The Hahn-Banach Theorem in Type Theory

Jan Cederquist and Thierry Coquand
Department of Computing Science
University of Göteborg

and

Sara Negri
Department of Philosophy
University of Helsinki

Abstract

We give the basic definitions for pointfree functional analysis and present constructive proofs of the Alaoglu and Hahn-Banach theorems in the setting of formal topology.

1 Introduction

We present the basic concepts and definitions needed in a pointfree approach to functional analysis via formal topology. Our main results are the constructive proofs of localic formulations of the Alaoglu and Helly-Hahn-Banach[1] theorems.

Earlier pointfree formulations of the Hahn-Banach theorem, in a topos-theoretic setting, were presented by Mulvey and Pelletier (1987,1991) and by Vermeulen (1986). A constructive proof based on points was given by Bishop (1967). In the formulation of his proof, the norm of the linear functional is preserved to an arbitrary degree by the extension and a counterexample shows that the norm, in general, is not preserved exactly.

As usual in pointfree topology, our guideline is to define the objects under analysis as formal points of a suitable formal space. After this has been accomplished for the reals, we consider the formal topology $\mathcal{L}(A)$ obtained as follows: To the formal space of mappings from a normed vector space A to the reals, we add the linearity and norm conditions in the form of covering axioms. The linear functionals of norm ≤ 1 from A to the reals then correspond to the formal points of this formal topology.

Given a subspace M of A, the classical Helly-Hahn-Banach theorem says that the restriction mapping from the linear functionals on A of norm ≤ 1 to those on M is surjective. In terms of covers, conceived as deductive systems, it becomes a conservativity statement (cf. Mulvey and Pelletier 1991): Whenever a is an

[1] As explained by Hochstad (1980), the main idea in the usual proof of what is called the Hahn-Banach theorem is due to Helly. Since this is also the key idea in our derivation, we here rename the theorem in this way.

element and U is a subset of the base of the formal space $\mathcal{L}(M)$ and we have a derivation in $\mathcal{L}(A)$ of $a \lhd U$, then we can find a derivation in $\mathcal{L}(M)$ with the same conclusion.

With this formulation it is quite natural to look for a proof by induction on covers. Moreover, as already pointed out by Mulvey and Pelletier (1991), it is possible to simplify the problem greatly, since it is enough to prove it for coherent spaces of which $\mathcal{L}(A)$ and $\mathcal{L}(M)$ are retracts. Then, in a derivation of a cover, we can assume that only finite subsets occur on the right-hand side of the cover relation. A global proof transformation makes it possible to change a derivation in $\mathcal{L}(A)$ into a derivation in $\mathcal{L}(M)$, since only a finite-dimensional extension of the space M has to be taken into account. In consequence, as in the case for the classical proof for the one-dimensional extension, no use of Zorn's lemma is needed.

There exist already two pointfree proofs of Hahn-Banach's theorem. The proof of Mulvey and Pelletier (1991) shows Hahn-Banach's theorem in any Grothendieck topos. However, the argument relies on Barr's theorem, for which no constructive justification has been given so far. The proof of Vermeulen (1986) is done in the framework of topos theory with a natural number object, and thus, a priori, relies on the use of impredicative quantification.

Here as elsewhere (cf. Cederquist and Negri 1996; Coquand 1992, 1996; Negri and Valentini 1995) the use of formal topology allows for elementary and constructive proofs of pointfree formulations of classical results.

The two main contributions of this paper are the following:

- Our proof of the pointfree version of the Hahn-Banach theorem, following rather closely the original proof by Helly.
- This proof can actually be expressed in Martin-Löf's Type Theory (cf. Martin-Löf 1972). In fact, on the basis of our proof, the first author has done a formalisation of the Hahn-Banach theorem in an implementation of the intensional version of Type Theory with one universe and finitary inductive definitions (see Cederquist 1997).

2 Preliminaries

We recall here the definition of formal topology introduced by Per Martin-Löf and Giovanni Sambin (see Sambin 1987). We remark that, in contrast to the definition given by Sambin and without any substantial difference in the development of the theory, we do not require the base monoid to have a unit. Nor do we have the positivity predicate used by Sambin.

Definition 2.1. (Formal topology) *A formal topology over a set S is a structure $\langle S, \cdot, \lhd \rangle$, where $\langle S, \cdot \rangle$ is a commutative monoid, $\lhd$ is a relation, called cover, between elements and subsets of S such that, for any $a, b \in S$ and $U, V \subseteq S$, the following conditions hold:*

$$\textit{(reflexivity)} \qquad \frac{a \in U}{a \lhd U}$$

$$\textit{(transitivity)} \qquad \frac{a \lhd U \qquad U \lhd V}{a \lhd V} \qquad \textit{where} \quad U \lhd V \equiv (\forall u \in U)(u \lhd V)$$

$$\textit{(dot - left)} \qquad \frac{a \lhd U}{a \cdot b \lhd U}$$

$$\textit{(dot - right)} \qquad \frac{a \lhd U \qquad a \lhd V}{a \lhd U \cdot V} \qquad \textit{where} \quad U \cdot V \equiv \{u \cdot v \mid u \in U, v \in V\}.$$

For readability reasons, when a singleton set occurs we will sometimes omit curly brackets, and write $a \lhd b$ for $a \lhd \{b\}$, and $U \cdot b$ for $U \cdot \{b\}$.

Since we dropped the unit element and the positivity predicate in the definition of formal topology, the definition of formal points given by Sambin (1987) has to be revised as follows:

Definition 2.2 *Let $\langle S, \cdot, \lhd \rangle$ be a formal topology. A subset α of S is said to be a formal point if for all $a, b \in S$, $U \subseteq S$ the following conditions hold:*

(1) $(\exists a \in S)(a \in \alpha)$;

(2) $\dfrac{a \in \alpha \quad b \in \alpha}{a \cdot b \in \alpha}$;

(3) $\dfrac{a \in \alpha \quad a \lhd U}{(\exists b \in U)(b \in \alpha)}$.

In order to maintain the usual intuition on points, in the sequel we will write $\alpha \Vdash a$ (α forces a, or α is a point in a) in place of $a \in \alpha$.

As an instance of the abstract definition of formal topology the continuum can be defined as a formal space. The proofs that this definition satisfies the rules of a formal topology can all be found in Cederquist and Negri 1996. Further properties of the continuum as a formal space can also be found in Negri and Soravia 1996, where a slightly different definition adopting the unit is given.

