UNIVERSIDADE TÉCNICA DE LISBOA
INSTITUTO SUPERIOR TÉCNICO

# Behavioral Algebraization of Logics

Ricardo João Rodrigues Gonçalves
(Licenciado)

Dissertação para Obtenção do Grau de Doutor em Matemática

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# Algebrização Comportamental de Lógicas

**Nome:** Ricardo João Rodrigues Gonçalves

**Doutoramento em:** Matemática

**Orientador:** Doutor Carlos Manuel Costa Lourenço Caleiro

**Provas concluídas em:**

**Resumo:** A lógica algébrica abstracta (LAA) tem como objecto de estudo o mecanismo pelo qual uma classe de álgebras se pode associar a uma dada lógica. Apesar do seu sucesso, a LAA é um pouco restricta no que diz respeito à sua aplicabilidade. Nesta dissertação generaliza-se a noção de lógica algebrizável usando ferramentas da álgebra multigénero comportamental. Deste modo estendemos a aplicabilidade da teoria não só a lógicas multigénero, como também a lógicas unigénero tradicionalmente não-algebrizáveis. Neste sentido desenvolveram-se as fundações de uma teoria comportamental sólida da LAA, obtendo-se resultados de generalização da hierarquia de Leibniz e caracterizações alternativas de várias classes de lógicas. Do ponto de vista semântico, usaram-se duas abordagens: a semântica matricial e a semântica de valorações. Em qualquer dos casos, obtiveram-se resultados fortes de completude. Por fim, são estudados vários exemplos que ilustram a aplicação da teoria desenvolvida.

**Palavras-chave:** Lógica algébrica abstracta, álgebra multigénero, equivalência comportamental, hierarquia de Leibniz.

# Behavioral Algebraization of Logics

**Abstract:** The theory of abstract algebraic logic (AAL) studies the mechanism by which a class of algebras can be associated with a given logic. Despite of its success, the traditional tools of AAL have a rather limited scope of application. In this dissertation we generalize the notion of algebraizable logic using the tools of many-sorted behavioral algebra. In this way, we extended the scope of application of AAL not only to many-sorted logics, but also to single-sorted logic which were not algebraizable according to the standard notion. Pursuing this path we have developed the foundations of a solid behavioral theory of AAL, where results generalizing the Leibniz hierarchy were obtained, as well as alternative characterizations for several classes of logics. From the semantical point of view, we used two approaches: one based on matrix semantics and the other on valuation semantics. In both cases, strong completeness result were obtained. Finally, to illustrate the theory we developed, we studied several interesting examples.

**Keywords:** Abstract algebraic logic, multi-sorted algebra, behavioral equivalence, Leibniz hierarchy.

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# Contents

# Chapter 1

# Introduction

The general theory of *abstract algebraic logic* (AAL, from now on) was first introduced in [BP89] with the aim of extending the so-called *Lindenbaum-Tarski method*, as used for instance to establish the relationship between classical propositional logic and Boolean algebras, to the systematic study of the connection between a given logic and a suitable equational theory. Therefore, AAL can be seen as the theory that studies the connection between logic and algebra. This connection enables one to use the powerful tools of universal algebra to study the metalogical properties of the logic being algebraized, namely with respect to its axiomatizability, definability aspects, the deduction theorem, or interpolation properties [FJP03, CP99, BP98]. The theory of AAL has had a fruitful development having as main aim the generalization of the Lindenbaum-Tarski method in order to be applicable to a larger class of logics [Cze01, CJ00, FJ96, Her96, Her97].

Despite of its success the standard tools of AAL have a relatively limited scope of application. Logics with a many-sorted language are good examples of logics that fall out of their scope. It goes without saying that rich logics, with many-sorted languages, are essential to specify and reason about complex systems, as also argued and justified by the theory of combined logics [SSC99]. However, even in the class of propositional based (unsorted) logics many interesting examples simply fall out of the scope of the standard tools of AAL. In particular, there are well-known examples of logics that may be seen as resulting from the extension (by adding connectives and rules) of algebraizable logics that turn out not to be algebraizable [LMS91]. This is the case, for example, of certain non-truth-functional logics, herein understood as logics which are extensions of algebraizable logics by some new connectives not satisfying the congruence property with respect to the equivalence of the algebraizable fragment. With the proliferation of logical systems, with applications ranging from computer science, to mathematics and philosophy, the examples of non-algebraizable logics that, therefore, lack from a meaningful and insightful algebraic counterpart are expected to become more and more common.

Although the standard tools of AAL can associate a class of algebras to every logic, the connection between a non-algebraizable logic and the corresponding class of algebras is, of course, not very strong nor very interesting. This phenomenon is well-known and may happen for several reasons, and in different degrees, depending on whether the Leibniz operator, one of the main tools in AAL, will lack the properties of injectivity, monotonicity or commutation with inverse substitutions. The particular issue of non-injectivity, staying within the realm of protoalgebraic and equivalential logics, has been carefully studied in [FJ01], where the authors restrict the models of the logic by considering just the matrices with a so-called Leibniz filter. Although this is a very interesting approach, the resulting logic is, of course, different from the original one. Contrarily to what is done in [FJ01], we do not want, at all, to change the logic we start from. Our strategy is rather to change a bit the algebraic perspective. This is achieved by considering behavioral equivalence rather then equality as the basic concept. In this dissertation we propose and study a generalization of the standard tools of abstract algebraic logic obtained by substituting unsorted equational logic by many-sorted behavioral logic.

## 1.1 Background

The main building blocks of this thesis are AAL, and the behavioral theory of equational logic that emerged in the theory of algebraic specifications.

### 1.1.1 Abstract algebraic logic

Algebraic logic in the modern sense was born with the work of Tarski, in particular with his 1935 paper (see [Tar83]) where we can find for the first time some characteristic features of the subject we recognize today. In this paper, Tarski gives the precise connection between Boolean algebra and classical propositional logic. The key idea is to look at the set of formulas as an algebra with operators induced by the logical connectives. Tarski then observed that logical equivalence is a congruence on the formula algebra, and therefore a quotient algebra could be built. This is the so-called Lindenbaum-Tarski method. It turns out that the quotient algebra is a Boolean algebra, and the theorems coincide exactly with the formulas equivalent to $\top$.

Using this idea, a number of other logics were algebraized, namely the intuitionistic propositional logic of Brouwer and Heyting, the multiple-valued logics of Post and Lukasiewicz, and the modal logics $S_4$ and $S_5$ of Lewis [Ras81, RT94]. In contrast to Boolean, cylindric, polyadic and Wajsberg algebras, which were known before the Lindenbaum-Tarski method was first applied to generate them from the appropriate logics, Heyting algebras were first identified precisely by applying the Lindenbaum-Tarski method to intuitionistic propositional logic.

The investigation of particular classes of logics gave place to a systematic investigation of broad classes of logics in a more abstract context. The focus has turned to the process of algebraization itself rather than being centered on the algebraization of particular logics. The general theory of the algebraization of logics that has developed is called *Abstract Algebraic Logic* (AAL).

In the 1989 seminal monograph by Blok and Pigozzi [BP89] the concept of an algebraizable logic was given, the first time, a mathematically precise sense. The notion of logic adopted in AAL is the so-called *Tarskian logic*. In a very informal way, a logic $\mathcal{L}$ is algebraizable if there exists a strong representation between $\mathcal{L}$ and the equational consequence associated with a class $K$ of algebras. In this case, the class $K$ of algebras can be considered an algebraic counterpart of $\mathcal{L}$.

Although the focus of algebraic logic was on finding an algebraic counterpart for particular classes of logics, there was also interest, when this counterpart was found, in investigating the relationship between the metalogical properties of the logic and the algebraic properties of the corresponding class of algebras. These results are usually called bridge theorems and allow us to use powerful methods of modern algebra in the investigation of metalogical properties of algebraizable logics. The theory of AAL provides a general context in which bridge theorems relating metalogical properties of a logic to algebraic properties of its algebraic counterpart can be formulated precisely and in abstract [FJP03, CP99, BP98]. For example, it was known that there is a close connection between the deduction theorem and algebraic properties of the class of algebras, but it was only in the general context of AAL that this connection could be made precise [BP98]. Indeed, the quest for a general framework in which this connection could be more precisely stated was one of the motivations for the development of AAL.

One of the goals of AAL is to discover general criteria for a class of algebras to be the algebraic counterpart of a logic and to develop methods to obtain this algebraic counterpart. Another important goal of AAL is the classification of logics based on the algebraic properties of their algebraic counterpart. This can be very useful since general theorems can be formulated and applied to all the members of a specific class. When it is known that a given logic belongs to a particular class, these general theorems immediately provide important informations about its properties. A survey of most of the achievements of AAL and pointers for several open problems can be found in [Cze01, FJP03, FJ96].

### 1.1.2 Many-sorted behavioral logic

The motivation for our use of the term *behavioral* emerges from computer science, namely from the algebraic approach to the specification and verification of complex (namely, object oriented) systems, where abstract data types and object classes are defined by the properties of their associated operations. Algebras are considered as abstract machines where the programs are to be run [GM96]. Such systems constitute a challenge for traditional algebraic methods, since they very often provide mechanisms to encapsulate internal data in order to make the updating of programs easier, and the internal data protected. Consequently, the data should naturally be split into two categories: *visible data* which can be directly accessed, and *hidden data* that can only be accessed indirectly by analyzing the meaning (output) of programs with visible output, called *experiments*. The role of experiments is to access the relevant information encapsulated in a state. With the aim of generalizing many-sorted algebras to give an algebraic semantics for the object oriented paradigm, Goguen in [Gog91] introduced the notion of *hidden algebra*. In a hidden algebra the elements are naturally split into the visible and the hidden data. Since one cannot access the hidden data, it is not possible to reason directly about the equality of two hidden values. Hence, equational logic needs to be replaced by behavioral equational logic (sometimes called hidden equational logic) based on the notion of behavioral equivalence. Two values are said to *behaviorally equivalent* if they cannot be distinguished by the set of all experiments (as introduced by Reichel in [Rei85]). For practical reasons, hidden algebras, when they first appeared, were considered over restricted signatures. The behavioral aspects of modern software make hidden algebras more suitable than standard algebra for practice as abstract machine implementations. Consequently, there has been an increasing development in this field. Goguen and his collaborators have been improving their theory and applying it in more general settings [GM96, GR99]. In fact, in a subsequent work [GM00] the possibility that not all experiments are available to distinguish two values was considered. This restriction induces the notion of $\Gamma$-behavioral equivalence [GM00], where $\Gamma$ is a subset of the set of original operations. Two values are said to $\Gamma$-behaviorally equivalent if they cannot be distinguished by all the experiments that can be build with the operations in $\Gamma$. It can be shown that the $\Gamma$-behavioral equivalence is the largest $\Gamma$-congruence (equivalence relation compatible with the operations in Γ) whose visible part is the identity relation. This has a strong connection with the notion of Leibniz congruence, a fundamental tool of AAL. Some fruitful applications of AAL in the theory of algebraic specification have been established [Mar04, Mar07] using the Leibniz congruence as the main tool. Notably, the possibility of having a restricted set of experiments also accommodates the existence of non-congruent operations [Ros04].

## 1.2 Aims

The general goal of this thesis is the definition and study of a generalization of the notion of algebraizable logic using many-sorted behavioral logic, that may encompass some less orthodox logics while still associating to them meaningful and insightful algebraic counterparts.

First of all, we shall concentrate on the development of the right framework in which the envisaged generalization of the notion of algebraizable logic can be fulfilled. Once the framework is established we shall define a behavioral notion of algebraizable logic. Of course, we do not want just to introduce a wider notion of algebraizable logic in order to force some non-algebraizable logics to be algebraizable this new sense. Our aim is also to discuss its importance and its limits.

Taking into account our motivation to broaden the scope of applicability of the standard notion of algebraizable logic, we shall also provide the necessary tools to the study of particular examples of logics. In one hand, we aim at achieving sufficient conditions to prove that a concrete logic is behaviorally algebraizable, as well as the tools to obtain, in this case, its behaviorally algebraic counterpart. On the other hand, it is also our goal to provide tools to prove non-behavioral algebraizability of a given concrete logic, namely by disproving necessary conditions for behavioral algebraizability. Of course, all these tools should be easy to apply to concrete examples.

We shall base our approach on a solid theory generalizing standard key notions and results from AAL, from both the syntactic and the semantical point of view. We aim at achieving a generalization of the so-called Leibniz hierarchy of logics, along with their corresponding characterization results. With respect to the semantic point of view our, our aim is to generalize the fruitful results of the theory of logical matrices in AAL.

Finally, we shall study a number of meaningful examples supporting the theory. Our aim is not only to provide new algebraic counterparts to several of these examples, but also to explain the connection of some examples with existing proposals of algebraic counterparts for them.

## 1.3 Outline

This dissertation is organized in 5 more chapters. We present a brief outline of each of them.

