approx o(\xi_2))\&(o(\xi_3) \approx o(\xi_4)) \rightarrow (o(\xi_1 \vee \xi_3) \approx o(\xi_2 \vee \xi_4))$;

iii) $(o(\xi_1) \approx o(\xi_2))\&(o(\xi_3) \approx o(\xi_4)) \rightarrow (o(\xi_1 \wedge \xi_3) \approx o(\xi_2 \wedge \xi_4))$;

iv) $(o(\xi_1) \approx o(\xi_2))\&(o(\xi_3) \approx o(\xi_4)) \rightarrow (o(\xi_1 \Rightarrow \xi_3) \approx o(\xi_2 \Rightarrow \xi_4))$;

Since K^*_{S5} satisfies the above quasi-equations (i)-(iv), we can define $\neg^v : v \to v$, $\vee^v : vv \to v$, $\wedge^v : vv \to v$ and $\Rightarrow^v : vv \to v$ over every member $\mathbf{A} \in K^*_{S5}$, respectively as $\neg^v_{\mathbf{A}}(o_{\mathbf{A}}(a)) = o_{\mathbf{A}}(\neg_{\mathbf{A}} a)$, $o_{\mathbf{A}}(a) \vee^v o_{\mathbf{A}}(b) = o_{\mathbf{A}}(a \vee_{\mathbf{A}} b)$, $o_{\mathbf{A}}(a) \wedge^v o_{\mathbf{A}}(b) = o_{\mathbf{A}}(a \wedge_{\mathbf{A}} b)$ and $o_{\mathbf{A}}(a) \Rightarrow^v o_{\mathbf{A}}(b) = o_{\mathbf{A}}(a \Rightarrow_{\mathbf{A}} b)$. For ease of notation consider the following abbreviations: $o(\mathbf{f}) = \bot, o(\mathbf{t}) = \top, \neg^v = -, \wedge^v = \sqcap, \vee^v = \sqcup$ and $\Rightarrow^v = \sqsupset$.

Due to the careful choice of the subsignature Γ, and since K^*_{S5} satisfies the above quasi-equations (i)-(iv), we can obtain the following useful lemma.

LEMMA 21. *Given $\mathbf{A} \in K^*_{S5}$, an equation $\varphi \approx \psi$ and h an assignment then*

$$\mathbf{A}, h \Vdash\!\vdash_{\Gamma} \varphi \approx \psi \textit{ iff } \mathbf{A}, h \Vdash o(\varphi) \approx o(\psi).$$

PROOF. The fact that $\mathbf{A}, h \Vdash\!\vdash_{\Gamma} \varphi \approx \psi$ implies $\mathbf{A}, h \Vdash o(\varphi) \approx o(\psi)$ follows from $\xi \in C^{\Gamma}_{\Sigma_{\mathcal{C}_1},\phi}[\xi]$. The other direction follows from an easy induction on the structure of contexts, recalling that $\mathbf{A}$ satisfies the quasi-equations $i)$-$iv)$. ∎

PROPOSITION 22. *If $\mathbf{A} \in K^*_{S5}$ then $\langle A_v, \sqcup_{\mathbf{A}}, \sqcap_{\mathbf{A}}, \top_{\mathbf{A}}, \bot_{\mathbf{A}}, -_{\mathbf{A}} \rangle$ is a Boolean algebra.*

PROOF. This result is a consequence of Lemma 21 and the fact that $S5$ satisfies the usual axioms for Boolean connectives. ∎

Note that the behaviorally equivalent algebraic semantics we have proposed for $S5$ much resembles the standard class of Boolean algebras with operations, the usual equivalent algebraic semantics of normal modal logics.

3.3.6. Constructive logic with strong negation

Constructive logic N with strong negation was formulated by Nelson [31] in order to overcome some non-constructive properties of intuitionistic negation. The main criticism to intuitionistic negation is the fact that in Intuitionistic Propositional Logic (IPL), from the derivability of $\neg(\varphi \wedge \psi)$, it does not follow that at least one of the formulas $\neg\varphi$ or $\neg\psi$ is derivable in IPL. Thus, in order to obtain a constructive logic with this property, IPL was extended with an unary connective for strong negation satisfying the desired property. We closely follow the notation of Kracht [24] and denote by N the constructive logic with strong negation. It is well known that N is algebraizable according to the standard notion and that its equivalent algebraic semantics is the class of so-called N-lattices [43, 38] (also known as Nelson algebras [41] or quasi-pseudo-Boolean algebras [33]). The variety of N-lattices has been extensively studied [33, 43, 38, 24]. One important result is the characterization of N-lattices through Heyting algebras.

Herein, our goal is to show that our framework can be useful even when applied to logics that are algebraizable in the standard sense. The change of perspective can help to provide a better insight on the algebraic counterpart of a logic. In more concrete terms, we show that N can be behaviorally algebraized by choosing a subsignature Γ of the original signature. This subsignature is obtained by excluding strong negation from the original signature, thus maintaining just the intuitionistic connectives. We then study the behavioral algebraic counterpart of N and show that the characterization of N-lattices through Heyting algebras emerges explicitly, thus reinforcing the central role of Heyting algebras in the algebraic counterpart of N.

We start by presenting the language of N. It is obtained from a single-sorted signature $\Sigma_N = \langle S, F\rangle$ such that $S = \{\phi\}$, $F_{\epsilon\phi} = \emptyset$, $F_{\phi\phi} = \{\neg, \sim\}$, $F_{\phi^2\phi} = \{\rightarrow, \vee, \wedge\}$ and $F_{\phi^n\phi} = \emptyset$, for all $n > 2$. As usual, we can define $\bot = (\varphi \wedge (\neg\varphi))$ and $\top = (\varphi \rightarrow \varphi)$, where $\varphi \in L_{\Sigma_M}(X)$ is some fixed but arbitrary formula. The connective $\sim$ is intended to represent strong negation and the remainder connectives are intended to represent the usual intuitionistic connectives. We can define the intuitionistic equivalence as usual as $\xi_1 \leftrightarrow \xi_2 = (\xi_1 \rightarrow \xi_2) \wedge (\xi_2 \rightarrow \xi_1)$ and we can also define a *strong implication* $(\xi_1 \Rightarrow \xi_2) = (\xi_1 \rightarrow \xi_2) \wedge (\sim \xi_2 \rightarrow \sim \xi_1)$. The structural single-sorted deductive system of N consists of the following axioms:

i) $\xi_1 \rightarrow (\xi_2 \rightarrow \xi_1)$;

ii) $(\xi_1 \wedge \xi_2) \rightarrow \xi_1$;

iii) $(\xi_1 \wedge \xi_2) \rightarrow \xi_2$;

iv) $\xi_1 \rightarrow (\xi_2 \rightarrow (\xi_1 \wedge \xi_2))$;

v) $\xi_1 \rightarrow (\xi_1 \vee \xi_2)$;

vi) $\xi_2 \rightarrow (\xi_1 \vee \xi_2)$;

vii) $\neg\xi_1 \rightarrow (\xi_1 \rightarrow \xi_2)$;

viii) $(\xi_1 \rightarrow (\xi_2 \rightarrow \xi_3)) \rightarrow ((\xi_1 \rightarrow \xi_2) \rightarrow (\xi_1 \rightarrow \xi_3))$;

ix) $(\xi_1 \rightarrow \xi_3) \rightarrow ((\xi_2 \rightarrow \xi_3) \rightarrow ((\xi_1 \vee \xi_2) \rightarrow \xi_3))$;

x) $(\xi_1 \rightarrow \xi_2) \rightarrow ((\xi_1 \rightarrow \neg\xi_2) \rightarrow \neg\xi_1)$;

xi) $\sim (\xi_1 \rightarrow \xi_2) \leftrightarrow (\xi_1 \wedge \sim \xi_2)$;

xii) $\sim (\xi_1 \wedge \xi_2) \leftrightarrow (\sim \xi_1 \vee \sim \xi_2)$;

xiii) $\sim (\xi_1 \vee \xi_2) \leftrightarrow (\sim \xi_1 \wedge \sim \xi_2)$;

xiv) $(\sim \neg\xi_1) \leftrightarrow \xi_1$;

xv) $(\sim\sim \xi_1) \leftrightarrow \xi_1$;

xvi) $(\sim \xi_1 \vee \neg\xi_1) \leftrightarrow \neg\xi_1$;

and the rule

$$(MP) \qquad \frac{\xi_1 \quad \xi_1 \rightarrow \xi_2}{\xi_2}.$$

Note that the axioms *i*) - *x*) are the usual axioms for IPL. Axioms *xi*) - *xvi*) express the relation between strong negation and the other connectives. It is well known that N is algebraizable [33]. However, it is not the intuitionistic equivalence $\leftrightarrow$ that is used as the set of equivalence formulas in the standard algebraization of N. This is mainly due to the fact that $\leftrightarrow$ does not have the congruence property with respect to strong negation. The equivalence used to algebraize N is the strong equivalence $(\xi_1 \Leftrightarrow \xi_2) = (\xi_1 \Rightarrow \xi_2) \wedge (\xi_2 \Rightarrow \xi_1)$.

