Decidability and complexity of fibred logics without shared connectives*

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Abstract

Fibring is a powerful mechanism for combining logics, and an essential tool for designing and understanding complex logical systems. Abstract results about the semantics and proof-theory of fibred logics have been extensively developed, including general preservation results for metalogical properties like soundness and (sufficient conditions for) completeness. Decidability, however, a key ingredient for the automated support of the fibred logic, has not deserved similar attention.

In this paper, we extend the preliminary results obtained in [28] regarding theoremhood, and address the problem of deciding the consequence relation of fibred logics, assuming that the logics do not share connectives. Namely, under this assumption, we provide a thorough characterization of the mixed patterns of reasoning that can occur in the fibred logic, and use it to obtain the first general decidability preservation result for fibring. The complexity of the decision procedure we obtain is also analyzed.

1 Introduction

Fibring is a powerful and appealing mechanism for combining logics, proposed in its general form by Dov Gabbay in [18, 19]. Owing to its fundamental character, abstract formulation and compositional nature, fibring is a key ingredient of the general theory of universal logic [5, 6]. Due to the ubiquity of its underlying problems, fibring is also a valuable tool for the construction and analysis of complex logics, a subject of ever growing importance in application fields like software engineering and artificial intelligence (see, for instance, the FroCoS series of events and publications in [17]).

Given two logics $\mathcal{L}_1$ and $\mathcal{L}_2$, fibring combines $\mathcal{L}_1$ and $\mathcal{L}_2$ into the smallest logical system for the combined language which extends both $\mathcal{L}_1$ and $\mathcal{L}_2$ (see [9]). Despite the long track of work on fibred logics, leading to a substantial understanding of the semantics and proof-theory of combined logics (see [3, 8, 10, 34, 36, 41]), automated support for fibred logics is still non-existent. This happens, in particular, because decidability preservation results are lacking. The only general result related to (but still distant from) the decidability of fibred logics is [13], where the preservation by fibring of the semantic notion of *finite model property* is studied. There is also a proof of decidability preservation of fusion (fibring) of modal logics [40, 20], but which uses ideas and results from modal semantics that cannot be easily generalized.

Indeed, at first sight, one might be tempted to use semantic arguments to address the decidability of fibred logics. However, this semantic path is far from clear, as we do not know of a generally satisfactory semantic counterpart of fibring that naturally relates models of the component logics with models of the combined logic. Similarly, decision results about combined equational theories [31, 30, 37, 2, 38, 33] do not help, in general, first of all because the component logics do not need be algebraizable (as in [7]), and ultimately because the fibred logic may fail to be algebraizable even if the component logics are [25, 16].

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Therefore, we shall address the decidability preservation problem directly, by taking advantage of a complete characterization of the mixed patterns of reasoning that may emerge in fibred logics, which we develop also in this paper. For the purpose, and given the inherent difficulty of the problem, we restrict our attention to the simpler case of *disjoint fibring*, where the logics being combined do not share connectives, and obtain a complete description of what can be derived from any set of hypotheses in the fibred system, based on what can be derived in the component logics being fibred. This characterization extends the partial results of [28] about reasoning without hypotheses, and of [27] about reasoning from non-mixed hypotheses. Although this characterization result is at the very heart of our decidability preservation result, its importance goes far beyond this application, as we discuss in the concluding remarks. In order not to take the focus of the paper from its main subject, decidability preservation, we decided to leave the (very technical) proof of our mixed reasoning characterization result to the appendix. Still, owing to the relevance of the results, the appendix is written not as a technical annex but as a section *per se*.

The main contribution of this paper, building upon the above mentioned characterization of mixed reasoning, is a full account of the decidability of disjointly fibred logics. Our proof of decidability preservation is constructive, in the sense that we show how to put up a decision procedure for the combined logic using decision procedures for the component logics. Further, we study the complexity of the decision procedure thus obtained, and show that it only worsens the added complexity of the components by a polynomial factor. Additionally, the other way around, we also provide a complete analysis of the reflection of decidability from disjoint fibring to the component logics. This paper is the third of a series of papers [28, 27] where we attacked the problem of understanding the mechanism of fibring the *harder way*, by trying to capture more and more about how a disjointly fibred logic emerges from its component logics. In particular, our decidability preservation result directly generalizes the result of [28] about deciding theoremhood. Note, however, that deciding theoremhood is enough in logics whose consequence relation can be characterized by its theorems, e.g., via some form of *deduction theorem*.

In Section 2, we shall recall or introduce a number of relevant notions, and fix notation to be used in the remainder of the paper. Section 3 provides a short introduction to fibring, analyses mixed reasoning in fibred logics, and states our characterization result about mixed reasoning for disjoint fibring. Section 4 is devoted to our decidability preservation results, and to a detailed study of the complexity of the decision procedures obtained. In Section 5, we draw conclusions, and discuss directions for further research. Finally, in the Appendix, we present in detail the results that characterize mixed reasoning in disjointly fibred logics, of which our account of decidability can be seen as an application.

2 Preliminaries

In this section we recall the essential concepts that we are dealing with in this paper, and introduce some useful notions, notations, and simple results.

2.1 Syntax

A *signature* is a $\mathbb{N}_0$-indexed family $\Sigma = \{\Sigma^{(n)}\}_{n\in\mathbb{N}_0}$ of sets. The elements of $\Sigma^{(n)}$ are dubbed n-place *connectives*. Being indexed families of sets, the usual set-theoretic notions can be smoothly extended to signatures. We will sometimes abuse notation, and confuse Σ with the set $(\biguplus_{n\in\mathbb{N}_0}\Sigma^{(n)})$ of all its connectives, and write $c \in \Sigma$ when c is some n-place connective $c \in \Sigma^{(n)}$. For this reason, the *empty signature*, with no connectives at all, will be simply denoted by $\emptyset$.

Let Σ, Σ' be two signatures. We say that Σ is a *subsignature* of Σ', and write $\Sigma \subseteq \Sigma'$, whenever $\Sigma^{(n)} \subseteq \Sigma'^{(n)}$ for every $n \in \mathbb{N}_0$. Expectedly, we can also define the *intersection* $\Sigma\cap\Sigma' = \{\Sigma^{(n)}\cap\Sigma'^{(n)}\}_{n\in\mathbb{N}_0}$, *union* $\Sigma \cup \Sigma' = \{\Sigma^{(n)} \cup \Sigma'^{(n)}\}_{n\in\mathbb{N}_0}$, and *difference* $\Sigma' \setminus \Sigma = \{\Sigma'^{(n)} \setminus \Sigma^{(n)}\}_{n\in\mathbb{N}_0}$ of signatures. Clearly, $\Sigma \cap \Sigma'$ is the largest subsignature of both Σ and Σ', and contains the connectives *shared* by Σ and Σ'. When there are no shared connectives we have that $\Sigma \cap \Sigma' = \emptyset$. Analogously, $\Sigma \cup \Sigma'$ is the smallest signature that has both Σ and Σ' as subsignatures, and features all the connectives from both Σ and Σ' in a *combined signature*. Furthermore, $\Sigma' \setminus \Sigma$ is the largest subsignature of Σ' which does not share any connectives with Σ.

Given a signature Σ and a set P of *variables*, the generated set of *formulas* is the carrier set $L_\Sigma(P)$ of the free Σ-algebra generated by P. In the sequel, we shall assume that signatures are countable and sets of variables are denumerable. We assume fixed a denumerable set P of variables. If Σ is a countable signature then $L_\Sigma(P)$ is clearly denumerable.

We define the *set of variables occurring in* φ to be either $\mathsf{var}(\varphi) = \{p\}$ when $\varphi = p \in P$, or $\mathsf{var}(\varphi) = \bigcup_{i=1}^{n} \mathsf{var}(\varphi_n)$ when $\varphi = c(\varphi_1, \ldots, \varphi_n)$ for formulas $\varphi_1, \ldots, \varphi_n \in L_\Sigma(P)$ and $c \in \Sigma^{(n)}$. Analogously, we also define the *set of subformulas of a formula* φ to be such that $\mathsf{sub}(\varphi) = \{p\}$ when $\varphi = p \in P$, and $\mathsf{sub}(\varphi) = \{\varphi\} \cup \bigcup_{i=1}^{n} \mathsf{sub}(\varphi_n)$ when $\varphi = c(\varphi_1, \ldots, \varphi_n)$ for formulas $\varphi_1, \ldots, \varphi_n \in L_\Sigma(P)$ and $c \in \Sigma^{(n)}$. We extend these notations to sets of formulas in the obvious way.

As usual, we further define the *size* of a formula to be such that $\mathsf{size}(\varphi) = 1$ when $\varphi = p \in P$, and $\mathsf{sub}(\varphi) = 1 + \sum_{i=1}^{n} \mathsf{size}(\varphi_n)$ when $\varphi = c(\varphi_1, \ldots, \varphi_n)$ for formulas $\varphi_1, \ldots, \varphi_n \in L_\Sigma(P)$ and $c \in \Sigma^{(n)}$. For $\Gamma \subseteq L_\Sigma(P)$, we further define $\mathsf{size}(\Gamma) = \sum_{\varphi \in \Gamma} \mathsf{size}(\varphi)$.

If $\varphi \in L_\Sigma(P)$ then we define the *head of* φ to be either $\mathsf{head}(\varphi) = p$ when $\varphi = p \in P$, or $\mathsf{head}(\varphi) = c$ when $\varphi = c(\varphi_1, \ldots, \varphi_n)$ for formulas $\varphi_1, \ldots, \varphi_n \in L_\Sigma(P)$ and $c \in \Sigma^{(n)}$. Clearly, if $\Sigma \subseteq \Sigma'$ and $P \subseteq P'$ then $L_\Sigma(P) \subseteq L_{\Sigma'}(P')$. Of course, given $\psi \in L_{\Sigma'}(P')$, $\mathsf{head}(\psi)$ may not be in Σ nor P.

Let $\Sigma \subseteq \Sigma'$ and $P \subseteq P'$. A Σ'-*substitution* is a function $\sigma : P \to L_{\Sigma'}(P')$, which extends freely to a function $\sigma : L_\Sigma(P) \to L_{\Sigma'}(P')$. Given a formula $\varphi \in L_\Sigma(P)$, $\sigma(\varphi)$ is the *instance* of φ by σ, sometimes denoted simply by φ^σ, and is the result of uniformly replacing each variable $p \in P$ occurring in φ by $\sigma(p)$. When $\Gamma \subseteq L_\Sigma(P)$ we use Γ^σ to denote $\{\varphi^\sigma : \varphi \in \Gamma\}$. This allows us to define the composition of substitutions in the obvious way: given a substitution $\rho : P \to L_{\Sigma'}(P')$ the *composition* $\rho \circ \sigma$ is the substitution such that $(\rho \circ \sigma)(p) = \sigma(p)^\rho$ for each $p \in P$.

2.2 Logical consequence

A *logic (over signature* Σ*)* is a tuple $\mathcal{L} = \langle \Sigma, \_^{\vdash} \rangle$, where $\_^{\vdash} : 2^{L_\Sigma(P)} \to 2^{L_\Sigma(P)}$ is a consequence operator (see [39], for instance), that is, it satisfies the following properties:

$$\begin{array}{ll} \Gamma \subseteq \Gamma^{\vdash} & (\textit{extensiveness}) \\ \Gamma^{\vdash} \subseteq (\Gamma \cup \Delta)^{\vdash} & (\textit{monotonicity}) \\ (\Gamma^{\vdash})^{\vdash} \subseteq \Gamma^{\vdash} & (\textit{idempotence}) \\ (\Gamma^{\vdash})^{\sigma} \subseteq (\Gamma^{\sigma})^{\vdash} & (\textit{structurality}) \end{array}$$

for every $\Gamma, \Delta \subseteq L_\Sigma(P)$ and $\sigma : P \to L_\Sigma(P)$. Note that we do not require, in general, that the logic is *finitary*, i.e., it may happen that $\Gamma^{\vdash}$ properly contains the union of all $\Gamma_0^{\vdash}$ for finite $\Gamma_0 \subseteq \Gamma$. Meaningful examples of logics that will be used throughout the paper will be presented below.

As usual, we shall confuse the consequence operator with its induced Tarskian consequence relation. Thus, given $\varphi \in L_\Sigma(P)$, we will write $\Gamma \vdash \varphi$ whenever $\varphi \in \Gamma^{\vdash}$. When $\Gamma = \{\varphi_1, \ldots, \varphi_n\}$ is finite we write $\varphi_1, \ldots, \varphi_n \vdash \varphi$ instead of $\{\varphi_1, \ldots, \varphi_n\} \vdash \varphi$. Moreover, as usual, if $\Gamma = \emptyset$ we write $\vdash \varphi$ instead of $\emptyset \vdash \varphi$, and dub φ a *theorem* of $\mathcal{L}$. A formula φ that is not a theorem of $\mathcal{L}$ but such that $\psi \vdash \varphi$ for any $\psi \in L_\Sigma(P)$ is dubbed a *quasi-theorem*, or simply a *q-theorem*. Clearly, φ is a q-theorem of $\mathcal{L}$ provided that $\nvdash \varphi$ but $p \vdash \varphi$ for some $p \in P$ that does not occur in φ. It is immediate that a logic cannot both have theorems and q-theorems.

We shall call any $\Gamma \subseteq L_\Sigma(P)$ such that $\Gamma = \Gamma^{\vdash}$ a *theory* of $\mathcal{L}$, and denote the set of all theories of $\mathcal{L}$ by $\mathbf{Th}(\mathcal{L})$. It is well known that $\mathbf{Th}(\mathcal{L})$ constitutes a complete lattice under the inclusion ordering (see [39], for instance). The bottom theory of the lattice is $\emptyset^{\vdash}$, whereas the top theory is $L_\Sigma(P)$, also called the *inconsistent* theory. When $(\Gamma^\sigma)^{\vdash}$ is inconsistent for every substitution $\sigma : P \to L_\Sigma(P)$ we say that Γ is $\vdash$-*explosive*.

Note that the notion of explosion we consider is slightly more elaborate than is usually presented. Clearly, given $p \in P \setminus \mathsf{var}(\Gamma)$, we have that Γ is $\vdash$-explosive if and only if $\Gamma \vdash p$. However, our definition is subtly different when $P = \mathsf{var}(\Gamma)$. For instance, in a minimal logic $\langle \Sigma, \_^{\vdash} \rangle$ such that $\Gamma^{\vdash} = \Gamma$ for all $\Gamma \subseteq L_\Sigma(P)$, the set $L_\Sigma(P)$ is not $\vdash$-explosive despite the fact that $L_\Sigma(P)^{\vdash} = L_\Sigma(P)$ is inconsistent. This is due to the fact that the logic does not encompass a mechanism to infer an arbitrary formula. Clarifying this aspect, explosiveness can be given the following alternative characterization.

Lemma 2.1. *Let* $\mathcal{L} = \langle \Sigma, \_^{\vdash} \rangle$ *be a logic,* $p_0, p_1, p_2, \ldots$ *an enumeration of* P*,* $P \cap Y = \emptyset$*, and consider* *the substitution* $\mathsf{nxt} : P \cup Y \to L_\Sigma(P \cup Y)$ *such that* $\mathsf{nxt}(p_k) = p_{k+1}$ *and* $\mathsf{nxt}(y) = y$ *if* $y \in Y$.

If $\mathcal{L} = \langle \Sigma, \_^{\vdash} \rangle$ *is a logic and* $\Gamma \cup \{\psi\} \subseteq L_\Sigma(P \cup Y)$ *then* Γ *is* $\vdash$*-explosive if and only if* $\Gamma^{\mathsf{nxt}} \vdash p_0$.

Moreover, if Γ *is* $\vdash$*-explosive then* Γ^σ *is also* $\vdash$*-explosive, for every substitution* $\sigma : P \cup Y \to L_\Sigma(P \cup Y)$.

Additionally, we also have that $\Gamma \vdash \psi$ *if and only if* $\Gamma^{\mathsf{nxt}} \vdash \psi^{\mathsf{nxt}}$.

Proof. Let $\sigma : P \cup Y \to L_\Sigma(P \cup Y)$ be a substitution and $\varphi \in L_\Sigma(P \cup Y)$, and define $\mathsf{prv}_{\langle \varphi, \sigma \rangle} : P \cup Y \to L_\Sigma(P \cup Y)$ such that $\mathsf{prv}_{\langle \varphi, \sigma \rangle}(p_0) = \varphi$, $\mathsf{prv}_{\langle \varphi, \sigma \rangle}(p_{k+1}) = \sigma(p_k)$, and $\mathsf{prv}_{\langle \varphi, \sigma \rangle}(y) = y$ for $y \in Y$.

Concerning the first statement, if Γ is $\vdash$-explosive then, in particular, $\Gamma^{\mathsf{nxt}} \vdash p_0$. Reciprocally, let us assume that $\Gamma^{\mathsf{nxt}} \vdash p_0$ and take any substitution σ and formula $\varphi \in L_\Sigma(P)$. By structurality, we get that $(\Gamma^{\mathsf{nxt}})^{\mathsf{prv}_{\langle \varphi, \sigma \rangle}} \vdash \mathsf{prv}_{\langle \varphi, \sigma \rangle}(p_0)$. Easily, $(\Gamma^{\mathsf{nxt}})^{\mathsf{prv}_{\langle \varphi, \sigma \rangle}} = \Gamma^\sigma$, $\mathsf{prv}_{\langle \varphi, \sigma \rangle}(p_0) = \varphi$, and we conclude that $\Gamma^\sigma \vdash \varphi$.

Easily, $(\Gamma^\sigma)^\rho = \Gamma^{(\rho \circ \sigma)}$ for every substitution ρ. Thus, it is straightforward to conclude that Γ^σ is $\vdash$-explosive whenever Γ is.

If $\Gamma \vdash \psi$ then, by structurality of $\vdash$, we also have $\Gamma^{\mathsf{nxt}} \vdash \psi^{\mathsf{nxt}}$. Reciprocally, assume that $\Gamma^{\mathsf{nxt}} \vdash \psi^{\mathsf{nxt}}$ and let $\mathsf{prv} = \mathsf{prv}_{\langle \varphi, \mathsf{id} \rangle}$ where id is the identity substitution and φ is any formula. By structurality, we get that $(\Gamma^{\mathsf{nxt}})^{\mathsf{prv}} \vdash (\psi^{\mathsf{nxt}})^{\mathsf{prv}}$. Easily, $(\Gamma^{\mathsf{nxt}})^{\mathsf{prv}} = \Gamma$, $(\psi^{\mathsf{nxt}})^{\mathsf{prv}} = \psi$, as prv is a left inverse of nxt,and we conclude that $\Gamma \vdash \psi$. ☐

A logic $\mathcal{L} = \langle \Sigma, \_^{\vdash} \rangle$ is said to be *consistent* if $\emptyset^{\vdash} \neq L_\Sigma(P)$. Clearly, $\mathcal{L}$ is *inconsistent* (not consistent) precisely when $\vdash p$ for some $p \in P$, or alternatively when $\mathbf{Th}(\mathcal{L}) = \{L_\Sigma(P)\}$. $\mathcal{L}$ is said to be *trivial*, if for all non-empty $\Gamma \subseteq L_\Sigma(P)$ we have $\Gamma^{\vdash} = L_\Sigma(P)$. Equivalently, $\mathcal{L}$ is trivial if there exist distinct variables $p, q \in P$ such that $p \vdash q$. Another equivalent characterization is that $\mathcal{L}$ is trivial if $\mathbf{Th}(\mathcal{L}) \subseteq \{\emptyset, L_\Sigma(P)\}$. Of course, all inconsistent logics are trivial. Moreover, easily, a trivial logic is consistent if and only if it has a q-theorem, if and only if all formulas are q-theorems, if and only if it has no theorems.

We say that a logic $\mathcal{L}' = \langle \Sigma', \_^{\vdash'} \rangle$ *extends* $\mathcal{L} = \langle \Sigma, \_^{\vdash} \rangle$ if $\Sigma \subseteq \Sigma'$, and $\vdash \subseteq \vdash'$, in the sense that $\Gamma^{\vdash} \subseteq \Gamma^{\vdash'}$ for every $\Gamma \subseteq L_\Sigma(P)$. We say that the extension of $\mathcal{L}$ by $\mathcal{L}'$ is *conservative* if for all $\Gamma \subseteq L_\Sigma(P)$, $\Gamma^{\vdash} = \Gamma^{\vdash'} \cap L_\Sigma(P)$. It is perhaps more common to express these properties in terms of the induced consequence relations. Clearly, $\mathcal{L}'$ extends $\mathcal{L}$ when $\Gamma \vdash \varphi$ implies $\Gamma \vdash' \varphi$ for all $\Gamma \cup \{\varphi\} \subseteq L_\Sigma(P)$. Furthermore, the extension is conservative precisely when $\Gamma \vdash \varphi$ if and only if $\Gamma \vdash' \varphi$.