Definition 2.3 *The formal topology of formal reals is the structure*

$$\mathcal{R} \equiv \langle Q \times Q, \cdot, \lhd_{\mathcal{R}} \rangle,$$

where Q is the set of rational numbers. The monoid operation is defined by $(p, q) \cdot (r, s) \equiv (max(p, r), min(q, s))$, the cover $\lhd_{\mathcal{R}}$ by

$$(p, q) \lhd_{\mathcal{R}} U \equiv (\forall p', q')(p < p' < q' < q \to (p', q') \lhd_{\mathcal{R}_f} U)$$

where the relation $\lhd_{\mathcal{R}_f}$ is inductively defined by

(1) $\dfrac{q \leq p}{(p, q) \lhd_{\mathcal{R}_f} U}$;

(2) $\dfrac{(p,q) \in U}{(p,q) \lhd_{\mathcal{R}_f} U}$;

(3) $\dfrac{p \leq r < s \leq q \quad (p,s) \lhd_{\mathcal{R}_f} U \quad (r,q) \lhd_{\mathcal{R}_f} U}{(p,q) \lhd_{\mathcal{R}_f} U}$;

(4) $\dfrac{p' \leq p < q \leq q' \quad (p',q') \lhd_{\mathcal{R}_f} U}{(p,q) \lhd_{\mathcal{R}_f} U}$.

We denote with $Pt(\mathcal{R})$ the collection of formal points of $\mathcal{R}$, called *formal reals.*

In Cederquist and Negri 1996 it is proved that $\lhd_{\mathcal{R}_f}$ and $\lhd_{\mathcal{R}}$ are cover relations. Moreover, $\lhd_{\mathcal{R}_f}$ is a *Stone cover*, since we have:

Proposition 2.4 *If $(p,q) \lhd_{\mathcal{R}_f} U$, then there exists a finite subset U_0 of U such that*
$(p,q) \lhd_{\mathcal{R}_f} U_0$.

Proof: See Cederquist and Negri 1996. □

In Cederquist and Negri 1996 it is also proved that whenever $(p,q) \lhd_{\mathcal{R}} U$ and U is finite then $(p,q) \lhd_{\mathcal{R}_f} U$ and therefore $\lhd_{\mathcal{R}_f}$ is the *Stone compactification* of $\lhd_{\mathcal{R}}$ (cf. Sambin 1987; Negri 1996), but this stronger result will not be used in the sequel.

With $I = (p,q)$ and $J = (r,s)$, we write $I < J$ (resp. $I \leq J$) to express that $r < p < q < s$ (resp. $r \leq p < q \leq s$). Thus $I \lhd_{\mathcal{R}} U$ means $J \lhd_{\mathcal{R}_f} U$ for all $J < I$. Moreover, we use the notations $I + J$ for $(p+r, q+s)$, and tI for (tp, tq) when $t \geq 0$ and for (tq, tp) when $t < 0$.

The following lemma will be used in section 3.

Lemma 2.5 *If $J < I$ and $I \lhd_{\mathcal{R}_f} U$, then there exists a subset V of $Q \times Q$ such that $(\forall K \in V)(\exists L \in U)(K < L)$ and $J \lhd_{\mathcal{R}_f} V$.*

Proof: Straightforward by induction on the derivation of $I \lhd_{\mathcal{R}_f} U$. □

3 Formal linear functionals

In this section we define the space of linear functionals of norm ≤ 1 from a seminormed space to the reals. This space is obtained by means of an inductive definition giving rise to a formal topology: Basic neighbourhoods correspond to basic opens in the *weak* topology* (Brezis 1983) and formal points correspond to the linear functionals of norm ≤ 1 from the given seminormed space to the space of formal reals.

Moreover, the formal cover for this space is defined explicitly from a Stone cover, and therefore a simple proof of Alaoglu's theorem is obtained.

3.1 The dual of a seminormed space as a formal space

We start by defining a seminormed space as in Mulvey and Pelletier 1991. The seminorm is defined by means of formal open balls centred around the origin.

Definition 3.1 *A seminormed space A on the rationals Q is a linear space A on Q together with a mapping*

$$N : Q^+ \longrightarrow \mathcal{P}(A)$$

from the positive rationals to the subsets of A satisfying the following conditions for $x, x' \in A$, $q, q' \in Q^+$:

N1. $x \in N(q) \to (\exists q' < q)(x \in N(q'))$;

N2. $(\exists q)(x \in N(q))$;

N3. $x \in N(q)$ & $x' \in N(q') \to x + x' \in N(q + q')$;

N4. $x \in N(q') \to qx \in N(qq')$;

N5. $x \in N(q) \to -x \in N(q)$;

N6. $0 \in N(q)$.

We limit ourselves to the presentation of the formal space $\mathcal{L}(A)$ of linear functionals of norm ≤ 1. The basic opens are finite sets of the form

$$w \equiv \{\langle x_1 \in I_1\rangle, \ldots, \langle x_n \in I_n\rangle\} ,$$

where $x_1, \ldots, x_n \in A$ and $I_1, \ldots, I_n$ are rational intervals. The intuitive reading of a basic open is that of a neighbourhood of functionals in the weak* topology. We will in the sequel use the notation $\langle x_1 \in I_1, \ldots, x_n \in I_n\rangle$ for $\{\langle x_1 \in I_1\rangle, \ldots, \langle x_n \in I_n\rangle\}$.

We obtain with the operation

$$w_1 w_2 \equiv w_1 \cup w_2$$

a commutative and idempotent monoid with unit given by the empty set. We will denote with $S_{\mathcal{L}(A)}$ such a base of $\mathcal{L}(A)$.

Let $w \equiv \langle x_1 \in I_1, \ldots, x_n \in I_n\rangle$, then define

$$\begin{aligned} w \leq \langle x \in I\rangle \quad \equiv \quad & (\exists \langle x_{i_1} \in I_{i_1}\rangle, \ldots, \langle x_{i_p} \in I_{i_p}\rangle \in w) \\ & (x_{i_1} = \ldots = x_{i_p} = x \;\&\; I_{i_1} \cdot \ldots \cdot I_{i_p} \leq I) \end{aligned}$$

and

$$w \leq w' \;\equiv\; (\forall \langle x \in I\rangle \in w')(w \leq \langle x \in I\rangle).$$

Then, without assuming decidability of equality in A, $\leq$ is a reflexive and transitive relation.

Equality between basic neighbourhoods is subset equality

$$w = w' \equiv (\forall \langle x \in I\rangle)(\langle x \in I\rangle \in w \Leftrightarrow \langle x \in I\rangle \in w') .$$

Notice that $w = w'$ implies $w \leq w'$. This defines an equivalence relation on the type of basic neighbourhoods. We follow Bishop (1967) in working systematically with types with an equivalence relation. Whenever we define a predicate, we have to be careful to check that this predicate respects the equivalence relation.

The finitary cover $\lhd_f$ for $\mathcal{L}(A)$ is inductively defined as follows:

$$C1\ \frac{w \in U}{w \lhd_f U}\ ;$$

$$C2\ \frac{w \leq w' \quad w' \lhd_f U}{w \lhd_f U}\ ;$$

$$C3\ \frac{V\mathit{finite} \quad I \lhd_{\mathcal{R}_f} V \quad (\forall J \in V)(\langle x \in J\rangle w' \lhd_f U)}{\langle x \in I\rangle w' \lhd_f U}\ ;$$

$$C4\ \frac{\langle x + y \in I + J\rangle w' \lhd_f U}{\langle x \in I, y \in J\rangle w' \lhd_f U}\ ;$$

$$C5\ \frac{r \neq 0 \quad \langle rx \in rI\rangle w' \lhd_f U}{\langle x \in I\rangle w' \lhd_f U}\ ;$$

$$C6\ \frac{x \in N(1) \quad \langle x \in (-1,1)\rangle w \lhd_f U}{w \lhd_f U}\ .$$

A motivation for the above definition can be given as follows: conditions *C1-C3* define formal functionals from A to the formal reals, *C4* and *C5* impose linearity and *C6* says that we only consider functionals of norm ≤ 1.