In what follows we describe the equivalent algebraic semantics of N, the class of *N-lattices*. Let $\mathcal{N}$ be the class of all Σ_N-algebras $\mathbf{A}$ such that:

- $\langle \mathbf{A}, \wedge_{\mathbf{A}}, \vee_{\mathbf{A}}, \top_{\mathbf{A}}, \bot_{\mathbf{A}} \rangle$ is a bounded distributive lattice;

and it also satisfies the following equations:

- $\xi_1 \to (\xi_2 \to \xi_1) \approx \top$;
- $(\xi_1 \to (\xi_2 \to \xi_3)) \to ((\xi_1 \to \xi_2) \to (\xi_1 \to \xi_3)) \approx \top$;
- $(\xi_1 \wedge \xi_2) \to \xi_1 \approx \top$;
- $(\xi_1 \wedge \xi_2) \to \xi_2 \approx \top$;
- $\xi_1 \to (\xi_2 \to (\xi_1 \wedge \xi_2)) \approx \top$;
- $\xi_1 \to (\xi_1 \vee \xi_2) \approx \top$;
- $\xi_2 \to (\xi_1 \vee \xi_2) \approx \top$;
- $(\xi_1 \to \xi_3) \to ((\xi_2 \to \xi_3) \to ((\xi_1 \vee \xi_2) \to \xi_3)) \approx \top$;
- $(\xi_1 \to \xi_2) \to ((\xi_1 \to \neg\xi_2) \to \neg\xi_1) \approx \top$;
- $\neg\xi_1 \to (\xi_1 \to \xi_2) \approx \top$;
- $\sim (\xi_1 \to \xi_2) \leftrightarrow (\xi_1 \wedge \sim \xi_2) \approx \top$;
- $\sim (\xi_1 \wedge \xi_2) \leftrightarrow (\sim \xi_1 \vee \sim \xi_2) \approx \top$;
- $\sim (\xi_1 \vee \xi_2) \leftrightarrow (\sim \xi_1 \wedge \sim \xi_2) \approx \top$;
- $(\sim \neg\xi_1) \leftrightarrow \xi_1 \approx \top$;
- $(\sim\sim \xi_1) \leftrightarrow \xi_1 \approx \top$;
- $(\sim \xi_1 \vee \neg\xi_1) \leftrightarrow \neg\xi_1 \approx \top$.

We briefly recall some important properties of N-lattices, namely with respect to their connection with Heyting algebras. We just present the results that are useful for our study. For the reader interested in a more detailed study of N-lattices we point to [33, 43, 38]. In [43] Vakarelov introduces a construction of N-lattices from Heyting algebras. The algebras obtained by this construction are called *twist algebras*. We now introduce the precise notion of twist algebra and present some interesting results connecting N-lattices and twist algebras.

Let $\Gamma = \langle S, F' \rangle$ be the subsignature of Σ_N such that $F'_{\phi\phi} = \{\neg\}$ and $F'_{ws} = F_{ws}$ for every $ws \in S^*$ such that $ws \neq \phi\phi$. Note that the subsignature Γ is nothing but the intuitionistic reduct of the signature Σ_N. Given a Γ-algebra $\mathbf{A}$, consider the set $A^{\bowtie} = \{\langle a, b\rangle : a, b \in A \text{ and } a \wedge_{\mathbf{A}} b = \bot_{\mathbf{A}}\}$. We can define a Σ-algebra $\mathbf{A}^{\bowtie} = \langle A^{\bowtie}, \wedge_{\mathbf{A}^{\bowtie}}, \vee_{\mathbf{A}^{\bowtie}}, \to_{\mathbf{A}^{\bowtie}}, \neg_{\mathbf{A}^{\bowtie}}, \sim_{\mathbf{A}^{\bowtie}}, \bot_{\mathbf{A}^{\bowtie}}, \top_{\mathbf{A}^{\bowtie}}\rangle$ over the set $A^{\bowtie}$ by defining the operations as follows:

- $\langle a_1, b_1\rangle \wedge_{\mathbf{A}^{\bowtie}} \langle a_2, b_2\rangle = \langle a_1 \wedge_{\mathbf{A}} a_2,\ b_1 \vee_{\mathbf{A}} b_2\rangle$;
- $\langle a_1, b_1\rangle \vee_{\mathbf{A}^{\bowtie}} \langle a_2, b_2\rangle = \langle a_1 \vee_{\mathbf{A}} a_2,\ b_1 \wedge_{\mathbf{A}} b_2\rangle$;
- $\langle a_1, b_1\rangle \to_{\mathbf{A}^{\bowtie}} \langle a_2, b_2\rangle = \langle a_1 \to_{\mathbf{A}} a_2,\ a_1 \wedge_{\mathbf{A}} b_2\rangle$;
- $\neg_{\mathbf{A}^{\bowtie}}\langle a, b\rangle = \langle \neg_{\mathbf{A}} a,\ a\rangle$;
- $\sim_{\mathbf{A}^{\bowtie}} \langle a, b\rangle = \langle b,\ a\rangle$;
- $\top_{\mathbf{A}^{\bowtie}} = \langle \top_{\mathbf{A}}, \bot_{\mathbf{A}}\rangle$;
- $\bot_{\mathbf{A}^{\bowtie}} = \langle \bot_{\mathbf{A}}, \top_{\mathbf{A}}\rangle$.

The algebra $\mathbf{A}^{\bowtie}$ is called a *full twist algebra over* $\mathbf{A}$. A *twist algebra* is a subalgebra of a full twist algebra. The following Theorem is due to Vakarelov [43].

THEOREM 23. *If* $\mathbf{A}$ *is a Heyting algebra then* $\mathbf{A}^{\bowtie}$ *is a* N*-lattice.*

Given a N-lattice $\mathbf{A}$ we can consider the equivalence relation $\theta_{\mathbf{A}}$ over $\mathbf{A}$ defined as $\langle a, b\rangle \in \theta_{\mathbf{A}}$ iff $(a \leftrightarrow_{\mathbf{A}} b) = \top_{\mathbf{A}}$. It is well known that this equivalence relation, that corresponds to intuitionistic equivalence in $\mathbf{A}$, is not a congruence relation, in general. This is due to the fact that the congruence condition might fail for strong negation. Despite this fact, $\theta_{\mathbf{A}}$ is compatible with all the intuitionistic operations and is therefore a Γ-congruence. We can then consider the Γ-algebra $\mathbf{A}_{\bowtie} = (\mathbf{A}_{|\Gamma})/_{\theta}$. Sendlewski [39] proves that $\mathbf{A}_{\bowtie}$ is a Heyting algebra and that it is the least Heyting algebra that can be obtained by factorization. It is usually called the *Heyting algebra associated with* $\mathbf{A}$ or the *untwist algebra of* $\mathbf{A}$. For more results concerning the constructions $(.)^{\bowtie}$ and $(.)_{\bowtie}$ we point to [43, 39, 24].

We proceed by studying the Γ-behavioral algebraizability of N. Recall that Γ is the subsignature of Σ_N representing the intuitionistic reduct. Intuitively, we are taking the strong negation out of the original signature, thus keeping just the intuitionistic connectives. Therefore, the intuitionistic equivalence will play a key role in the Γ-behavioral algebraization of N.

Theorem 24. *N is Γ-behaviorally algebraizable.*

Proof. Recall that Theorem 40 gives a sufficient condition for a logic to be Γ-behaviorally algebraizable. In this proof we use $\Delta = \{(\xi_1 \leftrightarrow \xi_2)\}$. The following conditions are all well-known to hold in IPL, and therefore in every axiomatic extension of IPL, which is the case of N.

i) $\vdash_N \delta_1 \Delta \delta_1$;

ii) $\delta_1 \Delta \delta_2 \vdash_N \delta_2 \Delta \delta_1$;

iii) $\delta_1 \Delta \delta_2, \delta_2 \Delta \delta_3 \vdash_N \delta_1 \Delta \delta_3$;

iv) $\delta_1 \Delta \delta_2 \vdash_N (\neg \delta_1) \Delta (\neg \delta_2)$;

v) $\delta_1 \Delta \delta_2, \delta_3 \Delta \delta_4 \vdash_N (\delta_1 \rightarrow \delta_3) \Delta (\delta_2 \rightarrow \delta_4)$;

vi) $\delta_1 \Delta \delta_2, \delta_3 \Delta \delta_4 \vdash_N (\delta_1 \wedge \delta_3) \Delta (\delta_2 \wedge \delta_4)$;

vii) $\delta_1 \Delta \delta_2, \delta_3 \Delta \delta_4 \vdash_N (\delta_1 \vee \delta_3) \Delta (\delta_2 \vee \delta_4)$;

viii) $\delta_1, \delta_1 \Delta \delta_2 \vdash_N \delta_2$;

ix) $\delta_1, \delta_2 \vdash_N \delta_1 \Delta \delta_2$.

Recall that, in this case, the set of defining equations can be defined as $\Theta(\xi) = \{\xi \approx (\xi \leftrightarrow \xi)\}$. ∎

We now describe the Γ-behaviorally equivalent algebraic semantics of N, the class K_N^Γ. Recall that K_N^Γ is a class of algebras over the extended two-sorted signature $\Sigma_N^o = \langle \{\phi, v\}, F^o \rangle$ obtained from Σ_N. This class K_N^Γ can be described using Theorem 11 together with the construction presented at the end of Section 3.2. Although Σ_N^o does not have operations on the sort v, we can define operations that correspond to the operations in Γ, in every algebra of K_N^Γ. In this particular case, we can define the operations $\wedge^o, \vee^o, \rightarrow^o$ $, \neg^o, \top^o, \bot^o$ on the sort v that correspond to the intuitionistic connectives. For the sake of notation we denote them by $\sqcap, \sqcup, \sqsupset, -, 1, 0$ respectively.
The class K_N^Γ is constituted by all Σ_N^o-algebras $\mathbf{B}$ such that:

$\langle B_v, \sqcap_{\mathbf{B}}, \sqcup_{\mathbf{B}}, \sqsupset_{\mathbf{B}}, -_{\mathbf{B}}, 1_{\mathbf{B}}, 0_{\mathbf{B}} \rangle$ is a Heyting algebra

and $\mathbf{B}$ Γ-behaviorally satisfies the following axioms:

1) $\sim (\xi_1 \rightarrow \xi_2) \ \approx \ (\xi_1 \wedge \sim \xi_2)$;

2) $\sim(\xi_1 \wedge \xi_2) \ \approx\ (\sim \xi_1 \vee \sim \xi_2)$;

3) $\sim(\xi_1 \vee \xi_2) \ \approx\ (\sim \xi_1 \wedge \sim \xi_2)$;

4) $(\sim \neg\xi_1) \ \approx\ \xi_1$;

5) $(\sim\sim \xi_1) \ \approx\ \xi_1$;

6) $(\sim \xi_1 \vee \neg\xi_1) \ \approx\ \neg\xi_1$.