2.3 Decidable logics

There are several different flavors of *decidability* that make sense when applied to a logic $\mathcal{L} = \langle \Sigma, \_^{\vdash} \rangle$, some more standard than others (see, for instance, [39]). Herein, we say that $\mathcal{L}$ is *decidable* if there is an algorithm D such that, for each finite set $\Gamma \subseteq L_\Sigma(P)$ outputs an algorithm $\mathsf{D}(\Gamma)$, which given any formula $\psi \in L_\Sigma(P)$ always terminates and outputs:

$$\mathsf{D}(\Gamma)(\psi) = \begin{cases} \text{yes}, & \text{if } \Gamma \vdash \psi, \\ \text{no}, & \text{if } \Gamma \not\vdash \psi. \end{cases}$$

Of course, with this definition, we are deciding only the finitary part of $\mathcal{L}$.

There many well known decidable logics, like classical or intuitionistic propositional logics. Any trivial logic is also (trivially) decidable. However, many interesting logics are undecidable, such as Anderson and Belnap's logic of *entailment* $\mathbf{E}$ [39, 1], Belnap's system $\mathbf{R}$ of *relevant implication* [4], or several many-dimensional modal logics, e.g., $\mathbf{K}^3 = \mathbf{K} \times \mathbf{K} \times \mathbf{K}$ or $[\mathbf{K4}, \mathbf{K4}] = \mathbf{K4} \times \mathbf{K4}$ [26, 20, 22].

2.4 (Trans)finite sequences

Along the paper, we will need to deal with (not necessarily finite) sequences of objects. Typically, these objects will be formulas of some logic, and the sequences will correspond to deductions in that logic. Let A be a set (of objects). Given an ordinal η, we use $\overline{a} = \langle a_\kappa \rangle_{\kappa < \eta}$ to denote a η-long sequence of elements of A, or simply a η-sequence, understood as a function from $\{\kappa : \kappa < \eta\}$ to A. As usual, if $\tau \leq \eta$, the sequence $\langle a_\kappa \rangle_{\kappa < \tau}$ will be dubbed a *prefix* of $\overline{a}$.

Note that when η is a limit ordinal, a η-sequence does not have a last element. On the contrary, if η is a successor ordinal, and in particular a finite ordinal, then a η-sequence $\overline{a} = \langle a_\kappa \rangle_{\kappa < \eta}$ can be understood as $a_0, a_1, \ldots, a_{\eta-1}$, and may also be represented by $\langle a_\kappa \rangle_{\kappa \leq \eta - 1}$. The 0-sequence (*empty* sequence) is simply not represented.

The η-sequence $\overline{a} = \langle a_\kappa \rangle_{\kappa<\eta}$ is said to be *injective/surjective* if it is injective/surjective as a function. We will dub as $\overline{A}$ any injective and surjective sequence based on A.

2.5 Hilbert calculi

A *Hilbert calculus* is a pair $\mathscr{H} = \langle \Sigma, R \rangle$ where Σ is a signature, and $R \subseteq 2^{L_\Sigma(P)} \times L_\Sigma(P)$ is a set of *inference rules*. Given $\langle \Delta, \psi \rangle \in R$, we refer to Δ as the set of *premises* and to ψ as the *conclusion* of the rule. When the set of premises is empty, ψ is dubbed an *axiom*. A rule is said to be *finitary* if it has a finite set of premises, and $\mathscr{H}$ is said to be *finitary* of all the rules in R are finitary. Note that we do not impose finitariness, in this paper, though most of the examples used are finitary. An inference rule $\langle \Delta, \psi \rangle \in R$ is often denoted by $\frac{\Delta}{\psi}$, or simply by $\frac{\psi_1 \;\ldots\; \psi_n}{\psi}$ if $\Delta = \{\psi_1, \ldots, \psi_n\}$ is finite, or by $\frac{}{\psi}$ if $\Delta = \emptyset$.

Given $\Sigma \subseteq \Sigma'$ and $P \subseteq P'$, a Hilbert calculus $\mathscr{H} = \langle \Sigma, R \rangle$ induces a consequence operator $\_^{\vdash_{\mathscr{H}}}$ on $L_{\Sigma'}(P')$ such that, for each $\Gamma \subseteq L_{\Sigma'}(P')$, $\Gamma^{\vdash_{\mathscr{H}}}$ is the least set that contains Γ and is closed for all applications of instances of the inference rules in R, that is, if $\frac{\Delta}{\psi} \in R$ and $\sigma : P \to L_{\Sigma'}(P')$ is such that $\Delta^\sigma \subseteq \Gamma^{\vdash_{\mathscr{H}}}$ then $\psi^\sigma \in \Gamma^{\vdash_{\mathscr{H}}}$. Of course, this definition induces a logic $\langle \Sigma', \_^{\vdash_{\mathscr{H}}} \rangle$ for each $\Sigma \subseteq \Sigma'$. By abuse of notation, we dub all such logics as $\mathcal{L}_{\mathscr{H}}$. Note that $\mathcal{L}_{\mathscr{H}}$ is a finitary logic whenever $\mathscr{H}$ is finitary.

The definition of $\mathcal{L}_{\mathscr{H}}$ above is arguably too abstract, as it does not highlight the sequence of rule applications that leads one to conclude that $\Gamma \vdash_{\mathscr{H}} \varphi$, when that is the case. Let us be more detailed. Given $\Sigma \subseteq \Sigma'$, $P \subseteq P'$, and $\Gamma \subseteq L_{\Sigma'}(P')$, a *$\mathscr{H}$-derivation from* Γ is a (not necessarily finite) sequence $\overline{\varphi} = \langle \varphi_\kappa \rangle_{\kappa<\eta}$ of formulas in $L_{\Sigma'}(P')$, for some ordinal η, such that, for each $\kappa < \eta$, either $\varphi_\kappa \in \Gamma$, or there is $\frac{\Delta}{\psi} \in R$ and $\sigma : P \to L_{\Sigma'}(P')$ with $\psi^\sigma = \varphi_\kappa$ and $\Delta^\sigma \subseteq \{\varphi_\tau : \tau < \kappa\}$.

The fact that $\overline{\varphi}$ is a $\mathscr{H}$-derivation from Γ is denoted by $\Gamma \vdash_{\mathscr{H}} \overline{\varphi}$. We say that such a derivation is a *$\mathscr{H}$-proof from* Γ of each of its formulas, as it is clear that any prefix of a $\mathscr{H}$-derivation from Γ is also a $\mathscr{H}$-derivation from Γ.

Clearly, $\Gamma \vdash_{\mathscr{H}} \varphi$ precisely if φ has a $\mathscr{H}$-proof from Γ, that is, there exists some $\mathscr{H}$-derivation $\langle \varphi_\kappa \rangle_{\kappa<\eta}$ from Γ such that $\varphi = \varphi_\kappa$ for some $\kappa < \eta$. Of course, in that case, $\langle \varphi_\iota \rangle_{\iota\leq\kappa}$ is a $\mathscr{H}$-proof of φ from Γ ending in φ.

Given a logic $\mathcal{L} = \langle \Sigma, \_^{\vdash} \rangle$, we can easily associate with it a Hilbert calculus $\mathscr{H}_{\mathcal{L}} = \langle \Sigma, \vdash \rangle$, where the consequence operator $\_^{\vdash}$ in the former is replaced by the induced consequence relation $\vdash$ (seen as a set of rules) in the latter. It is easy to check that $\mathcal{L}_{\mathscr{H}_{\mathcal{L}}} = \mathcal{L}$ (see [39], for instance).

When a Hilbert system is identified with a subscript $\mathscr{H} = \mathscr{H}_{\mathsf{name}}$ we drop the $\mathscr{H}$ in $\vdash_{\mathscr{H}_{\mathsf{name}}}$ and write just $\vdash_{\mathsf{name}}$. Analogously, we will use $\mathcal{L}_{\mathsf{name}}$ to denote the logic $\mathcal{L}_{\mathscr{H}_{\mathsf{name}}}$.

3 Fibred logics

In this section we recall the definition of fibred logics, explore their mixed syntax, and state our characterization result for mixed reasoning in the case of disjoint fibring.

3.1 Fibring

Let $\mathcal{L}_1 = \langle \Sigma_1, \_^{\vdash_1} \rangle$ and $\mathcal{L}_2 = \langle \Sigma_2, \_^{\vdash_2} \rangle$ be two logics. The *fibring* of $\mathcal{L}_1$ and $\mathcal{L}_2$ is the smallest logic $\mathcal{L}_1 \bullet \mathcal{L}_2$ over the joint signature $\Sigma_{12} = \Sigma_1 \cup \Sigma_2$ that extends both $\mathcal{L}_1$ and $\mathcal{L}_2$. A direct characterization of this fibred logic can be most easily given by first defining the fibring of Hilbert calculi.

Given Hilbert calculi $\mathscr{H}_1 = \langle \Sigma_1, R_1 \rangle$ and $\mathscr{H}_2 = \langle \Sigma_2, R_2 \rangle$ let their *fibring* be the Hilbert calculus

$$\mathscr{H}_1 \bullet \mathscr{H}_2 = \langle \Sigma_{12}, R_1 \cup R_2 \rangle.$$

Clearly, besides joining the given signatures, which will allow us to build so-called *mixed formulas*, the fibring of the two calculi consists in simply putting together their rules, thus allowing a form of *mixed reasoning*.

We can now give a simple characterization of the fibring of two logics $\mathcal{L}_1$ and $\mathcal{L}_2$:

$$\mathcal{L}_1 \bullet \mathcal{L}_2 = \mathcal{L}_{\mathscr{H}_{\mathcal{L}_1} \bullet \mathscr{H}_{\mathcal{L}_2}}.$$

This mean that if $\mathcal{L}_1 \bullet \mathcal{L}_2 = \langle \Sigma_{12}, \_^{\vdash_{12}} \rangle$ then, given $\Gamma \subseteq L_{\Sigma_{12}}(P)$, $\Gamma^{\vdash_{12}}$ is obtained by a (possibly transfinite) sequence of alternate applications of $\vdash_1$ and $\vdash_2$ using substitutions $\sigma : P \to L_{\Sigma_{12}}(P)$.

Both for logics and Hilbert calculi, when there are no shared connectives, i.e. $\Sigma_1 \cap \Sigma_2 = \emptyset$, the fibring is usually said to be *unconstrained*, or *disjoint*.

It is worth taking a look at some simple examples in order to better understand how mixed reasoning emerges in fibred logics.

Example 3.1. Let us first consider an example of disjoint fibring, starting from the Hilbert calculi:

- $\mathscr{H}_{\mathsf{neg}} = \langle \Sigma_{\mathsf{neg}}, R_{\mathsf{neg}} \rangle$, where Σ_{neg} has a unique 1-place connective $\neg$, and R_{neg} has the rules

$$\frac{p}{\neg\neg p} \qquad \frac{\neg\neg p}{p} \qquad \frac{p \quad \neg p}{q},$$

- $\mathscr{H}_{\mathsf{cnj}} = \langle \Sigma_{\mathsf{cnj}}, R_{\mathsf{cnj}} \rangle$, where Σ_{cnj} has a unique 2-place connective $\wedge$, and R_{cnj} has the rules

$$\frac{p \wedge q}{p} \qquad \frac{p \wedge q}{q} \qquad \frac{p \quad q}{p \wedge q}.$$

Clearly, $\mathcal{L}_{\mathsf{neg}}$ and $\mathcal{L}_{\mathsf{cnj}}$ correspond, respectively, to the negation-only and conjunction-only fragments of classical logic. Their (disjoint) fibring $\mathcal{L}_{\mathsf{neg}} \bullet \mathcal{L}_{\mathsf{cnj}} = \langle \Sigma_{\mathsf{neg}} \cup \Sigma_{\mathsf{cnj}}, \_^{\vdash} \rangle$ is, however, a strict subsystem of classical logic. For instance, one has $\neg\neg(p \wedge (\neg\neg q)) \vdash q \wedge (\neg\neg p)$, but $\neg\neg\neg(p \wedge p) \nvdash \neg p$. Checking that indeed $\neg\neg(p \wedge (\neg\neg q)) \vdash q \wedge (\neg\neg p)$ is quite straightforward. Let us look at the mixed-derivation below, in the center, as well as at how one can view the derivation from the local perspectives of $\mathcal{L}_{\mathsf{neg}}$ and $\mathcal{L}_{\mathsf{cnj}}$, in grey, on the sides.

$\mathcal{L}_{\mathsf{neg}}$		$\mathcal{L}_{\mathsf{neg}} \bullet \mathcal{L}_{\mathsf{cnj}}$		$\mathcal{L}_{\mathsf{cnj}}$	
$\neg\neg\, x_{p \wedge (\neg\neg q)}$	H	$\neg\neg(p \wedge (\neg\neg q))$	H	$x_{\neg\neg(p \wedge (\neg\neg q))}$	H
$x_{p \wedge (\neg\neg q)}$	$\neg$-R	$p \wedge (\neg\neg q)$	$\neg$-R	$p \wedge x_{\neg\neg q}$	H*
p	H*	p	$\wedge$-R	p	$\wedge$-R
$\neg\neg q$	H*	$\neg\neg q$	$\wedge$-R	$\neg\neg q$	$\wedge$-R
$\neg\neg p$	$\neg$-R	$\neg\neg p$	$\neg$-R	$x_{\neg\neg p}$	H*
q	$\neg$-R	q	$\neg$-R	q	H*
$x_{q \wedge (\neg\neg p)}$	H*	$q \wedge (\neg\neg p)$	$\wedge$-R	$q \wedge x_{\neg\neg p}$	$\wedge$-R

The proof sequence at the center is simple, and reaches the conclusion by departing from the given hypothesis, indicated by H, using interleaved applications of rules from $\mathcal{L}_{\mathsf{neg}}$ and $\mathcal{L}_{\mathsf{cnj}}$, indicated by $\neg$-R and $\wedge$-R, respectively.

If one thinks only from the side of $\mathcal{L}_{\mathsf{neg}}$, however, it is clear that conjunction formulas correspond to unknowns, which we represent in skeletal form by using x_φ instead of φ. More concretely, x_φ represents an atomization of φ, understood as a monolith, in the sense that φ's head connective does not belong to Σ_{neg}, and therefore no $\neg$-R rule can take advantage of φ's structure. As a consequence, each step of the proof sequence that corresponds to a $\wedge$-R simply introduces an additional hypothesis, indicated by H*. The same thing happens, *mutatis mutandis*, on the $\mathcal{L}_{\mathsf{cnj}}$-side with respect to negated formulas and $\neg$-R rules.

Showing that $\neg\neg\neg(p \wedge p) \nvdash \neg p$ is expectedly harder. One could use a number of more or less common semantic tricks, but our aim here is to be able to prove such negative properties directly. For the moment, just note that, precisely because of the monolithical nature of a formula with head connective from one logic when looked at from the point of view of the other, the hypothesis $\neg\neg\neg(p \wedge p)$ will essentially unfold to $\neg^{2n+1}(p \wedge p)$ with $n \in \mathbb{N}_0$, and conjunctions thereof, which is clearly insufficient to access the $(p \wedge p)$ that will remain protected by at least one negation. Below, we will provide a rigorous account of this fact, which will take advantage, in an essential way, of the fact that the signatures of the component logics are disjoint. △

Disjointness is the key ingredient that guarantees that, as the example above suggests, the interaction between the component logics is only relevant with respect to monoliths (subformulas) of the formulas given. To emphasize this fact, let us analyze a different example.

Example 3.2. For a simple but interesting example of non-disjoint fibring, let us consider the signature Σ_{tonk} with a unique 2-place connective tonk, and the following Hilbert calculi:

- $\mathscr{H}_{\mathsf{tonk(intr)}} = \langle \Sigma_{\mathsf{tonk(intr)}}, R_{\mathsf{tonk(intr)}} \rangle$, where $\Sigma_{\mathsf{tonk(intr)}} = \Sigma_{\mathsf{tonk}}$ and $R_{\mathsf{tonk(intr)}}$ has the rule

$$\frac{p}{\mathsf{tonk}(p,q)},$$

- $\mathscr{H}_{\mathsf{tonk(elim)}} = \langle \Sigma_{\mathsf{tonk(elim)}}, R_{\mathsf{tonk(elim)}} \rangle$, where $\Sigma_{\mathsf{tonk(elim)}} = \Sigma_{\mathsf{tonk}}$ and $R_{\mathsf{tonk(elim)}}$ has the rule

$$\frac{\mathsf{tonk}(p,q)}{q}.$$

Easily, $\mathcal{L}_{\mathsf{tonk}} = \mathcal{L}_{\mathsf{tonk(intr)}} \bullet \mathcal{L}_{\mathsf{tonk(elim)}}$ is Prior's tonk system, inducing a consistent but trivial logic. Namely, $p \vdash_{\mathsf{tonk}} q$ as can be easily observed from the mixed-derivation below.

$\mathcal{L}_{\mathsf{tonk}}$	
p	H
$\mathsf{tonk}(p,q)$	$\mathsf{tonk(intr)}$-R
q	$\mathsf{tonk(elim)}$-R

Contrarily to Example 3.1, a clearcut syntactical separation of the mixed-reasoning between $\mathcal{L}_{\mathsf{tonk(intr)}}$ and $\mathcal{L}_{\mathsf{tonk(elim)}}$ in this derivation is not possible, as tonk is a shared connective. Indeed, the essential interaction between the component logics, in this case, is provided precisely by the formula $\mathsf{tonk}(p,q)$, which is not a subformula of p nor q. △

The previous example illustrates the most essential difficulty related to non-disjoint fibring: an arbitrarily complex shared formula (or in a general a formula whose head is a shared connective) may be obtained on one of the component logics and then its structure explored by the other logic.

3.2 Mixed syntax: monoliths and skeletons

As illustrated above, in the context of fibring, it is useful to discriminate the structure of mixed formulas with respect to the syntax of each of the component logics.

Let $\Sigma \subseteq \Sigma'$ be signatures. We shall call a Σ-*monolith of* $\psi \in L_{\Sigma'}(P)$ to any outermost subformula of ψ whose head is in $\Sigma' \setminus \Sigma$. The set $\mathsf{Mon}_\Sigma(\psi)$ of all Σ-monoliths of ψ is defined as follows:

$$\mathsf{Mon}_\Sigma(\psi) = \begin{cases} \emptyset & \text{if } \psi \in P, \\ \bigcup_{i=1}^n \mathsf{Mon}_\Sigma(\psi_i) & \text{if } \psi = c(\psi_1,\ldots,\psi_n) \text{ and } c \in \Sigma^{(n)}, \\ \{\psi\}, & \text{otherwise.} \end{cases}$$

We extend the notation also to sets of formulas, using $\mathsf{Mon}_\Sigma(\Delta)$ to denote $\bigcup_{\psi\in\Delta} \mathsf{Mon}_\Sigma(\psi)$, given $\Delta \subseteq L_{\Sigma'}(P)$. Clearly, if $\Gamma \subseteq L_\Sigma(P)$ then $\mathsf{Mon}_\Sigma(\Gamma) = \emptyset$.

We shall now consider a reasonable way of defining the perspective, from the point of view of Σ, that one may have of a formula in $L_{\Sigma'}(P)$, by exchanging monoliths with skeletal variables. For the purpose, we use a denumerable set

$$X = \{x_\psi : \psi \in L_{\Sigma'}(P)\}$$

of additional propositional variables, disjoint from P. We define the function $\mathsf{skel}_\Sigma : L_{\Sigma'}(P) \to L_\Sigma(P \cup X)$ as follows:

$$\mathsf{skel}_\Sigma(\psi) = \begin{cases} \psi & \text{if } \psi \in P, \\ c(\mathsf{skel}_\Sigma(\psi_1),\ldots,\mathsf{skel}_\Sigma(\psi_n)) & \text{if } \psi = c(\psi_1,\ldots,\psi_n) \text{ and } c \in \Sigma^{(n)}, \\ x_\psi, & \text{otherwise.} \end{cases}$$

We call $\mathsf{skel}_\Sigma(\psi)$ the Σ-*skeleton of* ψ. Clearly, $\mathsf{skel}_\Sigma(\psi)$ is obtained from ψ by substituting each of its Σ-monoliths ϕ by the variable x_ϕ. Of course, in this context, x_ϕ is only useful if $\mathsf{head}(\phi) \in \Sigma' \setminus \Sigma$.

Example 3.3. Recall Example 3.1. Taking the mixed formula $\psi = (\neg p \wedge q) \wedge \neg(\neg\neg q \wedge \neg p)$ we have that

$$\mathsf{Mon}_{\Sigma_{\mathsf{cnj}}}(\psi) = \{\neg p, \neg(\neg\neg q \wedge \neg p)\}.$$

Note, in particular, that the subformula $\neg\neg q$ is not a Σ_{cnj}-monolith of ψ because it occurs inside the (outermost) monolith $\neg(\neg\neg q \wedge \neg p)$. For the same reason, $\neg p$ is only a Σ_{cnj}-monolith of ψ because it also occurs outside $\neg(\neg\neg q \wedge \neg p)$. Hence, we have that $\mathsf{skel}_{\Sigma_{\mathsf{cnj}}}(\psi) = (x_{\neg p} \wedge q) \wedge x_{\neg(\neg\neg q \wedge \neg p)}$. △

Of course, the operation of taking skeletons of formulas is revertible using an appropriate substitution.

Lemma 3.4. *If* $\Sigma \subseteq \Sigma'$ *are signatures and* $\psi \in L_{\Sigma'}(P')$ *then* $\mathsf{skel}_\Sigma(\psi)^{\mathsf{unsk}} = \psi$, *where the substitution* $\mathsf{unsk} : P \cup X \to L_{\Sigma'}(P \cup X)$ *is such that* $\mathsf{unsk}(p) = p$ *if* $p \in P$, *and* $\mathsf{unsk}(x_\psi) = \psi$ *for* $x_\psi \in X$.