### 1.3.1 Preliminaries

In Chapter 2 we introduce some preliminary notions and results that will be necessary for the remainder of the thesis. Most of the material presented in this chapter is a rephrasal, to our many-sorted behavioral setting, of well-known notions and results that can be found in textbooks on logic, namely [Wój88, MP96], on universal algebra, namely [BS81, MT92] and on abstract algebraic logic, namely [FJP03, Cze01]. We start by focusing on the central notion of logic, and some of this properties. In this dissertation we adopt the Tarskian notion of logic [Tar30]. We introduce the notion of deductive system, which is very useful to present particular examples of logics. To study the relationship between logics, we introduce a suitable notion of map between logics and study some general results. Strengthening the conditions in the definition of map we obtain the notions of conservative map and the fundamental notion of strong representation. Strong representations capture the connection between an algebraizable logic and the equational consequence of its equivalent algebraic semantics [Cze01]. We then pave the way towards our many-sorted framework by introducing the notion of many-sorted signature. A many-sorted logic is then introduced as a logic whose language is obtained from a many-sorted signature with a distinguished sort $\phi$ of formulas, and such that it further satisfies a structurality condition. We give the many-sorted notion of deductive system and use it to introduce some examples of many-sorted logics. Since this dissertation is in the area of algebraic logic, another central notion is the one of algebra. Along with the definition of many-sorted algebra over a many-sorted signature, we recall several usual constructions of many-sorted universal algebra [MT92]. We introduce many-sorted equational logic associated with a class of many-sorted algebras [EM85, GM85]. We present the notions of variety and quasivariety of logics along with some important characterizations theorems [BS81, Gor98]. A key tool in our dissertation is the notion of behavioral equivalence. We recall the notion of hidden many-sorted signature as a many-sorted signature which is divided in a visible and a hidden part. Two hidden elements of an algebra are behaviorally equivalent if they cannot be distinguished by any visible operation. Substituting, in the hidden part, the role of equality by behavioral equivalence we obtain behavioral versions of the notions of universal algebra introduced before [Mar04, Ros00, GM00]. We end the chapter by surveying some of the standard notions and results of the theory of abstract algebraic logic [FJP03, Cze01]. The fundamental notion of algebraizable logic is presented along with some characterizations results and also some sufficient and necessary conditions. We recall the Leibniz operator and present the Leibniz hierarchy which is built using characterization results involving properties of the Leibniz operator. We also present the notions of protoalgebraic logic and of weakly algebraizable logic along with several of their characterization results. Some of the key results in the semantic study of AAL are introduced, all developed around the central notion of logical matrix. We end by presenting some well-known examples in AAL.

### 1.3.2 Behavioral abstract algebraic logic

In Chapter 3 we introduce and study a generalization of the standard tools of AAL obtained by using many-sorted behavioral logic in the role traditionally played by unsorted equational logic. We start by setting up the framework for our many-sorted behavioral approach. We then introduce the central notion of $\Gamma$-behaviorally algebraizable logic, where $\Gamma$ is a subsignature of the original signature of the logic. The subsignature $\Gamma$ is a parameter and, once fixed, it means that the algebraic part of the behavioral algebraization process is built over the notion of $\Gamma$-behavioral equivalence. We then introduce the notion of $\Gamma$-behaviorally equivalential and use it in some necessary conditions for a logic to be behaviorally algebraizable. We prove that the novel notion of behaviorally algebraizable logic is not as broad as it becomes trivial, by proving that it is in the class of standard protoalgebraic logics, which is considered the largest class of logics amenable to the methods of AAL. We continue by introducing a behavioral version of the Leibniz operator and engage on a generalization of the Leibniz hierarchy. We introduce the behavioral versions of protoalgebraic logic and of weakly algebraizable logic along with several characterization results. Besides the results involving the Leibniz operator itself, we have also results involving the notion of set of behavioral equivalence formulas. Characterization results for the class of behaviorally algebraizable and behaviorally equivalential logics are also obtained. We end the chapter with some intrinsic and sufficient conditions that are very useful in practice to show that a given logic is behavioral algebraizable.

### 1.3.3 BAAL - semantical considerations

In Chapter 4 we continue the effort towards the generalization of the standard notions and results of AAL to the behavioral setting, now in a semantical perspective. We start by characterizing the class of algebras that our behavioral approach canonically associates with a given behaviorally algebraizable logic. We prove that a unicity result with respect to the algebraic counterpart of a behaviorally algebraizable logic can be obtained. We prove also a result that allows to produce the axiomatization of the algebraic counterpart of a behaviorally algebraizable logic $\mathcal{L}$ from the deductive system of $\mathcal{L}$. Matrix semantics is the standard tool for semantical investigations in AAL [Cze01]. The generalization of this tool to the behavioral setting is not straightforward and can lead to two different approaches. We start by exploring the most natural approach, the one centered on the standard notion of logical matrix. We generalize some of the results of the theory of logical matrices, ultimately aiming at bridging results, relating metalogical properties of a logic with algebraic properties of its associated class of algebras. We introduce a class $Alg_\Gamma$ of algebras generalizing the standard class $Alg$ of algebraic reducts of reduced matrices. Moreover, we prove that, in the case of a behaviorally algebraizable logic $\mathcal{L}$, the class $Alg_\Gamma(\mathcal{L})$ coincides with the largest behaviorally equivalent algebraic semantics. Given a logic $\mathcal{L}$ which is algebraizable in the standard sense and it also $\Gamma$-behaviorally algebraizable for some subsignature $\Gamma$ of the original signature, we study then the relationship between the classes $Alg_\Gamma(\mathcal{L})$ and $Alg(\mathcal{L})$. We establish relations between the classes of equations and quasi-equations satisfied by these two classes of algebras. We then develop the second approach to the generalization of the standard notion of logical matrix. This approach is strongly connected with the theory of valuation semantics [dCB94]. We introduce an algebraic version of valuation, the notion of $\Gamma$-valuation, and we prove a completeness theorem with respect to the class $Mod_\Gamma(\mathcal{L})$ of all $\Gamma$-valuation models. We prove also a result relating a metalogical property of a logic $\mathcal{L}$ and an algebraic property of $Mod_\Gamma(\mathcal{L})$. We end by showing how to extract a class $\mathcal{M}_K$ of $\Gamma$-valuations that is complete with respect to $\mathcal{L}$, from the algebraic counterpart $K$ of a $\Gamma$-behaviorally algebraizable logic $\mathcal{L}$.

### 1.3.4 Worked examples

In Chapter 5 we present some examples to further illustrate the relevance of our new approach to the algebraization of logics. In the first example, we show that our behavioral approach is indeed an extension of the existing tools of AAL [FJP03, CG07]. In the many-sorted case we also present some non-behavioral many-sorted definitions and results that can be useful when applying the theory to particular examples of logics. We proceed with the example of paraconsistent logic $\mathcal{C}_1$ of da Costa, whose non-algebraizability in the standard sense is well-known [dC74, Mor80, LMS91]. We show that $\mathcal{C}_1$ is behaviorally algebraizable and, moreover, we give an algebraic counterpart for it. Recall that, although the standard non-algebraizability of $\mathcal{C}_1$ is well-known, there have been some proposals of algebraic counterparts of $\mathcal{C}_1$, namely the class of so-called da Costa algebras and the non-truth-functional bivaluation semantics. Of course, since $\mathcal{C}_1$ is not algebraizable, their precise connection with $\mathcal{C}_1$ could never be established at the light of the standard tools of AAL. We prove that both the class of da Costa algebras and the class of bivaluations can now be obtained from the class of algebras that our approach canonically associates with $\mathcal{C}_1$, thus explaining their precise connection with $\mathcal{C}_1$. We also study the example of the Carnap-style presentation of modal logic $S5$, whose non-algebraizability in the standard sense is again well-known [BP89]. We prove that $S5$ is behaviorally algebraizable and we propose an algebraic counterpart for it. We continue by briefly analyzing the example of first order logic $FOL$, whose standard algebraization is well-studied [BP89, ANS01]. Our approach can be useful to shed light on the essential distinction between terms and formulas. Next, in the example of global logic we follow the exogenous semantic approach for enriching a logic [MSS05] and present a sound a complete deductive system for global logic $GL(\mathcal{L})$ over a given local logic $\mathcal{L}$. We also prove that $GL(\mathcal{L})$ is behaviorally algebraizable independently of $\mathcal{L}$. Moreover, we prove that in the cases where $\mathcal{L}$ is algebraizable we are able to recover the algebraic counterpart of $\mathcal{L}$ from the algebraic counterpart of $GL(\mathcal{L})$. Still following the exogenous semantic approach for enriching a logic we present the example of exogenous propositional probability logic EPPL. We prove that EPPL is behaviorally algebraizable and provide an algebraic counterpart for it. We proceed by exemplifying the power of our approach, by showing that it can be directly applied to study the algebraization of $k$-deductive systems [BP98, Mar04]. Finally, we study the example of Nelson's logic $N$, which is algebraizable according to the standard definition [Ras81], but its behavioral algebraization can help to give an extra insight on the role of Heyting algebras in the algebraic counterpart of $N$.

### 1.3.5 Conclusion

In Chapter 6 we make some final remarks and revise the contributions of this dissertation. We also point out related future directions of research.

## 1.4 Claim of contributions

The following are the contributions obtained in the scope of this dissertation that we would like to stress:

- the identification of many-sorted behavioral logic as the correct framework to engage on the envisaged generalization of the tools of AAL;
- the proposal of a non-trivial generalization of the notion of algebraizable logic broadening the range of application of the standard notion;
- the generalization to the behavioral setting of several of the standard key notions and results of AAL;
- the construction of a behavioral Leibniz hierarchy of logics and its comparison with the standard Leibniz hierarchy;
- the behavioral generalization of the notion of matrix semantics along with several of standard semantical results;
- the application of the behavioral approach to several examples, not only providing new algebraic counterparts to them, but also explaining their connection with existing proposals in the literature.

# Chapter 2

# Preliminaries

Our work is in the intersection of two main areas of research: algebraic logic and specification theory. One of the main goals is to use tools and techniques of specification theory, in particular of many-sorted behavioral logic, in the area of algebraic logic.

In this chapter we introduce some preliminary notions and results that will be necessary for the remainder of the thesis. The purpose is to get the reader acquainted with the subject of abstract algebraic logic given in a many-sorted perspective, paving the way to the generalization of the theory that we will present in the subsequent chapters. The reader already familiar with algebraic logic or with the theory of behavioral logic will find the thesis easier to read.

For the interested reader we refer to [FJP03, Cze01, Ros00] for more on these subjects and for the proofs of the results presented in this chapter.

## 2.1 Basic notions

In this section we introduce some relevant notation and results for understanding the thesis. It is not our intention to dwell on the discussion about what is a basic notion, but rather to let the reader get acquainted with our notations, terminology, conventions and mathematical language.

### 2.1.1 Logics

Herein, we introduce some important concepts and results around the central notion of *logic*.

#### 2.1.1.1 Logical consequence

The answer to the question of what is a *logic* is not consensual. It is not our aim to go into a philosophical discussion on this subject but rather to adopt a definition of logic general enough for our purposes. So, let us fix the notion of logic we will work with. This is usually called a Tarskian logic.

**Definition 2.1.1.** A *logic* is a pair $\mathcal{L} = \langle L, \vdash \rangle$, where $L$ is a set (of *formulas*) and $\vdash \subseteq 2^L \times L$ is a *consequence relation* satisfying, for all $\Psi \cup \Phi \cup \{\varphi, \psi\} \in 2^L$, the following conditions [Tar83]:

***Reflexivity:*** if $\varphi \in \Psi$ then $\Psi \vdash \varphi$;

***Cut:*** if $\Psi \vdash \varphi$ for all $\varphi \in \Phi$, and $\Phi \vdash \psi$ then $\Psi \vdash \psi$;

***Weakening:*** if $\Psi \vdash \varphi$ and $\Psi \subseteq \Phi$ then $\Phi \vdash \varphi$.

We consider only these three conditions, though more conditions could be imposed. Namely the condition of *finitariness* which was present on the original Tarski's proposal.

***Finitariness:*** if $\Psi \vdash \varphi$ then $\Psi' \vdash \varphi$ for some finite $\Psi' \subseteq \Psi$.

Even though we do not assume Finitariness, our definition of logic embodies all the spirit of Tarski's original proposal. Moreover, if finitariness was an a priori requirement, then a lot of important logics would be ruled out, namely those introduced by semantic means. In the first steps of the theory of Abstract Algebraic Logic (AAL), namely in the seminal paper by Blok and Pigozzi [BP89], the logics under consideration are always finitary. Nevertheless, it was soon realized that this is too restrictive and the theory of AAL was then generalized to possible non-finitary logics by Herrmann [Her93, Her96, Her97] and Czelakowski [Cze92].

When considering several logics, to avoid confusion, we attach the name of a logic to the respective consequence relation by writing $\vdash_{\mathcal{L}}$ instead of $\vdash$. In the sequel if $\Psi, \Phi \subseteq L$, we shall write $\Psi \vdash \Phi$ whenever $\Psi \vdash \varphi$ for all $\varphi \in \Phi$. We say that $\varphi$ and $\psi$ are *interderivable*, which is denoted by $\varphi \dashv\vdash \psi$, if $\varphi \vdash \psi$ and $\psi \vdash \varphi$. Similarly, given $\Psi, \Phi \subseteq L$ we say that $\Psi$ and $\Phi$ are *interderivable*, if $\Psi \vdash \Phi$ and $\Phi \vdash \Psi$.

**Lemma 2.1.2.** *Reflexivity and Cut together imply Weakening.*

*Proof.* Let $\Psi \subseteq \Phi \subseteq L$ and $\varphi \in L$. Suppose also that $\Psi \vdash \varphi$. We want to prove that $\Phi \vdash \varphi$. Let $\psi \in \Psi$, then, because $\Psi \subseteq \Phi$, we have that $\psi \in \Phi$. By reflexivity we know that $\Phi \vdash \psi$ for all $\psi \in \Psi$. Using the hypothesis that $\Psi \vdash \varphi$ and using Cut, we conclude that $\Phi \vdash \varphi$.

□

Despite the fact that Reflexivity and Cut together imply Weakening, we have kept Weakening in the definition for methodological reasons, thus explicitly excluding non-monotonic logics from this context.