We have observed that, in some sense, the class of algebra K_N^Γ explicitly describes the well-known relation between N-lattices and Heyting algebras. To be more specific, we prove that we can canonically define a N-lattice $\mathbf{B}^{\bowtie}$, given $\mathbf{B} \in K_N^\Gamma$. We can also define, for every N-lattice $\mathbf{A}$, a Σ_N^o-algebra $\mathbf{A}_{\bowtie}$ such that $\mathbf{A}_{\bowtie} \in K_N^\Gamma$. The abuse of notation when we write $\mathbf{B}^{\bowtie}$ and $\mathbf{A}_{\bowtie}$ is, as we will see, well justified by the key role of the constructions $(.)^{\bowtie}$ and $(.)_{\bowtie}$ in the definitions of $\mathbf{B}^{\bowtie}$ and $\mathbf{A}_{\bowtie}$, respectively.

Let $\mathbf{B} \in K_N^\Gamma$ and recall that $\equiv_\Gamma$ denotes the Γ-behavioral equivalence over $\mathbf{B}$. We can then define a Σ_N-algebra

$$\mathbf{B}^{\bowtie} = \langle B^{\bowtie}, \wedge_{\mathbf{B}^{\bowtie}}, \vee_{\mathbf{B}^{\bowtie}}, \to_{\mathbf{B}^{\bowtie}}, \neg_{\mathbf{B}^{\bowtie}}, \sim_{\mathbf{B}^{\bowtie}}, \top_{\mathbf{B}^{\bowtie}}, \bot_{\mathbf{B}^{\bowtie}} \rangle$$

where

$$B^{\bowtie} = \{\langle [a]_{\equiv_\Gamma}, [\sim_{\mathbf{B}} a]_{\equiv_\Gamma} \rangle : a \in B_\phi\}$$

and such that the operations are defined as in the construction $(.)^{\bowtie}$.

THEOREM 25. *Given $\mathbf{B} \in K_N^\Gamma$ we have that $\mathbf{B}^{\bowtie}$ is a N-lattice.*

PROOF. Let $h^* : X \to B^{\bowtie}$ be an assignment. Take $h : X \to B_\phi$ such that $h(x) = a$ where $h^*(x) = \langle [a]_{\equiv_\Gamma}, [\sim_{\mathbf{B}} a]_{\equiv_\Gamma} \rangle$. Using induction on the structure of a formula, it is easy to prove that $h^*(\varphi) = \langle [h(\varphi)]_{\equiv_\Gamma}, [\sim_{\mathbf{B}} h(\varphi)]_{\equiv_\Gamma} \rangle$, for every formula φ. Since $\mathbf{B} \in K_N^\Gamma$ we have that $\mathbf{B} \ |||\!\!- \ \varphi \approx \top$ for every theorem φ of N. So, for every assignment h' over B_ϕ, we can conclude that $[h'(\varphi)]_{\equiv_\Gamma} = \top_{\mathbf{B}}$. Therefore, given an axiom φ of N we have that $h^*(\varphi) = \langle [h(\varphi)]_{\equiv_\Gamma}, [\sim_{\mathbf{B}} h(\varphi)]_{\equiv_\Gamma} \rangle = \langle \top_{\mathbf{B}}, \bot_{\mathbf{B}} \rangle$, which is the unit.

All that remains to prove is that $\mathbf{B}^{\bowtie} = \langle B^{\bowtie}, \wedge_{\mathbf{B}^{\bowtie}}, \vee_{\mathbf{B}^{\bowtie}}, \top_{\mathbf{B}^{\bowtie}}, \bot_{\mathbf{B}^{\bowtie}} \rangle$ is a bounded distributive lattice. This is matter of a direct verification. ■

Consider now given a N-lattice $\mathbf{A}$. Recall that we can consider the Γ-congruence $\theta_{\mathbf{A}}$ over $\mathbf{A}$ defined as $\langle a, b \rangle \in \theta_{\mathbf{A}}$ iff $(a \leftrightarrow_{\mathbf{A}} b) = \top_{\mathbf{A}}$.

We can then consider the Σ_N^o-algebra $\mathbf{A}^{\bowtie}$ such that:

- $(\mathbf{A}^{\bowtie})_v = (\mathbf{A}_{|\Gamma})/_{\theta_{\mathbf{A}}}$;

- $(\mathbf{A}^{\bowtie})_\phi = \mathbf{A}$;
- $o_{\mathbf{A}^{\bowtie}}(a) = [a]_{\theta_\mathbf{A}}$ for every $a \in A$.

THEOREM 26. *If* $\mathbf{A}$ *is a* N*-lattice then* $\mathbf{A}^{\bowtie} \in K_N^\Gamma$.

PROOF. First of all, note that it is well known that $(\mathbf{A}_{|\Gamma})/_{\theta_\mathbf{A}}$ is a Heyting algebra [43, 24].

From the definition of N-lattice we can conclude that $[\sim(\xi_1 \to \xi_2)]_{\theta_\mathbf{A}} = [(\xi_1 \wedge \sim \xi_2)]_{\theta_\mathbf{A}}$, $[\sim (\xi_1 \wedge \xi_2)]_{\theta_\mathbf{A}} = [(\sim \xi_1 \vee \sim \xi_2)]_{\theta_\mathbf{A}}$, $[\sim (\xi_1 \vee \xi_2)]_{\theta_\mathbf{A}} = [(\sim \xi_1 \wedge \sim \xi_2)]_{\theta_\mathbf{A}}$, $[(\sim \neg\xi_1)]_{\theta_\mathbf{A}} = [\xi_1]_{\theta_\mathbf{A}}$, $[(\sim\sim \xi_1)]_{\theta_\mathbf{A}} = [\xi_1]_{\theta_\mathbf{A}}$ and $[(\sim \xi_1 \vee \neg\xi_1)]_{\theta_\mathbf{A}} = [\neg\xi_1]_{\theta_\mathbf{A}}$. Therefore, the result follows from the observation that, by construction, $\theta_\mathbf{A}$ is indeed $\equiv_\Gamma$, the Γ-behavioral equivalence on $\mathbf{A}^{\bowtie}$. ∎

We end this example with some conclusions. The first one is that with our approach we are able to make explicit the key role that Heyting algebras play in the algebraic counterpart of N. The algebras obtained by behavioral algebraization can be seen as N-lattices in a different perspective. Furthermore, our goal is not to provide an alternative to N-lattices, but only to provide one more tool for the study of the system N and, in particular, to the study of N-lattices.

Note that this example is just a first example of the application of our behavioral theory to the study of algebraizable logics. Of course, due to the large amount of research on N-lattices, we did not arrive at any novel major result or conclusion. Nevertheless, in logics with less studied semantics, our approach can help to unveil some interesting algebraic results and moreover to shed some light on the relation between different equivalences in a given logic, as it was the case of intuitionistic equivalence and strong equivalence.

4. Behavioral AAL

One of the goals of AAL is to discover general criteria for a class of algebras (or for a class of mathematical objects closely related to algebra, such as logical matrices) to be the algebraic counterpart of a logic, and to develop the methods for obtaining it. Another important goal of AAL is a classification of logics based on the properties of their algebraic counterparts. Ideally, when it is known that a given logic belongs to a particular group in the classification, one will have general theorems providing important information about its properties. Following these goals, we propose in this section a behavioral generalization of some of the standard notions and results of AAL.

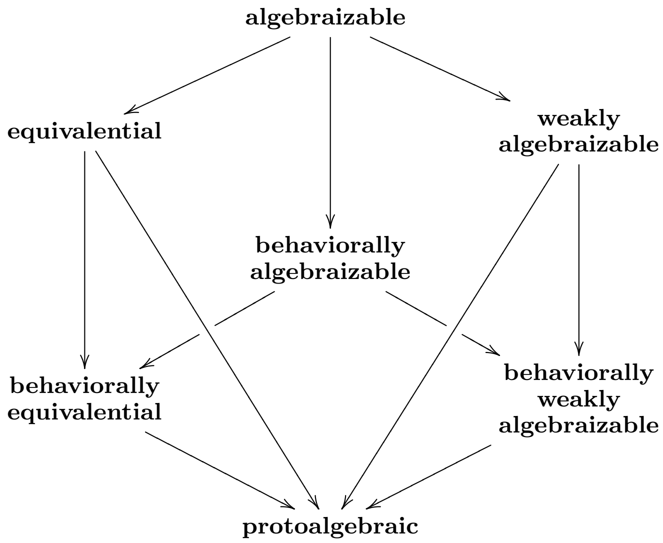

Figure 1. A view of the behavioral Leibniz hierarchy.