Proof. Immediate, as the operation of instantiation by unsk is a left inverse of skel_Σ. □

3.3 Mixed reasoning and disjoint fibring

Let $\mathcal{L}_1 = \langle \Sigma_1, \_^{\vdash_1} \rangle$ and $\mathcal{L}_2 = \langle \Sigma_2, \_^{\vdash_2} \rangle$ be two logics, which we assume fixed for the remaining of this section, and $\mathcal{L}_1 \bullet \mathcal{L}_2 = \langle \Sigma_{12}, \_^{\vdash_{12}} \rangle$ be their fibring. Assume also that the fibring is disjoint, that is, $\Sigma_1 \cap \Sigma_2 = \emptyset$.

The following definition is in hand, with the purpose of using the variables in X, which we used above to define skeletons, in order to represent contextual information regarding the alternation between uses of $\vdash_1$ and $\vdash_2$ in $\vdash_{12}$-derivations. For convenience, below, we work with

$$X_* = \{x_*\} \cup X,$$

where the extra variable x_* will be used to represent in $\vdash_1$ some generic provable formula in $\vdash_2$, or vice-versa.

Definition 3.5. Let $\Gamma \cup \{\psi\} \subseteq L_{\Sigma_{12}}(P)$ and $i \in \{1, 2\}$. We define:

- $\Gamma_\omega = \bigcup_{n \in \mathbb{N}_0} \Gamma_n$, where $\Gamma_n = \begin{cases} \Gamma & \text{if } n = 0, \\ \{\varphi \in \mathsf{sub}(\Gamma) : \Gamma_{n-1} \vdash_1 \varphi \text{ or } \Gamma_{n-1} \vdash_2 \varphi\} & \text{otherwise;} \end{cases}$
- $\Gamma^i = \begin{cases} \mathsf{skel}_{\Sigma_i}(\Gamma_\omega) \cup \{x_*\} & \text{if } \Gamma \neq \emptyset \text{ or } \mathcal{L}_1 \text{ has theorems or } \mathcal{L}_2 \text{ has theorems,} \\ \emptyset & \text{otherwise;} \end{cases}$
- $X^i_\Gamma(\psi) = \{x_\phi \in X : \phi \in \mathsf{Mon}_{\Sigma_i}(\psi) \text{ and } \Gamma \vdash_{12} \phi\}$.

Γ_ω is a *saturation* of Γ consisting of all subformulas of Γ that can be proved from Γ in the fibred logic, using only a finite number of alternations between the component logics, where each alternation only produces other subformulas of Γ as lemmas. Although the definition of Γ_ω may seem involved, it turns out that it coincides, as a consequence of the proof of Proposition 3.8 below, with $\{\phi \in \mathsf{sub}(\Gamma) : \Gamma \vdash_{12} \phi\}$. Our definition is however better suited for practical use, as it depends only on the component logics.

Γ^i is a local view of Γ_ω with respect to $\mathcal{L}_i$. Γ^i contains the Σ_i-skeleton of each formula in Γ_ω, plus possibly the ghost variable x_*. It is worth noting that $\Gamma^i = \emptyset$ if and only if $x_* \notin \Gamma^i$ if and only if $\Gamma = \emptyset$ and $\mathcal{L}_1, \mathcal{L}_2$ have no theorems, if and only if $\Gamma^{\vdash_1} = \Gamma^{\vdash_2} = \emptyset$ if and only if $\Gamma^{\vdash_{12}} = \emptyset$, which implies that $\Gamma_\omega = \emptyset$. At the light of Proposition 3.8 below, it is worth noting that variable x_* is only meaningful in the case when one of the component logics has theorems and the other has quasi-theorems and, even so, only when analyzing mixed reasoning without hypotheses.

$X^i_\Gamma(\psi)$ contains the Σ_i-skeletons of all the Σ_i-monoliths of ψ which are provable from Γ in the fibred logic. Obviously, the skeleton of a monolith over the same signature is simply a variable in X.

Example 3.6. Extending Examples 3.1, 3.3, the following equalities hold for the fibred logic $\mathcal{L}_{\mathsf{cnj}} \bullet \mathcal{L}_{\mathsf{neg}}$, for any $\Gamma \cup \{\psi\} \subseteq L_{\Sigma_{\mathsf{cnj}} \cup \Sigma_{\mathsf{neg}}}(P)$.

$$\begin{aligned}
\emptyset_\omega = \emptyset^{\mathsf{cnj}} = \emptyset^{\mathsf{neg}} &= \emptyset \\
\{p\}^{\mathsf{cnj}} = \{p\}^{\mathsf{neg}} &= \{p\}_\omega \cup \{x_*\} = \{p, x_*\} \\
\{p \wedge (\neg\neg(q \wedge \neg r))\}_\omega &= \{p \wedge (\neg\neg(q \wedge \neg r)), p, \neg\neg(q \wedge \neg r), q \wedge \neg r, q, \neg r\} \\
\{p \wedge (\neg\neg(q \wedge \neg r))\}^{\mathsf{cnj}} &= \{p \wedge x_{\neg\neg(q\wedge\neg r)}, p, x_{\neg\neg(q\wedge\neg r)}, q \wedge x_{\neg r}, q, x_{\neg r}, x_*\} \\
\{p \wedge (\neg\neg(q \wedge \neg r))\}^{\mathsf{neg}} &= \{x_{p\wedge(\neg\neg(q\wedge\neg r))}, p, \neg\neg x_{q\wedge\neg r}, x_{q\wedge\neg r}, q, \neg r, x_*\} \\
X^{\mathsf{cnj}}_{\emptyset}(\psi) = X^{\mathsf{neg}}_{\emptyset}(\psi) &= \emptyset \\
X^{\mathsf{cnj}}_{\Gamma}(p) = X^{\mathsf{neg}}_{\Gamma}(p) &= \emptyset \\
X^{\mathsf{cnj}}_{\{q\}}(p \wedge \neg\neg q) &= \{x_{\neg\neg q}\} \\
X^{\mathsf{neg}}_{\{p,q\}}(p \wedge q) &= \{x_{p\wedge q}\}
\end{aligned}$$

It is worth noting that while in the first and second lines $\emptyset_\omega$ and $\{p\}_\omega$ are computed in 0 steps, in the third line it takes 3 iterations to compute $\{p \wedge (\neg\neg(q \wedge \neg r))\}_\omega$, namely because we cannot extract q without first extracting $\neg\neg(q \wedge \neg r)$ in the $\mathcal{L}_{\mathsf{cnj}}$ side, and then $q \wedge \neg r$ in the $\mathcal{L}_{\mathsf{neg}}$ side. △

Next we prove a useful technical lemma about substitutions of the following form: given $\gamma \in L_{\Sigma_{12}}(P)$, let $\mathsf{unsk}_\gamma : P \cup X_* \to L_{\Sigma_{12}}(P \cup X)$ such that $\mathsf{unsk}_\gamma(x_*) = \gamma$, and $\mathsf{unsk}_\gamma(y) = \mathsf{unsk}(y)$ for $y \neq x_*$.

Lemma 3.7. *If $\Gamma \cup \{\gamma\} \subseteq L_{\Sigma_{12}}(P)$ and $i \in \{1, 2\}$, then $(\Gamma^i)^{\mathsf{unsk}_\gamma} = \Gamma_\omega \cup \{\gamma\}$ whenever $x_* \in \Gamma^i$.*

Moreover, if $\psi \in \mathsf{sub}(\Gamma)$ and $\Gamma^i \vdash_i \mathsf{skel}_{\Sigma_i}(\psi)$ then $\psi \in \Gamma_\omega$.

Proof. Easily, $(\mathsf{skel}_{\Sigma_i}(\Gamma_\omega))^{\mathsf{unsk}_\gamma} = (\mathsf{skel}_{\Sigma_i}(\Gamma_\omega))^{\mathsf{unsk}} = \Gamma_\omega$, as shown in Lemma 3.4, and $\mathsf{unsk}_\gamma(x_*) = \gamma$, which allows us to conclude that $(\Gamma^i)^{\mathsf{unsk}_\gamma} = \Gamma_\omega \cup \{\gamma\}$ whenever $x_* \in \Gamma^i$.

If $\psi \in \mathsf{sub}(\Gamma)$ we know that $\Gamma \neq \emptyset$ and thus $x_* \in \Gamma^i$. Let $\gamma \in \Gamma$ and consider the substitution unsk_γ. From $\Gamma^i \vdash_i \mathsf{skel}_{\Sigma_i}(\psi)$, by structurality of $\vdash_i$, we get that $\Gamma_\omega \vdash_i \psi$, since $\Gamma \subseteq \Gamma_\omega$ and $\mathsf{unsk}_\gamma(\mathsf{skel}_{\Sigma_i}(\psi)) = \psi$. Given that $\psi \in \mathsf{sub}(\Gamma)$ we conclude, using the fixed point characterization of Definition 3.5 and Lemma 4.4, that $\psi \in \Gamma_\omega$. □

The previous lemma shows, in particular, that x_* is essentially redundant whenever $\Gamma \neq \emptyset$.

In the next proposition we state our fundamental characterization of mixed reasoning in logics resulting from disjoint fibring, as already hinted by Example 3.1. Under the disjointness assumption, the result completely characterizes the behavior of a fibred logic based on the behavior of the logics being combined. We delay its proof to the Appendix, in the end of the paper, so that the focus on decidability is not lost.

Proposition 3.8. *Let $\mathcal{L}_1 = \langle \Sigma_1, \_^{\vdash_1} \rangle$ and $\mathcal{L}_2 = \langle \Sigma_2, \_^{\vdash_2} \rangle$ be logics such that $\Sigma_1 \cap \Sigma_2 = \emptyset$, and consider $\Gamma \cup \{\psi\} \subseteq L_{\Sigma_{12}}(P)$, as well as $i, j \in \{1, 2\}$ with $i \neq j$. Then,*

$$\Gamma \vdash_{12} \psi$$

if and only if

$$\Gamma^i, X^i_\Gamma(\psi) \vdash_i \mathsf{skel}_{\Sigma_i}(\psi) \text{ or } \Gamma^j \text{ is } \vdash_j\text{-explosive.}$$

Let us revisit Example 3.1 at the light of Proposition 3.8, for the sake of illustration.

Example 3.9. Regarding Example 3.1, with $\Gamma = \{\neg\neg\neg(p \wedge p)\}$ and $\psi = \neg p$, the following equalities hold.

$$\begin{aligned}
\Gamma_\omega &= \{\neg\neg\neg(p \wedge p), \neg(p \wedge p), x_*\} \\
\Gamma^{\mathsf{cnj}} &= \{x_{\neg\neg\neg(p\wedge p)}, x_{\neg(p\wedge p)}, x_*\} \\
\Gamma^{\mathsf{neg}} &= \{\neg\neg\neg x_{p\wedge p}, \neg x_{p\wedge p}, x_*\} \\
X^{\mathsf{cnj}}_\Gamma(\psi) = X^{\mathsf{neg}}_\Gamma(\psi) &= \emptyset
\end{aligned}$$

Using Proposition 3.8, we can justify the fact that $\Gamma \not\vdash \psi$ in the fibred logic $\mathcal{L}_{\mathsf{cnj}} \bullet \mathcal{L}_{\mathsf{neg}}$, as we claimed in Example 3.1, with two simple observations about fragments of classical logic:

- $\Gamma^{\mathsf{neg}}, X^{\mathsf{neg}}_{\Gamma}(\psi) \nvdash_{\mathsf{neg}} \mathsf{skel}_{\Sigma_{\mathsf{neg}}}(\psi)$, i.e., $\{\neg\neg\neg x_{p\wedge p}, \neg x_{p\wedge p}, x_*\} \nvdash_{\mathsf{neg}} \neg p$, as in fact p does not even occur in Γ^{neg}; and
- $\Gamma^{\mathsf{cnj}} = \{x_{\neg\neg\neg(p\wedge p)}, x_{\neg(p\wedge p)}, x_*\}$ is not $\vdash_{\mathsf{cnj}}$-explosive, namely because $\Gamma^{\mathsf{cnj}} \nvdash_{\mathsf{cnj}} p$, as p does not even occur in Γ^{cnj}.

Symetrically, we could also argue that:

- $\Gamma^{\mathsf{cnj}}, X^{\mathsf{cnj}}_{\Gamma}(\psi) \nvdash_{\mathsf{cnj}} \mathsf{skel}_{\Sigma_{\mathsf{cnj}}}(\psi)$, i.e., $\{x_{\neg\neg\neg(p\wedge p)}, x_{\neg(p\wedge p)}, x_*\} \nvdash_{\mathsf{cnj}} x_{\neg p}$; and
- $\Gamma^{\mathsf{neg}} = \{\neg\neg\neg x_{p\wedge p}, \neg x_{p\wedge p}, x_*\}$ is not $\vdash_{\mathsf{neg}}$-explosive. △

In order to emphasize how Proposition 3.8 depends in an essential way on the disjointness assumption, it is worth revisiting the tonk example.

Example 3.10. As we have seen in Example 3.2, we have that $p \vdash_{\mathsf{tonk}} q$. Since $\mathcal{L}_{\mathsf{tonk}} = \mathcal{L}_{\mathsf{tonk(intr)}} \bullet \mathcal{L}_{\mathsf{tonk(elim)}}$ is a fibred logic, one could be tempted to justify this fact using Proposition 3.8. As p and q have no strict subformulas (and even less monoliths, as $\Sigma_{\mathsf{tonk(intr)}} = \Sigma_{\mathsf{tonk(elim)}}$), we have

$$\{p\}_\omega = \{p\}^{\mathsf{tonk(intr)}} = \{p\}^{\mathsf{tonk(elim)}} = \{p, x_*\}$$
$$X^{\mathsf{tonk(intr)}}_{\{p\}}(q) = X^{\mathsf{tonk(elim)}}_{\{p\}}(q) = \emptyset$$

and it is easy to check that $\{p, x_*\} \nvdash_{\mathsf{tonk(intr)}} q$ and also that $\{p, x_*\}$ is not $\vdash_{\mathsf{tonk(elim)}}$-explosive; or *mutatis mutandis*, that $\{p, x_*\} \nvdash_{\mathsf{tonk(elim)}} q$ and also that $\{p, x_*\}$ is not $\vdash_{\mathsf{tonk(intr)}}$-explosive. △

4 Decidability and complexity

In this section we present and prove the main results of this paper. First, we prove decidability preservation for disjoint fibring, and analyze the complexity of the corresponding algorithms. Then, we also obtain a decidability reflection result which, excluding pathological cases, will allow us to conclude that $\mathcal{L}_1 \bullet \mathcal{L}_2$ is decidable if and only if both $\mathcal{L}_1$ and $\mathcal{L}_2$ are decidable, whenever the fibring is disjoint.

In order to compute monoliths and skeletons of formulas of $\mathcal{L}_1 \bullet \mathcal{L}_2$, we further assume that the signatures Σ_1, Σ_2 are decidable.

4.1 Deciding fibred logics

We say that decidability is *preserved* by fibring if $\mathcal{L}_1 \bullet \mathcal{L}_2$ is decidable whenever both $\mathcal{L}_1$ and $\mathcal{L}_2$ are decidable. Reciprocally, we say that decidability is *reflected* by fibring if $\mathcal{L}_1 \bullet \mathcal{L}_2$ decidable implies that $\mathcal{L}_1$ and $\mathcal{L}_2$ are both decidable.

It turns out that, in general, decidability is not preserved nor reflected by fibring, as the following examples illustrate.

Example 4.1. For the sake of illustration, let us consider the logics introduced by the following, more or less standard, Hilbert calculi:

- $\mathscr{H}_{\mathsf{imp}} = \langle \Sigma_{\mathsf{imp}}, R_{\mathsf{imp}} \rangle$, where Σ_{imp} has a unique 2-place connective $\to$, and R_{imp} has the rules

$$\frac{}{(p \to (q \to r)) \to ((p \to q) \to (p \to r))}$$

$$\frac{}{p \to (q \to p)} \qquad \frac{}{((p \to q) \to p) \to p} \qquad \frac{p \quad p \to q}{q},$$

- $\mathscr{H}_{\mathsf{botimp}} = \langle \Sigma_{\mathsf{botimp}}, R_{\mathsf{botimp}} \rangle$, where Σ_{botimp} equals Σ_{imp} plus a 0-place connective $\bot$, and R_{botimp} contains the unique rule

$$\frac{}{\bot \to p}.$$

Clearly, the system $\mathscr{H}_{\mathsf{imp}}$ induces the logic of *classical implication* $\mathcal{L}_{\mathsf{imp}}$, whereas $\mathcal{L}_{\mathsf{cls}} = \mathcal{L}_{\mathsf{imp}} \bullet \mathcal{L}_{\mathsf{botimp}}$ corresponds to full propositional classical logic, where other classical connectives can be obtained, as usual, by abbreviation, e.g., $\neg\varphi \triangleq \varphi \to \bot$.

It is also useful to consider modal logics:

- $\mathscr{H}_{\Box} = \langle \Sigma_{\Box}, R_{\Box} \rangle$, where $\Sigma_{\Box}$ has a unique 1-place connective $\Box$, and $R_{\Box}$ contains the rule

$$\frac{p}{\Box p}.$$

- $\mathscr{H}_{k_{\Box}} = \langle \Sigma_{k_{\Box}}, R_{k_{\Box}} \rangle$, where $\Sigma_{k_{\Box}} = \Sigma_{\mathsf{cls}} \cup \Sigma_{\Box}$, and $R_{k_{\Box}}$ contains just the $k_{\Box}$ axiom

$$\overline{\Box p \to \Box\Box p},$$

- $\mathscr{H}_{\mathbf{K}_{\Box}} = \langle \Sigma_{\mathbf{K}_{\Box}}, R_{\mathbf{K}_{\Box}} \rangle$, where $\Sigma_{\mathbf{K}_{\Box}} = \Sigma_{k_{\Box}}$, and $R_{\mathbf{K}_{\Box}}$ contains $R_{\mathsf{cls}} \cup R_{\Box} \cup R_{k_{\Box}}$ plus the rule

$$\overline{\Box(p \to q) \to (\Box p \to \Box q)}.$$

The system $\mathscr{H}_{\Box}$ is simply the $\Box$-syntactical fragment of the smallest normal modal logic $\mathbf{K}_{\Box}$, which is induced precisely by the system $\mathscr{H}_{\mathbf{K}_{\Box}}$. As usual, we shall use the abbreviation $\Diamond\varphi \triangleq \neg\Box\neg\varphi$.

Other modal and multi-modal axioms are also in hand:

- $\mathscr{H}_{4_{\Box}} = \langle \Sigma_{4_{\Box}}, R_{4_{\Box}} \rangle$, where $\Sigma_{4_{\Box}} = \Sigma_{k_{\Box}}$, and $R_{4_{\Box}}$ contains just the $4_{\Box}$ axiom

$$\overline{\Box p \to \Box\Box p},$$

- $\mathscr{H}_{\mathrm{tb}_{\Box}} = \langle \Sigma_{\mathrm{tb}_{\Box}}, R_{\mathrm{tb}_{\Box}} \rangle$, where $\Sigma_{\mathrm{tb}_{\Box}} = \Sigma_{k_{\Box}}$, and $R_{\mathrm{tb}_{\Box}}$ includes just the $t_{\Box}$ and $b_{\Box}$ axioms

$$\overline{\Box p \to p} \qquad \overline{p \to \Box\Diamond p},$$

- given $\Box_1 \neq \Box_2$, $\mathscr{H}_{\mathrm{ccr}_{\Box_1,\Box_2}} = \langle \Sigma_{\mathrm{ccr}_{\Box_1,\Box_2}}, R_{\mathrm{ccr}_{\Box_1,\Box_2}} \rangle$, where $\Sigma_{\mathrm{ccr}_{\Box_1,\Box_2}} = \Sigma_{k_{\Box_1}} \cup \Sigma_{k_{\Box_2}}$, and $R_{\mathrm{ccr}_{\Box_1,\Box_2}}$ includes just the commutativity and Church-Rosser axioms

$$\overline{\Box_1\Box_2 p \leftrightarrow \Box_2\Box_1 p} \qquad \overline{\Diamond_1\Box_2 p \to \Box_2\Diamond_1 p}.$$

Easily, $\mathbf{K}_{\Box} \bullet \mathcal{L}_{4_{\Box}}$ induces the common modal logic $\mathbf{K4}_{\Box}$. Additionally, $\mathbf{K4}_{\Box} \bullet \mathcal{L}_{\mathrm{TB}_{\Box}}$ corresponds to the well known modal system $\mathbf{S5}_{\Box}$. Moreover, given logics $\mathcal{L}_{\Box_i} = \mathcal{L}_{\mathscr{H}_i}$ with $\mathscr{H}_i = \langle \Sigma_{\mathbf{K}_{\Box_i}}, R_i \rangle$, for $i \in \{1, 2\}$, the system given by $(\mathcal{L}_{\Box_1} \bullet \mathcal{L}_{\Box_2}) \bullet \mathcal{L}_{\mathrm{CCR}_{\Box_1,\Box_2}}$ corresponds to the *commutator* $[\mathcal{L}_{\Box_1}, \mathcal{L}_{\Box_2}]$ of uni-modal logics $\mathcal{L}_{\Box_1}$ and $\mathcal{L}_{\Box_2}$ (that coincides with their product $\mathcal{L}_{\Box_1} \times \mathcal{L}_{\Box_2}$ whenever $\mathcal{L}_{\Box_1}$ and $\mathcal{L}_{\Box_2}$ are Horn-axiomatizable [20], which is actually the case for $\mathbf{K}_{\Box}$, $\mathbf{K4}_{\Box}$ and also $\mathbf{S5}_{\Box}$). △

Example 4.2. Decidability is, in general, not preserved by fibring. An interesting counterexample can be obtained by taking as component logics the commutator/product logic $[\mathbf{K}_{\Box_1}, \mathbf{K}_{\Box_2}]$, know from [21, 20] to be decidable, and the logic $\mathcal{L}_{4_{\Box_1}} \bullet \mathcal{L}_{4_{\Box_2}}$ which is straightforwardly decidable as it is defined by just two axiom schemata. It is easy to check that $[\mathbf{K}_{\Box_1}, \mathbf{K}_{\Box_2}] \bullet (\mathcal{L}_{4_{\Box_1}} \bullet \mathcal{L}_{4_{\Box_2}})$ coincides with the commutator/product logic $[\mathbf{K4}_{\Box_1}, \mathbf{K4}_{\Box_2}]$, know from [22] to be undecidable.