In the literature it is also usual to use the operator approach to introduce the notion of logic. In the operator approach a logic is introduced as a pair $\mathcal{L} = \langle L, {}^{\vdash} \rangle$ where $L$ is a set (of formulas) and ${}^{\vdash}$ is a *closure operator* on $L$ (in the sense of Kuratowski [MP96]), that is, ${}^{\vdash} : 2^L \to 2^L$ is a function that satisfies the following properties:

***Extensiveness:*** $\Phi \subseteq \Phi^{\vdash}$;

***Monotonicity:*** If $\Phi \subseteq \Psi$ then $\Phi^{\vdash} \subseteq \Psi^{\vdash}$;

***Idempotence:*** $(\Phi^{\vdash})^{\vdash} \subseteq \Phi^{\vdash}$.

Defined in this way, a logic is clearly a generalization of the notion of topological space provided by the Kuratowski closure axioms [MP96]. The difference is that two of the Kuratowski closure axioms are dropped since they are not desirable in logical terms: *groundness* and *additivity*. The groundness axiom states that $\emptyset^{\vdash} = \emptyset$ and it is not desirable because we do not want to assume that the set of theorems, $\emptyset^{\vdash}$, is always empty. The additivity axiom states that $(\Phi_1 \cup \Phi_2)^{\vdash} \subseteq (\Phi_1)^{\vdash} \cup (\Phi_2)^{\vdash}$ and it is not desirable because the interplay between formulas is an essential feature in logic.

**Lemma 2.1.3.** *It is equivalent to define a logic through a consequence relation or through a consequence operator.*

*Proof.* Let first $\mathcal{L} = \langle L, \vdash \rangle$ be a logic as given in Definition 2.1.1. Given a set $\Phi \subseteq L$, we can consider the set $\Phi^{\vdash} = \{\varphi \in L : \Phi \vdash \varphi\}$. This function on the powerset of $L$ is called the *consequence operator* of $\mathcal{L}$ and satisfies the Kuratowski axioms.

Extensiveness follows directly from the Reflexivity of $\vdash$. If $\varphi \in \Phi$ then by Reflexivity $\Phi \vdash \varphi$ and hence, by definition of $\Phi^{\vdash}$, $\varphi \in \Phi^{\vdash}$.

Monotonicity follows from the Weakening property of $\vdash$. Suppose that $\Phi \subseteq \Psi$ and $\varphi \in \Phi^{\vdash}$, that is, $\Phi \vdash \varphi$. By Weakening, $\Psi \vdash \varphi$, and so, $\varphi \in \Psi^{\vdash}$.

Idempotence follows from the Cut property of $\vdash$. Let $\varphi \in (\Phi^{\vdash})^{\vdash}$, that is, $\Phi^{\vdash} \vdash \varphi$. We also know that $\Phi \vdash \psi$ for all $\psi \in \Phi^{\vdash}$. Hence by Cut we have that $\Phi \vdash \varphi$ and this means that $\varphi \in \Phi^{\vdash}$.

Consider now given $\mathcal{L} = \langle L, {}^{\vdash}\rangle$ a logic where ${}^{\vdash}$ is a closure operator. Our aim is to prove that the relation $\vdash \subseteq 2^L \times L$, defined by $\Phi \vdash \varphi$ iff $\varphi \in \Phi^{\vdash}$, satisfies Reflexivity, Cut and Weakening.

Reflexivity follows from Extensiveness. If $\varphi \in \Phi$, then by Extensiveness $\varphi \in \Phi^{\vdash}$, that is, $\Phi \vdash \varphi$.

Cut follows from Idempotence and Monotonicity. Suppose that $\Phi \vdash \varphi$ for all $\varphi \in \Psi$, and $\Psi \vdash \psi$. We want to prove that $\Phi \vdash \psi$. By hypothesis we know that $\Psi \subseteq \Phi^{\vdash}$, and then by Monotonicity and Idempotence we have that $\Psi^{\vdash} \subseteq \Phi^{\vdash}$. Therefore $\psi \in \Phi^{\vdash}$, that is, $\Phi \vdash \psi$.

Weakening follows from the Monotonicity. Suppose that $\Phi \vdash \varphi$ and $\Phi \subseteq \Psi$. We want to prove that $\Psi \vdash \varphi$. We know that $\varphi \in \Phi^{\vdash}$ and so, by Monotonicity, we have that $\varphi \in \Psi^{\vdash}$, that is, $\Psi \vdash \varphi$.

□

This equivalence allows us to use in the sequel either the consequence relation or the consequence operator, interchangeably, whenever it is more convenient.

The *theorems* of $\mathcal{L}$ are the formulas $\varphi$ such that $\emptyset \vdash \varphi$. A *theory* of $\mathcal{L}$, or briefly a $\mathcal{L}$-*theory*, is a set $\Psi$ of formulas such that $\Psi$ is closed under the consequence relation $\vdash$, that is, such that if $\Psi \vdash \varphi$ then $\varphi \in \Psi$. Given a set $\Psi$, we can consider the set $\Psi^{\vdash} = \{\varphi \in L : \Psi \vdash \varphi\}$, the smallest theory containing $\Psi$. The set of all theories of $\mathcal{L}$ is denoted by $Th_{\mathcal{L}}$. Being a set of subsets of $L$, $Th_{\mathcal{L}}$ is naturally equipped with the binary relation of inclusion between its elements. The inclusion relation, $\subseteq$, can be seen as an ordering in $Th_{\mathcal{L}}$ and so we can study the properties of the pair $\langle Th_{\mathcal{L}}, \subseteq\rangle$ within order theory.

**Definition 2.1.4.** A *partial order* is a pair $\langle R, \leq\rangle$, where $R$ is a set and $\leq \subseteq R \times R$ is a relation satisfying the following properties:

***Reflexivity:*** $r \leq r$ for all $r \in R$;

***Transitivity:*** $r_1 \leq r_2$ and $r_2 \leq r_3$ implies $r_1 \leq r_3$ for all $r_1, r_2, r_3 \in R$;

***Antisymmetry:*** $r_1 \leq r_2$ and $r_2 \leq r_1$ implies $r_1 = r_2$ for all $r_1, r_2 \in R$.

If $\langle P, \leq \rangle$ is a partial order, then $P$ is called a *partial ordered set* or simply a *poset*. Let $A$ be a subset of a poset $P$. An element $p \in P$ is an *upper bound* for $A$ if $a \leq p$ for every $a$ in $A$. An element $p \in P$ is the *least upper bound of* $A$, or *supremum* of $A$ ($\bigvee A$) if $p$ is an upper bound of $A$, and $a \leq b$ for every $a \in A$ implies $p \leq b$ (i.e., $p$ is the smallest among the upper bounds of $A$). Similarly we can define what it means for $p$ to be a *lower bound of* $A$, and for $p$ to be the *greatest lower bound of* $A$, also called the *infimum* of $A$ ($\bigwedge A$).

**Definition 2.1.5.** A *lattice* is a pair $\langle R, \leq \rangle$ such that $\langle R, \leq \rangle$ is a poset and, for every $a, b \in R$, both $\bigvee\{a, b\}$ and $\bigwedge\{a, b\}$ exist in $R$.

**Definition 2.1.6.** Given two partial orders $\langle A, \leq_A \rangle$ and $\langle B, \leq_B \rangle$, a *map from* $\langle A, \leq_A \rangle$ *to* $\langle B, \leq_B \rangle$ is just a function $h : A \to B$.
Moreover, the map $h$ is *monotone* if for all $a_1, a_2 \in A$,

$$a_1 \leq_A a_2 \text{ implies } h(a_1) \leq_B h(a_2).$$

**Definition 2.1.7.** A poset $P$ is *complete* if for every subset $A$ of $P$ both $\bigvee A$ and $\bigwedge A$ always exist (in $P$).

**Definition 2.1.8.** A lattice $L$ is *complete* if it is complete as a poset.

**Definition 2.1.9.** A *map between two complete posets* $A$ and $B$ is a function $h : A \to B$ such that, for all $A_1 \subseteq A$,

$$h(\bigvee A_1) = \bigvee h[A_1].$$

This condition of preserving arbitrary supremums is enough to guarantee monotonicity.

**Lemma 2.1.10.** *A map between complete partial orders is always monotone.*

*Proof.* Suppose that $h : \langle A, \leq_A \rangle \to \langle B, \leq_B \rangle$ is a map between two complete partial orders and consider $a_1, a_2 \in A$ such that $a_1 \leq_A a_2$. Then clearly we have that $\bigvee\{a_1, a_2\} = a_2$. Because $h$ is sup-preserving we have that $h(a_2) = h(\bigvee\{a_1, a_2\}) = \bigvee h[\{a_1, a_2\}] = \bigvee\{h(a_1), h(a_2)\}$, which means that $h(a_1) \leq_B h(a_2)$.

□

Returning to the set $Th_{\mathcal{L}}$ of theories of a logic $\mathcal{L}$ we can now see that $\langle Th_{\mathcal{L}}, \subseteq \rangle$ is indeed a complete partial order.

**Lemma 2.1.11.** *The tuple $\langle Th_{\mathcal{L}}, \subseteq \rangle$ is a complete partial order, where the supremum and the infimum of a set $T$ of theories are respectively, $\bigvee^{\mathcal{L}} T = (\cup_{\Psi \in T} \Psi)^{\vdash}$ and $\bigwedge^{\mathcal{L}} T = \bigcap_{\Psi \in T} \Psi$.*

*Proof.* By the properties of $\subseteq$ is trivial to verify that $\langle Th_{\mathcal{L}}, \subseteq \rangle$ is a partial order. Hence, all we need to prove is that it is a complete partial order, that is, for every $T \subseteq Th_{\mathcal{L}}$ the supremum and the infimum of $T$ exist in $Th_{\mathcal{L}}$. First of all let us prove that $\bigvee^{\mathcal{L}} T$ and $\bigwedge^{\mathcal{L}} T$ are well-defined, that is, they are still in $Th_{\mathcal{L}}$. By definition, $\bigvee^{\mathcal{L}} T$ is always a theory and hence belongs to $Th_{\mathcal{L}}$. We now prove that the arbitrary intersection of theories is still a theory. Let $I = \bigcap_{\Psi \in T} \Psi$. If $I \vdash \varphi$ then by monotonicity $\Psi \vdash \varphi$ for every $\Psi \in T$. Because each $\Psi \in T$ is a theory, we have that $\varphi \in \Psi$ for every $\Psi \in T$, and hence $\varphi \in I$. So, we conclude that $I \in Th_{\mathcal{L}}$.

We now prove that in fact $\bigvee^{\mathcal{L}} T$ and $\bigwedge^{\mathcal{L}} T$ are, respectively, the supremum and the infimum of $T$. By definition, $\bigvee^{\mathcal{L}} T$ is clearly a theory that contains all $\Psi \in T$. Let $\Phi \in Th_{\mathcal{L}}$ such that $\Psi \subseteq \Phi$ for every $\Psi \in T$. Then we have that $\bigcup_{\Psi \in T} \Psi \subseteq \Phi$ and by Monotonicity we can conclude that $\bigvee^{\mathcal{L}} T = (\bigcup_{\Psi \in T} \Psi)^{\vdash} \subseteq (\Phi)^{\vdash} = \Phi$.

By definition, $\bigwedge^{\mathcal{L}} T$ is clearly a theory such that $\bigwedge^{\mathcal{L}} T \subseteq \Psi$ for all $\Psi \in T$. Let $\Phi \in Th_{\mathcal{L}}$ such that $\Phi \subseteq \Psi$ for all $\Psi \in T$. Then, trivially, we have that $\Phi \subseteq \bigcap_{\Psi \in T} \Psi = \bigwedge^{\mathcal{L}} T$.

□

We have seen that a logic is constituted by a set of formulas and a consequence relation over the formulas. So, when we want to introduce a particular logic we have to describe this consequence relation. What is usual, and in fact very useful, is to introduce a particular logic as a *deductive system*.

**Definition 2.1.12.** A *deductive system* is a pair $\mathcal{D} = \langle L, R \rangle$ where $L$ is a set (of *formulas*), and $R = \{\langle \Psi_i, \varphi_i \rangle : i \in I\}$ where $\Psi_i \subseteq L$ is a finite set and $\varphi_i \in L$, for each $i \in I$.

Each element $r = \langle \Psi, \varphi \rangle$ of $R$ is called an *inference rule*. We say that the (finite) set $\Psi$ is the set of *premises* of $r$, which we denote by $Prem(r)$, and that $\varphi$ is the *conclusion* of $r$, which we denote by $Conc(r)$. If $Prem(r) = \emptyset$, $r$ is said to be an *axiom*, as well as $Conc(r)$.

We said that the notion of deductive system is useful to introduce a particular logic. In fact, given a deductive system, a logic is obtained through the usual notion of derivability.

**Definition 2.1.13.** Given a *deductive system* $\mathcal{D} = \langle L, R \rangle$, a formula $\varphi \in L$ is *derivable from a set of formulas* $\Psi \subseteq L$ in $\mathcal{D}$, denoted by $\Psi \vdash_{\mathcal{D}} \varphi$, if there exists a sequence $\gamma_1, \ldots, \gamma_m \in L$ such that:

- $\gamma_m$ is $\varphi$;
- for each $i = 1, \ldots, m$, the formula $\gamma_i$ is either:
  - an element of $\Psi$, or
  - there exists a rule $r \in R$ such that $\gamma_i = Conc(r)$ and $Prem(r) \subseteq \{\gamma_1, \ldots, \gamma_{i-1}\}$.

The logic associated with a deductive system $\mathcal{D}$ is then $\mathcal{L}_{\mathcal{D}} = \langle L, \vdash_{\mathcal{D}} \rangle$ when $\vdash_{\mathcal{D}}$ is the consequence relation defined above. It is interesting to note that, since all rules have finite sets of premises, the logic $\mathcal{L}_{\mathcal{D}}$ is always finitary.

As a final and important remark in this introductory section about logic, we refer that the adopted notion of logic does not have necessarily a syntactical character. Consequence relations introduced by semantical means fit as perfectly in our notion of logic as those introduced by means of a deductive system, which clearly have a syntactical flavor.