This is basically a systematic continuation of the effort that was already started in the previous section.

We start by drawing a behavioral Leibniz hierarchy that generalizes part of the standard Leibniz hierarchy. We then present an useful intrinsic characterization of the notion of behaviorally algebraizable logic and, as corollary, a sufficient condition.

4.1. The behavioral Leibniz hierarchy

Until now we have focused on generalizing the notion of algebraizable logic. To further support our methodology, we now show how to extend other standard notions and results of AAL to the behavioral setting. Recall that one of the main tools of AAL is the Leibniz operator. It can be used to draw the so-called Leibniz hierarchy, briefly depicted in Figure 1, which will serve as a roadmap for the results of this section. It shows in a clear way the relationship between the standard and the behavioral hierarchies (the relations between behavioral classes are assumed to be drawn over the same subsignature Γ).

First, we need to introduce the behavioral variant of the notion of Leibniz operator. For the purpose, let us define the notion of Γ-congruence. Consider given a signature $\Sigma = \langle S, F \rangle$ and a subsignature Γ of Σ. A Γ-*congruence* over a Σ-algebra $\mathbf{A}$ is an equivalence relation θ over $\mathbf{A}$ such that:

$$\text{if } \langle a_1, b_1\rangle \in \theta_{s_1}, \ldots, \langle a_n, b_n\rangle \in \theta_{s_n} \text{ and } f : s_1 \ldots s_n \to s \in \Gamma,$$
$$\text{then } \langle f_{\mathbf{A}}(a_1, \ldots, a_n), f_{\mathbf{A}}(b_1, \ldots, b_n)\rangle \in \theta_s.$$

We will denote the set of all Γ-congruences over a Σ-algebra $\mathbf{A}$ by $Con^{\Sigma}_{\Gamma}(\mathbf{A})$. The difference between a Γ-congruence and a congruence over $\mathbf{A}$ is that a Γ-congruence is assumed to satisfy the congruence property just for contexts generated from the subsignature Γ. The contexts outside Γ do not necessarily satisfy the congruence property. Easily, a congruence is just a Σ-congruence in our setting, that is, we just have to take $\Gamma = \Sigma$.

A Γ_ϕ-congruence θ over a Σ-algebra $\mathbf{A}$ is a ϕ reduct of a Γ-congruence over $\mathbf{A}$, that is, an equivalence relation over A_ϕ that satisfies the condition that, if $\langle a_1, b_1\rangle \in \theta, \ldots, \langle a_n, b_n\rangle \in \theta$ and $f : \phi^n \to \phi \in Der_{\Gamma,\phi^n\phi}$, then $\langle f_{\mathbf{A}}(a_1, \ldots, a_n), f_{\mathbf{A}}(b_1, \ldots, b_n)\rangle \in \theta$. The set of all Γ_ϕ-congruences of $\mathbf{A}$ will be denoted by $Con^{\Sigma}_{\Gamma,\phi}(\mathbf{A})$. In what follows, the importance of Γ_ϕ-congruences reflects the distinguished role that the sort ϕ plays in our theory.

The next lemma is a generalization for Γ-congruences of a well known [5, 29] result for congruences. Its proof much resembles the standard proof and will thus be omitted.

Lemma 27. *Given a signature Σ and a subsignature Γ of Σ, $Con^{\Sigma}_{\Gamma}(\mathbf{A})$ is a complete sublattice of $Eqv^{\Sigma}(\mathbf{A})$ the complete lattice of equivalences on $\mathbf{A}$.*

It is easy to see that $Con^{\Sigma}_{\Gamma,\phi}(\mathbf{A})$ is a complete sublattice of $Eqv^{\Sigma_{|\phi}}(\mathbf{A}_{|\phi})$. The fact that every theory of $\vDash^{K,\Gamma}_{\Sigma,bhv}$ is a Γ-congruence over $\mathbf{T_\Sigma}(\mathbf{X})$ is an easy exercise and generalizes the well known relation between $\vDash_K$ and $Con^{\Sigma}_{\Sigma}(\mathbf{T_\Sigma}(\mathbf{X}))$. A Γ-congruence θ over a Σ-algebra $\mathbf{A}$ is compatible with a set $\Phi \subseteq A_\phi$ if for every $a_1, a_2 \in A_\phi$, if $\langle a_1, a_2\rangle \in \theta_\phi$ and $a_1 \in \Phi$ then $a_2 \in \Phi$.

Recall that the Leibniz congruence is the largest congruence compatible with a given $\mathcal{L}$-theory. The following lemma asserts the existence of the largest Γ-congruence over $\mathbf{T_\Sigma}(\mathbf{X})$ compatible with a given $\mathcal{L}$-theory T, thus generalizing the standard existence result.

Lemma 28. *Let $\mathcal{L} = \langle \Sigma, \vdash\rangle$ be a many-sorted logic and Γ a subsignature of Σ. For each $T \in Th_{\mathcal{L}}$, there is a largest Γ-congruence compatible with T.*

Proof. Let $T \in Th_{\mathcal{L}}$ and consider the binary relation Φ_T over $\mathbf{T_\Sigma}(\mathbf{X})$ defined, for every $s \in S$, as follows:

$$\langle t_1, t_2\rangle \in \Phi_{T,s} \text{ iff for every } c(x{:}s, x_1{:}s_1, \ldots, x_n{:}s_n) \in \mathcal{C}^{\Gamma}_{\Sigma,\phi}[x{:}s]$$
$$\text{and every } u_1 \in T_{\Sigma,s_1}(X), \ldots, u_n \in T_{\Sigma,s_n}(X) \text{ we have that}$$
$$c[t_1, u_1, \ldots, u_n] \in T \text{ iff } c[t_2, u_1, \ldots, u_n] \in T.$$

It is now an easy exercise to prove that Φ_T is indeed a Γ-congruence compatible with T and moreover it is the largest one. ■

Now that we have proved that, given a $\mathcal{L}$-theory T, the largest Γ-congruence compatible with T exists, we can use this result to extend the notion of Leibniz congruence to this behavioral setting. The *Γ-behavioral Leibniz congruence associated with a theory T* is the largest Γ-congruence compatible with T. The term Leibniz congruence was introduced in [3] but the concept appears much early. The characterization of the Leibniz Γ-congruence given in the proof of Lemma 28 justifies the use of the term Leibniz. The famous Leibniz second order criterion says that two objects in the universe of discourse are equal if they share all the properties that can be expressed in the language of discourse. In our behavioral generalization, we assume that the language of discourse includes only the contexts obtained from a given subsignature.

Definition 29. (Behavioral Leibniz operator)
Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic and Γ a subsignature of Σ. The *behavioral Leibniz operator* on the term algebra,

$$\Omega_\Gamma^{bhv} : Th_{\mathcal{L}} \to Con_\Gamma^\Sigma(\mathbf{T_\Sigma}(\mathbf{X}))$$

is such that, for each $T \in Th_{\mathcal{L}}$, $\Omega_\Gamma^{bhv}(T)$ is the largest Γ-congruence over $\mathbf{T_\Sigma}(\mathbf{X})$ compatible with T.

Note that, as before, this definition is parametrized by the choice of Γ. The behavioral Leibniz operator plays a central role in our approach. As we will see, some important classes of logics can be characterized by its properties. These properties include monotonicity, injectivity and commutation with inverse substitutions, where the last one means that given a substitutions σ over Σ and a theory $T \in Th_{\mathcal{L}}$ we have that $\Omega_\Gamma^{bhv}(\sigma^{-1}(T)) = \sigma^{-1}(\Omega_\Gamma^{bhv}(T))$. Using the behavioral Leibniz operator we can define a behavioral version of the notion of protoalgebraic logic. Consider given a subsignature Γ of Σ.

Definition 30. (Γ-behaviorally protoalgebraic logic)
A many-sorted logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ is Γ-behaviorally protoalgebraic if for every $T \in Th_{\mathcal{L}}$ and $\varphi, \psi \in L_\Sigma(X)$

$$\text{if } \langle \varphi, \psi \rangle \in \Omega_\Gamma^{bhv}(T) \text{ then } T, \varphi \vdash \psi \text{ and } T, \psi \vdash \varphi.$$

We will now prove equivalent characterizations of the notion of behaviorally protoalgebraic logic. These equivalent characterizations are behavioral versions of the standard results for protoalgebraic logics. Some of them will be useful to show the interesting result that the standard and the behavioral notions of protoalgebraic logic coincide.

Consider given a many-sorted logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ and a subsignature Γ of Σ. Now consider the set $T^{\mathcal{L},\Gamma}_{\xi_1,\xi_2} = \{\varphi \in L_\Gamma(X) : \vdash \sigma_{\xi_1 \to \xi_2}(\varphi)\}$, where $\sigma_{\xi_1 \to \xi_2}$ is the substitution that sends ξ_2 to ξ_1, that is, $\sigma_{\xi_1 \to \xi_2,\phi}(\xi_2) = \xi_1$, and leaves the remaining variables fixed. When the logic $\mathcal{L}$ is clear from the context, we will write just $T^{\Gamma}_{\xi_1,\xi_2}$ instead of $T^{\mathcal{L},\Gamma}_{\xi_1,\xi_2}$. The non-behavioral unsorted analogue of this set was used by Herrmann in [23] as a fundamental tool in the development of his theory. It is also an important tool in our framework and, in particular, it can be used to give an alternative characterization of the notion of behavioral protoalgebraizability. The following lemma asserts some simple but very useful properties of $T^{\Gamma}_{\xi_1,\xi_2}$. The proof is omitted since it is a straightforward generalization of the standard result [11].