Alternatively, we could also have obtained $[\mathbf{K4}_{\Box_1}, \mathbf{K4}_{\Box_2}]$ as the fibring $(\mathbf{K}_{\Box_1} \bullet \mathbf{K}_{\Box_2}) \bullet \mathcal{L}_{\mathrm{CCR}_{\Box_1,\Box_2}}$. Notably, $\mathbf{K}_{\Box_1} \bullet \mathbf{K}_{\Box_2}$ corresponds to the fusion of two decidable logics, and is therefore decidable [40], and $\mathcal{L}_{\mathrm{CCR}_{\Box_1,\Box_2}}$ is also straightforwardly decidable as it is defined by just two axiom schemata. △

Example 4.3. Perhaps more surprising, is the fact that decidability is, in general, also not reflected by fibring. A simple counterexample consists of taking any undecidable logic $\mathcal{U}$ and any trivial logic $\mathcal{T}$. Obviously, $\mathcal{T}$ is decidable. It is a simple fact that $\mathcal{U} \bullet \mathcal{T}$ is also trivial, and therefore also decidable, despite the fact that the component logic $\mathcal{U}$ is undecidable.

A more interesting counterexample, where both component logics are undecidable but still their fibring is decidable, can be obtained by considering any two undecidable logics $\mathcal{U}_1$ and $\mathcal{U}_2$ whose signatures do not contain the tonk 1-place connective, and taking the fibring $(\mathcal{U}_1 \bullet \mathcal{L}_{\mathsf{tonk(intr)}}) \bullet (\mathcal{U}_2 \bullet \mathcal{L}_{\mathsf{tonk(elim)}})$. It is easy to see that the component logics $\mathcal{U}_1 \bullet \mathcal{L}_{\mathsf{tonk(intr)}}$ and $\mathcal{U}_2 \bullet \mathcal{L}_{\mathsf{tonk(elim)}}$ are both still undecidable, but their fibring is decidable, as $(\mathcal{U}_1 \bullet \mathcal{L}_{\mathsf{tonk(intr)}}) \bullet (\mathcal{U}_2 \bullet \mathcal{L}_{\mathsf{tonk(elim)}}) = (\mathcal{U}_1 \bullet \mathcal{U}_2) \bullet \mathcal{L}_{\mathsf{tonk}}$, which extends the logic $\mathcal{L}_{\mathsf{tonk}}$ of Example 3.2, and is therefore also trivial.

Decidability reflection may fail even when the resulting fibred logic is not trivial. For such a counterexample one can simply take as component logics the commutator/product $[\mathbf{K4}_{\Box_1}, \mathbf{K4}_{\Box_2}]$, known from [22] to be undecidable, and the logic $\mathcal{L}_{\mathrm{TB}_{\Box_1}} \bullet \mathcal{L}_{\mathrm{TB}_{\Box_2}}$. It is routine to check that their fibring $[\mathbf{K4}_{\Box_1}, \mathbf{K4}_{\Box_2}] \bullet (\mathcal{L}_{\mathrm{TB}_{\Box_1}} \bullet \mathcal{L}_{\mathrm{TB}_{\Box_2}})$ coincides with the commutator/product logic $[\mathbf{S5}_{\Box_1}, \mathbf{S5}_{\Box_2}]$, known from [35, 20] to be decidable. △

Despite the difficulties illustrated above, the panorama with respect to the preservation and reflection of decidability is quite distinct when one restricts attention to disjoint fibring.

4.2 Decidability preservation for disjoint fibring

Our goal here is to show that disjoint fibring always preserves decidability. The essential idea is to take advantage of Proposition 3.8. In order to decide whether $\Gamma \vdash_{12} \psi$, in a logic obtained by disjoint fibring, the idea is to pick $i \in \{1, 2\}$ (it makes sense to choose i such that $\mathsf{head}(\psi) \in \Sigma_i$) and check, equivalently, whether $\Gamma^i, X^i_\Gamma(\psi) \vdash_i \mathsf{skel}_{\Sigma_i}(\psi)$ or Γ^{3-i} is $\vdash_{3-i}$-explosive. If both $\mathcal{L}_1$ and $\mathcal{L}_2$ are decidable, one just needs to be able to compute Γ_ω, skeletons of formulas, and also $X^i_\Gamma(\psi)$. In order to verify if Γ^{3-i} is $\vdash_{3-i}$-explosive it suffices to test if $\Gamma^{3-i} \vdash_{3-i} q$ where q is a variable not occurring in Γ^{3-i}.

Obtaining skeletons of formulas is easy by syntactic analysis. The set Γ_ω is also effectively calculable when $\mathcal{L}_1$ and $\mathcal{L}_2$ are decidable, as we show below.

Lemma 4.4. *Let $\Gamma \subseteq L_{\Sigma_{12}}(P)$. Then, $\Gamma_{|\mathsf{sub}(\Gamma)|} = \Gamma_\omega$.*

Proof. The result follows easily from the fact that $|\mathsf{sub}(\Gamma)| \leq \omega$ by observing that, for all $n \in \mathbb{N}_0$, $\Gamma_{n+1} \setminus \Gamma_n \subseteq \mathsf{sub}(\Gamma)$, and also that if $\Gamma_{n+1} = \Gamma_n$ then $\Gamma_n = \Gamma_\omega$. □

Lemma 4.4 tells us that the number of iterations needed to compute Γ_ω is bounded by the cardinality of the set $\mathsf{sub}(\Gamma)$, being of course finite when Γ is finite. Furthermore, the result justifies the ω bound in the definition, since we are assuming that we are dealing with denumerable languages. It is important to note that computing Γ_ω requires listing all the subformulas of formulas in Γ, which we can do, since the signatures are decidable, whenever Γ is finite.

Finally, our decidability preservation result can be presented. It uses an inductive argument that will allow us to compute $X^i_\Gamma(\psi)$.

Proposition 4.5. *Disjoint fibring preserves decidability, i.e., if $\mathcal{L}_1 = \langle \Sigma_1, \_^{\vdash_1} \rangle$ and $\mathcal{L}_2 = \langle \Sigma_2, \_^{\vdash_2} \rangle$ are both decidable, and $\Sigma_1 \cap \Sigma_2 = \emptyset$, then the problem of determining whether $\Gamma \vdash_{12} \psi$ for finite $\Gamma \cup \{\psi\} \subseteq L_{\Sigma_1 \cup \Sigma_2}(P)$ is decidable.*

Proof. For the sake of simplicity, we shall assume without loss of generality that $p_0 \notin \mathsf{var}(\Gamma)$. This assumption is easily justified by Lemma 2.1, since we could decide equivalently whether $\Gamma^{\mathsf{nxt}} \vdash_{12} \psi^{\mathsf{nxt}}$ and $p_0 \notin \mathsf{var}(\Gamma^{\mathsf{nxt}})$.

Let $\mathcal{L}_1$ be decidable with decision algorithm D^1, and $\mathcal{L}_2$ be decidable with decision algorithm D^2.

First, we consider the subroutine S that calculates Γ_ω defined in Figure 1. As stated in Lemma 4.4 the recursion in S has at most $|\mathsf{sub}(\Gamma)|$ steps, which is bounded by $\mathsf{size}(\Gamma)$, and therefore converges always to Γ_ω, as Γ is assumed to be finite.

```
S:  input Γ ⊆ L_{Σ12}(P)
    Γ′ ← Γ ∪ {φ ∈ sub(Γ) : D^1(Γ)(φ) = yes or D^2(Γ)(φ) = yes}
    if Γ′ = Γ then output Γ′ else output S(Γ′)
```

Figure 1: Saturation subroutine S to compute $\mathsf{S}(\Gamma) = \Gamma_\omega$.

Now, we can already consider deciding the fibred logic in the case when at least one of the component logics, $\mathcal{L}_1$ or $\mathcal{L}_2$, has theorems. Under such an assumption, the decision algorithm D for the fibred logic is defined in Figure 2.

```
D:  input Γ ⊆ L_{Σ12}(P)
    Γ_ω ← S(Γ)
    Γ¹ ← skel_{Σ1}(Γ_ω) ∪ {x_*}
    Γ² ← skel_{Σ2}(Γ_ω) ∪ {x_*}
    if D¹(Γ¹)(p_0) = yes or D²(Γ²)(p_0) = yes
    then output  ┌───────────────────────────────┐
                 │ D_yes: input ψ ∈ L_{Σ12}(P)   │
                 │ output yes                    │
                 └───────────────────────────────┘
    else output  ┌────────────────────────────────────────────┐
                 │ D(Γ¹,Γ²): input ψ ∈ L_{Σ12}(P)             │
                 │ if head(ψ) ∉ Σ2 then i ← 1 else i ← 2      │
                 │ M ← Mon_{Σi}(ψ)                            │
                 │ χ ← {x_φ : φ ∈ M, D(Γ¹,Γ²)(φ) = yes}       │
                 │ output D^i(Γ^i ∪ χ)(skel_{Σi}(ψ))          │
                 └────────────────────────────────────────────┘
```

Figure 2: Decision algorithm D for $\mathcal{L}_1 \bullet \mathcal{L}_2$ with theorems.

Assuming that at least one of the component logics, $\mathcal{L}_1$ or $\mathcal{L}_2$, has theorems, we now show that for all finite $\Gamma \cup \{\psi\} \subseteq L_{\Sigma_{12}}(P)$, $\mathsf{D}(\Gamma)(\psi)$ terminates, and

$$\mathsf{D}(\Gamma)(\psi) = \begin{cases} \mathsf{yes} & \text{if } \Gamma \vdash_{12} \psi, \\ \mathsf{no} & \text{if } \Gamma \nvdash_{12} \psi. \end{cases}$$

As we said above, we know that S converges always to Γ_ω in at most $\mathsf{size}(\Gamma)$ steps. Furthermore, as at least one of the component logics has theorems, we know that $x_* \in \Gamma^1$ and $x_* \in \Gamma^2$. Hence, the lines $\Gamma^1 \leftarrow \mathsf{skel}_{\Sigma_1}(\Gamma_\omega) \cup \{x_*\}$ and $\Gamma^2 \leftarrow \mathsf{skel}_{\Sigma_2}(\Gamma_\omega) \cup \{x_*\}$ correctly set the values of Γ^1 and Γ^2, following Definition 3.5.

If either $\mathsf{D}^1(\Gamma^1)(p_0) = \mathsf{yes}$ or $\mathsf{D}^2(\Gamma^2)(p_0) = \mathsf{yes}$ then, since $p_0 \notin \mathsf{var}(\Gamma^1)$ and $p_0 \notin \mathsf{var}(\Gamma^2)$ as a consequence of the assumption that $p_0 \notin \mathsf{var}(\Gamma)$, we can conclude that Γ^1 is $\vdash_1$-explosive or Γ^2 is $\vdash_2$-explosive, which in any case implies that Γ is $\vdash_{12}$-explosive. Hence $\Gamma \vdash_{12} \psi$ for all $\psi \in L_{\Sigma_{12}}(P)$, which justifies the fact that the output algorithm $\mathsf{D}(\Gamma) = \mathsf{D}_{\mathsf{yes}}$ always returns yes.

If neither of the Γ^i is $\vdash_i$-explosive, then the output algorithm $\mathsf{D}(\Gamma) = \mathsf{D}(\Gamma^1, \Gamma^2)$. We show, by induction on the structure of $\psi \in L_{\Sigma_{12}}(P)$, that $\mathsf{D}(\Gamma^1, \Gamma^2)(\psi)$ terminates and outputs

$$\mathsf{D}(\Gamma^1, \Gamma^2)(\psi) = \begin{cases} \mathsf{yes} & \text{if } \Gamma \vdash_{12} \psi, \\ \mathsf{no} & \text{if } \Gamma \nvdash_{12} \psi. \end{cases}$$

For the induction base, let $\psi = p \in P$. Obviously $\mathsf{head}(p) = p \notin \Sigma_2$ so $i = 1$. Hence, $\Gamma^1 = \{x_*\}$. As $M = \mathsf{Mon}_{\Sigma_1}(p) = \emptyset$ (so $\chi = X^i_\Gamma(p) = \emptyset$) and $\mathsf{skel}_{\Sigma_1}(p) = p$, one gets in the last line that $\mathsf{D}^1(\Gamma^1 \cup \chi)(p)$ tests precisely whether $\Gamma^1 \vdash_1 p$. Since $\mathcal{L}_1$ is decidable with D^1, the algorithm terminates. As we know that Γ^2 is not $\vdash_2$-explosive, Proposition 3.8 guarantees that the answer is yes if $\Gamma \vdash_{12} p$, and no if $\Gamma \nvdash_{12} p$.
For the induction step, let $\psi = c(\psi_1, \ldots, \psi_n)$ for some n-place connective $c = \mathsf{head}(\psi) \in \Sigma_{12}$. Clearly $i = 1$ if $c \in \Sigma_1$, and $i = 2$ if $c \in \Sigma_2$. Moreover, $M = \mathsf{Mon}_{\Sigma_i}(\psi) = \bigcup_{j=1}^n \mathsf{Mon}_{\Sigma_i}(\psi_j)$. By induction hypothesis, for each $\phi \in M$, $\mathsf{D}(\Gamma^1, \Gamma^2)(\phi)$ terminates and answers yes precisely if $\Gamma \vdash_{12} \phi$ and so, easily, $\chi = \{x_\phi : \phi \in M, \mathsf{D}(\Gamma^1, \Gamma^2)(\phi) = \mathsf{yes}\} = X^i_\Gamma(\psi)$. Hence, in the last line, $\mathsf{D}^i(\Gamma^i \cup \chi)(\mathsf{skel}_{\Sigma_i}(\psi))$ tests precisely whether $\Gamma^i, X^i_\Gamma(\psi) \vdash_i \mathsf{skel}_{\Sigma_i}(\psi)$. Since $\mathcal{L}_i$ is decidable with D^i, the algorithm terminates. As Γ^{3-i} is assumed not to be $\vdash_{3-i}$-explosive, Proposition 3.8 guarantees that the answer is yes if $\Gamma \vdash_{12} \psi$, and no if otherwise.

D': input $\Gamma \subseteq L_{\Sigma_{12}}(P)$
if $\Gamma = \emptyset$
then output

$\mathsf{D_{no}}$: input $\psi \in L_{\Sigma_{12}}(P)$ output no

else output $\mathsf{D}(\Gamma)$

Figure 3: Decision algorithm D' for $\mathcal{L}_1 \bullet \mathcal{L}_2$ without theorems.

When none of the component logics $\mathcal{L}_1$ and $\mathcal{L}_2$ has theorems, we need the slightly more involved algorithm D' described in Figure 3.

When none of $\mathcal{L}_1$, $\mathcal{L}_2$ has theorems, D'(Γ) first checks whether $\Gamma = \emptyset$. If that is the case, then it is immediate that $\mathcal{L}_1 \bullet \mathcal{L}_2$ also has no theorems and therefore $\Gamma \nvdash_{12} \varphi$, for all $\varphi \in L_{\Sigma_{12}}(P)$, which justifies the fact that the output algorithm is D'$(\Gamma) = \mathsf{D_{no}}$ always returns no. If $\Gamma \neq \emptyset$ then $x_* \in \Gamma^1$ and $x_* \in \Gamma^2$, and clearly $\mathsf{D}(\Gamma)$ decides $\mathcal{L}_1 \bullet \mathcal{L}_2$, using the same line of argument as above. Overall, when none of the component logics has theorems, for all finite $\Gamma \cup \{\psi\} \subseteq L_{\Sigma_{12}}(P)$, D'$(\Gamma)(\psi)$ terminates, and

$$\mathsf{D'}(\Gamma)(\psi) = \begin{cases} \mathsf{yes} & \text{if } \Gamma \vdash_{12} \psi, \\ \mathsf{no} & \text{if } \Gamma \nvdash_{12} \psi \end{cases}$$

as we wanted to prove. □

It is useful to have a look at a few application examples.

Example 4.6. Let us recover the logic $\mathcal{L}_{\mathsf{neg}} \bullet \mathcal{L}_{\mathsf{cnj}} = \langle \Sigma_{\mathsf{neg}} \cup \Sigma_{\mathsf{cnj}}, \_^{\vdash} \rangle$ introduced in Example 3.1, and run the decision algorithm on the formulas in Examples 3.1 and 3.9. Since $\mathcal{L}_{\mathsf{neg}} \bullet \mathcal{L}_{\mathsf{cnj}}$ has no theorems we shall be using D'.

- Let $\Gamma = \{\neg\neg(p \wedge (\neg\neg q))\}$ and $\psi = q \wedge (\neg\neg p)$. We know that $\Gamma \vdash \psi$.

  We call D'$(\Gamma)(\psi)$. The first step of D'(Γ) is to check whether $\Gamma = \emptyset$, which clearly is not the case. Hence, we proceed as $\mathsf{D}(\Gamma)(\psi)$. In turn $\mathsf{D}(\Gamma)$, calls $\mathsf{S}(\Gamma)$ and assigns its output to Γ_ω. $\mathsf{S}(\Gamma)$ starts by listing all the subformulas of Γ, $\mathsf{sub}(\Gamma) = \{\neg\neg(p \wedge (\neg\neg q)), \neg(p \wedge (\neg\neg q)), p \wedge (\neg\neg q), p, \neg\neg q, \neg q, q\}$. Then, the algorithm recursively saturates Γ, obtaining Γ_k in run k, until it reaches a fixed point. In this case, we have:

  - $\Gamma_0 = \Gamma$;
  - $\Gamma_1 = \Gamma_0 \cup \{p \wedge (\neg\neg q)\}$, since $\Gamma_0 \vdash_{\mathsf{neg}} p \wedge (\neg\neg q)$;
  - $\Gamma_2 = \Gamma_1 \cup \{p, \neg\neg q\}$, since $\Gamma_1 \vdash_{\mathsf{cnj}} p$ and $\Gamma_1 \vdash_{\mathsf{cnj}} \neg\neg q$;
  - $\Gamma_3 = \Gamma_2 \cup \{q\}$, since $\Gamma_2 \vdash_{\mathsf{neg}} q$;
  - $\Gamma_4 = \Gamma_3$.

  After computing $\Gamma_\omega = \{\neg\neg(p \wedge (\neg\neg q)), p \wedge (\neg\neg q), p, \neg\neg q, q\}$, $\mathsf{D}(\Gamma)$ applies $\mathsf{skel}_{\Sigma_{\mathsf{neg}}}$ and $\mathsf{skel}_{\Sigma_{\mathsf{cnj}}}$ to obtain Γ^{cnj}, Γ^{neg}. At this point, we have

  - $\Gamma^{\mathsf{neg}} = \{\neg\neg x_{p \wedge (\neg\neg q)}, x_{p \wedge (\neg\neg q)}, p, \neg\neg q, q, x_*\}$, and
  - $\Gamma^{\mathsf{cnj}} = \{x_{\neg\neg(p \wedge (\neg\neg q))}, p \wedge x_{\neg\neg q}, p, x_{\neg\neg q}, q, x_*\}$.

  Next, $\mathsf{D}(\Gamma)$ checks if either $\Gamma^{\mathsf{neg}} \vdash_{\mathsf{neg}} p_0$ or $\Gamma^{\mathsf{cnj}} \vdash_{\mathsf{cnj}} p_0$, which is clearly not the case. As a result, one now needs to compute $\mathsf{D}(\Gamma^{\mathsf{cnj}}, \Gamma^{\mathsf{neg}})(\psi)$. After checking that $\mathsf{head}(\psi) = \wedge \in \Sigma_{\mathsf{cnj}}$, the algorithm picks $i = \mathsf{cnj}$ and sets $M = \mathsf{Mon}_{\Sigma_{\mathsf{cnj}}}(\psi) = \{\neg\neg p\}$. In order to calculate χ, the algorithm then recursively calls $\mathsf{D}(\Gamma^{\mathsf{cnj}}, \Gamma^{\mathsf{neg}})(\phi)$ to each $\phi \in M$, that is, to $\phi = \neg\neg p$.