#### 2.1.1.2 Maps of logics

In mathematics, when we introduce a class of mathematical objects, it is usual and fundamental to study also the relationship between them. We have introduced in the previous section the notion of logic and explored some of its properties. In this section we study the relationship between different logics using a notion of *map* between logics. In fact, the relations between different logics play a fundamental role in AAL as we will see later on.

In the sequel, let us consider $\mathcal{L} = \langle L, \vdash \rangle$ and $\mathcal{L}' = \langle L', \vdash' \rangle$ two fixed but arbitrary logics.

**Definition 2.1.14.** A *map* $\theta$ from $\mathcal{L}$ to $\mathcal{L}'$ is a function $\theta : L \to 2^{L'}$ such that, for every $\Psi \cup \{\varphi\} \subseteq L$, the following holds:

$$\text{if} \quad \Psi \vdash \varphi \quad \text{then} \quad (\bigcup_{\gamma \in \Psi} \theta(\gamma)) \vdash' \theta(\varphi).$$

In the literature one can find definitions of maps between logics that can, at first sight, differ from the one presented here. It is usual, for example, to add the restriction that $\theta(\varphi)$ should be a singleton set for every $\varphi \in L$, or that it should be a finite set for every $\varphi \in L$. Note that, although important in some specific contexts, these are nothing but particular cases of the above definition of map.

For the sake of notation we use $\theta[\Psi] = \bigcup_{\gamma \in \Psi} \theta(\gamma)$. Using this notation, the condition of a map can be rewritten in a more simple form:

$$\text{if } \Psi \vdash \varphi \text{ then } \theta[\Psi] \vdash' \theta(\varphi).$$

Although the existence of a map between $\mathcal{L}$ and $\mathcal{L}'$ induces a strong connection between the consequence relations of the two logics, it does not allow us to use $\mathcal{L}'$ to reason about formulas of $\mathcal{L}$. By strengthening the relation between the consequence relations of the two logics we obtain the notion of *conservative map*.

**Definition 2.1.15.** A map $\theta$ from $\mathcal{L}$ to $\mathcal{L}'$ is *conservative* when, for every $\Psi \cup \{\varphi\} \subseteq L$, we have that:

$$\Psi \vdash \varphi \quad \text{iff} \quad \theta[\Psi] \vdash' \theta(\varphi).$$

Now the existence of a conservative map $\theta$ from a logic $\mathcal{L}$ to a logic $\mathcal{L}'$ allows us to use $\mathcal{L}'$ to reason about formulas of $\mathcal{L}$. Given $\Psi \cup \{\varphi\} \subseteq L$, suppose we want to see if it is the case that $\Psi \vdash \varphi$. Then, we only need to establish $\theta[\Psi] \vdash' \theta(\varphi)$.

When we use maps to compare logics we could say, roughly speaking, that if there is a conservative map $\theta : \mathcal{L} \to \mathcal{L}'$ then $\mathcal{L}'$ is stronger than $\mathcal{L}$. So, the existence of two conservative maps, one from $\mathcal{L}$ to $\mathcal{L}'$ and another from $\mathcal{L}'$ to $\mathcal{L}$, would mean that $\mathcal{L}$ and $\mathcal{L}'$ have the same expressive power. If to this we add the condition that these maps should be somehow inverse of each other, we would then obtain the notion of *strong representation*.

**Definition 2.1.16.** A *strong representation* of $\mathcal{L}$ in $\mathcal{L}'$ is a pair $(\theta, \tau)$ of maps $\theta : \mathcal{L} \to \mathcal{L}'$ and $\tau : \mathcal{L}' \to \mathcal{L}$ such that:

*i*) For all $\varphi \in L$ we have that $\varphi \dashv\vdash \tau[\theta(\varphi)]$;

*ii*) For all $\varphi' \in L'$ we have that $\varphi' \dashv\vdash' \theta[\tau(\varphi')]$

*iii*) $\theta$ is conservative;

*iv*) $\tau$ is conservative.

Noting the symmetry in the above definition we can easily conclude that, whenever we have a strong representation $(\theta, \tau)$ of $\mathcal{L}$ in $\mathcal{L}'$, we also have a strong representation $(\tau, \theta)$ of $\mathcal{L}'$ in $\mathcal{L}$.

Although we have presented four conditions in this definition of strong representation, there are strong dependences between them. They are established in the next lemma.

**Lemma 2.1.17.** *Let $\theta : \mathcal{L} \to \mathcal{L}'$ and $\tau : \mathcal{L}' \to \mathcal{L}$ be two maps and consider the following four conditions:*

*i*) *for all $\varphi \in L$ we have that $\varphi \dashv\vdash \tau[\theta(\varphi)]$;*

*ii*) *for all $\varphi' \in L'$ we have that $\varphi' \dashv\vdash' \theta[\tau(\varphi')]$;*

*iii*) *$\theta$ is conservative;*

*iv*) *$\tau$ is conservative.*

*Then, we have the following:*

*1) if conditions iii) and iv) hold then condition i) is equivalent to condition ii);*

*2) conditions ii) plus iii) are equivalent to conditions i) plus iv).*

*Proof.* To prove statement 1) suppose that $\theta$ and $\tau$ are both conservative maps. Suppose now that we have *i*) and consider $\varphi' \in L'$. Then, because $\tau(\varphi') \in L$, by *i*), we have that $\tau(\varphi') \dashv\vdash \tau(\theta(\tau(\varphi')))$. By conservativeness of $\tau$ we can conclude that $\varphi' \dashv\vdash' \theta(\tau(\varphi'))$. Suppose now that condition *ii*) holds and consider $\varphi \in L$. Then, because $\theta(\varphi) \subseteq L'$, by *ii*), we have that for all $\psi \in \theta(\varphi)$, $\psi \dashv\vdash' \theta[\tau(\psi)]$. Then

$\theta(\varphi) \dashv\vdash' \theta[\tau[\theta(\varphi)]]$. By conservativeness of $\theta$ we can conclude that $\varphi \dashv\vdash \tau[\theta(\varphi)]$.

We now prove statement 2). Suppose that conditions *ii*) and *iii*) hold, that is, $\theta$ is conservative and for all $\varphi \in L$ we have that $\varphi \dashv\vdash \tau[\theta(\varphi)]$. We aim to prove that condition *i*) and *iv*) also hold, that is, $\tau$ is conservative and for all $\varphi' \in L'$ we have $\varphi' \dashv\vdash' \theta[\tau(\varphi')]$. First we prove that $\tau$ is conservative, that is, for all $\Psi' \cup \{\varphi'\} \subseteq L'$ we have that $\Psi' \vdash' \varphi'$ iff $\tau[\Psi'] \vdash \tau(\varphi')$. Suppose first that $\Psi' \vdash' \varphi'$. By *ii*) we have $\theta[\tau[\Psi']] \vdash' \theta[\tau(\varphi')]$ and by the conservativeness of $\theta$ we get that $\tau[\Psi'] \vdash \tau(\varphi')$. Suppose now that $\tau[\Psi'] \vdash \tau(\varphi')$. Because $\theta$ is conservative we get that $\theta[\tau[\Psi']] \vdash' \theta[\tau(\varphi')]$. Using 1) we get that $\Psi' \vdash' \varphi'$. Now that we have proved that $\tau$ is also conservative and condition *ii*) holds, we can use statement 2) to conclude that condition *i*) also holds.

The inverse direction, that condition *i*) and *iv*) jointly imply conditions *ii*) and *iii*), is proved with a symmetric argument.

□

There are some proposals in the literature for the notion of equivalence between logics [CG05, Pol98, PU03]. It is very interesting to note that our notion of strong representation can be seen as an abstraction that captures the common part of these proposals. The main difference between existing notions of equivalence is in the way the maps take into account an existing structure in the formulas.

Given a map $\theta : \mathcal{L} \to \mathcal{L}'$ we can extend it to theories thus obtaining a function between complete partial orders $\theta^{Th} : Th_{\mathcal{L}} \to Th_{\mathcal{L}'}$, defined by $\theta^{Th}(\Psi) = \theta[\Psi]^{\vdash'}$. We can also consider the function between complete partial orders $\theta^{-1} : Th_{\mathcal{L}'} \to Th_{\mathcal{L}}$ defined by $\theta^{-1}(\Psi) = \{\varphi \in L : \theta(\varphi) \subseteq \Psi\}$. Let us prove that $\theta^{-1}$ is indeed well-defined, that is, that $\theta^{-1}(\Psi)$ is a $\mathcal{L}$-theory for all $\mathcal{L}'$-theory $\Psi$. Suppose that $\theta^{-1}(\Psi) \vdash \varphi$. We want to see that $\varphi \in \theta^{-1}(\Psi)$. Since $\theta$ is a map we have that $\theta[\theta^{-1}(\Psi)] \vdash' \theta(\varphi)$. Since $\theta[\theta^{-1}(\Psi)] \subseteq \Psi$, by monotonicity we get that $\Psi \vdash' \theta(\varphi)$. Then $\theta(\varphi) \subseteq \Psi$ since $\Psi$ is a $\mathcal{L}'$-theory. By definition of $\theta^{-1}(\Psi)$ we have that $\varphi \in \theta^{-1}(\Psi)$.

When a function is extended naturally from a set to the powerset of that set, the extended function is always monotone.

**Proposition 2.1.18.** *Suppose* $\theta : \mathcal{L} \to \mathcal{L}'$ *is a map. Then both* $\theta^{Th} : Th_{\mathcal{L}} \to Th_{\mathcal{L}'}$ *and* $\theta^{-1} : Th_{\mathcal{L}'} \to Th_{\mathcal{L}}$ *are monotone.*

*Proof.* Let $T, S \in Th_{\mathcal{L}}$ such that $T \subseteq S$. Let us first prove that $\theta^{Th}(T) \subseteq \theta^{Th}(S)$. Clearly, $\theta[T] \subseteq \theta[S]$, so $\theta^{Th}(T) = (\theta[T])^{\vdash'} \subseteq (\theta[S])^{\vdash'} = \theta^{Th}(S)$.

We now want to show that $\theta^{-1}[T_1] \subseteq \theta^{-1}[T_2]$. Let $\varphi \in \theta^{-1}[T_1]$, that is, $\varphi \in \bigcup_{\Phi_1 \in T_1} \theta^{-1}[\Phi_1]$. Then, since $T_1 \subseteq T_2$, we can conclude that $\varphi \in \bigcup_{\Phi_2 \in T_2} \theta^{-1}[\Phi_2]$, that is, $\varphi \in \theta^{-1}[T_2]$.

□

The following proposition shows that, although not precisely the inverse of each other, $\theta^{Th}$ and $\theta^{-1}$ have a strong relation between them.

**Proposition 2.1.19.** *Suppose $\theta : \mathcal{L} \to \mathcal{L}'$ is a map. Then, for every $\mathcal{L}$-theories $\Psi$ and $\Phi$ we have that:*

$$\theta^{-1}(\theta^{Th}(\Psi)) \supseteq \Psi \qquad \textit{and} \qquad \theta^{Th}(\theta^{-1}(\Phi)) \subseteq \Phi$$

*Proof.* We first prove that $\theta^{-1}(\theta^{Th}(\Psi)) \supseteq \Psi$.

$$\begin{aligned} \theta^{-1}(\theta^{Th}(\Psi)) &= \{\varphi \in L : \theta(\varphi) \subseteq \theta^{Th}(\Psi)\} \\ &= \{\varphi \in L : \theta(\varphi) \subseteq \theta[\Psi]^{\vdash'}\} \\ &= \{\varphi \in L : \theta[\Psi] \vdash' \theta(\varphi)\}. \end{aligned}$$

As an immediate consequence we have that $\theta^{-1}(\theta^{Th}(\Psi)) \supseteq \Psi$.

Let us now prove that $\theta^{Th}(\theta^{-1}(\Phi)) \subseteq \Phi$.

$$\begin{aligned} \theta^{Th}(\theta^{-1}(\Phi)) &= \theta^{Th}(\{\varphi \in L : \theta(\varphi) \subseteq \Phi\}) \\ &= (\textstyle\bigcup_{\varphi \in \theta^{-1}(\Phi)} \theta(\varphi))^{\vdash'} \\ &\subseteq \Phi^{\vdash'} \\ &= \Phi. \end{aligned}$$

□

If in Proposition 2.1.19 we assume furthermore that $\theta : \mathcal{L} \to \mathcal{L}'$ is conservative then we have that $\theta^{-1}$ and $\theta^{Th}$ are precisely inverse of each other.

**Proposition 2.1.20.** *Suppose $\theta : \mathcal{L} \to \mathcal{L}'$ is a conservative map. Then, for every $\mathcal{L}$-theory, we have that:*

$$\theta^{-1}(\theta^{Th}(\Psi)) = \Psi.$$

*Proof.* Let us prove the two inclusions. The inclusion $\theta^{-1}(\theta^{Th}(\Psi)) \supseteq \Psi$ is an immediate consequence of Proposition 2.1.20. To prove that $\theta^{-1}(\theta^{Th}(\Psi)) \subseteq \Psi$ consider

$\varphi \in \theta^{-1}(\theta^{Th}(\Psi))$. As we have already saw $\theta^{-1}(\theta^{Th}(\Psi)) = \{\varphi \in L : \theta[\Psi] \vdash' \theta(\varphi)\}$, so $\theta[\Psi] \vdash' \theta(\varphi)$. Then, by the conservativeness of $\theta$, we get that $\Psi \vdash \varphi$. Since $\Psi$ is a theory, we have that $\varphi \in \Psi$.