Notice that, by *C2*, we have $w \lhd_f U$ whenever $w = w'$ and $w' \lhd_f U$.

In order to get a finitary inductive definition, the subset V in clause *C3* has to be finite. This however is not a restriction since $\lhd_{\mathcal{R}_f}$ is a Stone cover and such a finite set can always be found.

We have:

Proposition 3.2 *The relation $\lhd_f$ is a cover.*

Proof: Reflexivity holds by definition, *dot-left* follows from *C2* since $w_1w_2 \leq w_1$, *transitivity* and *localization* are straightforward by induction on the derivation of $w \lhd_f U$. □

Then $\lhd$ is defined by

$$\begin{aligned}&\langle x_1 \in I_1, \ldots, x_n \in I_n\rangle \lhd U \equiv \\ &\qquad (\forall J_1 < I_1, \ldots, J_n < I_n)(\langle x_1 \in J_1, \ldots, x_n \in J_n\rangle \lhd_f U).\end{aligned}$$

Next, we prove that $\lhd$ is a cover, but before that some lemmas are needed.

Lemma 3.3 *If $\langle x_1 \in K_1, \ldots, x_n \in K_n\rangle \leq \langle y_1 \in L_1, \ldots, y_m \in L_m\rangle$ and $J_1 < K_1, \ldots, J_n < K_n$, then $(\exists L'_1 < L_1, \ldots, L'_m < L_m)(\langle x_1 \in J_1, \ldots, x_n \in J_n\rangle \leq \langle y_1 \in L'_1, \ldots, y_m \in L'_m\rangle)$.*

Proof: Starting from the hypotheses $\langle x_1 \in K_1, \ldots, x_n \in K_n\rangle \leq \langle y_1 \in L_1, \ldots, y_m \in L_m\rangle$ and $J_1 < K_1, \ldots, J_n < K_n$, the intervals L'_i $(1 \leq i \leq m)$ are constructed in the following way. By definition, for each $i \leq m$ there exists $p_1, \ldots, p_k$ such that $x_{p_1} = \ldots = x_{p_k} = y_i$ and $K_{p_1} \cdot \ldots \cdot K_{p_k} \leq L_i$. Since $J_1 < K_1, \ldots, J_n < K_n$ we have $J_{p_1} \cdot \ldots \cdot J_{p_k} < L_i$. Thus we put $L'_i = J_{p_1} \cdot \ldots \cdot J_{p_k}$. □

The following lemma makes the definition of $\lhd$ legitimate.

Lemma 3.4 *The relation $\lhd$ respects equality between basic neighbourhoods.*

Proof: We will show that *C2* also holds for $\lhd$. The conclusion then follows since $w = w'$ implies $w \leq w'$.

Suppose $\langle x_1 \in I_1, \ldots, x_n \in I_n\rangle \leq \langle y_1 \in J_1, \ldots, y_m \in J_m\rangle$ and $\langle y_1 \in J_1, \ldots, y_m \in J_m\rangle \lhd U$. Let $I'_1 < I_1, \ldots, I'_n < I_n$. Then, by lemma 3.3, there exist $J'_1 < J_1, \ldots, J'_m < J_m$ such that $\langle x_1 \in I'_1, \ldots, x_n \in I'_n\rangle \leq \langle y_1 \in J'_1, \ldots, y_m \in J'_m\rangle$. For such $J'_1, \ldots, J'_m$ we have $\langle y_1 \in J'_1, \ldots, y_m \in J'_m\rangle \lhd_f U$ and, by *C2*, $\langle x_1 \in I'_1, \ldots, x_n \in I'_n\rangle \lhd_f U$. Therefore $\langle x_1 \in I_1, \ldots, x_n \in I_n\rangle \lhd U$. $\Box$

Lemma 3.5 *If $w \lhd_f U$ then $w \lhd U$.*

Proof: Suppose $\langle x_1 \in I_1, \ldots, x_n \in I_n\rangle \lhd_f U$ and let $J_1 < I_1, \ldots, J_n < I_n$. Then $\langle x_1 \in J_1, \ldots, x_n \in J_n\rangle \leq \langle x_1 \in I_1, \ldots, x_n \in I_n\rangle$ so, by *C2*, $\langle x_1 \in J_1, \ldots, x_n \in J_n\rangle \lhd_f U$. Therefore $\langle x_1 \in I_1, \ldots, x_n \in I_n\rangle \lhd U$. $\Box$

The following lemma is used in the proof of *transitivity* for $\lhd$.

Lemma 3.6 *If $\langle x_1 \in K_1, \ldots, x_n \in K_n\rangle \lhd_f U$, $J_1 < K_1, \ldots, J_n < K_n$ and $U \lhd V$, then $\langle x_1 \in J_1, \ldots, x_n \in J_n\rangle \lhd_f V$.*

Proof: The proof is by induction on the derivation of $\langle x_1 \in K_1, \ldots, x_n \in K_n\rangle \lhd_f U$.

1. If $\langle x_1 \in K_1, \ldots, x_n \in K_n\rangle \in U$ then $\langle x_1 \in K_1, \ldots, x_n \in K_n\rangle \lhd V$, thus by definition, $\langle x_1 \in J_1, \ldots, x_n \in J_n\rangle \lhd_f V$.

2. If $\langle x_1 \in K_1, \ldots, x_n \in K_n\rangle \lhd_f U$ is derived by *C2* from $\langle x_1 \in K_1, \ldots, x_n \in K_n\rangle \leq \langle y_1 \in L_1, \ldots, y_n \in L_m\rangle$ and $\langle y_1 \in L_1, \ldots, y_n \in L_m\rangle \lhd_f U$, then by lemma 3.3 there exist $L'_1 < L_1, \ldots, L'_m < L_m$ such that $\langle x_1 \in J_1, \ldots, x_n \in J_n\rangle \leq \langle y_1 \in L'_1, \ldots, y_n \in L'_m\rangle$. For such $L'_1, \ldots, L'_m$, by induction hypothesis, $\langle y_1 \in L'_1, \ldots, y_n \in L'_m\rangle \lhd_f V$. Then, by *C2*, $\langle x_1 \in J_1, \ldots, x_n \in J_n\rangle \lhd_f V$.