  - Locally, on the run of $\mathsf{D}(\Gamma^{\mathsf{cnj}}, \Gamma^{\mathsf{neg}})(\neg\neg p)$, after checking that $\neg \notin \Sigma_{\mathsf{cnj}}$, the algorithm picks $i = \mathsf{neg}$ and sets $M = \mathsf{Mon}_{\Sigma_{\mathsf{neg}}}(\neg\neg p) = \emptyset$. As a consequence, $\chi = \emptyset$ and the algorithm tests whether $\Gamma^{\mathsf{neg}} \vdash_{\mathsf{neg}} \mathsf{skel}_{\Sigma_{\mathsf{neg}}}(\neg\neg p)$. As $\mathsf{skel}_{\Sigma_{\mathsf{neg}}}(\neg\neg p) = \neg\neg p$ and $p \in \Gamma^{\mathsf{neg}}$ the algorithm returns yes.

Returning to the main algorithm, we get $\chi = \{x_{\neg\neg p}\}$ and finally test whether $\Gamma^{\mathsf{cnj}} \cup \chi \vdash_{\mathsf{cnj}} \mathsf{skel}_{\Sigma_{\mathsf{cnj}}}(\psi)$. As $\mathsf{skel}_{\Sigma_{\mathsf{cnj}}}(\psi) = q \wedge x_{\neg\neg p}$, $q \in \Gamma^{\mathsf{cnj}}$ and $x_{\neg\neg p} \in \chi$, the algorithm returns yes.

- $\Gamma = \{\neg\neg\neg(p \wedge p)\}$ and $\psi = \neg p$. We know that $\neg\neg\neg(p \wedge p) \not\vdash \neg p$.

  We call $\mathsf{D}'(\Gamma)(\psi)$. The first step of $\mathsf{D}'(\Gamma)$ is to check whether $\Gamma = \emptyset$, which clearly is not the case. Hence, we proceed as $\mathsf{D}(\Gamma)(\psi)$. In turn $\mathsf{D}(\Gamma)$, calls $\mathsf{S}(\Gamma)$ to compute Γ_ω. As calculated in Example 3.9, $\Gamma_\omega = \{\neg\neg\neg(p \wedge p), \neg(p \wedge p), x_*\}$. Afterwards, $\mathsf{D}(\Gamma)$ applies $\mathsf{skel}_{\Sigma_{\mathsf{neg}}}$ and $\mathsf{skel}_{\Sigma_{\mathsf{cnj}}}$ to obtain Γ^{cnj}, Γ^{neg}. At this point, as obtained in Example 3.9, we have

  - $\Gamma^{\mathsf{neg}} = \{\neg\neg\neg x_{p\wedge p}, \neg x_{p \wedge p}, x_*\}$, and
  - $\Gamma^{\mathsf{cnj}} = \{x_{\neg\neg\neg(p\wedge p)}, x_{\neg(p\wedge p)}, x_*\}$.

  Next, $\mathsf{D}(\Gamma)$ checks if either $\Gamma^{\mathsf{neg}} \vdash_{\mathsf{neg}} p_0$ or $\Gamma^{\mathsf{cnj}} \vdash_{\mathsf{cnj}} p_0$, which is again not the case. As a result, one now needs to compute $\mathsf{D}(\Gamma^{\mathsf{cnj}}, \Gamma^{\mathsf{neg}})(\psi)$. After checking that $\mathsf{head}(\psi) = \wedge \notin \Sigma_{\mathsf{cnj}}$, the algorithm picks $i = \mathsf{neg}$ and sets $M = \mathsf{Mon}_{\Sigma_{\mathsf{neg}}}(\psi) = \emptyset$. As a consequence, we get $\chi = \emptyset$ and finally test whether $\Gamma^{\mathsf{neg}} \vdash_{\mathsf{neg}} \mathsf{skel}_{\Sigma_{\mathsf{neg}}}(\psi)$. We have that $\mathsf{skel}_{\Sigma_{\mathsf{neg}}}(\psi) = \psi = \neg p$, and it is not hard to see that the algorithm returns no as $\Gamma^{\mathsf{neg}} \not\vdash_{\mathsf{neg}} \neg p$. △

4.2.1 Complexity

Having shown above that disjoint fibring preserves decidability, it is worth analyzing the complexity of the decision algorithms obtained. For the sake of simplicity, we shall further assume that the signatures Σ_1, Σ_2 of the component logics are not only decidable, but decidable in constant time $\mathcal{O}(1)$. This assumption fits perfectly with the cases when the signatures are finite, as in all the examples we analyze. A thorougher analysis, including possibly non-constant decision procedures for membership in Σ_1, Σ_2 is of course possible, but would lead to dealing with additional terms involving their running time functions.

Proposition 4.7. *Let $\mathcal{L}_1 = \langle \Sigma_1, \_^{\vdash_1} \rangle$ and $\mathcal{L}_2 = \langle \Sigma_2, \_^{\vdash_2} \rangle$ be such that $\Sigma_1 \cap \Sigma_2 = \emptyset$. Let also $\Gamma \cup \{\psi\} \in L_{\Sigma_{12}}(P)$, $m = \mathsf{size}(\Gamma)$, and $n = \mathsf{size}(\psi)$.*

If logic $\mathcal{L}_1$ is decidable in time bounded by $f_1(m,n)$ and logic $\mathcal{L}_2$ is decidable in time bounded by $f_2(m,n)$, then $\mathcal{L}_1 \bullet \mathcal{L}_2$ is decidable in time bounded by $f(m,n)$ such that

$$f(m,n) \leq \mathcal{O}(m^2+n) \times \max(f_1(m^2+n,n), f_2(m^2+n,n)).$$

Proof. Let algorithms D^1, D^2 decide the component logics $\mathcal{L}_1$, $\mathcal{L}_2$ in time bounded by f_1, f_2. We analyze the running time of different parts of the algorithms[1].

(a) For a given finite $\Gamma \cup \{\psi\} \in L_{\Sigma_{12}}(P)$, depending on whether at least one of the component logics $\mathcal{L}_1$, $\mathcal{L}_2$ has theorems or not, we must consider algorithm D or algorithm D', from the proof of Proposition 4.5. However, algorithm D' just checks, in one go, if $\Gamma = \emptyset$, and then, in the worst case, proceeds as D.

(b) Then, independently of ψ, algorithm D uses the subroutine S to compute $\mathsf{S}(\Gamma) = \Gamma_\omega$. For that purpose, it must list all the subformulas of formulas in Γ, taking $\mathcal{O}(\mathsf{size}(\Gamma)) = \mathcal{O}(m)$ steps, and then run the recursion coded in S. In each recursive round k, it calls $\mathsf{D}^1(\Gamma_{k-1})(\varphi)$ and $\mathsf{D}^2(\Gamma_{k-1})(\varphi)$ for each $\varphi \in \mathsf{sub}(\Gamma)$, calculating Γ_k, until it gets to a fixed-point. We know from Lemma 4.4, that the recursion needs at most $|\mathsf{sub}(\Gamma)| \leq m$ rounds, and so the number of calls to each D^1, D^2 is bounded by m^2. Of course, $\mathsf{size}(\varphi) \leq m$ for each $\varphi \in \mathsf{sub}(\Gamma)$, and it is not hard to see that for every k, $\mathsf{size}(\Gamma_k) \leq \mathsf{size}(\mathsf{sub}(\Gamma)) \leq |\mathsf{sub}(\Gamma)| \times \mathsf{size}(\Gamma) \leq \mathsf{size}(\Gamma)^2 = m^2$.

(c) In order to calculate Γ^1 and Γ^2, the algorithm applies skel_{Σ_i} to every formula in Γ_ω. For each formula in Γ_ω, this operation takes at most as many steps as the size of the formula. In total, it takes time $\mathcal{O}(m^2)$. There are two additional steps to add x_* to each of the resulting sets. Note that $2m^2+2 = 2(m^2+1)$, and m^2+1 is precisely an upper-bound for the size of both Γ^1 and Γ^2. Then, the algorithm calls $\mathsf{D}^1(\Gamma^1)(p_0)$ and $\mathsf{D}^2(\Gamma^2)(p_0)$. If any of the calls returns yes it stops.

[1]Throughout the analysis, we assume, without loss of generality, that all the running time functions take positive integer values and are monotonic in each argument. The same is to say that, for instance, that $f^1(m,n)$ is an upper-bound for the running time of D^1 over all inputs with sizes bounded by m and n.

(d) However, in the worst case, both calls will return no, and the algorithm will further call $\mathsf{D}(\Gamma^1, \Gamma^2)(\psi)$.

Hence, in the worst case scenario, and collecting the contributions of each of the four phases discriminated above, we have that

$$\begin{array}{rcl} f(m,n) & \leq & \mathcal{O}(1)+ \\ & & \mathcal{O}(m) + 2m^2(f_1(m^2,m) + f_2(m^2,m))+ \\ & & \mathcal{O}(2(m^2+1)) + f_1(m^2,1) + f_2(m^2,1)+ \\ & & g_{(\Gamma^1,\Gamma^2)}(n) \\ & \leq & \mathcal{O}(m^2) + (2m^2+1)(f_1(m^2,m) + f_2(m^2,m)) + g_{(\Gamma^1,\Gamma^2)}(n) \\ & = & \mathcal{O}(m^2) + \mathcal{O}(m^2) \times (f_1(m^2,m) + f_2(m^2,m)) + g_{(\Gamma^1,\Gamma^2)}(n) \\ & = & \mathcal{O}(m^2) \times (f_1(m^2,m) + f_2(m^2,m)) + g_{(\Gamma^1,\Gamma^2)}(n) \end{array}$$

where $g_{(\Gamma^1,\Gamma^2)}(n)$ is the running time of $\mathsf{D}(\Gamma^1, \Gamma^2)(\psi)$.

We now prove, by induction, $g_{(\Gamma^1,\Gamma^2)}(n) \leq \mathcal{O}(n) \times \max(f_1(m^2+n,n), f_2(m^2+n,n))$. Just note that under our assumption of constant time for deciding membership in the component signatures, checking whether $\mathsf{head}(\psi) \in \Sigma_2$ and calculating $M = \mathsf{Mon}_{\Sigma_i}(\psi)$ will take time bounded by $\mathcal{O}(1) + \mathcal{O}(\mathsf{size}(\psi))$.

For the induction base, let $n = 1$. If $\mathsf{size}(\psi) = n = 1$ then either $\psi = p \in P$ or $\psi = c \in \Sigma_i^{(0)}$ for some $i \in \{1, 2\}$. When $\psi = p \in P$, the algorithm sets $i = 1$. Then, in both cases, the algorithm gets $M = \mathsf{Mon}_{\Sigma_i}(\psi) = \emptyset$, and computes $\mathsf{skel}_{\Sigma_i}(\psi) = \psi$. Hence, we need only consider the running time of $\mathsf{D}^i(\Gamma^i)(\psi)$, that is,

$$\begin{array}{rcl} g_{(\Gamma^1,\Gamma^2)}(1) & = & \mathcal{O}(1) + \mathcal{O}(1) + f_i(m^2,1) \\ & \leq & \mathcal{O}(1) + f_i(m^2+1,1) \\ & \leq & \mathcal{O}(1) + \max(f_1(m^2+1,1), f_2(m^2+1,1)) \\ & = & \mathcal{O}(1) + 1 \times \max(f_1(m^2+1,1), f_2(m^2+1,1)) \\ & = & \mathcal{O}(1) \times \max(f_1(m^2+1,1), f_2(m^2+1,1)) \\ & = & \mathcal{O}(n) \times \max(f_1(m^2+n,n), f_2(m^2+n,n)). \end{array}$$

For the induction step, let $n > 1$. If $\mathsf{size}(\psi) = n$ then $\psi = c(\psi_1, \ldots, \psi_k)$ for some k-place connective $c = \mathsf{head}(\varphi) \in \Sigma_1 \cup \Sigma_2$ and $\mathsf{size}(\psi_1) + \cdots + \mathsf{size}(\psi_k) = n - 1$. Note that $\mathsf{D}(\Gamma^1, \Gamma^2)(\psi)$ sets $i = 1$ if $c \in \Sigma_1$ and $i = 2$ if $c \in \Sigma_2$. Note also that $\mathsf{size}(M) \leq n$ and $\mathsf{size}(\mathsf{skel}_{\Sigma_i}(\psi)) \leq n$. Further, it follows that $n_\phi = \mathsf{size}(\phi) \leq n$ for every $\phi \in M = \mathsf{Mon}_{\Sigma_i}(\psi)$, and $\mathsf{size}(\Gamma^i \cup \{x_\phi : \phi \in M, \mathsf{D}(\Gamma^1,\Gamma^2)(\phi) = \mathsf{yes}\}) \leq \mathsf{size}(\Gamma^i) + \mathsf{size}(\chi) \leq m^2 + n$. As we have to consider the running times of all $\mathsf{D}(\Gamma^1, \Gamma^2)(\phi)$, for $\phi \in M$, and also of $\mathsf{D}^i(\Gamma^i \cup \chi)(\mathsf{skel}_{\Sigma_i}(\psi))$, using the induction hypothesis, the total running time of $\mathsf{D}(\Gamma^1, \Gamma^2)(\psi)$ is such that

$$\begin{array}{rcl} g_{(\Gamma^1,\Gamma^2)}(n) & \leq & \mathcal{O}(1) + \mathcal{O}(n) + [\sum_{\phi \in M} g_{(\Gamma^1,\Gamma^2)}(n_\phi)] + f_i(m^2+n,n) = \\ & \leq & \mathcal{O}(n) + [\sum_{\phi \in M} \mathcal{O}(n_\phi) \times \max(f_1(m^2+n_\phi,n_\phi), f_2(m^2+n_\phi,n_\phi))] + f_i(m^2+n,n) \\ & \leq & \mathcal{O}(n) + [\sum_{\phi \in M} \mathcal{O}(n_\phi) \times \max(f_1(m^2+n,n), f_2(m^2+n,n))] + f_i(m^2+n,n) \\ & = & \mathcal{O}(n) + [[\sum_{\phi \in M} \mathcal{O}(n_\phi)] \times \max(f_1(m^2+n,n), f_2(m^2+n,n))] + f_i(m^2+n,n) \\ & \leq & \mathcal{O}(n) + [\mathcal{O}(\mathsf{size}(M)) \times \max(f_1(m^2+n,n), f_2(m^2+n,n))] + f_i(m^2+n,n) \\ & \leq & \mathcal{O}(n) + [\mathcal{O}(n) \times \max(f_1(m^2+n,n), f_2(m^2+n,n))] + f_i(m^2+n,n) \\ & \leq & \mathcal{O}(n) + \mathcal{O}(n+1) \times \max(f_1(m^2+n,n), f_2(m^2+n,n)) \\ & = & \mathcal{O}(n) + \mathcal{O}(n) \times \max(f_1(m^2+n,n), f_2(m^2+n,n)) \\ & = & \mathcal{O}(n) \times \max(f_1(m^2+n,n), f_2(m^2+n,n)). \end{array}$$

Concluding the argument, we have that

$$\begin{array}{rcl} f(m,n) & \leq & \mathcal{O}(m^2) \times (f_1(m^2,m) + f_2(m^2,m)) + g_{(\Gamma^1,\Gamma^2)}(n) \\ & \leq & \mathcal{O}(m^2) \times (f_1(m^2,m) + f_2(m^2,m)) + \mathcal{O}(n) \times \max(f_1(m^2+n,n), f_2(m^2+n,n)) \\ & \leq & \mathcal{O}(m^2+n) \times \max(f_1(m^2+n,n), f_2(m^2+n,n)), \end{array}$$

as we wanted to prove.

Additionally, just note that if $p_0 \in \mathsf{var}(\Gamma)$, our complexity expression must be added an additional linear term corresponding to the calculation of Γ^{nxt} and ψ^{nxt}, as explained above. Of course, this linear term does not change the upper-bound obtained. $\square$

The complexity upper-bound obtained above shows that the decision problem for the fibred logic reduces polynomially to the worst decision problem of the given logics. In particular, if the decision problems for $\mathcal{L}_1$ and $\mathcal{L}_2$ are both in the complexity class $\mathcal{C}$ then the decision problem for $\mathcal{L}_1 \bullet \mathcal{L}_2$ is also in $\mathcal{C}$, for any class $\mathcal{C}$ that contains P and is closed for composition with polynomials, such as the common classes P, NP, coNP, PSPACE, EXPTIME, and so on.

Example 4.8. Consider the unconstrained fibring $\mathcal{L}_{\mathsf{cnj}} \bullet \mathcal{L}_{\mathsf{neg}}$ that we have used in Examples 3.1, 3.3, 3.9 and 4.6. It is clear that $\mathcal{L}_{\mathsf{cnj}}$ is decidable in quadratic time (its decision procedure boils down to set inclusion), and that $\mathcal{L}_{\mathsf{neg}}$ is decidable in linear time. As a direct application of Propositions 4.5 and 4.7, we can conclude that $\mathcal{L}_{\mathsf{cnj}} \bullet \mathcal{L}_{\mathsf{neg}}$ is also decidable, in polynomial time, with a time bound polynomial of degree four. We already knew from Examples 3.1 and 4.6 that $\mathcal{L}_{\mathsf{cnj}} \bullet \mathcal{L}_{\mathsf{neg}}$ does not coincide with classical logic, but we can anyway confirm it is the case if we assume the usual conjecture P $\neq$ NP, since the decision problem for classical logic is known to be in coNP.

As another application of Propositions 4.5 and 4.7, let us consider the unconstrained fibring $\mathcal{L}_{\mathsf{imp}} \bullet \mathcal{L}_{\mathsf{neg}}$. As the decision problem for $\mathcal{L}_{\mathsf{imp}}$ is known to be in coNP, we can conclude that $\mathcal{L}_{\mathsf{imp}} \bullet \mathcal{L}_{\mathsf{neg}}$ is also decidable and its decision problem in coNP. Note, however, that $\mathcal{L}_{\mathsf{imp}} \bullet \mathcal{L}_{\mathsf{neg}}$ is still strictly weaker than classical logic, which can be shown with the aid of Proposition 3.8.

Finally, let us consider the Hilbert calculus

- $\mathscr{H}_{\mathsf{top}} = \langle \Sigma_{\mathsf{top}}, R_{\mathsf{top}} \rangle$, where Σ_{top} has a unique 0-place connective $\top$, and R_{top} has the unique rule

$$\frac{}{\top}\,.$$

presenting the logic of classical (or intuitionistic) top (*verum*). It is clear that $\mathcal{L}_{\mathsf{top}}$ is decidable in linear time. Hence, Proposition 4.7 gives us an upper-bound for the complexity of the decision procedure of $\mathcal{L}_{\mathsf{cnj}} \bullet \mathcal{L}_{\mathsf{top}}$ which is a polynomial of degree four. This shows that our upper-bound is not tight, as it is relatively simple to check that $\mathcal{L}_{\mathsf{cnj}} \bullet \mathcal{L}_{\mathsf{top}}$ can be decided in quadratic time, as $\mathcal{L}_{\mathsf{cnj}}$ is. $\triangle$

4.3 Decidability reflection for disjoint fibring

We now show that disjoint fibring also reflects decidability, in all but trivial pathological cases.

Proposition 4.9. *Let $\mathcal{L}_1 = \langle \Sigma_1, \_^{\vdash_1} \rangle$ and $\mathcal{L}_2 = \langle \Sigma_2, \_^{\vdash_2} \rangle$ be such that $\Sigma_1 \cap \Sigma_2 = \emptyset$.*

If $\mathcal{L}_1 \bullet \mathcal{L}_2$ is decidable then $\mathcal{L}_1$ and $\mathcal{L}_2$ are both decidable, unless one of the component logics is trivial (in which case the other component logic may be undecidable).

Proof. Our proof relies in an essential way on the notion of conservative extension, as defined in Subsection 2.2.

Let us assume that $\mathcal{L}_1 \bullet \mathcal{L}_2$ is decidable and none of the component logics $\mathcal{L}_1$, $\mathcal{L}_2$ is trivial. It is clear that if $\mathcal{L}_1 \bullet \mathcal{L}_2$ extends conservatively $\mathcal{L}_i$ with $i \in \{1, 2\}$, then $\mathcal{L}_i$ is also decidable. We are left with checking what happens when $\mathcal{L}_1 \bullet \mathcal{L}_2$ does not extend conservatively some of the component logics $\mathcal{L}_i$. According to the main result of [27] about conservativity of disjoint fibring, this may happen in only two situations:

(a) $\mathcal{L}_i$ is not trivial but $\mathcal{L}_{3-i}$ is trivial, or

(b) $\mathcal{L}_i$ has q-theorems and $\mathcal{L}_{3-i}$ has theorems.