Lemma 31.
Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic and Γ a subsignature of Σ. Then,

i) if σ is a substitution over Γ such that $\sigma_{\xi_1 \to \xi_2}(\sigma\xi_1) = \sigma_{\xi_1 \to \xi_2}(\sigma\xi_2)$ then $T^{\Gamma}_{\xi_1,\xi_2}$ is closed under σ, that is, $\sigma[T^{\Gamma}_{\xi_1,\xi_2}] \subseteq T^{\Gamma}_{\xi_1,\xi_2}$;

ii) $\langle \xi_1, \xi_2 \rangle \in \Omega^{bhv}_{\Gamma,\phi}((T^{\Gamma}_{\xi_1,\xi_2})^{\vdash})$;

iii) $\Delta(\xi_1, \xi_2) \subseteq L_\Gamma(X)$ is a Γ-behavioral equivalence iff $\Delta \subseteq T^{\Gamma}_{\xi_1,\xi_2}$ and $\Delta^{\vdash} = (T^{\Gamma}_{\xi_1,\xi_2})^{\vdash}$.

The following notion of behavioral protoequivalence system of formulas is the basis of a characterization of behavioral protoalgebraizability. It generalizes the concept of (many-sorted) protoequivalence system given in [27].

Definition 32. (Behavioral protoequivalence system)
Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic and Γ a subsignature of Σ. A a set $\Delta(\xi_1, \xi_2, \underline{z}) \subseteq L_\Gamma(X)$ where $\underline{z} = \langle z_1 : s_1, z_2 : s_2, \ldots \rangle$ is a set of *parametric variables* with sort different from ϕ and at most one variable of each sort is said a Γ-*protoequivalence system for* $\mathcal{L}$ if it satisfies the following conditions:

(R) $\vdash \Delta(\xi, \xi, \underline{z})$;

(MP) $\xi_1, \Delta(\xi_1, \xi_2, \underline{z}) \vdash \xi_2$.

The following theorem is a behavioral version of well known characterizations of the standard notion of protoalgebraic logic [11].

THEOREM 33. *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic and Γ a subsignature of Σ. Then, the following conditions are equivalent:*

i) $\mathcal{L}$ is Γ-behaviorally protoalgebraic;

ii) $\Omega^{bhv}_{\Gamma,\phi}$ is monotone;

iii) $\xi_1, T^{\Gamma}_{\xi_1,\xi_2} \vdash \xi_2$;

iv) there exists a Γ-protoequivalence system for $\mathcal{L}$.

PROOF. $i) \Rightarrow ii)$: Assume $\mathcal{L}$ is Γ-behaviorally protoalgebraic. Let $T_1, T_2 \subseteq Th_{\mathcal{L}}$ such that $T_1 \subseteq T_2$. We will prove that $\Omega^{bhv}_{\Gamma}(T_1)$ is compatible with T_2. For, let $\varphi \in T_2$ and $\langle \varphi, \psi \rangle \in \Omega^{bhv}_{\Gamma,\phi}(T_1)$. Hence $T_1, \varphi \dashv\vdash \psi, T_1$ by Γ-protoalgebraizability. Since $T_1 \subseteq T_2$ we have that $T_2, \varphi \dashv\vdash \psi, T_2$. So, since $\varphi \in T_2$ and T_2 is a theory, we can conclude that $\psi \in T_2$. Now that have proved that $\Omega^{bhv}_{\Gamma}(T_1)$ is compatible with T_2, we can conclude that $\Omega^{bhv}_{\Gamma,\phi}(T_1) \subseteq \Omega^{bhv}_{\Gamma,\phi}(T_2)$ since $\Omega^{bhv}_{\Gamma}(T_2)$ is the largest Γ-congruence compatible with T_2.

$ii) \Rightarrow iii)$: By Lemma 31 we have that $\langle \xi_1, \xi_2 \rangle \in \Omega^{bhv}_{\Gamma,\phi}((T^{\Gamma}_{\xi_1,\xi_2})^{\vdash})$. Since $\Omega^{bhv}_{\Gamma,\phi}$ is monotone we have $\langle \xi_1, \xi_2 \rangle \in \Omega^{bhv}_{\Gamma,\phi}((\{\xi_1\} \cup T^{\Gamma}_{\xi_1,\xi_2})^{\vdash})$, and by compatibility we can conclude that $\xi_2 \in (\{\xi_1\} \cup T^{\Gamma}_{\xi_1,\xi_2})^{\vdash}$, that is, $\xi_1, T^{\Gamma}_{\xi_1,\xi_2} \vdash \xi_2$.

$iii) \Rightarrow iv)$: Take $\Delta = \sigma T^{\Gamma}_{\xi_1,\xi_2}$ where σ is a substitution such that $\sigma_{\phi}(\xi_1) = \xi_1$ and $\sigma\phi(\xi) = \xi_2$ for every $\xi \neq \xi_1$ and, for every $s \neq \phi$, $\sigma_s(x) = x_0$ for every $x \in X_s$, where x_0 is a fixed variable of sort s. So, the conditions over the variables are verified. To verify (R) and (MP) note first that, since σ is a substitution over Γ and $\sigma_{\xi_1 \to \xi_2}(\sigma\xi_1) = \sigma_{\xi_1 \to \xi_2}(\sigma\xi_2)$ we have, using Lemma 31, that $\sigma T^{\Gamma}_{\xi_1,\xi_2} \subseteq T^{\Gamma}_{\xi_1,\xi_2}$. So, (R) is satisfied. In turn, (MP) follows from from $iii)$ and structurality.

$iv) \Rightarrow i)$: Suppose that there exists a Γ-protoequivalence set $\Delta(\xi_1, \xi_2, \underline{z})$ for $\mathcal{L}$. Let $\varphi, \psi \in L_{\Sigma}(X)$ and let T be a theory of $\mathcal{L}$ such that $\langle \varphi, \psi \rangle \in \Omega^{bhv}_{\Gamma,\phi}(T)$. So, for every $\delta(\xi_1, \xi_2) \in \Delta$, we have that $\langle \delta(\varphi, \psi), \delta(\varphi, \varphi) \rangle \in \Omega^{bhv}_{\Gamma,\phi}(T)$. So, by compatibility and using (R) we have that $\Delta(\varphi, \psi) \subseteq T$. So, using (MP) we have that $T, \varphi \vdash \psi$. In the same way we have that $T, \psi \vdash \varphi$. So $T, \varphi \dashv\vdash \psi, T$. ■

Note that, using condition $iv)$ of the above theorem, we have that if a logic is Γ-behaviorally equivalential then it is also Γ-behaviorally protoalgebraic. More interestingly, condition $iv)$ also allows us to conclude that if a logic is behaviorally protoalgebraic then it is also protoalgebraic in the standard sense. This is an important fact since it means that all our behavioral hierarchy is contained in the class of protoalgebraic logics, the class of logics that is widely considered to be the largest class amenable to the tools of AAL and standard matrix semantics.

After focusing on behavioral protoalgebraizability, we turn our attention to other notions of the Leibniz hierarchy, such as weak algebraizability.

DEFINITION 34. (Γ-behaviorally weakly algebraizable logic)
Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic and Γ a subsignature of Σ. Then, $\mathcal{L}$ is Γ-*behaviorally weakly algebraizable* if there exists a class K of Σ^o-algebras, a set $\Theta(\xi, \underline{z}) \subseteq Comp_{\Sigma}^{K,\Gamma}(X)$ of ϕ-equations and a set $\Delta(\xi_1, \xi_2, \underline{w}) \subseteq T_{\Gamma,\phi}(\{\xi_1, \xi_2, \underline{w}\})$ of formulas such that, for every $T \cup \{t\} \subseteq L_{\Sigma}(X)$ and for every set $\Phi \cup \{t_1 \approx t_2\}$ of ϕ-equations,

i) $T \vdash t$ iff $\Theta[\langle T \rangle] \vDash_{\Sigma,bhv}^{K,\Gamma} \Theta(\langle t \rangle)$;

ii) $\Phi \vDash_{\Sigma,bhv}^{K,\Gamma} t_1 \approx t_2$ iff $\Delta[\langle \Phi \rangle] \vdash \Delta(\langle t_1, t_2 \rangle)$;

iii) $\xi \dashv\vdash \Delta[\langle \Theta(\langle \xi \rangle) \rangle]$;

iv) $\xi_1 \approx \xi_2 =\!\mid\!\vDash_{\Sigma,bhv}^{K,\Gamma} \Theta[\langle \Delta(\langle \xi_1, \xi_2 \rangle) \rangle]$;

The difference between the notion of Γ-behaviorally weakly algebraizable logic and the notion of Γ-behaviorally algebraizable logic is the fact that, in the former, both the equivalence set of formulas and the defining set of equations have parametric variables. We are able to generalize to our behavioral setting the standard characterization of weakly algebraizable logics using the Leibniz operator.

THEOREM 35. *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted signature and Γ a subsignature of Σ. Then $\mathcal{L}$ is Γ-behaviorally weakly algebraizable if $\Omega_{\Gamma,\phi}^{bhv}$ is monotone and injective.*

We will not present here the proof of this result since the techniques involved are all contained in the proofs of Theorems 33 and 38. The reason for presenting this notion here was to show that it is easy to obtain a (many-sorted) behavioral version of the notion of weakly algebraizable logic, and also that it is easy to generalize the well known characterization result using the Leibniz operator. Nevertheless, it is not our intention to study this notion in detail, for now.