3. Let $\langle x_1 \in K_1, \ldots, x_n \in K_n\rangle \lhd_f U$ be derived by *C3* from $K_1 \lhd_{\mathcal{R}_f} V'$ and $(\forall J' \in V')(\langle x_1 \in J', x_2 \in K_2, \ldots, x_n \in K_n\rangle \lhd_f U)$, where V' is finite. Since $J_1 < K_1$ and $K_1 \lhd_{\mathcal{R}_f} V'$, by lemma 2.5, we can construct a finite subset V'' such that $J_1 \lhd_{\mathcal{R}_f} V''$ and $(\forall J'' \in V'')(\exists J' \in V')(J'' < J')$. Then, by induction hypothesis, $(\forall J'' \in V'')(\langle x_1 \in J'', x_2 \in J_2, \ldots, x_n \in J_n\rangle \lhd_f V)$ and, by *C3*, $\langle x_1 \in J_1, \ldots, x_n \in J_n\rangle \lhd_f V$.

4. Let $\langle x_1 \in K_1, \ldots, x_n \in K_n\rangle \lhd_f U$ be derived by *C4* from $\langle x_1 + x_2 \in K_1 + K_2, x_3 \in K_3 \ldots, x_n \in K_n\rangle \lhd_f U$. Then $J_1 + J_2 < K_1 + K_2$ so, by induction hypothesis, $\langle x_1 + x_2 \in J_1 + J_2, x_3 \in J_3 \ldots, x_n \in J_n\rangle \lhd_f V$ and, by *C4*, $\langle x_1 \in J_1, \ldots, x_n \in J_n\rangle \lhd_f V$.

5. Similar to 4.

6. Let $\langle x_1 \in K_1, \ldots, x_n \in K_n\rangle \lhd_f U$ be derived by *C6* from $\langle x \in (-1, 1), x_1 \in K_1 \ldots, x_n \in K_n\rangle \lhd_f U$ and $x \in N(1)$. By *N1* there exists an $r < 1$ such that $x \in N(r)$. For such an r, by induction hypothesis, $\langle x \in (-r, r), x_1 \in J_1 \ldots, x_n \in J_n\rangle \lhd_f V$. Now the claim follows by *C6*. $\Box$

We are now ready to prove:

Proposition 3.7 *The relation $\lhd$ is a cover.*

Proof: *Reflexivity*: Immediate from *reflexivity* of $\lhd_f$ and lemma 3.5.

Dot-left, dot-right: Immediate from the definition of $\lhd$ and the corresponding properties of $\lhd_f$.

Transitivity: Suppose $\langle x_1 \in I_1, \ldots, x_n \in I_n\rangle \lhd U$ and $U \lhd V$. If $J_1 < I_1, \ldots, J_n < I_n$, then we can find $K_1, \ldots, K_n$ such that $J_1 < K_1 < I_1, \ldots, J_n < K_n < I_n$ and, by definition of $\langle x_1 \in I_1, \ldots, x_n \in I_n\rangle \lhd U$, we get $\langle x_1 \in K_1, \ldots, x_n \in K_n\rangle \lhd_f U$. Lemma 3.6 now gives $\langle x_1 \in J_1, \ldots, x_n \in J_n\rangle \lhd_f V$. Thus $\langle x_1 \in I_1, \ldots, x_n \in I_n\rangle \lhd V$. □

Finally, the following lemma will be useful in section 4.

Lemma 3.8 *The axioms for $\lhd_f$ are valid rules for $\lhd$.*

Proof: 1: Given $w \in U$, $w \lhd U$ is immediate from *C1* and lemma 3.5.

2: See the proof of lemma 3.4.

3: Suppose $I_1 \lhd_R V$ and $(\forall J \in V)(\langle x_1 \in J\rangle\langle x_2 \in I_2, \ldots, x_n \in I_n\rangle \lhd U)$. Let $I_1' < I_1, \ldots, I_n' < I_n$. Then there exists I'' such that $I_1' < I'' < I_1$. For such an I'', $I'' \lhd_{\mathcal{R}_f} V$ so by lemma 2.5 there exists a subset V' such that $I_1' \lhd_{\mathcal{R}_f} V'$ and $(\forall L' \in V')(\exists L \in V)(L' < L)$. But then, since $\lhd_{\mathcal{R}_f}$ is a Stone cover, there exists a finite subset $V_0' \subseteq V'$ such that $I_1' \lhd_{\mathcal{R}_f} V_0'$ and $(\forall L' \in V_0')(\exists L \in V)(L' < L)$. Now, by definition of $\lhd$, $(\forall J' \in V_0')(\langle x_1 \in J'\rangle\langle x_2 \in I_2', \ldots, x_n \in I_n'\rangle \lhd_f U)$ and, by *C3*, $\langle x_1 \in I_1', \ldots, x_n \in I_n'\rangle \lhd_f U$. Therefore $\langle x_1 \in I_1, \ldots, x_n \in I_n\rangle \lhd_f U$.

4 and 5 follows easily from the definition of $\lhd_f$, *C4* and *C5*, respectively.

6: Suppose $x \in N(1)$ and $\langle x \in (-1,1), x_1 \in I_1, \ldots, x_n \in I_n\rangle \lhd U$. Let $J_1 < I_1, \ldots, J_n < I_n$. By *N1*, there exists $r < 1$ such that $x \in N(r)$. For such an r, $\frac{1}{r}x \in N(1)$. Now, by *C5* and the definition of $\lhd$, $\langle \frac{1}{r}x \in (-1,1), x_1 \in J_1, \ldots, x_n \in J_n\rangle \lhd_f U$ and, by *C6*, $\langle x_1 \in J_1, \ldots, x_n \in J_n\rangle \lhd_f U$. Therefore $\langle x_1 \in I_1, \ldots, x_n \in I_n\rangle \lhd U$. □

3.2 Linear functionals as formal points

In this section, which is not needed for the proof of the Hahn-Banach theorem, we will show how linear functionals of norm ≤ 1 from A to the formal space of reals are obtained in our setting. We denote the collection of formal points of $\mathcal{L}(A)$ with $Pt(\mathcal{L}(A))$. Given $F \in Pt(\mathcal{L}(A))$ and $x \in A$, let

$$F^*(x) \equiv \{(p,q) \in Q \times Q : F \Vdash \langle x \in (p,q)\rangle\}.$$

The following propositions are easily proved from the definitions (see also Mulvey and Pelletier 1987):

Proposition 3.9 *F^* is a linear functional from A to $Pt(\mathcal{R})$ of norm ≤ 1, that is:*

(1) *For all $x \in A$, $F^*(x) \in Pt(\mathcal{R})$;*
(2) *$F^*(x+y) = F^*(x) + F^*(y)$;*
(3) *$F^*(tx) = tF^*(x)$;*
(4) *$x \in N(1) \Rightarrow -1 < F^*(x) < 1$.*

Conversely, given a linear functional f from A to $Pt(\mathcal{R})$, that is, a map satisfying *1–4* as in proposition 3.9, let f° be the subset of $S_{\mathcal{L}(A)}$ defined by

$$f^\circ \equiv \{\langle x_1 \in I_1, \ldots, x_n \in I_n\rangle : f(x_1) \Vdash I_1, \ldots, f(x_n) \Vdash I_n\}.$$

Then we have:

Proposition 3.10 $f^\circ \in Pt(\mathcal{L}(A))$.