Situation (a) cannot be the case, as one of the component logics would have to be trivial, contradicting our assumption. We are thus left with the situation (b). We assume, without loss of generality that $i = 1$, i.e., $\mathcal{L}_1 \bullet \mathcal{L}_2$ does not extend conservatively $\mathcal{L}_1$ precisely because $\mathcal{L}_1$ has q-theorems and $\mathcal{L}_2$ has

A^1: input $\Gamma \subseteq L_{\Sigma_1}(P)$
if $\Gamma = \emptyset$
then output

A_{no}: input $\psi \in L_{\Sigma_1}(P)$ output no

else output

A': input $\psi \in L_{\Sigma_1}(P)$ output $\mathsf{A}(\Gamma)(\psi)$

Figure 4: Decision algorithm A^1 for $\mathcal{L}_1$ with q-theorems.

theorems. In this case, we claim that algorithm A^1 described in Figure 4 decides $\mathcal{L}_1$, where A^1 uses any given algorithm A deciding the fibred logic $\mathcal{L}_1 \bullet \mathcal{L}_2$.

We need to show that for all finite $\Gamma \cup \{\psi\} \subseteq L_{\Sigma_1}(P)$, $\mathsf{A}^1(\Gamma)(\psi)$ terminates, and

$$\mathsf{A}^1(\Gamma)(\psi) = \begin{cases} \mathsf{yes} & \text{if } \Gamma \vdash_1 \psi, \\ \mathsf{no} & \text{if } \Gamma \nvdash_1 \psi. \end{cases}$$

Termination of the algorithm, in all cases, is straightforward, as we assume that algorithm A decides the fibred logic.

If $\Gamma = \emptyset$ then $\mathsf{A}^1(\Gamma)(\psi)$ always outputs no. In fact, as we know that $\mathcal{L}_1$ has q-theorems it easily follows that $\mathcal{L}_1$ does not have theorems, and thus $\nvdash_1 \psi$.

If $\Gamma \neq \emptyset$, then the output $\mathsf{A}^1(\Gamma)(\psi) = \mathsf{A}(\Gamma)(\psi)$ is yes if and only if $\Gamma \vdash_{12} \psi$. Using Proposition 3.8, $\Gamma \vdash_{12} \psi$ if and only if $\Gamma^1, X^1_\Gamma(\psi) \vdash_1 \mathsf{skel}_{\Sigma_1}(\psi)$ or Γ^2 is $\vdash_2$-explosive.

Since $\Gamma \subseteq L_{\Sigma_1}(P)$, it happens that also $\Gamma_\omega \subseteq L_{\Sigma_1}(P)$, and thus $x_* \in \Gamma^2 \subseteq X_* \cup P$. As Γ is finite, it is obvious that Γ^2 is also finite, and we can pick a variable $q \notin \Gamma^2$. If Γ^2 would be $\vdash_2$-explosive, we would have $\Gamma^2 \vdash_2 q$ and therefore also $p \vdash_2 q$, by using the structurally of $\vdash_2$, which would contradict the assumption that $\mathcal{L}_2$ is not trivial.

Furthermore, as $\psi \in L_{\Sigma_1}(P)$, it follows that $X^1_\Gamma(\psi) = \emptyset$ and $\mathsf{skel}_{\Sigma_1}(\psi) = \psi$. Moreover, as we already saw that $\Gamma_\omega \subseteq L_{\Sigma_1}(P)$ it also follows that $\Gamma^1 = \Gamma_\omega$.

Hence, we have that $\mathsf{A}^1(\Gamma)(\psi)$ outputs yes if and only if $\Gamma_\omega \vdash_1 \psi$. As $\Gamma \subseteq \Gamma_\omega$, in order to close the proof, we just need to show that $\Gamma_\omega \subseteq \Gamma^{\vdash_1}$. We use an inductive argument on the construction of Γ_ω.

For the base case, it is clear that $\Gamma_0 = \Gamma \subseteq \Gamma^{\vdash_1}$.

For the induction step, let us assume that $\Gamma_n \subseteq \Gamma^{\vdash_1}$ and $\varphi \in \Gamma_{n+1} \setminus \Gamma_n$. If $\Gamma_n \vdash_1 \varphi$ then it is obvious that $\varphi \in \Gamma^{\vdash_1}$. We just need to see that the alternative case, when $\Gamma_n \vdash_2 \varphi$, cannot happen. If it did, as $\varphi \in \mathsf{sub}(\Gamma)$, it would follow that also $\varphi \in L_{\Sigma_1}(P)$. Therefore, the Σ_2-skeletons of not only the formulas in Γ_n, but also of φ, would be variables. Moreover, as different formulas have different skeletons, it would also be the case that $\mathsf{skel}_{\Sigma_2}(\varphi) \notin \mathsf{skel}_{\Sigma_2}(\Gamma_n)$. Finally, using Proposition A.3, we would be able to conclude that $\Gamma_n \vdash_2 \varphi$ is equivalent to $\mathsf{skel}_{\Sigma_2}(\Gamma_n) \vdash_2 \mathsf{skel}_{\Sigma_2}(\varphi)$, and would once again contradict the assumption that $\mathcal{L}_2$ is not trivial, using the same argument as above. □

We can now strengthen Proposition 4.9, using the decidability preservation result from Proposition 4.5, into the following corollary.

Corollary 4.10. *Let $\mathcal{L}_1 = \langle \Sigma_1, \_^{\vdash_1} \rangle$ and $\mathcal{L}_2 = \langle \Sigma_2, \_^{\vdash_2} \rangle$ be non-trivial logics such that $\Sigma_1 \cap \Sigma_2 = \emptyset$.*

Then, $\mathcal{L}_1 \bullet \mathcal{L}_2$ is decidable if and only if $\mathcal{L}_1$ and $\mathcal{L}_2$ are decidable.

5 Conclusion

We have studied in detail the decision problem for fibred logics, and shown that decidability is preserved and (almost) reflected by disjoint fibring. The results we obtained extend the preliminary results of [28] about deciding theoremhood, and are the first of their kind, thus opening the way to formal tool support for fibred logics in a neat and modular way. Although distinct in nature, it is worth mentioning that our decision algorithms bear some similarities with the Nelson-Oppen approach to deciding joint equational theories [30].

Owing to their importance, a special word is due with respect to the complexity results obtained. Indeed, we have proved that, in the disjoint case, the complexity of the decision problem we obtain for the fibred logic is only slightly worse than the complexity of the hardest component logic, the slow-down being polynomial. The result thus entails that complexity in fibred logics is necessarily a by-product of sharing connectives and, as a consequence, that the complexity of an arbitrary logic can be ultimately broken down to the complexity of its connectives in isolation, and the complexity of its mixed axioms and rules (pertaining to more than one connective).

In practice, it is common and useful to study also the *satisfiability problem (SAT)* for a given logic $\mathcal{L} = \langle \Sigma, \_^{\vdash} \rangle$, usually presented as a semantical notion: given a formula $\varphi \in L_\Sigma(P)$ is there a model for $\mathcal{L}$ satisfying it? Our development here is completely independent of semantics, but it is a well known fact (e.g., [39]) that, for Tarskian logics such as the ones we consider, logical matrices do provide a suitable notion of model. Furthermore, as a consequence, it is straightfoward to check that a formula φ is thus satisfiable (in some matrix model of $\mathcal{L}$) if and only if φ is not $\vdash$-explosive, or equivalently, if $\varphi \nvdash p_0$ for some variable $p_0 \notin \mathsf{var}(\varphi)$. Hence, if $\mathcal{L}$ is decidable then so is its SAT problem. The converse is in general not true, as Example 5.1 illustrates. Thus, the preservation of SAT by disjoint fibring (and fibring in general) remains an open problem, except for the cases where the component logics do not only have a decidable SAT problem but are themselves decidable.

Example 5.1. Let $\mathbb{M}$ denote the set of all Turing machines, and $\eta : \mathbb{N}_0 \to \mathbb{M}$ be an enumeration of $\mathbb{M}$. Consider the logic $\mathcal{L}_{\mathsf{halt}}$ defined by the Hilbert calculus

- $\mathscr{H}_{\mathsf{halt}} = \langle \Sigma_{\mathsf{halt}}, R_{\mathsf{halt}} \rangle$, where Σ_{halt} has the 0-place connective all, and the 1-place connective halt, and R_{halt} has the rules

$$\frac{}{\mathsf{halt}^i(\mathsf{all})} \quad \text{if machine } \eta(i) \text{ halts for all inputs,}$$

where $\mathsf{halt}^0(\varphi) = \varphi$ and $\mathsf{halt}^{k+1}(\varphi) = \mathsf{halt}^k(\mathsf{halt}(\varphi))$, as usual.

The SAT problem for $\mathcal{L}_{\mathsf{halt}}$ is trivial, as no formula explodes (every formula is satisfiable), but the decidability problem for $\mathcal{L}_{\mathsf{halt}}$ is obviously undecidable. △

The decidability and complexity results we obtained depend, in a fundamental way, of the key result of Proposition 3.8, which fully characterizes disjoint fibring in terms of its component logics, and that generalizes the partial characterization results obtained in [28, 27]. Although this result was intentionally secondarized in the exposition, in order to keep the focus on decidability, it is a very rich tool whose range of applicability goes well beyond. For instance, in [27], a weaker form of Proposition 3.8 was used to study the conservativity problem for fibred logics, and also to show that fibring does not preserve finite-valuedness. We expect the result to be useful in further studies of fibring, including a deeper and much needed understanding of its semantics. Besides fibring, the result of Proposition 3.8 can be understood as one of the rare meaningful results about the proof-theory of Hilbert calculi.

There are several lines of future work that we would like to follow.

An obvious research direction is to pursue a version of Proposition 3.8 that could allow us to tackle non-disjoint fibring. Of course, we cannot expect to obtain a general (effective) description of mixed patterns of reasoning, such as the one obtained here for disjoint fibring, as we know and have shown that decidability is not always preserved by fibring. Moreover, such a generalization seems to need new insights to deal with shared connectives in an appropriate way, but would certainly provide us with a much deeper understanding of the fibring mechanism, and could potentially have a myriad of practical applications. There is some hope, however, in obtaining partial results of this kind that may be applicable to particular forms of fibring with sufficiently well-behaved shared connectives. We are currently investigating this possibility, namely by exploring the lessons learnt with fusions of modal logics and all the interesting results known about them [20], where a basic classical logic is shared. In a wider perspective, it also worth investigating the relationship between our goals and recent work aimed at building, using syntactic criteria, proof-calculi for suitably well behaved logics [12, 11, 14, 24].

Another possibility we are considering, is to try and generalize our decidability results to a wider range of fibred logics. Of course, effective generalizations of Proposition 3.8 as discussed in the previous paragraph, would necessarily entail more general decidability results. However, we can also attack the decidability problem more directly. An idea would be to explore situations where the fibred logic is a conservative extension of its components, which could lead to decidability preservation under the proviso of obtaining suitable translations from each of the component logics to their shared fragment.

Our result has also potential applications beyond disjoint fibring. Suppose that a certain logic $\mathcal{L} = \langle \Sigma, \_^{\vdash} \rangle$ can be split into disjoint fragments $\mathcal{L}_1$ and $\mathcal{L}_2$, plus a finite number of interaction axioms. Then, we have that $\Gamma \vdash \varphi$ if and only if $\Gamma, \mathrm{Ax} \vdash_{12} \varphi$, where Ax contains all instances of the interaction axioms. Now, this is still not enough to decide the logic $\mathcal{L}$, in general, even if $\mathcal{L}_1$ and $\mathcal{L}_2$ are both decidable, as the set Ax is typically infinite. However, it may happen that for each finite Γ and φ there exists a finite $A_{\Gamma,\varphi} \subseteq \mathrm{Ax}$ such that $\Gamma, \mathrm{Ax} \vdash_{12} \varphi$ if and only if $\Gamma, A_{\Gamma,\varphi} \vdash_{12} \varphi$, e.g., when the logics are finitary. If the set $A_{\Gamma,\varphi}$ happens to be computable, then we are able to recover the decidability of $\mathcal{L}$. Furthermore, we can still evaluate the complexity of the decision problem for $\mathcal{L}$ on the basis of the complexity of computing $A_{\Gamma,\varphi}$ in each case. Note also that, taking into account our complexity results, when we know the complexity of the logic $\mathcal{L}$ we may be able to draw interesting conclusions about the size and complexity of computing $A_{\Gamma,\varphi}$, or even about the complexity of the fragments $\mathcal{L}_1$ and $\mathcal{L}_2$.

Take, for instance, classical logic $\mathcal{L}_{\mathsf{cls}}$, and consider its fragment $\mathcal{L}_{\mathsf{bot}}$ as defined in Example 5.2. It is easy to see that $\mathcal{L}_{\mathsf{cls}} = \mathcal{L}_{\mathsf{imp}} \bullet \mathcal{L}_{\mathsf{bot}} \bullet \mathcal{L}_{\mathsf{botimp}}$, and thus $\Gamma \vdash_{\mathsf{cls}} \varphi$ is equivalent to $\Gamma, \mathrm{Ax} \vdash_{12} \varphi$ if we take $\mathcal{L}_1 = \mathcal{L}_{\mathsf{imp}}$, $\mathcal{L}_2 = \mathcal{L}_{\mathsf{bot}}$ and $\mathrm{Ax} = \{\bot \to \psi : \psi \in L_{\Sigma_{\mathsf{cls}}}(P)\}$. Even better, using the results available for propositional classical logic, it is not hard to check that $\Gamma \vdash_{\mathsf{cls}} \varphi$ if and only if $\Gamma, A_{\Gamma,\varphi} \vdash_{12} \varphi$ where $A_{\Gamma,\varphi} = \{\bot \to \varphi\}$. Note that, in this case, $A_{\Gamma,\varphi}$ is computable in linear time from φ. This observation, together with our complexity result, allows us to have an alternative view on the interplay between the complexities of the fragments of classical logic involved. Namely, knowing that the complexity of deciding $\mathcal{L}_{\mathsf{bot}}$ is linear, and that the problem of deciding $\mathcal{L}_{\mathsf{cls}}$ is coNP-complete, we immediately obtain that deciding $\mathcal{L}_{\mathsf{imp}}$ must also be coNP-complete (a result that is well known, of course, but of which we just obtain an alternative simple proof). We look forward to analyze, in the future, further (more interesting) instances of this phenomenon, and moreover explore its possible connections with the finite model property.

Example 5.2. Consider the logic $\mathcal{L}_{\mathsf{bot}}$ defined by the Hilbert calculus

- $\mathscr{H}_{\mathsf{bot}} = \langle \Sigma_{\mathsf{bot}}, R_{\mathsf{bot}} \rangle$, where Σ_{bot} has only the 0-place connective $\bot$, and R_{bot} has the unique rule

$$\frac{\bot}{p}.$$

Easily, $\mathcal{L}_{\mathsf{bot}}$ is the logic of classical (or intuitionistic) bottom (*falsum*). △

Last but not least, refining the idea above, a particularly appealing application of our results seems to be in approximate reasoning in the sense used also in [23, 15]. In the scenario above, even when for each Γ and φ there is a finite $A_{\Gamma,\varphi} \subseteq \mathrm{Ax}$ such that $\Gamma, \mathrm{Ax} \vdash_{12} \varphi$ if and only if $\Gamma, A_{\Gamma,\varphi} \vdash_{12} \varphi$, it will often be the case that this set $A_{\Gamma,\varphi}$ is too big, or not effectively/efficiently calculable. In general, it will not be an easy task to characterize such a set $A_{\Gamma,\varphi}$, but knowing the complexity of the logics involved may help us graft some of its properties, as our results allow us to set a lower bound on the complexity of computing $A_{\Gamma,\varphi}$. Moreover, even if $A_{\Gamma,\varphi}$ is not computable or too hard to compute, it is still reasonable to look for efficient approximations of the set. Hence, if we manage to come up with a strictly increasing sequence of efficiently computable sets $A^0_{\Gamma,\varphi} \subsetneq A^1_{\Gamma,\varphi} \subsetneq A^2_{\Gamma,\varphi} \subsetneq \cdots \subsetneq A_{\Gamma,\varphi}$, one may end up with a sequence of workable approximations of the resulting logic by considering whether $\Gamma, A^i_{\Gamma,\varphi} \vdash_{12} \varphi$ for $i \in \mathbb{N}_0$, under complexity bounds that are completely determined by our results. Such approximations seem to be very much specific of each logic we consider, but could definitely have a big impact on practical applications.

As an illustration, let us consider propositional normal modal logic suitably decomposed as $\mathbf{K}_\Box = \mathcal{L}_{\mathsf{cls}} \bullet \mathcal{L}_\Box \bullet \mathcal{L}_{k_\Box}$. It is well known that the problem of deciding $\mathbf{K}_\Box$ is PSPACE-complete. Moreover, deciding $\mathcal{L}_{\mathsf{cls}}$ is coNP-complete, whereas it is not difficult to conclude that the decision problem for $\mathcal{L}_\Box$ is polynomial. Hence, assuming that NP≠PSPACE (and therefore coNP≠PSPACE), our complexity results allow us to conclude that there cannot be a polynomial time algorithm to compute a suitable set $A_{\Gamma,\varphi} \subseteq \{\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) : \varphi, \psi \in L_{\Sigma_{\mathbf{K}_\Box}}(P)\}$. Still, it would be specially interesting to look for efficient approximating sequences $\{A^i_{\Gamma,\varphi}\}_{i \in \mathbb{N}_0}$ to $A_{\Gamma,\varphi}$ in the spirit of [29, 32].

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A Appendix

In this appendix, which should actually be seen as an additional section of the paper, we present the details of the result on which our decidability and complexity results depend. We emphasise that this appendix should not be understood as a mere technical annex, not because it is not relatively involved in technical terms, which it is, but because it presents a fundamental result which is important *per se*: our final aim is to obtain a proof of the full characterization result of disjoint fibring with respect to the component logics presented in Proposition 3.8.

A.1 Monoliths and substitutions

It is useful to consider forms of replacement of formulas by formulas.

Let $\Sigma \cup \Sigma'$ be signatures. Given two η-sequences $\overline{\alpha}$ and $\overline{\beta}$ of $L_{\Sigma'}(P)$ formulas, with $\overline{\alpha}$ injective, we write $\psi[\overline{\alpha}/\overline{\beta}]_\Sigma$ to denote the formula obtained by replacing each occurrence of α_i as a Σ-monolith of ψ by β_i, for all $i < \eta$. We dub this notion as *shallow monolithic-substitution*. It is not difficult to check that $\psi[\overline{\alpha}/\overline{\beta}]_\Sigma = (\mathsf{skel}_\Sigma(\psi))^\sigma$ where σ is a substitution $\sigma : P \cup X \to L_{\Sigma'}(P)$ such that $\sigma(x_{\alpha_i}) = \beta_i$ for all $i < \eta$ and $\sigma(y) = y$ for $y \in P \cup (X \setminus \{x_{\alpha_i} : i < \eta\})$. We extend the notation to sets, or sequences, of formulas $\overline{\psi}$ in the natural way.

Clearly, the notion of substitution that we have just defined is useful in the context of fibring, whenever we consider two signatures Σ_1 and Σ_2 such that $\Sigma \in \{\Sigma_1, \Sigma_2\}$ and $\Sigma' = \Sigma_1 \cup \Sigma_2$. When we further work in the context of disjoint fibring, i.e., when $\Sigma_1 \cap \Sigma_2 = \emptyset$, the following more complex notion of replacement is also useful.

Given $\psi, \alpha, \beta \in L_{\Sigma_{12}}(P)$ such that $\mathsf{head}(\alpha) \in \Sigma_i$ for $i \in \{1, 2\}$, we write $\psi\{\alpha/\beta\}_{\Sigma_1,\Sigma_2}$ to denote the *deep monolithic-substitution* of the formula α by β in ψ, obtained by replacing by β any occurrence of α as a Σ_{3-i}-monolith of a subformula of ψ whose head is not in Σ_i. Formally:

$$\psi\{\alpha/\beta\}_{\Sigma_1,\Sigma_2} = \begin{cases} \beta & \text{if } \psi = \alpha, \\ \psi[\overline{M}/\overline{M}\{\alpha/\beta\}_{\Sigma_1,\Sigma_2}]_{\Sigma_j} & \text{if } \psi \neq \alpha, \mathsf{head}(\psi) \notin \Sigma_{3-j} \text{ and } M = \mathsf{Mon}_{\Sigma_j}(\psi). \end{cases}$$

We shall write $\psi\{\alpha/\beta\}$ instead of $\psi\{\alpha/\beta\}_{\Sigma_1,\Sigma_2}$, for simplicity, whenever the signatures Σ_1 and Σ_2 are clear in the context. The curly bracket substitution notations for deep monolithic-substitution is also extended to sets, or sequences, of formulas in the obvious manner.

Though involved, specially the notion of deep substitution, these notions are quite simple to use, as illustrated by the following example.

Example A.1. For a change, let us consider Σ_{neg} and Σ_{imp}, as defined in Examples 3.1 and 4.1.