In a previous section we have introduced the notion of behaviorally equivalential logic. In the next proposition we group two interesting properties regarding behaviorally equivalential logics and the behavioral Leibniz operator. The first one generalizes the well known criterion for equivalentiality due to Herrmann [23]. The second property generalizes the intimate connection between an equivalence set and the Leibniz congruence.

PROPOSITION 36. *Let* $\mathcal{L} = \langle \Sigma, \vdash \rangle$ *a many-sorted logic and* Γ *a subsignature of* Σ. *Let* $\Delta(\xi_1, \xi_2) \subseteq L_\Gamma(\{\xi_1, \xi_2\})$ *a set of formulas. Then,*

i) if $\Delta(\xi_1, \xi_2)$ *is a* Γ*-behavioral equivalence set for* $\mathcal{L}$ *then, for every* $T \in Th_\mathcal{L}$ *and* $\varphi, \psi \in L_\Sigma(X)$, *we have that* $\langle \varphi, \psi \rangle \in \Omega^{bhv}_{\Gamma,\phi}(T)$ *iff* $\Delta(\varphi, \psi) \subseteq T$.

ii) Herrmann's Test: suppose $\mathcal{L}$ *is* Γ*-behaviorally protoalgebraic. Then,* $\Delta(\xi_1, \xi_2)$ *is an* Γ*-behavioral equivalence set for* $\mathcal{L}$ *iff* $\Delta(\xi_1, \xi_2) \subseteq T^\Gamma_{\xi_1,\xi_2}$ *and it satisfies* $\langle \xi_1, \xi_2 \rangle \in \Omega^{bhv}_{\Gamma,\phi}(\Delta(\xi_1, \xi_2)^\vdash)$;

PROOF. *i*) First let $\langle \varphi, \psi \rangle \in \Omega^{bhv}_{\Gamma,\phi}(T)$. Then, by compatibility, we have that $\Delta(\varphi, \psi) \subseteq T$ iff $\Delta(\varphi, \varphi) \subseteq T$. Since Δ satisfies (R) we can conclude that $\Delta(\varphi, \psi) \subseteq T$.

On the other direction, suppose that $\Delta(\varphi, \psi) \subseteq T$. So, for every $c \in \mathcal{C}^\Gamma_{\Sigma,\phi}[\xi]$ we have that $\Delta(c[\varphi], c[\psi]) \subseteq T$. So, using (MP) we can conclude that $c[\varphi] \in T$ iff $c[\psi] \in T$. So, we have that $\langle \varphi, \psi \rangle \in \Omega^{bhv}_{\Gamma,\phi}(T)$.

ii): Suppose first that Δ is a Γ-behavioral equivalence for $\mathcal{L}$. Since $\vdash \Delta(\xi_1, \xi_1)$ we have that $\Delta \subseteq T^\Gamma_{\xi_1,\xi_2}$. By Lemma 31 we have that $\Delta^\vdash = {T^\Gamma_{\xi_1,\xi_2}}^\vdash$. Again by Lemma 31 we have that $\langle \xi_1, \xi_2 \rangle \in \Omega^{bhv}_{\Gamma,\phi}((T^\Gamma_{\xi_1,\xi_2})^\vdash) = \Omega^{bhv}_{\Gamma,\phi}(\Delta(\xi_1, \xi_2)^\vdash)$. Now suppose that $\Delta(\xi_1, \xi_2) \subseteq T^\Gamma_{\xi_1,\xi_2}$ and $\langle \xi_1, \xi_2 \rangle \in \Omega^{bhv}_{\Gamma,\phi}(\Delta(\xi_1, \xi_2)^\vdash)$. Then $\Delta(\xi_1, \xi_2)^\vdash \subseteq (T^\Gamma_{\xi_1,\xi_2})^\vdash$. To prove the reverse inclusion, let $\varphi \in T^\Gamma_{\xi_1,\xi_2}$. By definition of $T^\Gamma_{\xi_1,\xi_2}$ we have that $\varphi(\xi_1, \xi_1)$. So $\varphi(\xi_1, \xi_1) \in \Delta(\xi_1, \xi_2)^\vdash$. Since $\langle \xi_1, \xi_2 \rangle \in \Omega^{bhv}_{\Gamma,\phi}(\Delta(\xi_1, \xi_2)^\vdash)$ we have by compatibility that $\varphi(\xi_1, \xi_2) \in \Delta(\xi_1, \xi_2)^\vdash$ iff $\varphi(\xi_1, \xi_1) \in \Delta(\xi_1, \xi_2)^\vdash$. So, $\varphi(\xi_1, \xi_2) \in \Delta(\xi_1, \xi_2)^\vdash$. So, we have that $(T^\Gamma_{\xi_1,\xi_2})^\vdash \subseteq \Delta(\xi_1, \xi_2)^\vdash$ and we can conclude that $(T^\Gamma_{\xi_1,\xi_2})^\vdash = \Delta(\xi_1, \xi_2)^\vdash$. By Lemma 31 we have that Δ is a Γ-behavioral equivalence. ■

We will now show that the notion of behavioral equivalentiality can also be characterized by properties of the behavioral Leibniz operator. This result also generalizes a well-known standard result.

THEOREM 37. *Let* $\mathcal{L} = \langle \Sigma, \vdash \rangle$ *be a many-sorted logic and* Γ *a subsignature of* Σ. *Suppose* $\mathcal{L}$ *is* Γ*-standard, then the following are equivalent:*

i) $\mathcal{L}$ *is* Γ*-behaviorally equivalential;*

ii) $\Omega^{bhv}_{\Gamma,\phi}$ *is monotone and commutes with inverse substitutions;*

iii) $\Omega^{bhv}_{\Gamma,\phi}$ *is monotone and* $\sigma\Omega^{bhv}_{\Gamma,\phi}(T) \subseteq \Omega^{bhv}_{\Gamma,\phi}((\sigma T)^{\vdash})$*, for all substitutions and* $\mathcal{L}$*-theories* T*.*

Proof. *i)* ⇒ *ii)*: Suppose that $\mathcal{L}$ is Γ-behaviorally equivalential and let $\Delta(\xi_1, \xi_2)$ be a equivalence set for $\mathcal{L}$. As we already said, since $\mathcal{L}$ is Γ-behaviorally equivalential, then it is Γ-behaviorally protoalgebraic. By Theorem 33 we can conclude that $\Omega^{bhv}_{\Gamma,\phi}$ is monotone. To prove that $\Omega^{bhv}_{\Gamma,\phi}$ commutes with inverse substitutions, consider given a $T \in Th_{\mathcal{L}}$ and a substitution σ. Now, we have the following sequence of equivalent sentences: $\langle t_1, t_2\rangle \in \sigma^{-1}\Omega^{bhv}_{\Gamma,\phi}(T)$ iff $\langle \sigma t_1, \sigma t_2\rangle \in \Omega^{bhv}_{\Gamma,\phi}(T)$ iff $\Delta(\sigma t_1, \sigma t_2) \subseteq T$ iff $\sigma\Delta(t_1, t_2) \subseteq T$ iff $\Delta(t_1, t_2) \subseteq \sigma^{-1}T$ iff $\langle t_1, t_2\rangle \in \Omega^{bhv}_{\Gamma,\phi}(\sigma^{-1}T)$.

ii) ⇒ *iii)*: Let $T \in Th_{\mathcal{L}}$ and let σ be a substitution over Σ. Let $T_0 = (\sigma T)^{\vdash}$. It is obvious that $T \subseteq \sigma`-1T_0$ and hence $\Omega^{bhv}_{\Gamma,\phi}(T) \subseteq \Omega^{bhv}_{\Gamma,\phi}(\sigma^{-1}T_0)$. Since $\Omega^{bhv}_{\Gamma,\phi}$ commutes with inverse substitutions we have that $\Omega^{bhv}_{\Gamma,\phi}(\sigma^{-1}T_0) = \sigma^{-1}\Omega^{bhv}_{\Gamma,\phi}(T_0)$. Thus, $\Omega^{bhv}_{\Gamma,\phi}(T) \subseteq \sigma^{-1}\Omega^{bhv}_{\Gamma,\phi}(T_0)$. This yields $\sigma\Omega^{bhv}_{\Gamma,\phi}(T) \subseteq \Omega^{bhv}_{\Gamma,\phi}((\sigma T)^{\vdash})$.

iii) ⇒ *i)*: Suppose *iii)*. By Proposition 36, $\mathcal{L}$ is equivalential provided some $\Delta(\xi_1, \xi_2) \subseteq L_\Sigma(\{xi_1, \xi_2\})$ satisfies $\Delta \subseteq T^{\Gamma}_{\xi_1,\xi_2}$ and $\langle \xi_1, \xi_2\rangle \in \Omega^{bhv}_{\Gamma}((\Delta)^{\vdash}$. Recall that since $\mathcal{L}$ is Γ-standard there exists a closed term over Γ for each sort $s \in S$. Let σ be a substitution such that $\sigma_\phi(\xi_1) = \xi_1$ and $\sigma_\phi(\xi)) = \xi_2$ for every $\xi \in X_\phi$ and, for every $s \in S$ and every $x \in X_s$, $\sigma_s(x) = t_s$ where t_s is a closed term of sort s. Now take $\Delta(\xi_1, \xi_2) = \sigma T^{\Gamma}_{\xi_1,\xi_2}$. So, $\Delta \subseteq L_\Gamma(\{\xi_1, \xi_2\})$. Since $\sigma_{\xi_1\to\xi_2}(\sigma\xi_1) = \sigma_{\xi_1\to\xi_2}(\sigma\xi_2)$ we have that, by Lemma 31 we have that $\Delta = \sigma T^{\Gamma}_{\xi_1,\xi_2} \subseteq T^{\Gamma}_{\xi_1,\xi_2}$. We know that $\langle \xi_1, \xi_2\rangle \in \Omega^{bhv}_{\Gamma,\phi}((T^{\Gamma}_{\xi_1,\xi_2})^{\vdash})$ by Lemma 31. So, $\langle \sigma\xi_1, \sigma\xi_2\rangle \in \sigma\Omega^{bhv}_{\Gamma,\phi}((T^{\Gamma}_{\xi_1,\xi_2})^{\vdash})$. By hypothesis, $\langle \xi_1, \xi_2\rangle \in \Omega^{bhv}_{\Gamma,\phi}((\sigma T^{\Gamma}_{\xi_1,\xi_2})^{\vdash})$. So, $\langle \xi_1, \xi_2\rangle \in \Omega^{bhv}_{\Gamma,\phi}((T^{\Gamma}_{\xi_1,\xi_2})^{\vdash})$. By Proposition 36 we have that Δ is an equivalence. ■