The correspondence stated above is biunivocal since we have, with the notation used above, $(f^\circ)^* = f$ and $(F^*)^\circ = F$.

3.3 Alaoglu's theorem

The cover $\lhd_f$ is a Stone cover, since we have:

Proposition 3.11 *If $w \lhd_f U$, then there exists a finite subset U_0 of U such that $w \lhd_f U_0$.*

Proof: The proof is straightforward by induction on the derivation of $w \lhd_f U$. We will only consider the case when $w \lhd_f U$ is obtained by *C3* from $w \equiv \langle x \in I\rangle w'$, $I \lhd_{\mathcal{R}_f} V$ and $(\forall J \in V)(\langle x \in J\rangle w' \lhd_f U)$, for a finite subset V. By induction hypothesis, given an element $J \in V$, there exists a finite subset U_J of U such that $\langle x \in J\rangle w' \lhd_f U_J$. Let U_0 be the union of all such U_J's. Then by *C3*, $w \lhd_f U_0$. □

As an immediate consequence of the definition of the cover for $\mathcal{L}(A)$, we obtain Alaoglu's theorem, asserting that the unit ball of the space of linear and continuous functionals is compact in the weak topology.

Corollary 3.12 *The formal space $\mathcal{L}(A)$ is compact.*

Proof: Given $\langle\ \rangle \lhd U$, by definition of $\lhd$, $\langle\ \rangle \lhd_f U$. Then, since $\lhd_f$ is a Stone cover, $\langle\ \rangle \lhd_f U_0$ for some finite subset U_0 of U. Therefore, by lemma 3.5, $\langle\ \rangle \lhd U_0$. □

4 The Helly-Hahn-Banach theorem

In this section we prove the localic version of the Helly-Hahn-Banach theorem in the form of a conservativity result of the theory for the cover in $\mathcal{L}(A)$ over that in $\mathcal{L}(M)$. At the end of the section we also provide an informal motivation why this result is the localic statement of the usual point-set theorem.

The definition of $\lhd$ in terms of $\lhd_f$ allows us to replace $\lhd$ with $\lhd_f$ for proving conservativity. If M is a linear subspace of A, we say that $\lhd_f$ for $\mathcal{L}(A)$ is *conservative* over $\lhd_f$ for $\mathcal{L}(M)$ if, whenever w is an element and U is a subset of the base of $\mathcal{L}(M)$ and $w \lhd_f U$ in $\mathcal{L}(A)$, then $w \lhd_f U$ in $\mathcal{L}(M)$. Conservativity of $\lhd$ for $\mathcal{L}(A)$ over $\lhd$ for $\mathcal{L}(M)$ is defined in the same way. Then we have:

Proposition 4.1 *If $\lhd_f$ for $\mathcal{L}(A)$ is conservative over $\lhd_f$ for $\mathcal{L}(M)$, then $\lhd$ for $\mathcal{L}(A)$ is conservative over $\lhd$ for $\mathcal{L}(M)$.*

Proof: If $\langle x_1 \in I_1, \ldots, x_n \in I_n\rangle \lhd U$ in $\mathcal{L}(A)$, then, for all $J_1 < I_1, \ldots, J_n < I_n$, $\langle x_1 \in J_1, \ldots, x_n \in J_n\rangle \lhd_f U$, hence by conservativity for $\lhd_f$, for all

$J_1 < I_1, \ldots, J_n < I_n$, $\langle x_1 \in J_1, \ldots, x_n \in J_n \rangle \lhd_f U$ in $\mathcal{L}(M)$ and hence $\langle x_1 \in I_1, \ldots, x_n \in I_n \rangle \lhd U$ in $\mathcal{L}(M)$. □

The localic statement of the Helly-Hahn-Banach theorem in the framework of formal topologies is, "*$\lhd$ for $\mathcal{L}(A)$ is conservative over $\lhd$ for $\mathcal{L}(M)$*". By the above proposition it thus reduces to the following:

Theorem 4.2 *$\lhd_f$ for $\mathcal{L}(A)$ is conservative over $\lhd_f$ for $\mathcal{L}(M)$.*

Since $\lhd_f$ is a Stone cover, we can transform any derivation of $a \lhd_f U$ in $\mathcal{L}(A)$ into one in which only finite subsets occur on the right-hand side of the cover relation. Therefore we can assume that only a finite number of elements not belonging to the base of $\mathcal{L}(M)$ are involved. Thus, arguing by induction, the problem is reduced to showing that if $a \lhd_f U$ is derived in $\mathcal{L}(M')$, where $M' \equiv [M + x]$ and x is an element in the base of $\mathcal{L}(A)$, then there exists also a derivation of $a \lhd_f U$ in $\mathcal{L}(M)$. The claim of theorem 4.2 thus becomes:

Proposition 4.3
$\lhd_f$ for $\mathcal{L}(M')$ is conservative over $\lhd_f$ for $\mathcal{L}(M)$.

We will use the notations $w \lhd_{M_f} U$ and $w \lhd_{M'_f} U$ to express that $w \lhd_f U$ in $\mathcal{L}(M)$ and $\mathcal{L}(M')$, respectively. Before proving the proposition, some auxiliary lemmas will be needed.

Lemma 4.4 $\langle y + z \in (p, q)\rangle \lhd \{\langle y \in (r, s), z \in (r', s')\rangle : p \leq r + r' < s + s' \leq q\}$

Proof: Since $(p'', q'') < (p, q)$ implies that

$$\langle y + z \in (p'', q'')\rangle \in \{\langle y + z \in (p', q')\rangle : p < p' < q' < q\} \ ,$$

we have by definition of $\lhd$ that

$$\langle y + z \in (p, q)\rangle \lhd \{\langle y + z \in (p', q')\rangle : p < p' < q' < q\}.$$

By *N2* and 3.8(6) we have

$$\langle\ \rangle \lhd \{\langle y \in (r, s)\rangle : r < s\} \ ,$$

and similarly

$$\langle\ \rangle \lhd \{\langle z \in (r', s')\rangle : r' < s'\} \ ,$$

so by stability

$$\begin{aligned}\langle y + z \in (p, q)\rangle \lhd \{\langle y + z \in (p', q'), \quad & y \in (r, s), z \in (r', s')\rangle : \\ & p < p' < q' < q, r < s, r' < s'\} \ .\end{aligned}$$