If we take $\psi = (\neg p) \to ((\neg(q \to \neg p)) \to \neg\neg p) \in L_{\Sigma_{\mathsf{neg}} \cup \Sigma_{\mathsf{imp}}}(P)$, we have that

$$\psi[\neg p/\beta]_{\Sigma_{\mathsf{imp}}} = \beta \to ((\neg(q \to \neg p)) \to \neg\neg p)$$

$$\psi\{\neg p/\beta\}_{\Sigma_{\mathsf{neg}},\Sigma_{\mathsf{imp}}} = \beta \to ((\neg(q \to \beta)) \to \neg\neg p).$$

In order to understand why, it is useful to analyze each of the three occurrences of $\neg p$ in ψ. The leftmost occurrence of $\neg p$ is a Σ_{imp}-monolith of ψ and is thus replaced in both the shallow and deep substitutions. The second occurrence of $\neg p$ is strictly inside another Σ_{imp}-monolith, that is, $\neg(q \to \neg p)$. Thus, it is not touched by the shallow substitution. However, $\neg p$ is a Σ_{imp}-monolith of the subformula $q \to \neg p$ inside, and is therefore replaced by the deep substitution. Finally, the third and rightmost occurrence of $\neg p$ is strictly inside the Σ_{imp}-monolith $\neg\neg p$. However, $\neg p$ is never a Σ_{imp}-monolith of a subformula of $\neg\neg p$ whose head is not $\neg$ (there is no such subformula), and thus both the shallow and the deep substitution leave the rightmost $\neg p$ untouched. △

Next, we borrow a very useful lemma from [28]. We include its proof, for the sake of self-containment, but also because it is small but may help the reader to understand what is happening, and hopefully work as a warm up for what comes next.

Lemma A.2. *Let $\Sigma \subseteq \Sigma'$ and $\Gamma \subseteq L_\Sigma(P)$. Then, for every $\sigma : P \to L_{\Sigma'}(P)$, and every two η-sequences $\overline{\alpha}$ and $\overline{\beta}$ of formulas in $L_{\Sigma'}(P)$, with $\overline{\alpha}$ injective, there exists $\rho : P \to L_{\Sigma'}(P)$ such that*

$$\Gamma^\rho = \Gamma^\sigma[\overline{\alpha}/\overline{\beta}]_\Sigma.$$

Proof. One should observe, to start with, that $\mathsf{Mon}_\Sigma(\Gamma) = \emptyset$. Thus, if $\alpha_\kappa \in \mathsf{Mon}_\Sigma(\varphi^\sigma)$ for some $\varphi \in \Gamma$, then there must exist a variable $p \in P$ occurring in φ such that $\alpha_\kappa \in \mathsf{Mon}_\Sigma(\sigma(p))$. Hence, the substitution defined by $\rho(q) = \sigma(q)[\overline{\alpha}/\overline{\beta}]_\Sigma$ for every $q \in P$ satisfies the conditions of the lemma. □

Note that the lemma reflects the fact that the occurrence of Σ-monoliths in instances of $L_\Sigma(P)$ formulas is only possible if they are brought about by the substitution. As a corollary, we obtain the following result.

Proposition A.3. *Let $\mathcal{L} = \langle \Sigma, \_^{\vdash} \rangle$, $\Sigma \subseteq \Sigma'$ and $\Delta \cup \{\varphi\} \subseteq L_{\Sigma'}(P)$. Then, the following properties hold:*

- *$\Delta \vdash \varphi$ if and only if $\mathsf{skel}_\Sigma(\Delta) \vdash \mathsf{skel}_\Sigma(\varphi)$, and*
- *Δ is $\vdash$-explosive if and only if $\mathsf{skel}_\Sigma(\Delta)$ is $\vdash$-explosive.*

Proof. We prove each of the properties.

- If $\Delta \vdash \varphi$ then, by definition, there exist $\Gamma \cup \{\psi\} \subseteq L_\Sigma(P)$ and $\sigma : P \to L_{\Sigma'}(P)$ such that $\Gamma \vdash \psi$, $\Gamma^\sigma = \Delta$ and $\psi^\sigma = \varphi$. Let $\overline{a} = \overline{\mathsf{Mon}_\Sigma(\Delta \cup \{\varphi\})}$ be any injective and surjective sequence based on $\mathsf{Mon}_\Sigma(\Delta \cup \{\varphi\})$, and define $\overline{\beta}$ to be the same length sequence such that each $\beta_\iota = x_{\alpha_\iota}$. Note that $\varphi'[\overline{\alpha}/\overline{\beta}]_\Sigma = \mathsf{skel}_\Sigma(\varphi')$ for every $\varphi' \in \Delta \cup \{\varphi\}$. The left-to-right implication follows simply by applying Lemma A.2 to $\Gamma \cup \{\psi\}$, $\overline{\alpha}$ and $\overline{\beta}$, and then the structurality of $\vdash$ under the resulting substitution ρ.

  Conversely, the fact that $\mathsf{skel}_\Sigma(\Delta) \vdash \mathsf{skel}_\Sigma(\varphi)$ implies $\Delta \vdash \varphi$ follows easily from the structurality of $\vdash$ by considering the substitution unsk as defined in Lemma 3.4.

- If Δ is $\vdash$-explosive then we know that $\Delta^{\mathsf{nxt}} \vdash p_0$. Using the first property, already proved above, we get that $\mathsf{skel}_\Sigma(\Delta^{\mathsf{nxt}}) \vdash p_0$. Consider the substitution $\rho : P \cup X \to L_{\Sigma'}(P \cup X)$ such that $\rho(p) = p$ if $p \in P$, $\rho(x_\phi) = x_{\phi^{\mathsf{prv}}}$, with prv defined as in the proof of Lemma 2.1. Easily, we have $\mathsf{skel}_\Sigma(\Delta^{\mathsf{nxt}})^\rho = \mathsf{skel}_\Sigma(\Delta)^{\mathsf{nxt}}$, and by structurality we conclude that $\mathsf{skel}_\Sigma(\Delta)^{\mathsf{nxt}} \vdash p_0$ and $\mathsf{skel}_\Sigma(\Delta)$ is $\vdash$-explosive.

  Reciprocally, assume that $\mathsf{skel}_\Sigma(\Delta)$ is $\vdash$-explosive, and consider again the substitution unsk. Of course, $\mathsf{skel}_\Sigma(\Delta)^{\mathsf{unsk}}$ is also $\vdash$-explosive, and it follows from Lemma 3.4 that $\mathsf{skel}_\Sigma(\Delta)^{\mathsf{unsk}} = \Delta$.

□

The next lemma plays, for deep monolithic-substitution, the same role played by Lemma A.2 for shallow substitution.

Lemma A.4. *Let Σ_1, Σ_2 be disjoint signatures, $\Gamma \subseteq L_{\Sigma_i}(P)$ for some $i \in \{1, 2\}$, and $\alpha, \beta \in L_{\Sigma_{12}}(P)$ with $\mathsf{head}(\alpha) \in \Sigma_i$. Then, for every $\sigma : P \to L_{\Sigma_{12}}(P)$ with $\alpha \notin \Gamma^\sigma$, there exists $\rho : P \to L_{\Sigma_{12}}(P)$ such that $\Gamma^\rho = \Gamma^\sigma\{\alpha/\beta\}$.*

Proof. Letting $M = \mathsf{Mon}_{\Sigma_i}(\Gamma^\sigma)$, we have that $\Gamma^\sigma\{\alpha/\beta\} = \{\psi\{\alpha/\beta\} : \psi \in \Gamma^\sigma\} = \{\psi[\overline{M}/\overline{M}\{\alpha/\beta\}]_{\Sigma_i} : \psi \in \Gamma^\sigma\} = \Gamma^\sigma[\overline{M}/\overline{M}\{\alpha/\beta\}]_{\Sigma_i}$, since $\alpha \notin \Gamma^\sigma$. Therefore, by Lemma A.2, there exist ρ such that $\Gamma^\rho = \Gamma^\sigma[\overline{M}/\overline{M}\{\alpha/\beta\}]_{\Sigma_i} = \Gamma^\sigma\{\alpha/\beta\}$, as we wanted to prove. □

A.2 The interplay between the components in disjoint fibring

Let us now have a technical look at the patterns of mixed reasoning that occur in fibred logics, formalizing the ideas in Example 3.1. We proceed by characterizing the irrelevance of certain monoliths in derivations in logics obtained by disjoint fibring, by taking advantage of the deep monolithic-substitutions introduced above. Note that the disjointness of the signatures is instrumental for this purpose. Before stating and proving the result, let us analyze a few concrete cases.

Example A.5.

Consider the fibred logic $\mathcal{L}_{\mathsf{imp}} \bullet \mathcal{L}_{\mathsf{neg}} = \langle \Sigma, \_^{\vdash} \rangle$. Let $\Gamma = \{\neg\neg\neg p, \neg p \to (\neg t \to p), (\neg t \to p) \to (p \to \neg t)\} \subseteq L_\Sigma(P)$ and $\beta \in L_\Sigma(P)$, and consider the following four 9-sequences.

$\kappa < 9$	$\overline{\psi}$	$\overline{\psi}\{\neg t/\beta\}$	$\overline{\psi}[\neg t/\beta]_{\Sigma_{\mathsf{imp}}}$	$\overline{\psi}\{\neg p/r\}$
0	$\neg\neg\neg p$	$\neg\neg\neg p$	$\neg\neg\neg p$	$\neg\neg\neg p$
1	$\neg p \to (\neg t \to p)$	$\neg p \to (\beta \to p)$	$\neg p \to (\beta \to p)$	$r \to (\neg t \to p)$
2	$(\neg t \to p) \to (p \to \neg t)$	$(\beta \to p) \to (p \to \beta)$	$(\beta \to p) \to (p \to \beta)$	$(\neg t \to p) \to (p \to \neg t)$
3	$\neg p$	$\neg p$	$\neg p$	r
4	$\neg t \to p$	$\beta \to p$	$\beta \to p$	$\neg t \to p$
5	$p \to \neg t$	$p \to \beta$	$p \to \beta$	$p \to \neg t$
6	$(p \to \neg t) \to (\neg q \to (p \to \neg t))$	$(p \to \beta) \to (\neg q \to (p \to \beta))$	$(p \to \beta) \to (\neg q \to (p \to \beta))$	$(p \to \neg t) \to (\neg q \to (p \to \neg t))$
7	$\neg q \to (p \to \neg t)$	$\neg q \to (p \to \beta)$	$\neg q \to (p \to \beta)$	$\neg q \to (p \to \neg t)$
8	$\neg\neg(\neg q \to (p \to \neg t))$	$\neg\neg(\neg q \to (p \to \beta))$	$\neg\neg(\neg q \to (p \to \neg t))$	$\neg\neg(\neg q \to (p \to \neg t))$

It is straightforward to check that $\Gamma \vdash \overline{\psi}$, in the leftmost column of the table. The other three sequences are the result of applying monolithic substitutions to $\overline{\psi}$. These sequences illustrate how deep monolithic substitution provides the adequate notion to express the irrelevance of certain monoliths in a derivation. Moreover, they suggest a criterion for the (ir)relevance of a certain monolith.

It is not hard to see that if we use deep monolithic substitutions to replace all the monolithic occurrences of $\neg t$ with any other formula β, we still obtain a proof in $\mathcal{L}_{\mathsf{imp}} \bullet \mathcal{L}_{\mathsf{neg}}$ from $\Gamma\{\neg t/\beta\}$. That is, $\Gamma\{\neg t/\beta\} \vdash \overline{\psi}\{\neg t/\beta\}$.

The fact that $\Gamma[\neg t/\beta]_{\Sigma_{\mathsf{imp}}} \nvdash \overline{\psi}[\neg t/\beta]_{\Sigma_{\mathsf{imp}}}$ highlights the difference between deep and shallow monolithic substitutions. The point is that shallow substitution fails to reach the monoliths inside other monoliths, as is the case of $\neg t$ in $\neg\neg(\neg q \to (p \to \neg t))$ at the last step of $\overline{\psi}[\neg t/\beta]_{\Sigma_{\mathsf{imp}}}$.

However, it may happen that deep substitutions also fail to preserve the derivation. This is illustrated in the last column of the table above, by taking the Σ_{imp}-monolith $\neg p$ and $\beta = r \in P$. Clearly, $\Gamma\{\neg p/r\} \nvdash \overline{\psi}\{\neg p/r\}$ as the formula r at step 3 of the sequence is neither an hypothesis nor can it be justified by a rule applied to the previous formulas.

We may now say that the Σ_{imp}-monoliths $\neg t$ and $\neg p$ have different roles and relevance in the derivation $\Gamma \vdash \overline{\psi}$. If a formula appears in a derivation, as is the case of $\neg p = \psi_3$, we cannot hope, in general, to be able to replace its occurrences by a given formula. However, as we shall see below, the fact that a monolith does not appear in a derivation, such as in the case of $\neg t$ in $\overline{\psi}$, will be enough to ensure that it can be deep monolithically substituted by any formula. △

For the remainder of this appendix, we assume fixed two logics $\mathcal{L}_1 = \langle \Sigma_1, \_^{\vdash_1} \rangle$ and $\mathcal{L}_2 = \langle \Sigma_2, \_^{\vdash_2} \rangle$ such that $\Sigma_1 \cap \Sigma_2 = \emptyset$. We let also $\mathscr{H}_{\mathcal{L}_1} \bullet \mathscr{H}_{\mathcal{L}_2} = \langle \Sigma_{12}, \vdash_{12} \rangle$.

The next lemma, whose content is illustrated by Example A.5, extends similar partial results about shallow substitution and disjoint fibring, but concerning only $\Gamma \subseteq P$ as obtained in [28], or $\Gamma \subseteq L_{\Sigma_1}(P) \cup L_{\Sigma_2}(P)$ as obtained in [27].

Lemma A.6. *Let η be an ordinal, $\Gamma \cup \{\psi_\kappa : \kappa < \eta\} \subseteq L_{\Sigma_{12}}(P)$ and $\overline{\psi} = \langle \psi_\kappa \rangle_{\kappa < \eta}$.*

If $\Gamma \vdash_{12} \overline{\psi}$ and $\alpha \in L_{\Sigma_{12}}(P)$, then we have that either

- *$\alpha = \psi_\kappa$ for some $\kappa < \eta$, or*
- *$\Gamma\{\alpha/\beta\} \vdash_{12} \overline{\psi}\{\alpha/\beta\}$ for every $\beta \in L_{\Sigma_{12}}(P)$.*

Proof. Let us assume that $\alpha \neq \psi_\kappa$ for every $\kappa < \eta$. The proof of the second condition follows by complete transfinite induction on the size η of the derivation. It suffices to assume, by induction hypothesis, that $\Gamma\{\alpha/\beta\} \vdash_{12} \langle \psi_\kappa\{\alpha/\beta\} \rangle_{\kappa < \tau}$ for every $\tau < \eta$, and show that it implies $\Gamma\{\alpha/\beta\} \vdash_{12} \langle \psi_\kappa\{\alpha/\beta\} \rangle_{\kappa < \eta}$.

If $\eta = 0$ the result is trivial, as the derivation is empty. If η is a limit ordinal the result is immediate, by definition of derivation. If η is a successor ordinal, we have to consider two cases.

(1) $\psi_{\eta-1} \in \Gamma$.

If $\psi_{\eta-1} \in \Gamma$ then $\psi_{\eta-1}\{\alpha/\beta\} \in \Gamma\{\alpha/\beta\}$ and so the result follows.

By induction hypothesis we have that $\Gamma\{\alpha/\beta\} \vdash_{12} \langle\psi_\kappa\{\alpha/\beta\}\rangle_{\kappa<\eta-1}$ and so, by definition of derivation, we also have $\Gamma\{\alpha/\beta\} \vdash_{12} \langle\psi_\kappa\{\alpha/\beta\}\rangle_{\kappa<\eta}$.

(2) $\psi_{\eta-1} = \varphi^\sigma$ and $\Delta^\sigma \subseteq \{\psi_\kappa : \kappa < \eta - 1\}$, for some $i \in \{1, 2\}$ and $\Delta \cup \{\varphi\} \subseteq L_{\Sigma_i}(P)$ such that $\Delta \vdash_i \varphi$.

Since we are assuming that $\alpha \notin (\Delta \cup \varphi)^\sigma$, we can apply Lemma A.4 to $\Delta \cup \{\varphi\} \subseteq L_{\Sigma_i}(P)$, σ, α and β, and get that there exists ρ such that $\varphi^\rho = \varphi^\sigma\{\alpha/\beta\} = \psi_{\eta-1}\{\alpha/\beta\}$, and also $\Delta^\rho = \Delta^\sigma\{\alpha/\beta\} \subseteq \{\psi_\kappa\{\alpha/\beta\} : \kappa < \eta - 1\}$.

By induction hypothesis we have that $\Gamma\{\alpha/\beta\} \vdash_{12} \langle\psi_\kappa\{\alpha/\beta\}\rangle_{\kappa<\eta-1}$ and so, by definition of derivation, we also have $\Gamma\{\alpha/\beta\} \vdash_{12} \langle\psi_\kappa\{\alpha/\beta\}\rangle_{\kappa<\eta}$. □

The following lemma provides sufficient conditions for a consequence to hold in a logic obtained by disjoint fibring, which already amount to half of Proposition 3.8.

Lemma A.7. *Let $\Gamma \cup \{\psi\} \subseteq L_{\Sigma_{12}}(P)$. Given $i, j \in \{1, 2\}$ with $i \neq j$,*

$$\Gamma^i, X^i_\Gamma(\psi) \vdash_i \mathsf{skel}_{\Sigma_i}(\psi) \textit{ or } \Gamma^j \textit{ is } \vdash_j \textit{-explosive}$$

implies

$$\Gamma \vdash_{12} \psi.$$

Proof. Recall from Definition 3.5 that $x_* \in \Gamma^i$ if and only if $\Gamma^{\vdash_{12}} \neq \emptyset$. Let σ be the substitution such that $\sigma = \mathsf{unsk}$ if $\Gamma^{\vdash_{12}} = \emptyset$, and otherwise $\sigma = \mathsf{unsk}_\gamma$ with $\gamma \in \Gamma^{\vdash_{12}} \neq \emptyset$.

Now, on one hand, if we have that $\Gamma^i, X^i_\Gamma(\psi) \vdash_i \mathsf{skel}_{\Sigma_i}(\psi)$ then using structurality we obtain $(\Gamma^i)^\sigma, (X^i_\Gamma(\psi))^\sigma \vdash_i (\mathsf{skel}_{\Sigma_i}(\psi))^\sigma$. However, clearly, $(\Gamma^i \cup X^i_\Gamma(\psi))^\sigma \subseteq \Gamma^{\vdash_{12}}$ and $(\mathsf{skel}_{\Sigma_i}(\psi))^\sigma = \psi$, by using Lemmas 3.4 and 3.7, and we conclude that $\Gamma \vdash_{12} \psi$.

If, on the other hand, we know that Γ^j is $\vdash_j$-explosive, then we have $\Gamma^j \vdash_j \psi$. However, we also have $(\Gamma^j)^\sigma \subseteq \Gamma^{\vdash_{12}}$ and $\psi^\sigma = \psi$, and we conclude again that $\Gamma \vdash_{12} \psi$. □

Before we state and prove Proposition 3.8, we need an additional lemma, which takes advantage of the fact that one can always work under the assumption that there is a fresh variable in P.

Lemma A.8. *Given $\Gamma \cup \{\psi\} \subseteq L_{\Sigma_{12}}(P)$, let $\Theta = \Gamma^{\mathsf{nxt}}\{\psi^{\mathsf{nxt}}/p_0\}$. Then, $(\Theta_\omega)^{\mathsf{prv}_{\langle\psi,\mathsf{id}\rangle}} \subseteq \Gamma_\omega$.*

Proof. We prove, by induction on $n \in \mathbb{N}_0$, that $(\Theta_n)^{\mathsf{prv}_{\langle\psi,\mathsf{id}\rangle}} \subseteq \Gamma_n$.

The base case is straightforward, as $(\Theta_0)^{\mathsf{prv}_{\langle\psi,\mathsf{id}\rangle}} = \Theta^{\mathsf{prv}_{\langle\psi,\mathsf{id}\rangle}} = (\Gamma^{\mathsf{nxt}}\{\psi^{\mathsf{nxt}}/p_0\})^{\mathsf{prv}_{\langle\psi,\mathsf{id}\rangle}} = \Gamma = \Gamma_0$, because $\mathsf{prv}_{\langle\psi,\mathsf{id}\rangle}(p_0) = \psi$, $\mathsf{prv}_{\langle\psi,\mathsf{id}\rangle}(p_{k+1}) = p_k$, and $p_0 \notin \mathsf{var}(\Gamma^{\mathsf{nxt}})$.

For the step, let $\varphi \in \Theta_n$ and $\varphi \in \mathsf{sub}(\Theta)$, and assume by induction hypothesis that $(\Theta_{n-1})^{\mathsf{prv}_{\langle\psi,\mathsf{id}\rangle}} \subseteq \Gamma_{n-1}$. Since $\varphi \in \Theta_n$, we know that $\Theta_{n-1} \vdash_i \varphi$ for some $i \in \{1, 2\}$, and by structurality we get that $(\Theta_{n-1})^{\mathsf{prv}_{\langle\psi,\mathsf{id}\rangle}} \vdash_i \varphi^{\mathsf{prv}_{\langle\psi,\mathsf{id}\rangle}}$. Finally, monotonicity allows us to conclude that $\Gamma_{n-1} \vdash_i \varphi^{\mathsf{prv}_{\langle\psi,\mathsf{id}\rangle}}$ and therefore $\varphi^{\mathsf{prv}_{\langle\psi,\mathsf{id}\rangle}} \in \Gamma_n$. □

We can finally tackle the envisaged characterization result. It will be useful to consider the following additional substitutions: given $\Lambda \subseteq L_{\Sigma_{12}}(P)$, let $\mathsf{ast}_\Lambda : P \cup X_* \to L_{\Sigma_{12}}(P \cup X_*)$ be such that $\mathsf{ast}_\Lambda(x_\phi) = x_*$ if $\phi \notin \Lambda$, and $\mathsf{ast}_\Lambda(y) = y$ otherwise. It is straightforward to check that if $Y \subseteq X$ then $Y^{\mathsf{ast}_\Lambda} \subseteq \{x_*\} \cup \{x_\phi \in Y : \phi \in \Lambda\}$.