We will now go towards the main theorem of this section: the characterization of behavioral algebraizability using properties of the behavioral Leibniz operator. At this point it is important to recall that quotient algebras are an essential ingredient of AAL. However, since we are now working with Γ-congruences instead of congruences, we cannot perform quotients directly. This is where algebras over the extended signature Σ^o will play a key role. We are able to construct from a given Γ-congruence θ over $\mathbf{T_\Sigma(X)}$ a congruence over $\mathbf{T^o_\Sigma(X^o)}$ that keeps the relevant information of the original Γ-congruence. We can consider a relation $\theta^o = \{\theta^o_s\}_{s\in S^o}$ over $\mathbf{T^o_\Sigma(X^o)}$ such that $\theta^o_v = \{\langle o(\varphi), o(\psi)\rangle : \langle \varphi, \psi\rangle \in \theta_\phi\} \cup \{\langle t, t\rangle : t \in T_{\Sigma^o,v}(X^o)\}$ and, for every $s \neq v$, θ^o_s is the identity relation over $T_{\Sigma^o,s}(X^o)$. It is easy to verify that θ^o is indeed a congruence on $\mathbf{T}^{\mathbf{o}}_{\mathbf{\Sigma}}(\mathbf{X}^{\mathbf{o}})$.

We can now proceed to the characterization of behavioral algebraizability using the behavioral Leibniz operator. The result generalizes the well know standard result. In the proof, we will follow a methodology closely related to the one used by Herrmann in [23]. The techniques used there are easier to adapt to the behavioral setting than, for example, those used in the proof given by Blok and Pigozzi [3].

THEOREM 38. *Let $\mathcal{L} = \langle \Sigma, \vdash\rangle$ be a Γ-standard many-sorted logic, where Γ is a subsignature of Σ. Then, $\mathcal{L}$ is Γ-behaviorally algebraizable iff $\Omega^{bhv}_{\Gamma,\phi}$ is injective, monotone and commutes with inverse substitutions.*

PROOF. First assume that $\mathcal{L}$ is Γ-behaviorally algebraizable and let Δ and Θ be the sets of equivalence formulas and of defining equations respectively. So, it is equivalential, and therefore $\Omega^{bhv}_{\Gamma,\phi}$ is monotone and commutes with inverse substitutions. To prove that it is also injective let T_1, T_2 such that $\Omega^{bhv}_{\Gamma,\phi}(T_1) = \Omega^{bhv}_{\Gamma,\phi}(T_2)$. Now consider the following sequence of equivalent sentences: $\varphi \in T_1$ iff $\Delta[\Theta(\varphi)] \subseteq T_1$ iff $\Theta(\varphi) \subseteq \Omega^{bhv}_{\Gamma,\phi}(T_1)$ iff $\Theta(\varphi) \subseteq \Omega^{bhv}_{\Gamma,\phi}(T_2)$ iff $\Delta[\Theta(\varphi)] \subseteq T_2$ iff $\varphi \in T_2$. So, $T_1 = T_2$, showing that $\Omega^{bhv}_{\Gamma,\phi}$ is injective.

Assume now that $\Omega^{bhv}_{\Gamma,\phi}$ is injective, monotone and commutes with inverse substitutions. So, by Theorem 37 $\mathcal{L}$ is Γ-behaviorally equivalential. Let $\Delta(\xi_1, \xi_2)$ be an equivalence for $\mathcal{L}$.

Take $K = \{\mathbf{T}_{\mathbf{\Sigma^o}}(\mathbf{X^o})/_{(\Omega^{bhv}_{\Gamma}(T))^o} : T \in Th_{\mathcal{L}}\}$ a class of Σ^o-algebras. Using Lemma 36 and having in mind the definition of K, it an easy exercise to prove that, for every set Φ of ϕ-equations and $\varphi, \psi \in L_\Sigma(X)$, we have $\Phi \vDash^{K,\Gamma}_{\Sigma,bhv} \varphi \approx \psi$ iff $\Delta[\Phi] \vdash \Delta(t_1, t_2)$. Let us now prove that $\xi \dashv\vdash \Delta[\Theta(\xi)]$ for some set $\Theta(\xi)$ of ϕ-equations with just the variable ξ.

Let $T_\xi = \{\xi\}^\vdash$ and take a substitution σ such that $\sigma_\phi(\xi') = \xi$ for every $\xi' \in X_\phi$ and, for every $s \in S$ and $s \neq \phi$, we have that $\sigma_s(x) = t_s$ where t_s is a closed term of sort s. Take $\Theta(\xi) = \sigma\Omega^{bhv}_{\Gamma,\phi}(T_\xi)$. So, $\Theta(\xi) \subseteq Eq_\Sigma(\{\xi\})$. Since $\sigma(\Delta(\Omega^{bhv}_{\Gamma,\phi}(T_\xi))) = \Delta(\sigma\Omega^{bhv}_{\Gamma,\phi}(T_\xi)) = \Delta(\Theta(\xi))$, it suffices to show that $\xi \dashv\vdash \Delta[\Omega^{bhv}_{\Gamma,\phi}(T_\xi)]$, or equivalently that $T_\xi = (\Delta[\Omega^{bhv}_{\Gamma,\phi}(T_\xi)])^\vdash$. For that, consider the following sequence of equivalent sentences: $\langle \varphi, \psi\rangle \in \Omega^{bhv}_{\Gamma,\phi}((\Delta[\Omega^{bhv}_{\Gamma,\phi}(T_\xi)])^\vdash)$ iff $\Delta(\varphi, \psi) \subseteq (\Delta[\Omega^{bhv}_{\Gamma,\phi}(T_\xi)])^\vdash$ iff $(\Delta[\Omega^{bhv}_{\Gamma,\phi}(T_\xi)]) \vdash \Delta(\varphi, \psi)$ iff $\Omega^{bhv}_{\Gamma,\phi}(T_\xi) \vDash^{K,\Gamma}_{\Sigma,bhv} \varphi \approx \psi$ iff $\langle \varphi, \psi\rangle \in \Omega^{bhv}_{\Gamma,\phi}(T_\xi)$.

So, $\Omega^{bhv}_{\Gamma,\phi}(T_\xi) = \Omega^{bhv}_{\Gamma}((\Delta[\Omega^{bhv}_{\Gamma,\phi}(T_\xi)])^\vdash)$. By injectivity of $\Omega^{bhv}_{\Gamma,\phi}$ we have that $T_\xi = (\Delta[\Omega^{bhv}_{\Gamma,\phi}(T_\xi)])^\vdash$. ■

4.2. Intrinsic and sufficient characterizations

From the point of view of its definition, the notion of behaviorally algebraizable logic may seem impure, since it depends on the *a priori* existence of a behavioral equivalent algebraic semantics. The characterization of behavioral algebraizability using the behavioral Leibniz operator already shows that it is in fact an intrinsic property of a logic. We now provide a second intrinsic characterization of behavioral algebraizability and, as a corollary, we will be able to obtain a useful sufficient condition.

We have seen that a necessary condition for a many-sorted logic to be Γ-behaviorally algebraizable is that it be Γ-behaviorally equivalential. The following theorem shows that, by adding a natural assumption, we get a necessary and sufficient condition for Γ-behavioral algebraizability.

THEOREM 39. *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic and Γ a subsignature of Σ. Then we have that $\mathcal{L}$ is Γ-behaviorally algebraizable iff it is Γ-behaviorally equivalential with equivalence set $\Delta(\xi_1, \xi_2)$ and there exists a set $\Theta(\xi) \subseteq Eq_{\Sigma,\phi}(\{\xi\})$ of ϕ-equations such that $\xi \dashv\vdash \Delta[\Theta(\xi)]$.*

PROOF. Suppose first that $\mathcal{L}$ is Γ-behaviorally algebraizable. Then, using Proposition 8, we have that $\mathcal{L}$ is Γ-behaviorally equivalential. The existence of the set $\Theta(\xi)$ of ϕ-equations such that $\xi \dashv\vdash \Delta[\Theta(\xi)]$ is immediate from the definition of behaviorally algebraizable.