Since every interval can be covered by arbitrarily small intervals, using 3.8(3), we get

$$\begin{aligned}\langle y + z \in (p, q)\rangle \lhd \{\langle y + z \in (p', q'), \quad & y \in (r, s), z \in (r', s')\rangle : \\ & p < p' < q' < q, r < s, r' < s', \\ & s - r \leq min(p' - p, q - q')/2, \\ & s' - r' \leq min(p' - p, q - q')/2\}\end{aligned}$$

and by 3.8(4)

$$\begin{aligned}\langle y+z \in (p,q)\rangle \lhd \{\langle y+z \in (p',q'), \quad & y \in (r,s), z \in (r',s'), y+z \in (r+r', s+s')\rangle : \\ & p < p' < q' < q, r < s, r' < s', \\ & s - r \leq min(p'-p, q-q')/2, \\ & s'-r' \leq min(p'-p, q-q')/2\} \ .\end{aligned}$$

Now, if $r + r' < p$ then $s + s' < p'$ and, by 3.8(2),

$$\langle y+z \in (p',q'), y+z \in (r+r', s+s')\rangle \lhd \langle y+z \in (p', s+s')\rangle.$$

Then, since $s + s' < p'$, the right-hand side is covered by anything, in particular the empty set. If $q < s + s'$ we obtain symmetrically $\langle y+z \in (p',q'), y+z \in (r+r', s+s')\rangle \lhd \langle y+z \in (r+r', q')\rangle$, where $q' < r + r'$. We thus obtain

$$\langle y+z \in (p,q)\rangle \lhd \{\langle y \in (r,s), z \in (r',s')\rangle : p \leq r+r' < s+s' \leq q\}. \ \Box$$

Let I be a finite index set, $\{a_n\}$ and $\{t_n\}$ sequences of A and Q, respectively, such that $(\forall i \in I)(a_i \in M \ \& \ a_i + t_i x \in N(1))$. Then, for rational q, define

$$P_q = \langle \ldots, a_i \in (-1 + t_i q, 1 + t_i q), \ldots \rangle.$$

The following lemma is the core of our proof. Intuitively it says that we can find a rational approximation for the value of $u(x)$, where u is a generic linear functional. We observe that the argument is similar to the one used by Helly.

Lemma 4.5 $\langle\ \rangle \lhd_{M_f} \{P_q : q \in Q\}$.

Proof: First observe that for all $i \in I$ such that $t_i = 0$, we can apply *C6* and *dot-right*. So, in the rest of this proof, we can assume that $t_i \neq 0$ for all i. For any $i, j \in I$, by the rules of N we have

$$\begin{aligned}\left\{\begin{array}{l} a_i + t_i x \in N(1) \\ a_j + t_j x \in N(1)\end{array}\right. \quad & \Rightarrow \quad \left\{\begin{array}{r} a_i/t_i + x \in N(1/|t_i|) \\ -a_j/t_j - x \in N(1/|t_j|)\end{array}\right. \\ & \Rightarrow \quad a_i/t_i - a_j/t_j \in N(1/|t_i| + 1/|t_j|) \ .\end{aligned}$$

By the rules for $\lhd_{M_f}$ we get

$$\langle\ \rangle \lhd_{M_f} \langle a_i/t_i - a_j/t_j \in (-1/|t_i| - 1/|t_j|, 1/|t_i| + 1/|t_j|)\rangle,$$

by the lemmas 3.5 and 4.4

$$\begin{aligned}\langle\ \rangle \quad \lhd_M \quad & \{\langle a_i/t_i \in (r_i, s_i), a_j/t_j \in (r_j, s_j)\rangle : \\ & -1/|t_i| - 1/|t_j| \leq r_i - s_j < s_i - r_j \leq 1/|t_i| + 1/|t_j|\}.\end{aligned}$$

From the definition of $\lhd$ we obtain the same for $\lhd_f$, and by *dot-right*

$$\begin{aligned}\langle\ \rangle \quad \lhd_{M_f} \quad & \{\langle \ldots, a_i/t_i \in (r_i, s_i), \ldots \rangle : \\ & (\forall i, j \in I)(-1/|t_i| - 1/|t_j| \leq r_i - s_j < s_i - r_j \leq 1/|t_i| + 1/|t_j|)\}\end{aligned}$$

$$\Rightarrow$$

$$\langle\ \rangle \lhd_{M_f} \{\langle \dots, a_i/t_i \in (r_i, s_i), \dots\rangle : (\forall i, j \in I)(s_i - 1/|t_i| \leq r_j + 1/|t_j|)\}$$

$$\Rightarrow$$

$$\langle\ \rangle \lhd_{M_f} \{\langle \dots, a_i/t_i \in (r_i, s_i), \dots\rangle : \bigvee_{i \in I}(s_i - 1/|t_i|) \leq \bigwedge_{i \in I}(r_i + 1/|t_i|)\}.$$

Now there are two cases depending on whether t_i is positive or negative; we only consider the case when t_i is positive.

$$\langle\ \rangle \lhd_{M_f} \{\langle \dots, a_i \in (t_i r_i, t_i s_i), \dots\rangle : \bigvee_{i \in I}(s_i - 1/|t_i|) \leq \bigwedge_{i \in I}(r_i + 1/|t_i|)\}.$$

If $q = \bigvee_{i \in I}(s_i - 1/|t_i|)$, then $-1 + qt_i \leq t_i r_i < t_i s_i \leq 1 + qt_i$, because $s_i - 1/|t_i| \leq q \leq r_i + 1/|t_i|$, so we get

$$\langle\ \rangle \lhd_{M_f} \{P_q : q \equiv \bigvee_{i \in I}(s_i - 1/|t_i|) \leq \bigwedge_{i \in I}(r_i + 1/|t_i|)\}.$$

and *a fortiori* $\langle\ \rangle \lhd_{M_f} \{P_q : q \in Q\}$. $\Box$

Let I be a finite set as above. We define $\lhd'_{M'_f}$ as $\lhd_{M'_f}$, where the axiom $C6$ is replaced by

$$C6' : \ \frac{i \in I \quad \langle a_i + t_i x \in (-1, 1)\rangle w \lhd'_{M'_f} U}{w \lhd'_{M'_f} U}.$$

Let $w \equiv \langle \dots, b_j + u_j x \in (r_j, s_j), \dots\rangle$ be an element in the base of M' and let q be a rational number. Then define

$$\bar{w} \equiv \langle \dots, b_j \in (r_j + u_j q, s_j + u_j q), \dots\rangle,$$

$$\bar{U} \equiv \{\bar{w} : w \in U\}.$$

Note that $\bar{w}$ depends on the proof of $w \in S_{\mathcal{L}(M')}$, and if $x \in M$ there are many ways of writing $b_j + u_j x$ and thus many proofs of $w \in S_{\mathcal{L}(M')}$.

We have:

Lemma 4.6 $w \lhd'_{M'_f} U \Rightarrow x \in M \ \vee \ (\forall q \in Q)(P_q \bar{w} \lhd_{M_f} \bar{U})$.

Proof: By induction on the derivation of $w \lhd'_{M'_f} U$. We only consider the cases *C1'* and *C6'*.