Proposition A.9. *Let η be an ordinal, $\Gamma \cup \{\psi_\kappa : \kappa \leq \eta\} \subseteq L_{\Sigma_{12}}(P)$ and $\overline{\psi} = \langle\psi_\kappa\rangle_{\kappa\leq\eta}$.*

If $\Gamma \vdash_{12} \overline{\psi}$ then the following properties hold:

(1) $\Gamma^i, X^i_\Gamma(\overline{\psi},\eta) \vdash_i \mathsf{skel}_{\Sigma_i}(\psi_\eta)$ *or* Γ^j *is* $\vdash_j$ *-explosive, for* $i,j \in \{1,2\}$ *with* $i \neq j$,
where $X^i_\Gamma(\overline{\psi},\eta) = \{x_\phi \in X^i_\Gamma(\psi_\eta) : \phi = \psi_\kappa$ *for some* $\kappa \leq \eta\}$, *and*

(2) $\psi_\eta \in \Gamma_\omega$ *whenever* $\psi_\eta \in \mathsf{sub}(\Gamma)$.

Proof. The proof follows by complete transfinite induction on the ordinal η. It suffices to assume, by induction hypothesis, that the statement holds for every derivation $\Theta \vdash_{12} \overline{\xi}$ of a sequence $\overline{\xi} = \langle \xi_\kappa \rangle_{\kappa \leq \tau}$ with $\tau < \eta$, and show that it implies properties (1) and (2) provided that $\Gamma \vdash_{12} \overline{\psi}$.

If $\Gamma \vdash_{12} \overline{\psi}$ then $\Gamma^{\vdash_{12}} \neq \emptyset$, and so $x_* \in \Gamma^1$ and $x_* \in \Gamma^2$.

If either Γ^1 is $\vdash_1$-explosive or Γ^2 is $\vdash_2$-explosive (1) immediately follows. To see that (2) also follows, let us assume that Γ^i is $\vdash_i$-explosive for some $i \in \{1,2\}$. Clearly, then, $\Gamma^i \vdash_i \mathsf{skel}_{\Sigma_i}(\psi_\eta)$. Hence, if $\psi_\eta \in \mathsf{sub}(\Gamma)$ we can apply Lemma 3.7 and conclude that $\psi_\eta \in \Gamma_\omega$.

We proceed by assuming that Γ^1 and Γ^2 are not explosive. We have to consider two cases.

- $\psi_\eta \in \Gamma$.

  By definition, $\Gamma \subseteq \Gamma_\omega$.

  (1) $\Gamma^i, X^i_\Gamma(\overline{\psi},\eta) \vdash_i \mathsf{skel}_{\Sigma_i}(\psi_\eta)$, because $\psi_\eta \in \Gamma \subseteq \Gamma_\omega$ implies $\mathsf{skel}_{\Sigma_i}(\psi_\eta) \in \Gamma^i$, for $i \in \{1,2\}$.

  (2) Of course $\psi_\eta \in \Gamma \subseteq \mathsf{sub}(\Gamma)$, but we also have $\psi_\eta \in \Gamma \subseteq \Gamma_\omega$.

- $\psi_\eta = \varphi^\sigma$ and $\Delta^\sigma \subseteq \{\psi_\kappa : \kappa < \eta\}$, for some $t \in \{1,2\}$ and $\Delta \cup \{\varphi\} \subseteq L_{\Sigma_t}(P)$ such that $\Delta \vdash_t \varphi$.

  From Proposition A.3, we know that $\{\mathsf{skel}_{\Sigma_t}(\psi_\kappa) : \kappa < \eta\} \vdash_t \mathsf{skel}_{\Sigma_t}(\psi_\eta)$.

  By induction hypothesis, as $\Gamma \vdash_{12} \langle \psi_\tau \rangle_{\tau \leq \kappa}$ for each $\kappa < \eta$, it follows that $\Gamma^t, \bigcup_{\kappa<\eta} X^t_\Gamma(\overline{\psi},\kappa) \vdash_t \mathsf{skel}_{\Sigma_t}(\psi_\eta)$. Consider the substitution ast_Λ with $\Lambda = \mathsf{sub}(\Gamma) \cup \mathsf{Mon}_{\Sigma_t}(\psi_\eta)$. Easily, if $\kappa < \eta$, we have that $(X^t_\Gamma(\overline{\psi},\kappa))^{\mathsf{ast}_\Lambda} \subseteq \Gamma^t \cup X^t_\Gamma(\overline{\psi},\eta)$. Indeed, by definition, $(X^t_\Gamma(\overline{\psi},\kappa))^{\mathsf{ast}_\Lambda} \subseteq \{x_*\} \cup \{x_\phi \in X^t_\Gamma(\overline{\psi},\kappa) : \phi \in \mathsf{sub}(\Gamma) \cup \mathsf{Mon}_{\Sigma_t}(\psi_\eta)\}$. Clearly, $x_* \in \Gamma^t$. Furthermore, if $x_\phi \in X^t_\Gamma(\overline{\psi},\kappa)$ then $\phi = \psi_{\kappa'}$ for some $\kappa' \leq \kappa$, $\phi \in \mathsf{Mon}_{\Sigma_t}(\psi_\kappa)$, and $\Gamma \vdash_{12} \phi$. If $\phi \in \mathsf{sub}(\Gamma)$ then $\phi \in \Gamma_\omega$, using the induction hypothesis from the fact that $\Gamma \vdash_{12} \langle \psi_\tau \rangle_{\tau \leq \kappa'}$, from where we can conclude that $\mathsf{skel}_{\Sigma_t}(\phi) = x_\phi \in \Gamma^t$; otherwise, $\phi \in \mathsf{Mon}_{\Sigma_t}(\psi_\eta)$ and thus $x_\phi \in X^t_\Gamma(\overline{\psi},\eta)$, as $\kappa' \leq \kappa < \eta$.
  Hence, $(\Gamma^t \cup (\bigcup_{\kappa<\eta} X^t_\Gamma(\overline{\psi},\kappa)))^{\mathsf{ast}_\Lambda} \subseteq \Gamma^t \cup X^t_\Gamma(\overline{\psi},\eta)$, and so

  $$\Gamma^t, X^t_\Gamma(\overline{\psi},\eta) \vdash_t \mathsf{skel}_{\Sigma_t}(\psi_\eta) \tag{*}$$

  by structurality and monoticity of $\vdash_t$, as it is clear from the definition of ast_Λ that $(\Gamma^t)^{\mathsf{ast}_\Lambda} = \Gamma^t$ and $(\mathsf{skel}_{\Sigma_t}(\psi_\eta))^{\mathsf{ast}_\Lambda} = \mathsf{skel}_{\Sigma_t}(\psi_\eta)$.

  Before we proceed, let us exclude some of the cases to be analyzed. Namely, we will show that one cannot simultaneously have

  (a) $\mathsf{head}(\psi_\eta) \notin \Sigma_t$,

  (b) $\psi_\eta \neq \psi_\kappa$ for all $\kappa < \eta$, and

  (c) $\psi_\eta \notin \Gamma_\omega$.

  Clearly, condition (a) is satisfied if either (a1) $\psi_\eta \in P$, or (a2) $\mathsf{head}(\psi_\eta) \in \Sigma_{3-t}$. We shall reach a contradiction, assuming either of these cases along with (b) and (c).

  (a1) $\psi_\eta = p \in P$.

  As $\psi_\eta \in P$ we have that $\mathsf{skel}_{\Sigma_t}(\psi_\eta) = p$ and $X^t_\Gamma(\overline{\psi},\eta) = \emptyset$. Therefore, as we know from (*) that $\Gamma^t, X^t_\Gamma(\overline{\psi},\eta) \vdash_t \mathsf{skel}_{\Sigma_t}(\psi_\eta)$, we obtain $\Gamma^t \vdash_t p$. Here we have to split the analysis in yet another two cases.

  * If $p \notin \mathsf{var}(\Gamma^t)$ then, by structurality of $\vdash_t$, we easily conclude that Γ^t is $\vdash_t$-explosive, which yields a contradiction.

* If $p \in \mathsf{var}(\Gamma^t)$ then $p \in \mathsf{var}(\Gamma) \subseteq \mathsf{sub}(\Gamma)$. Since $\Gamma^t \vdash_t p$ and $\mathsf{skel}_{\Sigma_t}(p) = p$, we can apply Lemma 3.7 and conclude that $p \in \Gamma_\omega$, thus contradicting assumption (c).

(a2) $\mathsf{head}(\psi_\eta) \in \Sigma_{3-t}$.

Let $\Theta = \Gamma^{\mathsf{nxt}}\{\psi_\eta^{\mathsf{nxt}}/p_0\}$ and $\overline{\xi} = \langle \psi_\kappa^{\mathsf{nxt}}\{\psi_\eta^{\mathsf{nxt}}/p_0\}\rangle_{k\leq\eta}$.

First of all, we note that $x_* \in \Theta^t$. Indeed, it is clearly the case if $\Theta \neq \emptyset$. Otherwise, as we know that $x_* \in \Gamma^t$, and $\Theta = \emptyset$ if and only if $\Gamma = \emptyset$, we easily obtain that $\Gamma^t = \Theta^t = \{x_*\}$.

Next, we show that it cannot be the case that Θ^t is $\vdash_t$-explosive. If $\Gamma = \emptyset$ then $\Theta^t = \Gamma^t = \{x_*\}$, and the $\vdash_t$-explosiveness of $\Gamma^t = \Theta^t$ follows. If $\Gamma \neq \emptyset$ then $\Theta \neq \emptyset$, and let $\theta \in \Theta$. By applying Lemma 3.7, and noting that $\Theta \subseteq \Theta_\omega$, we know that $(\Theta^t)^{\mathsf{unsk}_\theta} = \Theta_\omega$ is also $\vdash_t$-explosive. Immediately, this implies that $(\Theta_\omega)^{\mathsf{prv}_{\langle \psi_\eta, \mathsf{id}\rangle}}$ must also be $\vdash_t$-explosive which, using Lemma A.8, leads to the $\vdash_t$-explosiveness of Γ_ω. Therefore, $\Gamma^t = \mathsf{skel}_{\Sigma_t}(\Gamma) \cup \{x_*\}$ is also $\vdash_t$-explosive, by an immediate application of Proposition A.3. In any case we reach a contradiction, since we are assuming that none of Γ^1 and Γ^2 is explosive.

We proceed knowing that $x_* \in \Theta^t$ and Θ^t is not $\vdash_t$-explosive. Given that $\Gamma \vdash_{12} \overline{\psi}$ we have that $\Gamma^{\mathsf{nxt}} \vdash_{12} \overline{\psi}^{\mathsf{nxt}}$, and Lemma A.6 and the fact that property (b) holds, allow us to conclude that $\Theta \vdash_{12} \langle \xi_\kappa \rangle_{\kappa<\eta}$. Hence, by induction hypothesis, we get that $\Theta^t, X^t_\Theta(\overline{\xi}, \kappa) \vdash_t \mathsf{skel}_{\Sigma_t}(\xi_\kappa)$ for each $\kappa < \eta$. Consider the substitution $\nu : P \cup X \to L_{\Sigma_{12}}(P \cup X)$ such that $\nu(p_k) = p_{k+1}$, $\nu(x_{\psi_\eta}) = p_0$ and $\nu(x_\phi) = x_{\phi^{\mathsf{nxt}}\{\psi_\eta^{\mathsf{nxt}}/p_0\}}$ for $\phi \neq \psi_\eta$. Recall that $\{\mathsf{skel}_{\Sigma_t}(\psi_\kappa) : \kappa < \eta\} \vdash_t \mathsf{skel}_{\Sigma_t}(\psi_\eta)$. It is not difficult to check that $\mathsf{skel}_{\Sigma_t}(\psi_\kappa)^\nu = \mathsf{skel}_{\Sigma_t}(\xi_\kappa)$ for every $\kappa \leq \eta$, namely because we are assuming (a2). Therefore, we can conclude that $\Theta^t, \bigcup_{\kappa<\eta} X^t_\Theta(\overline{\xi}, \kappa) \vdash_t \mathsf{skel}_{\Sigma_t}(\xi_\eta)$, i.e.,

$$\Theta^t, \bigcup_{\kappa<\eta} X^t_\Theta(\overline{\xi}, \kappa) \vdash_t p_0$$

as $\xi_\eta = \psi_\eta^{\mathsf{nxt}}\{\psi_\eta^{\mathsf{nxt}}/p_0\} = p_0$.

Consider the substitution $\mathsf{ast}_{\mathsf{sub}(\Theta)}$. For each $\kappa < \eta$, we have that $(X^t_\Theta(\overline{\xi}, \kappa))^{\mathsf{ast}_{\mathsf{sub}(\Theta)}} \subseteq \Theta^t$. Indeed, by definition, $(X^t_\Theta(\overline{\xi}, \kappa))^{\mathsf{ast}_{\mathsf{sub}(\Theta)}} \subseteq \{x_*\} \cup \{x_\phi \in X^t_\Theta(\overline{\xi}, \kappa) : \phi \in \mathsf{sub}(\Theta)\}$. We know that $x_* \in \Theta^t$. Furthermore, if $x_\phi \in X^t_\Theta(\overline{\xi}, \kappa)$ then $\phi = \xi_{\kappa'}$ for some $\kappa' \leq \kappa$, $\phi \in \mathsf{Mon}_{\Sigma_t}(\xi_\kappa)$, and $\Theta \vdash_{12} \phi$. If $\phi \in \mathsf{sub}(\Theta)$ then $\phi \in \Theta_\omega$, using the induction hypothesis from the fact that $\Theta \vdash_{12} \langle \xi_\tau \rangle_{\tau \leq \kappa'}$, from where we can conclude that $\mathsf{skel}_{\Sigma_t}(\phi) = x_\phi \in \Theta^t$.

Additionally, it is easy to see that $(\Theta^t)^{\mathsf{ast}_{\mathsf{sub}(\Theta)}} = \Theta^t$ and $\mathsf{ast}_{\mathsf{sub}(\Theta)}(p_0) = p_0$, from where we obtain by structurality that

$$\Theta^t \vdash_t p_0.$$

From here, we can exclude the case that $p_0 \notin \mathsf{var}(\Theta^t)$, as it would imply a contradiction to our knowledge that Θ^t is not $\vdash_t$-explosive. Hence, we proceed by assuming that $p_0 \in \mathsf{var}(\Theta^t)$. In this case, $p_0 \in \mathsf{var}(\Theta) \subseteq \mathsf{sub}(\Theta)$. Therefore, since we know that $\Theta^t \vdash_t p_0$ and $\mathsf{skel}_{\Sigma_t}(p_0) = p_0$, we can apply Lemma 3.7 and conclude that $p_0 \in \Theta_\omega$. According to Lemma A.8 we have $(\Theta_\omega)^{\mathsf{prv}_{\langle \psi, \mathsf{id}\rangle}} \subseteq \Gamma_\omega$, and thus $\mathsf{prv}_{\langle \psi_\eta, \mathsf{id}\rangle}(p_0) = \psi_\eta \in \Gamma_\omega$, contradicting assumption (c).

We can finally establish properties (1) and (2).

(1) If $i = t$, the property corresponds precisely to (*), established above.

If $i \neq t = j$ then we show that the property holds once we assume the failure of any of the jointly contradictory cases (a), (b) or (c).

– If condition (a) fails, then $\mathsf{head}(\psi_\eta) \in \Sigma_j$. In that case, the property follows easily, as we have that $\mathsf{Mon}_{\Sigma_i}(\psi_\eta) = \{\psi_\eta\}$, and thus $\mathsf{skel}_{\Sigma_i}(\psi_\eta) = x_{\psi_\eta} \in X^i_\Gamma(\overline{\psi}, \eta)$, from which we conclude that $\Gamma^i, X^i_\Gamma(\overline{\psi}, \eta) \vdash_i \mathsf{skel}_{\Sigma_i}(\psi_\eta)$.

– If condition (b) fails, then it is the case that $\psi_\eta = \psi_\kappa$ for some $\kappa < \eta$. In that case, it is easy to check that $X^i_\Gamma(\overline{\psi}, \kappa) \subseteq X^i_\Gamma(\overline{\psi}, \eta)$. Since $\Gamma \vdash_{12} \langle \psi_\tau \rangle_{\tau \leq \kappa}$, we know by induction hypothesis that $\Gamma^i, X^i_\Gamma(\overline{\psi}, \kappa) \vdash_i \mathsf{skel}_{\Sigma_i}(\psi_\kappa)$, and $\mathsf{skel}_{\Sigma_i}(\psi_\eta) = \mathsf{skel}_{\Sigma_i}(\psi_\kappa)$, we can thus use the monotonicity of $\vdash_i$ to conclude that $\Gamma^i, X^i_\Gamma(\overline{\psi}, \eta) \vdash_i \mathsf{skel}_{\Sigma_i}(\psi_\eta)$.

– If condition (c) fails, then $\psi_\eta \in \Gamma_\omega$. Consequently, $\mathsf{skel}_{\Sigma_i}(\psi_\eta) \in \Gamma^i$, and therefore we also have $\Gamma^i, X^i_\Gamma(\overline{\psi}, \eta) \vdash_i \mathsf{skel}_{\Sigma_i}(\psi_\eta)$.

(2) Assume that $\psi_\eta \in \mathsf{sub}(\Gamma)$. As above, we will show that $\psi_\eta \in \Gamma_\omega$ by taking advantage of the fact that conditions (a), (b), (c) are jointly contradictory. In fact, what we want to prove is precisely that condition (c) must fail.

– If condition (a) fails then $\mathsf{head}(\psi_\eta) \in \Sigma_t$. Therefore, it is immediate that $\psi_\eta \notin \mathsf{Mon}_{\Sigma_t}(\psi_\eta)$ and so $x_{\psi_\eta} \notin X^t_\Gamma(\overline{\psi}, \eta)$. Hence, $x_\phi \in X^t_\Gamma(\overline{\psi}, \eta)$ implies that $\phi \in \mathsf{sub}(\psi_\eta) \subseteq \mathsf{sub}(\Gamma)$ and $\phi = \psi_\kappa$ for some $\kappa < \eta$, and the induction hypothesis ensures that $\phi \in \Gamma_\omega$, as $\Gamma \vdash_{12} \langle \psi_\tau \rangle_{\tau \leq \kappa}$. Easily, as one knows that $\phi \in \mathsf{Mon}_{\Sigma_t}(\psi_\eta)$, it follows that $\mathsf{skel}_{\Sigma_t}(\phi) = x_\phi \in \Gamma^t$. Thus, we get that $X^t_\Gamma(\overline{\psi}, \eta) \subseteq \Gamma^t$.

As we know from (*) that $\Gamma^t, X^t_\Gamma(\overline{\psi}, \eta) \vdash_t \mathsf{skel}_{\Sigma_t}(\psi_\eta)$, we can conclude that $\Gamma^t \vdash_t \mathsf{skel}_{\Sigma_t}(\psi_\eta)$. Given that $\psi_\eta \in \mathsf{sub}(\Gamma)$, we can apply Lemma 3.7 and conclude that $\psi_\eta \in \Gamma_\omega$.

– If condition (b) fails, then it is the case that $\psi_\eta = \psi_\kappa$ for some $\kappa < \tau$. Immediately, $\psi_\eta = \psi_\kappa \in \Gamma_\omega$, by induction hypothesis, since $\Gamma \vdash_{12} \langle \psi_\tau \rangle_{\tau \leq \kappa}$.

– If condition (c) fails, we know precisely that $\psi_\eta \in \Gamma_\omega$.

□

The following result is a simple corollary of Proposition A.9, already hinted at just after Definition 3.5.

Corollary A.10. *If* $\Gamma \subseteq L_{\Sigma_{12}}(P)$ *then* $\Gamma_\omega = \{\phi \in \mathsf{sub}(\Gamma) : \Gamma \vdash_{12} \phi\}$.

Proof. Immediate from Proposition A.9, property (2). □

As another consequence of Proposition A.9, we finally obtain the envisaged characterization result.

Proposition 3.8. Let $\mathcal{L}_1 = \langle \Sigma_1, \_^{\vdash_1} \rangle$ and $\mathcal{L}_2 = \langle \Sigma_2, \_^{\vdash_2} \rangle$ be logics such that $\Sigma_1 \cap \Sigma_2 = \emptyset$, and consider $\Gamma \cup \{\psi\} \subseteq L_{\Sigma_{12}}(P)$, as well as $i, j \in \{1, 2\}$ with $i \neq j$. Then,

$$\Gamma \vdash_{12} \psi$$

if and only if

$$\Gamma^i, X^i_\Gamma(\psi) \vdash_i \mathsf{skel}_{\Sigma_i}(\psi) \text{ or } \Gamma^j \text{ is } \vdash_j \text{-explosive.}$$

Proof. The result is immediate from Lemma A.7 and Proposition A.9, property (1). □