□

Given a logic $\mathcal{L}$, the theory space of $\mathcal{L}$, $Th_{\mathcal{L}}$, is a good measure of the expressive power of a logic. Indeed, given the theory space of a logic $\mathcal{L}$ we can recover its consequence operator, by defining, for every $\Psi \subseteq L$

$$\Psi^{\vdash} = \bigwedge \{\Phi \in Th_{\mathcal{L}} : \Psi \subseteq \Phi\}.$$

Recall that the existence of a strong representation $(\theta, \tau)$ between two logics $\mathcal{L}$ and $\mathcal{L}'$ can be seen as some sort of equivalence between $\mathcal{L}$ and $\mathcal{L}'$. In the sequel we prove that, in fact, the existence of a strong representation $(\theta, \tau)$ between two logics $\mathcal{L}$ and $\mathcal{L}'$ implies that the correspondent theory spaces are isomorphic as complete lattices.

Let us start by proving that, in this case, $\theta^{Th}$ is a map between complete partial orders, that is, it preserves supremums.

**Proposition 2.1.21.** *Suppose $(\theta, \tau)$ is a strong representation between the logics $\mathcal{L}$ and $\mathcal{L}'$. Then $\theta^{Th} : Th_{\mathcal{L}} \to Th_{\mathcal{L}'}$ is sup-preserving, that is, for every $T \subseteq Th_{\mathcal{L}}$ we have that:*

$$\theta^{Th}(\bigvee^{\mathcal{L}} T) = \bigvee^{\mathcal{L}'} \theta^{Th}[T]$$

*Proof.* Note that, using the definition of supremum on $Th_{\mathcal{L}}$ and the definition of $\theta^{Th}$, all we need to prove is that $(\theta[(\bigcup_{\Psi \in T} \Psi)^{\vdash}])^{\vdash'} = (\bigcup_{\Psi \in T} \theta[\Psi]^{\vdash'})^{\vdash'}$. So, let us prove the two inclusions.

First let $\varphi \in (\theta[(\bigcup_{\Psi \in T} \Psi)^{\vdash}])^{\vdash'}$, that is, $(\theta[(\bigcup_{\Psi \in T} \Psi)^{\vdash}]) \vdash' \varphi$. Since $(\theta, \tau)$ is a strong representation, we have that $(\theta[(\bigcup_{\Psi \in T} \Psi)^{\vdash}]) \vdash' \theta[\tau(\varphi)]$. Then, by the conservativeness of $\theta$, we have that $\bigcup_{\Psi \in T} \Psi \vdash \tau(\varphi)$. From this it follows that $\bigcup_{\Psi \in T} \theta[\Psi] \vdash' \varphi$. Since $\bigcup_{\Psi \in T} \theta[\Psi] \subseteq \bigcup_{\Psi \in T} \theta[\Psi]^{\vdash'}$ and using the monotonicity of $\vdash'$, we get that $\bigcup_{\Psi \in T} \theta[\Psi]^{\vdash'} \vdash' \varphi$, that is, $\varphi \in (\bigcup_{\Psi \in T} \theta[\Psi]^{\vdash'})^{\vdash'}$ as intended.

Suppose now that $\varphi \in (\bigcup_{\Psi \in T} \theta[\Psi]^{\vdash'})^{\vdash'}$, that is, $\bigcup_{\Psi \in T} \theta[\Psi]^{\vdash'} \vdash' \varphi$. Since clearly $\bigcup_{\Psi \in T} \theta[\Psi]^{\vdash'} \subseteq (\bigcup_{\Psi \in T} \theta[\Psi])^{\vdash'}$, we have, by the monotonicity of $\vdash'$, that $\bigcup_{\Psi \in T} \theta[\Psi] \vdash' \varphi$. From this, and by the fact that $(\theta, \tau)$ is a strong representation, we can conclude that $\bigcup_{\Psi \in T} \Psi \vdash \tau(\varphi)$. Using again the monotonicity of $\vdash'$, we have that $(\bigcup_{\Psi \in T} \Psi)^{\vdash} \vdash \tau(\varphi)$. Finally, we can conclude that $\theta[(\bigcup_{\Psi \in T} \Psi)^{\vdash}] \vdash' \varphi$, that is, $\varphi \in (\theta[(\bigcup_{\Psi \in T} \Psi)^{\vdash}])^{\vdash'}$ as intended.

□

**Proposition 2.1.22.** *Suppose $(\theta, \tau)$ is a strong representation of the logic $\mathcal{L}$ in the logic $\mathcal{L}'$. Then we have that:*

$$\theta^{-1} = \tau^{Th} \qquad \textit{and} \qquad \theta^{Th} = \tau^{-1}$$

*Proof.* Since $(\theta, \tau)$ is a strong representation, then $\theta$ and $\tau$ are both conservative maps and for all $\varphi \in L$ we have that $\varphi \dashv\vdash \tau[\theta(\varphi)]$, and for all $\varphi' \in L'$ we have that $\varphi' \dashv\vdash' \theta[\tau(\varphi')]$. First let us prove that $\theta^{-1} = \tau^{Th}$. For this we prove that, for all $\mathcal{L}'$-theory $\Phi$, $\theta^{-1}(\Phi) \subseteq \tau^{Th}(\Phi)$ and $\theta^{-1}(\Phi) \supseteq \tau^{Th}(\Phi)$. Let $\varphi \in \theta^{-1}(\Phi)$, then $\theta(\varphi) \subseteq \Phi$. By reflexivity we get that $\Phi \vdash' \theta(\varphi)$ and, since $\tau$ is a map, we get that $\tau[\Phi] \vdash \tau[\theta(\varphi)]$. By hypothesis we know that $\tau[\theta(\varphi)] \vdash \varphi$ and then by cut we have that $\tau[\Phi] \vdash \varphi$, that is, $\varphi \in \tau[\Phi]^{\vdash}$ which means that $\varphi \in \tau^{Th}(\Phi)$. We then conclude that $\theta^{-1}(\Phi) \subseteq \tau^{Th}(\Phi)$.

Let $\varphi \in \tau^{Th}(\Phi)$, that is, $\varphi \in \tau[\Phi]^{\vdash}$. Then $\tau[\Phi] \vdash \varphi$. By hypothesis $\varphi \vdash \tau[\theta(\varphi)]$. Then, using cut, we have that $\tau[\Phi] \vdash \tau[\theta(\varphi)]$. Using the conservativeness of $\tau$, we have $\Phi \vdash' \theta(\varphi)$. Since $\Phi$ is a theory, we conclude that $\theta(\varphi) \subseteq \Phi$, that is, $\varphi \in \theta^{-1}(\Phi)$.

The proof of the equality $\theta^{Th} = \tau^{-1}$ is analogue.

□

As an immediate corollary we have that the theory spaces of two logics connected by a strong representation are isomorphic.

**Corollary 2.1.23.** *Suppose $(\theta, \tau)$ is a strong representation of the logic $\mathcal{L}$ in the logic $\mathcal{L}'$. Then $\theta^{Th}$ is a bijection between the corresponding theory spaces.*

#### 2.1.1.3 Many-sorted signatures

The standard tools of AAL encompass only logics whose language can be built from a propositional signature and further satisfy the usual structurality condition. Propositional logics are, nevertheless, not expressive enough when one wants to reason about complex systems. To this purpose we need logics over richer languages where it is possible to distinguish elements by *sorts*. A paradigmatic example is First-order Logic (FOL) where there is a clear syntactic distinction between formulas and terms.

In our work we focus our attention on a wider class of logics than just the propositional based logics: those logics whose language can be built from a many-sorted signature.

**Definition 2.1.24.** A *many-sorted signature* is a pair $\Sigma = \langle S, F\rangle$ where $S$ is a set (of *sorts*) and $F = \{F_{ws}\}_{w\in S^*, s\in S}$ is an indexed family of sets (of *operations*).

We say that a many-sorted signature $\Sigma = \langle S, F\rangle$ is $n$-sorted if $n = |S|$. For simplicity, we write $f : s_1 \dots s_n \to s \in F$ for an element $f$ of $F_{s_1\dots s_n s}$. As usual, we denote by $T_\Sigma(X) = \{T_{\Sigma,s}(X)\}_{s\in S}$ the $S$-indexed family of carrier sets of the free $\Sigma$-algebra $\mathbf{T_\Sigma(X)}$ with generators taken from a sorted family $X = \{X_s\}_{s\in S}$ of variable sets. We denote by $x{:}s$ the fact that $x \in X_s$. An element of $T_{\Sigma,s}(X)$ is called a *term of sort $s$* or just a *$s$-term*. A term without variables is called a *closed term*.

A many-sorted signature $\Sigma = \langle S, F\rangle$ is called *standard* if, for every $s \in S$, there exists a closed $s$-term. Often we will need to write terms over a finite set of variables $t \in T_\Sigma(x_1 : s_1, \dots, x_n : s_n)$. For simplicity, we denote such a term by $t(x_1 : s_1, \dots, x_n : s_n)$. Moreover, if $T$ is a set whose elements are all terms of this form, we write $T(x_1{:}s_1, \dots, x_n{:}s_n)$.

In the sequel we consider fixed a sorted set $X$ of variables.

**Example 2.1.25** (FOL)**.** *The language of First-order Logic (FOL) can be obtained from a two-sorted signature. The language is divided in the sort of terms and the sort of formulas. The predicates can be viewed as operations that transform terms in formulas and the connectives as operations on formulas. The usual interpretation structures can be viewed as algebras over this two-sorted signature. In this two-sorted signature we have a set to interpret terms, the domain of interpretation of the structure, a set to interpret formulas, the set* $\{0, 1\}$*, the predicates, usually interpreted as relations, are functions from the domain of interpretation of the structure to* $\{0, 1\}$ *and the connectives with their usual interpretation as functions over* $\{0, 1\}$*.*

**Definition 2.1.26.** Given a many-sorted signature $\Sigma = \langle S, O\rangle$, a *(many-sorted) substitution over* $\Sigma$ is a $S$-indexed family of functions $\sigma = \{\sigma_s : X_s \to T_{\Sigma,s}(X)\}_{s\in S}$. As usual, $\sigma(t)$ denotes the term obtained by uniformly applying $\sigma$ to each variable in $t$.

Given a term $t(x_1 : s_1, \dots, x_n : s_n)$ and terms $t_1 \in T_{\Sigma,s_1}(X), \dots, t_n \in T_{\Sigma,s_n}(X)$, we write $t(t_1, \dots, t_n)$ to denote the term $\sigma(t)$ where $\sigma$ is a substitution such that $\sigma_{s_1}(x_1) = t_1, \dots, \sigma_{s_n}(x_n) = t_n$. Extending everything to sets, given $T(x_1{:}s_1, \dots, x_n{:}s_n)$ and $U \in T_{\Sigma,s_1}(X) \times \dots \times T_{\Sigma,s_n}(X)$, we use $T[U] = \bigcup_{\langle t_1,\dots,t_n\rangle\in U} T(t_1, \dots, t_n)$.

We can compose the original operations of the signature thus obtaining *derived operations*. A *derived operation of type* $s_1 \dots s_n \to s$ over $\Sigma$ is a term in

$T_{\Sigma,s}(\{x_1 : s_1, \ldots, x_n : s_n\})$ for some $n$. We denote by $Der_{\Sigma,s_1\ldots s_n s}$ the set of all derived operations of type $s_1 \ldots s_n \to s$ over $\Sigma$.

**Definition 2.1.27.** A *(general many-sorted) subsignature* of $\Sigma = \langle S, F\rangle$ is a many-sorted signature $\Gamma = \langle S, F'\rangle$ such that, for each $w \in S^*$, $F'_w \subseteq Der_{\Sigma,w}$.

#### 2.1.1.4 Many-sorted logics

The idea of a many-sorted logic is that its language is built from a many-sorted signature. Of course, although we have many syntactic sorts, we can only reason about formulas. In the sequel we assume fixed a signature $\Sigma = \langle S, F\rangle$ with a distinguished sort $\phi$ (the syntactic sort of formulas) and a $S$-sorted set $X$ of variables. Moreover, for the sake of notation, we assume that $X_\phi = \{\xi_i \mid i \in \mathbb{N}\}$ and simply write $\xi_k$ instead of $\xi_k : \phi$. Whenever $\Gamma$ is a subsignature of $\Sigma$, we say that $\Sigma$ is $\Gamma$-*standard* if, for every $s \in S$, there exists a closed $\Gamma$-term of sort $s$, that is, a $\Gamma$-term of sort $s$ without variables. We define the induced set of *formulas* $L_\Sigma(X)$ to be the carrier set of sort $\phi$ of the free algebra $\mathbf{T_\Sigma(X)}$ with generators taken from $X$.

We are now ready to introduce the class of logics that we study in our many-sorted behavioral approach.

**Definition 2.1.28.** A *(structural) many-sorted logic* is a tuple $\mathcal{L} = \langle \Sigma, \vdash\rangle$ where $\Sigma$ is a many-sorted signature and $\vdash \subseteq \mathcal{P}(L_\Sigma(X)) \times L_\Sigma(X)$, such that $\langle L_\Sigma(X), \vdash\rangle$ is a logic that satisfies, for every $T \cup \{\varphi\} \subseteq L_\Sigma(X)$ and every substitution $\sigma$:

***structurality:*** if $T \vdash \varphi$ then $\sigma[T] \vdash \sigma(\varphi)$.

An important remark to make here is that propositional-like logics appear as a particular case of many-sorted logics. They can be obtained by taking $\phi$ to be the only sort of the signature, that is, considering a signature $\Sigma = \langle S, F\rangle$ such that $S = \{\phi\}$ and are called *single-sorted logics*. So, we conclude that, at least from the point of view of scope, our theory is a generalization of the standard tools of the theory of algebraization. In fact, as we will see later on, our work is indeed a generalization of the standard tools of AAL, in the sense that, for the particular case of propositional logics, our non-behavioral definitions and results coincide with standard ones.

When we are dealing with many-sorted logics we can particularize the notion of deductive system, now using schematic rules. Again, this is particularly useful for introducing a logic.