C1': Given $w \in U$, since $U \subseteq S_{\mathcal{L}(M')}$ we have two proofs of $w \in S_{\mathcal{L}(M')}$ and two possibly different values of $\bar{w}$, $\langle \dots, a_i \in (r_i + t_i q, s_i + t_i q), \dots\rangle$ and $\langle \dots, b_i \in (r_i + u_i q, s_i + u_i q), \dots\rangle$. Then there are two cases; if $(\forall i)(t_i = u_i)$, then $(\forall i)(a_i = b_i)$ and $\bar{w} \in \bar{U}$, thus $(\forall q \in Q)(P_q \bar{w} \lhd_{M_f} \bar{U})$; if $t_i \neq u_i$, then $x = \frac{a_i - b_i}{u_i - t_i} \in M$.

C6′: By inductive hypothesis we have

$$x \in M \ \vee \ (\forall q \in Q)(P_q\langle a_i \in (-1+t_iq, 1+t_iq)\rangle\bar{w} \lhd_{M_f} \bar{U})$$

that gives, since $\langle a_i \in (-1+t_iq, 1+t_iq)\rangle \in P_q$,

$$x \in M \ \vee \ (\forall q \in Q)(P_q\bar{w} \lhd_{M_f} \bar{U}) \ . \ \square$$

Observe here that we do not require decidability of membership of M. In lemma 4.6 a large part of the effort is devoted for the possibility of different values of $\bar{w}$ for different proofs of $w \in S_{\mathcal{L}(M')}$. This problem only occurs when $x \in M$. So decidability would simplify the proof; if $x \in M$ then $M' = M$ and there is nothing to prove, and if $x \notin M$ then the formulation and the proof of lemma 4.6 are easier.

If w is an element and U is a subset of the base of $\mathcal{L}(M)$, then $\bar{w} = w$ and $\bar{U} = U$, so as a corollary to lemma 4.6 we obtain:

Corollary 4.7 *Let w be an element, U a subset of the base of $\mathcal{L}(M)$, then*

$$w \lhd'_{M'_f} U \Rightarrow x \in M \ \vee \ (\forall q \in Q)(P_qw \lhd_{M_f} U).$$

Proof of proposition 4.3: Suppose $w \lhd_{M'_f} U$, where w is an element and U a subset in the base of $\mathcal{L}(M)$. First, by examining the axioms of the form *C6* in the proof of $w \lhd_{M'_f} U$, we can find finite sequences $\{a_n\}$ and $\{t_n\}$ in A and in Q, respectively, such that $(\forall i)(a_i \in M \ \& \ a_i + t_ix \in N(1))$. Then, the proof of $w \lhd_{M'_f} U$ can be converted into a proof of $w \lhd'_{M'_f} U$. By corollary 4.7 we have $x \in M \vee (\forall q \in Q)(P_qw \lhd_{M_f} U)$. If $x \in M$, $M' = M$ and there is nothing to prove. If $(\forall q \in Q)(P_qw \lhd_{M_f} U)$ then $\{P_q : q \in Q\} \cdot w \lhd_{M_f} U$ and, since $\langle\ \rangle \lhd_{M_f} \{P_q : q \in Q\}$ (lemma 4.5), we obtain $w \lhd_{M_f} U$ by *localization* and *transitivity*. $\square$

We include a proof in this setting of the following application of the Helly-Hahn-Banach theorem (which is proved indirectly by Mulvey and Pelletier (1987), whereas Vermeulen (1986) gives a direct proof).

Proposition 4.8 *If $\langle\ \rangle \lhd \langle x \in (-1,1)\rangle$, then $x \in N(1)$.*

The above proposition is proved by interpreting the neighbourhoods of $\mathcal{L}([x])$ as subsets of $Q \times Q$. First we define, for $r \in Q$ and $I = (p,q)$,

$$r \in I \equiv p < r < q.$$

Then we put

$$\langle t_1x \in I_1, \ldots, t_nx \in I_n\rangle' \equiv \begin{cases} \emptyset & \text{if } (\exists i)(t_i = 0 \ \& \ 0 \notin I_i) \\ Q \times Q & \text{if } (\forall i)(t_i = 0 \ \& \ 0 \in I_i) \\ \{\bigwedge_{1 \le i \le n, t_i \neq 0} \frac{1}{t_i}I_i\} & \text{otherwise} \end{cases}$$

and let U' be the union of all w' such that $w \in U$.

Then we have:

Lemma 4.9 *If $w \lhd_f U$ in $\mathcal{L}([x])$, then*

$$(\exists r \in Q)(x \in N(r) \ \&\ (-r, r) \cdot w' \lhd_{\mathcal{R}_f} U').$$

Proof: The proof is by induction on the derivation of $w \lhd_f U$.

1. Let $w \lhd_f U$ be derived by *C1* from $w \in U$. Then, by the definition of $'$, $w' \subseteq U'$ so $w' \lhd_{\mathcal{R}_f} U'$. By *N2*, $(\exists r)(x \in N(r))$ and for such an r, $(-r, r) \cdot w' \lhd_{\mathcal{R}_f} U'$. Thus $(\exists r)(x \in N(r) \ \&\ (-r, r) \cdot w' \lhd_{\mathcal{R}_f} U')$.

For 2–6 below we argue by case analysis on the way $\langle t_1 x \in I_1, \ldots, t_n x \in I_n\rangle'$ is constructed. The cases $(\exists i)(t_i = 0 \ \&\ 0 \notin I_i)$ and $(\forall i)(t_i = 0 \ \&\ 0 \in I_i)$ are easy, so below we only consider the third case for which

$$\langle t_1 x \in I_1, \ldots, t_n x \in I_n\rangle' \ \equiv\ \{ \bigwedge_{1 \leq i \leq n, t_i \neq 0} \frac{1}{t_i} I_i \}\ .$$

2. Let $w_1 \lhd_f U$ be derived by *C2* from $w_1 \leq w_2$ and $w_2 \lhd_f U$. From the definition of $\leq$ and $'$, $w_1' = \{I\}$ and $w_2' = \{J\}$, for some $I \leq J$, thus $w_1' \lhd_{\mathcal{R}_f} w_2'$. By induction hypothesis $(\exists r)(x \in N(r) \ \&\ (-r, r) \cdot w_2' \lhd_{\mathcal{R}_f} U')$. For such an r, $(-r, r) \cdot w_1' \lhd_{\mathcal{R}_f} (-r, r) \cdot w_2'$ so, by *transitivity*, $(-r, r) \cdot w_1' \lhd_{\mathcal{R}_f} U'$. Thus the claim.