On the other direction, suppose that $\mathcal{L}$ is Γ-behaviorally equivalential and that there exists a set $\Theta(\xi) \subseteq Eq_{\Sigma,\phi}(\xi)$ of ϕ-equations such that $\xi \dashv\vdash \Delta[\Theta(\xi)]$. For each theory $T \in Th_{\mathcal{L}}$ we define a binary relation $\Omega_\Delta(T)$ over $\mathbf{T}_{\mathbf{\Sigma},\phi}(\mathbf{X})$ such that $(\Omega_\Delta(T)) = \{\langle \varphi_1, \varphi_2 \rangle : \Delta(\varphi_1, \varphi_2) \subseteq T\}$. By Proposition 36 we have that $\Omega_\Delta(T) = \Omega^{bhv}_{\Gamma,\phi}(T)$ for every $T \in Th_{\mathcal{L}}$.

We will now prove that $\Omega_\Delta : Th_{\mathcal{L}} \to Con_\phi(\mathbf{T}_{\mathbf{\Sigma}}(\mathbf{X}))$ is monotone, injective and commutes with inverse substitutions.

Let $T_1, T_2 \in Th_{\mathcal{L}}$ such that $T_1 \subseteq T_2$. Suppose that $\langle \varphi_1, \varphi_2 \rangle \in \Omega_\Delta(T_1)$. Then $\Delta(\varphi_1, \varphi_2) \subseteq T_1$. Since $T_1 \subseteq T_2$ we have that $\Delta(\varphi_1, \varphi_2) \subseteq T_2$ and so $\langle \varphi_1, \varphi_2 \rangle \in \Omega_\Delta(T_2)$. Thus Ω_Δ is monotone.

Suppose that $\Omega_\Delta(T_2) = \Omega_\Delta(T_1)$ and let $\varphi \in T_1$. Then, using the fact that $\varphi \dashv\vdash \Delta[\Theta(\varphi)]$, we have that $\Delta[\Theta(\varphi)] \subseteq T_1$ and hence $\langle \delta(\varphi), \epsilon(\varphi) \rangle \in \Omega_\Delta(T_1)$ for every $\delta \approx \epsilon \in \Theta$. Thus $\langle \delta(\varphi), \epsilon(\varphi) \rangle \in \Omega_\Delta(T_2)$ for every $\delta \approx \epsilon \in \Theta$ and so $\Delta[\Theta(\varphi)] \subseteq T_2$ and $\varphi \in T_2$ using the fact that $\varphi \dashv\vdash \Delta[\Theta(\varphi)]$. This shows that $T_1 \subseteq T_2$, and by symmetry we have that $T_1 = T_2$. Thus Ω_Δ is injective.

Let σ be a substitution over Σ. Then $\langle \varphi_1, \varphi_2 \rangle \in (\Omega_\Delta(\sigma^{-1}T))$ iff $\Delta(\varphi_1, \varphi_2) \subseteq \sigma^{-1}T$ iff $\sigma\Delta(\varphi_1, \varphi_2) \subseteq T$ iff $\Delta(\sigma\varphi_1, \sigma\varphi_2) \subseteq T$ iff $\langle \sigma\varphi_1, \sigma\varphi_2 \rangle \in (\Omega_\Delta(T))$ iff $\langle \varphi_1, \varphi_2 \rangle \in \sigma^{-1}(\Omega_\Delta(T))$. So $\Omega_\Delta(\sigma^{-1}T) =$

$\sigma^{-1}\Omega_\Delta(T)$, that is, Ω_Δ commutes with inverse substitutions. Since $\Omega_\Delta = \Omega^{bhv}_{\Gamma,\phi}$ we can apply Theorem 38 to conclude that $\mathcal{L}$ is Γ-behaviorally algebraizable. Note that Theorem 38 has the assumption that $\mathcal{L}$ is Γ-standard. This assumption is only used in the construction of the equivalence set Δ, to guarantee that Δ has no parametric variables of sorts different from ϕ. In this case, since we are assuming the existence of a set Δ with no parametric variables, we do not need to assume that $\mathcal{L}$ is Γ-standard. ■

As a corollary, we can provide a useful sufficient condition for a logic to be behaviorally algebraizable. The result extends a well-known standard sufficient condition [3].

COROLLARY 40. *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic and Γ a subsignature of Σ. A sufficient condition for $\mathcal{L}$ to be Γ-behaviorally algebraizable is that it is Γ-behaviorally equivalential with equivalence set $\Delta(\xi_1, \xi_2)$ satisfying also:*

(G) $\xi_1, \xi_2 \vdash \Delta(\xi_1, \xi_2)$.

In this case $\Delta(\xi_1, \xi_2)$ and $\Theta(\xi) = \{\xi \approx e(\xi, \xi) : e \in \Delta\}$ are, respectively, the equivalence formulas and defining equations for $\mathcal{L}$.

PROOF. Since $\Delta[\Theta(\xi)] = \Delta[\xi, \Delta(\xi, \xi)]$, and using the (G) we have that $\xi, \Delta(\xi, \xi) \vdash \Delta[\Theta(\xi)]$. Thus $\xi \vdash \Delta[\Theta(\xi)]$, since $\Delta(\xi, \xi)$ is a $\mathcal{L}$-theorem. On the other hand, $\Delta[\Theta(\xi)] \vdash \xi$ is a consequence of (MP) and using again the fact that $\Delta(\xi, \xi)$ is a $\mathcal{L}$-theorem. Since all conditions of Theorem 39 hold, we can conclude that $\mathcal{L}$ is Γ-behaviorally algebraizable. ■

5. Conclusions

We have proposed a novel extension of the theory of AAL, using many-sorted behavioral logic instead of unsorted equational logic, with the aim of broadening its range of applicability to richer and less orthodox logics. This leap was not only motivated by concrete examples, but is also well supported by the consistent development of behavioral logic, and by the fact that in a logic we can only observe the behavior of terms or other syntactic entities indirectly, through their influence on the logical value of the formulas where they appear. Pursuing this path, we have obtained behavioral versions of several of the standard key notions and results of AAL, including the Leibniz operator and the resulting behavioral Leibniz hierarchy. We have shown how behavioral algebraization indeed generalizes the standard notion, while further encompassing in a natural way logics whose algebraization was not possible before. Still, we have proved that the behavioral approach remains non-trivial, and actually within the range of protoalgebraizability.

Our results are encouraging in that they allow us to shed new light over logics like $\mathcal{C}_1$, whose algebraization was not possible before. In fact, there are in the literature other proposals of an algebraic counterpart for $\mathcal{C}_1$, namely the class of so-called *da Costa algebras* proposed by da Costa and later refined by Carnielli and de Alcantara [14, 16, 15, 10]. Still, the precise connection between $\mathcal{C}_1$ and this class of algebraic structures was never established by algebraic means. However, using the tools of behavioral logic, it is possible to recover, in our behavioral approach, this class of structures. A full treatment of these questions about $\mathcal{C}_1$, which falls clearly beyond the scope of this paper, can be found in [22, 9]. Furthermore, even the behavioral analysis of logics algebraizable in the standard sense seems to be useful. In this case there exists a strong connection between the equivalent algebraic semantics and the behavioral equivalent algebraic semantics, which can help to shed some new light on the algebraic counterpart of the logic. For instance, in the case of Nelson's logic [31], behavioral algebraization helps to understand better the connection between N-lattices and Heyting algebras.

Other interesting examples we have started studying are those in the family of exogenous (global, probabilistic and quantum) logics stemming from [28]. Indeed, in order to make the extended theory useful and assess its merits in full, a comprehensive treatment of interesting examples is essential. Important examples are those separating classes in the behavioral Leibniz hierarchy, one of the main tasks for future work. But this paper raises many other interesting questions, both technical and methodological.

With respect to the technical development, several issues should deserve a closer look. For instance, behaviorally, we can no longer guarantee uniqueness with respect to the possible distinct equivalences in a logic. We would like to engage in an exhaustive study of the relationship between existing possible equivalence sets for a given logic, and their impact on the distinct behavioral algebraizations of the same logic that can be obtained using distinct and non-interderivable equivalences. Another topic that deserves further analysis is the precise role of the parametric variables of the contexts. On the methodological side, this paper is just a starting point towards a full-blown behavioral theory of AAL. Such a development will need time and effort to be consolidated. Still, it is possible to put forth a few directions that should clearly be pursued. For example, the definition of behaviorally equivalential logic presented here is syntactic, but we are convinced that model theoretic characterizations (closure properties of the class of reduced models), similar to the standard ones, could be established. The theory certainly needs many more semantic results, namely involving metalogical properties of a given behaviorally algebraizable logic and algebraic properties of the class of algebras associated with it. With respect to the connections to matrix semantics, we have already obtained some important results, which we postpone to a forthcoming paper. Namely, we were able to show that a logic is behaviorally algebraizable with behaviorally equivalent algebraic semantics K if and only if the behavioral Leibniz operator is an isomorphism between the lattice of filters and the K-congruences. This result is very useful when one wants to show that a logic is not behaviorally algebraizable. We are also able to associate to a behaviorally algebraizable logic a class Alg^* which can be shown to coincide with K. Although promising, this semantic generalization is not completely smooth since we are not able to associate to a logic a class Mod^* of reduced matrix models. This is due to the fact that, since we are dealing with Γ-congruences, we cannot perform quotients. The best we can do is to use the extended signature and algebras over the extended signature to simulate the quotient. This mismatch clearly leads to the exploration of more suitable alternatives, including the theory of valuations [17], Avron's non-deterministic matrices [1], or perhaps even gaggles [18]. Some preliminary results on adopting a suitable algebraic version of valuations for semantical considerations in the behavioral setting are reported in [8]. In another direction, stemming from the worked examples but aiming more at the core of AAL, our approach seems to be a useful tool for studying in abstract terms the interplay between systems of equivalence and the detachment deduction theorem, and therefore contributing to a better understanding of the Fregean hierarchy.

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