**Definition 2.1.29.** A *structural deductive system* is a pair $\mathcal{D} = \langle \Sigma, R \rangle$ where $\Sigma$ is a many-sorted signature, and $R$ is a subset of $(\wp_{fin} L_\Sigma(X)) \times L_\Sigma(X)$.

**Definition 2.1.30.** Given a structural deductive system $\mathcal{D} = \langle \Sigma, R \rangle$, the *consequence relation associated with* $\mathcal{D}$, denoted by $\vdash_\mathcal{D} \subseteq 2^{L_\Sigma(X)} \times L_\Sigma(X)$, is the relation associated with the deductive system

$$\mathcal{D} = \langle L_\Sigma(X), \{\sigma(r) : r \in R \text{ and } \sigma \text{ is a substitution}\} \rangle.$$

The logic associated with $\mathcal{D}$ is $\mathcal{L}_\mathcal{D} = \langle L_\Sigma(X), \vdash_\mathcal{D} \rangle$. Note that the logic $\mathcal{L}_\mathcal{D}$ is always finitary since all rules have finite sets of premises.

#### 2.1.1.5 Examples

We now present some examples of many-sorted logics. As expected, they are introduced through the correspondent deductive system. The first two examples are single-sorted (propositional) logics, whereas the last one is an example of a truly many-sorted logic, with more than one sort.

**Example 2.1.31.** Classical propositional logic CPL

In this example we present the so-called classical propositional logic. Besides presenting a logic with an undeniable importance, the purpose of this example is also to introduce the reader to our notation in a well-known setting. This will help the reader not acquainted with these subjects.

The language of CPL is obtained from the single-sorted signature $\Sigma_{CPL} = \langle S, F \rangle$ such that:

- $S = \{\phi\}$
- $F_{\epsilon\phi} = \emptyset$
- $F_{\phi\phi} = \{\neg\}$
- $F_{\phi^2\phi} = \{\Rightarrow\}$

- $F_{\phi^n \phi} = \emptyset$, for all $n > 2$

As usual, the Boolean binary connectives $\wedge$ and $\vee$ can be introduced as derived connectives.

The consequence relation of CPL can be given by the structural deductive system composed of the following axioms:

A1) $\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_1)$;

A2) $((\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_3)) \Rightarrow ((\xi_1 \Rightarrow \xi_2) \Rightarrow (\xi_1 \Rightarrow \xi_3)))$;

A3) $((\neg\xi_1 \Rightarrow \neg\xi_2) \Rightarrow (\xi_2 \Rightarrow \xi_1))$;

and the rule of *modus ponens*:

(MP) $\xi_1, \xi_1 \Rightarrow \xi_2 \vdash \xi_2$

■

**Example 2.1.32.** Paraconsistent logic $\mathcal{C}_1$ (da Costa, 1963)

The language of $\mathcal{C}_1$ is generated by the single-sorted signature $\Sigma_{\mathcal{C}_1} = \langle S, F \rangle$ with unique sort $S = \{\phi\}$ and with the following operations:

- $\mathbf{t}, \mathbf{f} :\to \phi$
- $\neg : \phi \to \phi$
- $\wedge, \vee, \Rightarrow : \phi^2 \to \phi$

The consequence relation of $\mathcal{C}_1$ can be given by the structural deductive system composed of the following axioms:

A1) $\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_1)$

A2) $(\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_3)) \Rightarrow ((\xi_1 \Rightarrow \xi_2) \Rightarrow (\xi_1 \Rightarrow \xi_3))$

A3) $(\xi_1 \wedge \xi_2) \Rightarrow \xi_1$

A4) $(\xi_1 \wedge \xi_2) \Rightarrow \xi_2$

A5) $\xi_1 \Rightarrow (\xi_2 \Rightarrow (\xi_1 \wedge \xi_2))$

A6) $\xi_1 \Rightarrow (\xi_1 \vee \xi_2)$

A7) $\xi_2 \Rightarrow (\xi_1 \vee \xi_2)$

A8) $(\xi_1 \Rightarrow \xi_3) \Rightarrow ((\xi_2 \Rightarrow \xi_3) \Rightarrow ((\xi_1 \vee \xi_2) \Rightarrow \xi_3))$

A9) $\neg\neg\xi_1 \Rightarrow \xi_1$

A10) $\xi_1 \vee \neg\xi_1$

A11) $\xi_1^\circ \Rightarrow (\xi_1 \Rightarrow (\neg\xi_1 \Rightarrow \xi_2))$

A12) $(\xi_1^\circ \wedge \xi_2^\circ) \Rightarrow (\xi_1 \wedge \xi_2)^\circ$

A13) $(\xi_1^\circ \wedge \xi_2^\circ) \Rightarrow (\xi_1 \vee \xi_2)^\circ$

A14) $(\xi_1^\circ \wedge \xi_2^\circ) \Rightarrow (\xi_1 \Rightarrow \xi_2)^\circ$

A15) $\mathbf{t} \equiv (\xi_1 \Rightarrow \xi_1)$

A16) $\mathbf{f} \equiv (\xi_1^\circ \wedge (\xi_1 \wedge \neg\xi_1))$

and the rule of *modus ponens*:

(MP) $\xi_1, \xi_1 \Rightarrow \xi_2 \vdash \xi_2$

where $\varphi^\circ$ is an abbreviation of $\neg(\varphi \wedge \neg\varphi)$ and $\varphi \equiv \psi$ is an abbreviation of $(\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi)$.

■

As a last example we briefly introduce *classical global logic*. This is the first example of a logic whose language is built from a many-sorted signature with more than one sort.

**Example 2.1.33** (Classical global logic)**.**

Classical global logic (GL) was introduced in [MSS05] in the context of the exogenous approach for enriching a logic. The aim of this exogenous approach is to start from a base logic and, on top of it, build a logic for reason about sets of models of the base logic. In [MSS05] this is used as a first step in the direction of a quantum logic, that is, a logic for reasoning about quantum systems.

In this example we present the logic obtained from exogenously enriching classical propositional logic CPL using this globalization mechanism. The language is divided in two sorts, one for representing the formulas of the base logic and the other for representing the formulas of the resulting global logic. More precisely, the language of GL is obtained from the two-sorted signature $\Sigma_{GL} = \langle S, F\rangle$ such that $S = \{\phi, l\}$ and the with the following operations:

- $\neg : l \to l$;
- $\Rightarrow : l^2 \to l$;
- $\Box : l \to \phi$;
- $\top, \bot :\to \phi$
- $\boxminus : \phi \to \phi$
- $\sqsupset : \phi^2 \to \phi$

The Boolean connectives $\wedge, \vee$ (local conjunction and disjunction) and $\sqcap, \sqcup$ (global conjunction and disjunction) can be introduced as derived operations as usual from $\neg, \Rightarrow$ (local negation and implication) and $\boxminus, \sqsupset$ (global negation and implication), respectively.

The sort $l$ is the the sort of local formulas and $\phi$ is the sort of global formulas. Of course, we can only reason directly about terms of sort $\phi$, that is, about global formulas.

Let $\mathcal{V}$ be the set of all models of the base logic. Since the base logic is CPL, $\mathcal{V}$ is the class of all classical valuations, that is, functions $v : X_l \to \{0,1\}$ satisfying the usual classical conditions.

The key idea of global logic is to take sets of models of the local logic as models of the global logic. In this example, the models will be sets of classical valuations. We define the satisfaction of a global formula $\delta$ by the global model $V \subseteq \mathcal{V}$, denoted by $V \Vdash_g \delta$, inductively as follows:

- $V \Vdash_g \Box\varphi$ iff for every $v \in V$, we have that $v \Vdash \varphi$;
- $V \Vdash_g \boxminus\delta$ iff $V \nVdash_g \delta$;
- $V \Vdash_g \delta_1 \sqsupset \delta_2$ iff $V \nVdash_g \delta_1$ or $V \Vdash_g \delta_2$.

The consequence relation of global logic, denoted by $\vDash_g$, is defined as follows: for every $\Phi \cup \{\delta\} \subseteq L_{\Sigma_{GL}}(X)$, we have that $\Phi \vDash_g \delta$, if, for every global model $V$,

$$V \Vdash_g \delta \text{ whenever } V \Vdash_g \gamma \text{ for every } \gamma \in \Phi.$$

GL is a good example of a logic that is semantically defined, thus showing that a logic in our sense does not necessarily need to be introduced as a deductive system.

■

### 2.1.2 Algebra and equational logic

In this section we introduce some preliminary notation and notions concerning algebra. Following our many-sorted approach we focus on the many-sorted version of universal algebra.

#### 2.1.2.1 Sets, functions and relations

Given a set $A$, we let $A^*$ denote the set of finite strings with elements in $A$. Given two sets, $A$ and $B$, we let $[A \to B]$ denote the set of functions of source $A$ and target $B$.

If $f : A \to B$ is a function and $w \in A^*$ is the list $a_1 a_2 \ldots a_n$, then $f(w)$ is the list $f(a_1)f(a_2)\ldots f(a_n) \in B^*$. If $C \subseteq A$ then $f_{|C} : C \to B$ is the *restriction* of $f$ to $C$. If $f : A \to B$ and $g : B \to C$ are two functions, then we let $f \circ g : A \to C$ denote their *composition*. For those familiar with category theory, we let *Set* denote the category of sets and functions.

If $S$ is a set, $A = \{A_s : s \in S\}$ is a $S$-sorted, or $S$-indexed set. Given $w \in S^*$ such that $w = s_1 s_2 \ldots s_n$, then $A_w$ denotes the product $A_{s_1} \times A_{s_2} \times \ldots \times A_{s_n}$. Given $s \in S$ and $n \in \mathbb{N}$ we write $A_s^n$ to denote the product of $A_s$ by itself $n$ times.

If $A$ is a $S$-sorted set, then a *$S$-sorted $n$-ary relation* $R$ on $A$ is a $S$-sorted set of relations $\{R_s \subseteq A_s^n : s \in S\}$. We write simply $R \subseteq A^n$ to denote the fact that $R$ is a $n$-ary relation on $A$. A binary relation on $A$, $R \subseteq A^2$, is a *$S$-sorted equivalence* if each $R_s$ is an equivalence.

#### 2.1.2.2 Many-sorted algebras

The fundamental concept in universal algebra is the notion of algebra. Herein we concentrate on the many-sorted version of algebra.

**Definition 2.1.34.** Given a many-sorted signature $\Sigma = \langle S, F\rangle$, a $\Sigma$-*algebra* is a pair

$$\mathbf{A} = \langle \{A_s\}_{s\in S}, \_{\mathbf{A}}\rangle$$

where each $A_s$ is a non-empty set, the *carrier of sort* $s$, and $\_{\mathbf{A}}$ assigns to each operation $f : s_1 \ldots s_n \to s$ of $\Sigma$ a function

$$\underline{f}_{\mathbf{A}} : A_{s_1} \times \ldots \times A_{s_n} \to A_s.$$

The set of all $\Sigma$-algebras is denoted by $Alg_{\Sigma}$. When the signature $\Sigma$ is clear from the context we just write algebra instead of $\Sigma$-algebra.

Recall that we are assuming that $A_s \neq \emptyset$ for all $s \in S$. This assumption is usually assumed in the literature and it makes the metamathematics simpler since most results of many-sorted universal algebra hold in their usual form. Moreover, the assumption holds automatically if $\Sigma$ is a standard signature.

A $\Sigma$-algebra is *trivial* if each of its carriers contains exactly one element. We denote algebras by bold face roman letters and their universes by the corresponding italic letters. Let $\mathbf{A}$ be a $\Sigma$-algebra. A sorted subset $B$ of $A$ is a *subuniverse of* $\mathbf{A}$ if it is closed under the operations of $\mathbf{A}$. More precisely, for each operation $f : s_1 \ldots s_n \to s$ and $b_1 \in B_{s_1}, \ldots, b_n \in B_{s_n}$, we have that $\underline{f}_{\mathbf{A}}(b_1, \ldots, b_n) \in B_s$.

**Definition 2.1.35.** A $\Sigma$-algebra $\mathbf{B}$ is *a subalgebra of* $\mathbf{A}$, in symbols $\mathbf{B} \subseteq \mathbf{A}$, if $B$ is a non-empty subuniverse of $\mathbf{A}$ and for each operation $f : s_1 \ldots s_n \to s$ and $b_1 \in B_{s_1}, \ldots, b_n \in B_{s_n}$ we have that

$$\underline{f}_{\mathbf{B}}(b_1, \ldots, b_n) = \underline{f}_{\mathbf{A}}(b_1, \ldots, b_n).$$

It is easy to check that the intersection of any set of subuniverses of a $\Sigma$-algebra $\mathbf{A}$ is also a subuniverse of $\mathbf{A}$. Given a sorted subset $B$ of $A$, the subuniverse of $\mathbf{A}$ generated by $B$ is the intersection of all subuniverses of $\mathbf{A}$ containing $B$. If the intersection is non-empty, in particular if $B$ is non-empty, then the corresponding subalgebra of $\mathbf{A}$ is called the subalgebra generated by $B$ and we denote it by $\langle B\rangle_{\mathbf{A}}$. If $K$ is a class of $\Sigma$-algebras we say that $K$ is closed under subalgebras if whenever $\mathbf{A} \in K$ then, for every subalgebra $\mathbf{B}$ of $\mathbf{A}$, we have that $\mathbf{B} \in K$.