3. Suppose $\langle tx \in I\rangle w \lhd_f U$ is derived by *C3* from V *finite*, $I \lhd_f V$ and $(\forall J \in V)(\langle tx \in J\rangle w \lhd_f U)$. Here also we only consider the case for which w' is a singleton set of an interval, the other cases being easy. By induction hypothesis $(\forall J \in V)(\exists r)(x \in N(r) \ \&\ (-r, r) \cdot (\langle tx \in J\rangle w)' \lhd_{\mathcal{R}_f} U')$ and since V is finite we can take the smallest such r and obtain $x \in N(r) \ \&\ (\forall J \in V)((-r, r) \cdot (\langle tx \in J\rangle w)' \lhd_{\mathcal{R}_f} U')$. We assume V to be nonempty, since if V is empty, I is negative and the claim follows easily. Since $(\langle tx \in J\rangle w)' = \frac{1}{t} I \cdot w'$, $(\forall K \in (-r, r) \cdot \frac{1}{t} w' \cdot V)(K \lhd_{\mathcal{R}_f} U')$. Moreover $(-r, r) \cdot \frac{1}{t} w' \cdot I \lhd_{\mathcal{R}_f} (-r, r) \cdot \frac{1}{t} w' \cdot V$ so, by *transitivity*, $(-r, r) \cdot \frac{1}{t} w' \cdot I \lhd_{\mathcal{R}_f} U'$. Thus the claim.

4. Suppose $\langle t_1 x \in I_1, t_2 x \in I_2\rangle w \lhd_f U$ is derived by *C4* from $\langle t_1 x + t_2 x \in I_1 + I_2\rangle w \lhd_f U$. By induction hypothesis $(\exists r)(x \in N(r) \ \&\ (-r, r) \cdot (\langle t_1 x + t_2 x \in I_1 + I_2\rangle w)' \lhd_{\mathcal{R}_f} U')$. Then we just notice that $\langle t_1 x \in I_1, t_2 x \in I_2\rangle' \leq \langle t_1 x + t_2 x \in I_1 + I_2\rangle'$ and therefore the claim.

5. Similar to 4.

6. Suppose $w \lhd_f U$ is derived by *C6* from $tx \in N(1)$ and $\langle tx \in (-1, 1)\rangle w \lhd_f U$. Again, we only consider the case in which w' is a singleton set of an interval. By *N4* and *N5*, $x \in N(\frac{1}{|t|})$, and by induction hypothesis,

$$(\exists r)(x \in N(r) \ \&\ (-r, r) \cdot (\langle tx \in (-1, 1)\rangle w)' \lhd_{\mathcal{R}_f} U').$$

We have $(\langle tx \in (-1, 1)\rangle w)' = \frac{1}{t}(-1, 1) \cdot w'$. If $r \leq \frac{1}{|t|}$ then $(-r, r) \leq (-r, r) \cdot \frac{1}{t}(-1, 1)$ and therefore $(-r, r) \cdot w' \lhd_{\mathcal{R}_f} U'$; if $r > \frac{1}{|t|}$ we have $\frac{1}{t}(-1, 1) \cdot w' \lhd_{\mathcal{R}_f} U'$ and the claim follows from $x \in N(\frac{1}{|t|})$. Thus $(\exists r)(x \in N(r) \ \&\ (-r, r) \cdot w' \lhd_{\mathcal{R}_f} U')$. $\square$

Proof of proposition 4.8: Suppose $\langle\ \rangle \lhd_{\mathcal{L}(A)} \langle x \in (-1,1)\rangle$. Then, by the (localic) Helly-Hahn-Banach theorem, we have $\langle\ \rangle \lhd_{\mathcal{L}([x])} \langle x \in (-1,1)\rangle$ and, by definition of $\lhd$, $\langle\ \rangle \lhd_{\mathcal{L}([x])_f} \langle x \in (-1,1)\rangle$. Thus by, lemma 4.9, there exists an $r \in Q^+$ such that $x \in N(r)$ and $(-r,r) \lhd_{\mathcal{R}_f} (-1,1)$. Therefore (cf. Cederquist and Negri 1996, lemma 10) $(-r,r) \leq (-1,1)$, that is, $r \leq 1$, thus $x \in N(1)$. $\square$

The usual Hahn-Banach theorem states that the restriction mapping

$$\begin{array}{rcl} Pt(\mathcal{L}(A)) & \to & Pt(\mathcal{L}(M)) \\ f & \mapsto & \overline{f} \equiv f \cap S_{\mathcal{L}(M)} \end{array}$$

is surjective.

We can obtain this result from ours plus an assumption of extensionality, up to an arbitrarily good approximation. Namely we can prove that given g in $Pt(\mathcal{L}(M)$ and given $\omega \in g$ there exists $f \in Pt(\mathcal{L}(A))$ such that $\omega \in \overline{f}$: By contradiction, suppose that no such f exists. Then by extensionality $\omega \lhd_{\mathcal{L}(A)} \emptyset$ and therefore, by conservativity, $\omega \lhd_{\mathcal{L}(M)} \emptyset$, that contradicts $\omega \in g$.

5 Conclusion

The basic problem of the Helly-Hahn-Banach theorem can be formulated as follows. We have a normed vector space M, a linear functional u on M of norm ≤ 1, and we want to extend u to $[M + x_0]$. For this we consider two families

$$\begin{cases} A_x = u(x) - \|x - x_0\|\ , & x \in M \\ B_y = u(y) + \|y - x_0\|\ , & y \in M. \end{cases}$$

We have $A_x \leq B_y$ for all $x, y \in M$ and the core of the question is to find a real number r such that $A_x \leq r \leq B_y$ for all x and y. Classically this is possible by taking $r = sup(A_x)$. Intuitionistically, there is no reason why A_x should have a supremum. There are then two alternatives (we limit ourselves to the finite dimensional case):

Bishop (1967); Bridges and Richman (1987):

One finds instead an extension u' of u of norm $\leq 1 + \epsilon$ for any given $\epsilon > 0$. For this, given $\epsilon > 0$, we change A_x and B_y into

$$\begin{cases} A'_x = u(x) - (1+\epsilon)\|x - x_0\| \\ B'_y = u(y) + (1+\epsilon)\|y - x_0\|. \end{cases}$$

We still have $A'_x \leq B'_y$, but the difference is that now $sup(A'_x)$ exists, because A'_x goes to $-\infty$ when x is big, and we can therefore restrict $sup(A'_x)$ to a compact subset of M.

So here we build a linear functional that extends the given one, but with norm $\leq 1 + \epsilon$ instead of ≤ 1.

Formal topology:

One reduces the problem to a finite collection of x and y. We can then take $sup\ A_x$, over a finite subset of M.

We never build a functional, only a finite approximation.

From this analysis, it is quite unlikely that one can easily deduce the Bishop version from the formal one, nor do we think that the proof in formal topology follows from Bishop's formulation. On the other hand, the formal one may be as good as Bishop's result for applications, since in an application one will only need finite approximations of a functional.

As we said, our proof can be represented in Martin-Löf's type theory, and indeed, proof-theoretically, we use only the notion of finitary inductive definition. An interesting question at this point is the connection of our work to the work of Brown and Simpson (1986). For instance, our spaces are absolute (see Fourman and Grayson 1982) only for separable vector spaces, which explains the separability restriction, in Brown and Simpson 1986, that does not appear here.

One aim of this approach is to provide an alternative to Bishop's treatment of functional analysis in constructive mathematics. For this, it will be important to find concrete instances of the use of the Alaoglu and Hahn-Banach theorems, for which our proof can help to extract their computational content.

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