**Definition 2.1.36.** A *homomorphism* $h : \mathbf{A} \to \mathbf{B}$ from the $\Sigma$-algebra $\mathbf{A}$ to the $\Sigma$-algebra $\mathbf{B}$ is a $S$-sorted set $\{h_s : A_s \to B_s\}_{s\in S}$, such that for all $f \in F_{s_1\ldots s_n s}$, we have that

$$h_s(\underline{f}_A(a_1,\ldots,a_n)) = \underline{f}_B(h_{s_1}(a_1),\ldots,h_{s_n}(a_n)).$$

The set of all homomorphisms from $\mathbf{A}$ to $\mathbf{B}$ is denoted by $Hom(\mathbf{A},\mathbf{B})$. If a homomorphism $h : \mathbf{A} \to \mathbf{B}$ is surjective then $\mathbf{B}$ is said to be a *homomorphic image* of $\mathbf{A}$, and $h$ is called an *epimorphism.* A class $K$ of $\Sigma$-algebras is said to be closed under homomorphic images if whenever $\mathbf{A} \in K$ then $\mathbf{B} \in K$ for every $\mathbf{B}$ homomorphic image of $\mathbf{A}$.

By an *embedding* we mean an injective homomorphism. A homomorphism that is both injective and surjective is called an *isomorphism.* If there exists an isomorphism $h : \mathbf{A} \to \mathbf{B}$, then $\mathbf{A}$ and $\mathbf{B}$ are said to be *isomorphic.* A class $K$ of $\Sigma$-algebras is said to be closed under isomorphisms if whenever $\mathbf{A} \in K$ then $\mathbf{B} \in K$ for every $\mathbf{B}$ isomorphic to $\mathbf{A}$.

**Definition 2.1.37.** A *congruence* on a $\Sigma$-algebra $\mathbf{A}$ is a $S$-sorted set $\equiv\, = \{\equiv_s\}_{s\in S}$, such that, for every $s \in S$, $\equiv_s$ is an equivalence relation on $A_s$ and, for each $f : s_1 \ldots s_n \to s$, we have that

$$\text{if} \quad (a_1 \equiv_{s_1} b_1),\ldots,(a_n \equiv_{s_n} b_n) \quad \text{then} \quad \underline{f}_{\mathbf{A}}(a_1,\ldots,a_n) \equiv_s \underline{f}_{\mathbf{A}}(b_1,\ldots,b_n).$$

We denote by $Cong_{\mathbf{A}}$ the set of all congruences on the algebra $\mathbf{A}$. Given $\theta_1,\theta_2 \in Cong_{\mathbf{A}}$, we can define the operation $\circ : Cong_{\mathbf{A}} \times Cong_{\mathbf{A}} \to Cong_{\mathbf{A}}$, such that, for every sort $s \in S$, we have that $\langle a,b\rangle \in (\theta_1 \circ \theta_2)_s$ iff there exists $c \in A$ such that $\langle a,c\rangle \in (\theta_1)_s$ and $\langle c,b\rangle \in (\theta_2)_s$. The composition $r_1 \circ r_2 \circ \ldots \circ r_n$ is inductively defined by $(r_1 \circ r_2 \circ \ldots \circ r_{n-1}) \circ r_n$, as expected. It is not difficult to see that $\langle Cong_A, \subseteq\rangle$ is a complete partial order, where for $\{\theta_i\}_{i\in I} \subseteq Cong_A$, $\bigwedge\{\theta_i\}_{i\in I} = \bigcap_{i\in I}\theta_i$ and $\bigvee\{\theta_i\}_{i\in I} = \bigcup\{\theta_{i_1} \circ \theta_{i_2} \circ \ldots \circ \theta_{i_k} : i_1, i_2, \ldots, i_k \in I, k < \infty\}$ are, respectively, the infimum and the supremum.

Let $K$ be a class of $\Sigma$-algebras closed under isomorphisms. For a $\Sigma$-algebra $\mathbf{A}$, not necessarily in $K$, we define $Con_K(\mathbf{A})$ to be the set of all congruences $\theta$ on $\mathbf{A}$ such that $\mathbf{A}/_\theta \in K$. The members of $Con_K(\mathbf{A})$ are called $K$-congruences of $\mathbf{A}$.

**Definition 2.1.38.** Given an indexed family $\{\mathbf{A}_i\}_{i\in I}$ of $\Sigma$-algebras the *direct product* of this family is the $\Sigma$-algebra

$$\mathbf{\Pi}_{i\in I}\mathbf{A}_i = \langle \{\Pi_{i\in I}(A_i)_s\}_{s\in S}, \underline{\ \ }_{\mathbf{\Pi}_{i\in I}\mathbf{A}_i}\rangle$$

such that the operations are defined component wise, as usual.

If $I$ is the empty set then $\mathbf{\Pi}_{i\in I}\mathbf{A}_i$ is by definition a trivial algebra. For each $j \in I$, the *projection* $\pi_j : \mathbf{\Pi}_{i\in I}\mathbf{A}_i \to \mathbf{A}_j$ is a surjective homomorphism from $\mathbf{\Pi}_{i\in I}\mathbf{A}_i$ onto $\mathbf{A}_j$.

**Definition 2.1.39.** A $\Sigma$-algebra $\mathbf{B}$ is a *subdirect product of a family of* $\Sigma$*-algebras* $\{\mathbf{A}_i\}_{i\in I}$ if $\mathbf{B} \subseteq \mathbf{\Pi}_{i\in I}\mathbf{A}_i$ and, for each $i \in I$, we have that $\pi_i(B) = A_i$. We use $\mathbf{B} \subseteq_{SD} \mathbf{\Pi}_{i\in I}\mathbf{A}_i$ to denote a subdirect product.

An embedding $\alpha : \mathbf{A} \to \mathbf{\Pi}_{i\in I}\mathbf{A}_i$ is *subdirect* if $\alpha(\mathbf{A})$ is a subdirect product of the family $\mathbf{A}_i$, $i \in I$. A $\Sigma$-algebra $\mathbf{A}$ is *subdirectly irreducible* if for every subdirect embedding $\alpha : \mathbf{A} \to \mathbf{\Pi}_{i\in I}\mathbf{A}_i$ there is $i \in I$ such that $\pi_i \circ \alpha : \mathbf{A} \to \mathbf{A}_i$ is an isomorphism. A class $K$ of $\Sigma$-algebras is said to be closed under subdirect products if whenever $\mathbf{A}_i \in K$, for every $i \in I$, then $\mathbf{B} \in K$ for every $\mathbf{B} \subseteq_{SD} \mathbf{\Pi}_{i\in I}\mathbf{A}_i$.

**Definition 2.1.40.** Let $I$ be a non-empty set. A *filter on* $I$ is a family $\mathcal{F}$ of subsets of $I$ which satisfies the following conditions:

- $I \in \mathcal{F}$;
- if $J \in \mathcal{F}$ and $J \subseteq K \subseteq I$ then $K \in \mathcal{F}$;
- if $J, K \in \mathcal{F}$ then $J \cap K \in \mathcal{F}$.

A filter $\mathcal{F}$ that does not contain the empty set ( $\emptyset \notin \mathcal{F}$) is called a *proper* filter. A filter $\mathcal{F}$ such that there is no proper filter on $I$ strictly including $\mathcal{F}$ is called *maximal*. Using Zorn's lemma we can prove that every proper filter can be extended to a maximal filter $\mathcal{F}$. A proper filter $\mathcal{F}$ which is maximal is called an *ultrafilter*.

Sometimes it is useful to consider an equivalent characterization of ultrafilter: an ultrafilter over $I$ is a filter $\mathcal{U}$ over $I$ not containing the empty set and such that, for every $J \subseteq I$, either $J \in \mathcal{F}$ or $\overline{J} \in \mathcal{F}$. It is easy to see that it is not possible that a set and its complement both belong to an ultrafilter.

Given an indexed family of $\Sigma$-algebras $\{\mathbf{A}_i : i \in I\}$ and a filter $\mathcal{F}$ over $I$ we can define an equivalence relation $\sim_{\mathcal{F}} = \{(\sim_{\mathcal{F}})_s\}_{s\in S}$ on the direct product $\mathbf{\Pi}_{i\in I}\mathbf{A}_i$ such that, for every $a, b \in \Pi_{i\in I}(A_i)_s$, we have that:

$$a \sim_{\mathcal{F}} b \ \text{ iff } \ \{i \in I : a_i = b_i\} \in \mathcal{F}.$$

In fact, it is easy to prove that $\sim_{\mathcal{F}}$ is indeed a congruence relation on $\mathbf{\Pi}_{i\in I}\mathbf{A}_i$. So, we can consider the quotient algebra $(\mathbf{\Pi}_{i\in I}\mathbf{A}_i)/_{\sim_{\mathcal{F}}}$, called the *reduced product* of the family $\{\mathbf{A}_i : i \in I\}$ by the filter $\mathcal{F}$. When $\mathcal{U}$ is an ultrafilter over $I$ then $(\mathbf{\Pi}_{i\in I}\mathbf{A}_i)/_{\sim_{\mathcal{U}}}$ is called the *ultraproduct* of $\{\mathbf{A}_i : i \in I\}$ by $\mathcal{U}$.

Consider given a subsignature $\Gamma$ of $\Sigma$ and a $\Sigma$-algebra $\mathbf{A} = \langle (A_s)_{s\in S}, \_\mathbf{A}\rangle$. Then, the *reduct of* $\mathbf{A}$ *to* $\Gamma$, denoted by $\mathbf{A}_{|\Gamma}$, is a $\Gamma$-algebra $\mathbf{A}_{|\Gamma} = \langle (A_s)_{s\in S}, \_{\mathbf{A}_{|\Gamma}}\rangle$ where $\_{\mathbf{A}_{|\Gamma}}$ is just the restriction of $\_\mathbf{A}$ to the operations of $\Gamma$.

**Definition 2.1.41.** An *assignment* of $X$ over a $\Sigma$-algebra $\mathbf{A}$ is a family $h = \{h_s\}_{s\in S}$ such that, for every $s \in S$, $h_s : X_s \to A_s$.

Given an assignment $h$ of $X$ over $\mathbf{A}$, the *denotation or interpretation of terms* is just the free extension of $h$ to $\mathbf{T_\Sigma(X)}$, that we also denote by $h$. Since $\mathbf{T_\Sigma(X)}$ is the free $\Sigma$-algebra over $X$ and given $t \in T_{\Sigma,s}(X)$ for some $s \in S$, we have that $h$ depends only on the value it assigns to the variables occurring in $t$. So, an interpretation is nothing but a homomorphism $h : \mathbf{T_\Sigma(X)} \to \mathbf{A}$.

Given a term $t(x_1{:}s_1,\ldots,x_n{:}s_n)$ and $\langle a_1,\ldots,a_n\rangle \in A_{s_1}\times\ldots\times A_{s_n}$, then we denote by $t_\mathbf{A}(a_1,\ldots,a_n)$ the the denotation of $t$ in $\mathbf{A}$ when the variables $x_1,\ldots,x_n$ are interpreted by $a_1,\ldots,a_n$, respectively. Algebraically, $t_\mathbf{A}(a_1,\ldots,a_n) = h(t)$, where $h$ is any assignment such that $h(x_i) = a_i$ for all $i \leq n$.

#### 2.1.2.3 Many-sorted equational logic

Many-sorted equational logic is an example of a successful algebraic tool in computer science. Although very simple to work with, many-sorted equational logic possesses a high expressive power which has been used extensively, for example, to specify programs and also in the study of abstract data types. For more details about many-sorted equational logic and its applications to specification theory we point to [EM85] and [GM96].

Given a many-sorted signature $\Sigma$ we denote an *equation* over $\Sigma$ by $t_1 \approx t_2$ where $t_1, t_2 \in T_{\Sigma,s}(X)$ for some $s \in S$. Consider $Eq_\Sigma(X)$, the set of *equations* over $\Sigma$, defined by $Eq_\Sigma(X) = \{t_1 \approx t_2 : t_1, t_2 \in T_{\Sigma,s}(X)$ and s $\in S\}$. For each $s \in S$ we can consider the set $Eq_{\Sigma,s}(X) = \{t_1 \approx t_2 : t_1, t_2 \in T_{\Sigma,s}(X)\}$ of all equations over $\Sigma$ of sort $s$.

We use $((t_1 \approx r_1)\& \ldots \&(t_n \approx r_n)) \to (t \approx r)$ to denote a *quasi-equation* or *conditional equation* over $\Sigma$. The set of all quasi-equations over $\Sigma$ is denoted by $QEq_\Sigma(X)$.

Given a $\Sigma$-algebra $\mathbf{A}$, an assignment $h$ over $\mathbf{A}$ and $t_1 \approx t_2 \in Eq_\Sigma(X)$, we write $\mathbf{A}, h \Vdash t_1 \approx t_2$ if $h(t_1) = h(t_2)$. We also write $\mathbf{A} \Vdash t_1 \approx t_2$ whenever $\mathbf{A}, h \Vdash t_1 \approx t_2$ for every assignment $h$ over $\mathbf{A}$, and in this case we say that $\mathbf{A}$ *satisfies* $t_1 \approx t_2$ or that $\mathbf{A}$ is a model of $t_1 \approx t_2$.

Given a quasi-equation $((t_1 \approx r_1)\& \ldots \&(t_n \approx r_n)) \to (t \approx r)$ we say that $\mathbf{A}$ satisfies $((t_1 \approx r_1)\& \ldots \&(t_n \approx r_n)) \to (t \approx r)$, which is denoted by $\mathbf{A} \Vdash ((t_1 \approx r_1)\& \ldots \&(t_n \approx r_n)) \to (t \approx r)$, if for every assignment $h$ over $\mathbf{A}$ we have that $\mathbf{A}, h \Vdash t \approx r$ whenever $\mathbf{A}, h \Vdash t_i \approx r_i$ for every $i \in \{1, \ldots, n\}$.

Given a class $K$ of $\Sigma$-algebras we can consider the (semantical many-sorted) equational consequence associated with $K$, denoted by $\vDash_\Sigma^K \subseteq 2^{Eq_\Sigma(X)} \times Eq_\Sigma(X)$, defined by: $\Psi \vDash_\Sigma^K t_1 \approx t_2$ if for every $\mathbf{A} \in K$ and every assignment $h$ over $\mathbf{A}$, we have that

$$\mathbf{A}, h \Vdash t_1 \approx t_2 \quad \text{whenever} \quad \mathbf{A}, h \Vdash r_1 \approx r_2 \text{ for every } r_1 \approx r_2 \in \Psi.$$

It is well-known that $\vDash_\Sigma^{Alg_\Sigma}$ is precisely [GM85] the so-called free (Birkhoff) many-sorted equational logic defined, for every $s, w \in S$, and every $x, x_1, x_2, x_3 \in X_s$, and every $t \in T_{\Sigma,w}(X)$, by the rules:

reflexivity $$\frac{}{x \approx x};$$

symmetry $$\frac{x_1 \approx x_2}{x_2 \approx x_1};$$

transitivity $$\frac{x_1 \approx x_2 \quad x_2 \approx x_3}{x_1 \approx x_3};$$

congruence $$\frac{x_1 \approx x_2}{t(x \backslash x_1) \approx t(x \backslash x_2)};$$

substitution $$\frac{x_1 \approx x_2}{\sigma(x_1) \approx \sigma(x_2)}.$$

If we add additional axioms (equations and/or quasi-equations) to the free equational logic we obtain an applied equational logic. Consider, for example, the class $K_E$ of all $\Sigma$-algebras that satisfy a set $E$ of $\Sigma$-equations. Then, the semantical equational consequence associated with $K_E$ coincides with the applied equational logic obtained by adding the equations in $E$ to the free equational logic.

#### 2.1.2.4 Varieties and quasivarieties of algebras

A major theme in universal algebra is the study of classes of algebras closed under one or more constructions. In the sequel we consider fixed a many-sorted signature $\Sigma$.

**Definition 2.1.42.** A *many-sorted variety* is a class of $\Sigma$-algebras closed under subalgebras, homomorphic images and direct products.

A paradigmatic example of a (single-sorted) variety is the class of all Boolean algebras. We will usually write just variety instead of many-sorted variety. Given a class $K$ of $\Sigma$-algebras, the *variety generated by* $K$, denoted by $V(K)$, is the smallest variety containing $K$.

**Definition 2.1.43.** A class $K$ of $\Sigma$-algebras is an *equational class* if it is the class of all algebras which are models of some set $E$ of equations.

In such case, $E$ is said to be an equational axiomatization of the class $K$. Note that we did not require the class to be closed under isomorphisms since being closed under homomorphic images implies that it is closed under isomorphisms.

A fundamental result relating varieties with equational classes was proved by Birkhoff [BS81, Bir35] in the unsorted case. It states that every variety of unsorted algebras is an equational class. An analogous result to the Birkhoff characterization also holds in the many-sorted case [MT92].

**Theorem 2.1.44.** *A class of $\Sigma$-algebras $K$ is a many-sorted variety if and only if $K$ is an equational class.*

This theorem, sometimes called the Variety theorem, was first proved Birkhoff in the thirties in the context of one-sorted universal algebra. It was later generalized to the many-sorted case (see [MT92]). The main technical idea of the proof consists of the fact that every variety contains a free algebra.

The variety theorem is one of many results that characterize syntactic classes of formulas in terms of the closure of their classes of models under certain algebraic constructions.

**Definition 2.1.45.** A class of $\Sigma$-algebras is a *(many-sorted) quasivariety* if it is closed under ultraproducts, subalgebras, direct products and isomorphisms.

Given a class $K$ of $\Sigma$-algebras, the *quasivariety generated by* $K$, denoted by $Q(K)$, is the smallest quasivariety containing $K$. Note that any variety is a quasivariety since an ultraproduct of a class of algebras is nothing but a homomorphic image of direct product of these algebras.

**Definition 2.1.46.** A class $K$ of $\Sigma$-algebras is a *quasi-equational class* if it is the class of all models of a set $Q$ of quasi-equations.

In this case, $Q$ is said to be a *quasi-equational axiomatization* of the class $K$. An equational class is always a quasi-equational class since an equations is a particular case of a quasi-equation.

The theorem that relates quasivariesties with quasi-equational classes is due to Mal'cev [Mal73] (see also Grätzer [Gra08]). The result also holds in the many-sorted case and its proof can be found in [MT92].

**Theorem 2.1.47.** *A class of $\Sigma$-algebras $K$ is a many-sorted quasivariety if and only if it is a quasi-equational class.*

In a given variety the subdirectly irreducible algebras play a special role since they can describe all the other algebras in the variety. The following result is a variety version of a Birkhoff's theorem.

**Theorem 2.1.48.** *If $K$ is a many-sorted variety, then every member of $K$ is isomorphic to a subdirect product of subdirectly irreducible members of $K$.*

A variety of particular interest in the sequel is the well-known variety of Boolean algebras. In the particular variety, the subdirectly irreducible algebras can be exactly characterized. Indeed, we have the following result due to Stone [Hal63, BS81, Sto36].

**Theorem 2.1.49.** *The two element Boolean algebra* **2** *is, up to isomorphism, the only directly indecomposable Boolean algebra which is nontrivial.*

Combining the last two theorems we obtain the following important theorem also due to Stone.

**Corollary 2.1.50.** *Every Boolean algebra is isomorphic to a subdirect power of* **2**.

## 2.2 Behavioral reasoning

In this section we will concentrate in the notion of behavioral logic. Recall that one of our goals is to build a framework that is general enough to capture many-sorted and non-truth-functional logics. Behavioral reasoning in many-sorted equational logic will play a key role in our approach. It is not our intention to present the theory of many-sorted behavioral reasoning in full detail, but rather to focus on the definitions and tools from behavioral logic that are necessary for our purposes. Further details on this subject can be found, for example, in [Ros00].

### 2.2.1 Hidden signatures

The main distinction between many-sorted equational logic and many-sorted behavioral logic is that in the latter the set of sorts is explicitly split in two: the *visible* sorts and the *hidden* sorts.

**Definition 2.2.1.** A *hidden many-sorted signature* is a tuple $\langle\Sigma, V\rangle$ where $\Sigma = \langle S, F\rangle$ is a many-sorted signature and $V \subseteq S$ is the set of *visible* sorts. The subset $H = S\backslash V$ will be dubbed the set of *hidden* sorts.

When there is no risk of confusion we will denote a hidden many-sorted signature $\langle\Sigma, V\rangle$ by $\Sigma$. A *hidden subsignature* of a hidden many-sorted signature $\langle\Sigma, V\rangle$ is a hidden signature $\langle\Gamma, V\rangle$ such that $\Gamma$ is a many-sorted subsignature of $\Sigma$. In the remainder of this section we will consider fixed a hidden signature $\langle\Sigma, V\rangle$.

Let us now focus on the fundamental notion of *experiment*. Given the intuitive nature of visible and hidden sorts, the role of experiments is to access the hidden sort. We have argued that, in some cases, not all operations can be used to build the experiments. This leads to following definition.

**Definition 2.2.2.** Given a hidden subsignature $\Gamma$ of $\Sigma$, a $\Gamma$-*context for sort* $s$ is a term $t(x : s, x_1 : s_1, \dots, x_m : s_m) \in T_\Gamma(X)$, with a distinguished variable $x$ of sort $s$ and parametric variables $x_1, \dots, x_m$ of sorts $s_1, \dots, s_m$ respectively. The set of all $\Gamma$-contexts for sort $s$ will be denoted by $\mathcal{C}^\Gamma_\Sigma[x : s]$ (note that $x \in \mathcal{C}^\Gamma_\Sigma[x : s]$). The $\Gamma$-contexts whose sort is visible will be dubbed $\Gamma$-*experiments*. The set of $\Gamma$-experiments for sort $s \in S$ will be denoted by $\mathcal{E}^\Gamma_\Sigma[x{:}s]$.

When it is important to refer the sort of the contexts or experiments then we will follow the notation used for terms: $\mathcal{C}^\Gamma_{\Sigma,s'}[x{:}s]$ denotes the set of $\Gamma$-contexts of sort $s'$ for sort $s$, while $\mathcal{E}^{\Gamma}_{\Sigma,s'}[x:s]$ denotes the set of $\Gamma$-experiments of sort $s'$ for sort $s$. When $\Gamma$ is clear form the context we just write context instead of $\Gamma$-context. Given $c \in \mathcal{C}^{\Gamma}_{\Sigma,s'}[x:s]$ and $t \in T_{\Sigma,s}(X)$, we denote by $c[t]$ the term obtained from $c$ by replacing every occurrence of $x$ by $t$. Note that the interesting contexts and experiments are those for hidden sorts, that is, those with $s \in H$. Contexts of visible sort are allowed more for the sake of symmetry, to make the presentation smoother. Recall that the role of experiments is to access the hidden terms.

### 2.2.2 Behavioral equational logic

We are now ready to introduce the most distinctive feature of behavioral logic, *behavioral equivalence*. The intuition is that two terms are behaviorally equivalent if they cannot be distinguished by any experiment.

**Definition 2.2.3.** Assume given a $\Sigma$-algebra $\mathbf{A}$, a hidden subsignature $\Gamma$ of $\Sigma$ and a sort $s \in S$. Then $a, b \in A_s$ *are $\Gamma$-behaviorally equivalent*, in symbols $a \equiv_\Gamma b$, if for every $\epsilon(x{:}s, x_1{:}s_1, \dots, x_n{:}s_n) \in \mathcal{E}^{\Gamma}_{\Sigma}[x{:}s]$ and for all $\langle a_1, \dots, a_n\rangle \in A_{s_1} \times \dots \times A_{s_n}$, we have that

$$\epsilon_{\mathbf{A}}(a, a_1, \dots, a_n) = \epsilon_{\mathbf{A}}(b, a_1, \dots, a_n).$$

Now that we have defined behavioral equivalence, we can talk about behavioral satisfaction of an equation by a $\Sigma$-algebra.

Let $\mathbf{A}$ be a $\Sigma$-algebra, $h$ an assignment over $\mathbf{A}$ and $t_1 \approx t_2$ an equation of sort $s \in S$. We say that $\mathbf{A}$ and $h$ $\Gamma$-*behaviorally satisfy* the equation $t_1 \approx t_2$, in symbols $\mathbf{A}, h \Vvdash_\Gamma t_1 \approx t_2$ if $h(t_1) \equiv_\Gamma h(t_2)$. We say that $\mathbf{A}$ behaviorally satisfies $t_1 \approx t_2$, in symbols $\mathbf{A} \Vvdash_\Gamma t_1 \approx t_2$, if $\mathbf{A}, h \Vvdash_\Gamma t_1 \approx t_2$ for every assignment $h$ over $\mathbf{A}$.

Given a quasi-equation $((t_1 \approx r_1) \& \dots \& (t_n \approx r_n)) \to (t \approx r)$ we say that $\mathbf{A}$ behaviorally satisfies $((t_1 \approx r_1) \& \dots \& (t_n \approx r_n)) \to (t \approx r)$, which is denoted by $\mathbf{A} \Vvdash_\Gamma ((t_1 \approx r_1) \& \dots \& (t_n \approx r_n)) \to (t \approx r)$, if for every assignment $h$ over $\mathbf{A}$ we have that $\mathbf{A}, h \Vvdash_\Gamma t \approx r$ whenever $\mathbf{A}, h \Vvdash_\Gamma t_i \approx r_i$ for every $i \in \{1, \dots, n\}$.

**Definition 2.2.4.** Given a class $K$ of $\Sigma$-algebras, the *behavioral consequence over $\Sigma$ associated with $K$ and $\Gamma$*, $\models\!\!\models^{K,\Gamma}_{\Sigma} \subseteq 2^{Eq_\Sigma(X)} \times Eq_\Sigma(X)$, is such that

$$\Psi \models\!\!\models^{K,\Gamma}_{\Sigma} t_1 \approx t_2$$

if for every $\mathbf{A} \in K$ and every assignment $h$ over $\mathbf{A}$

$$\mathbf{A}, h \Vvdash_\Gamma t_1 \approx t_2 \text{ whenever } \mathbf{A}, h \Vvdash_\Gamma u_1 \approx u_2 \text{ for every } u_1 \approx u_2 \in \Psi.$$

Let $\models_{\Sigma}^{K,\Gamma}$ be the behavioral consequence associated with the class $K$ of $\Sigma$-algebras as defined above. Let $\Psi(x:s)$ be a set of equations with a distinguished variable $x$ of sort $s$. Then $\Psi$ is said to be a *compatible set of equations* if $x_1 \approx x_2, \Psi(x_1) \models_{\Sigma}^{K,\Gamma} \Psi(x_2)$. We will denote by $Comp_{\Sigma}^{K,\Gamma}(Y)$ the set of all compatible sets of equations for the consequence relation $\models_{\Sigma}^{K,\Gamma}$ whose variables are contained in $Y \subseteq X$.

### 2.2.3 Hidden varieties and quasivarieties

In this section we present the notion of hidden variety and quasivariety. We have seen above that the notion of variety and quasivariety can have two equivalent characterizations. One is related to the closure with respect to some algebraic constructions and the other is related to the satisfaction of equations and quasi-equations. If we generalize this last characterization by replacing the role of equational satisfiability by behavioral satisfiability we obtain the hidden version of variety and quasivariety.

**Definition 2.2.5.** Given a hidden signature $\langle \Sigma, V \rangle$, a class $K$ of $\Sigma$-algebras is a *hidden variety* if there exists a set $E \subseteq Eq_{\Sigma}(X)$ of equations such that $K$ contains exactly the $\Sigma$-algebras that behaviorally satisfy all equations in $E$.

Given a class $K$ of $\Sigma$-algebras, the *hidden variety generated by* $K$, denoted by $HV(K)$, is the smallest hidden variety containing $K$.

We similarly define the notion of *hidden quasivariety*, but now considering quasi-equations instead of just equations.

**Definition 2.2.6.** Given a hidden signature $\langle \Sigma, V \rangle$, a class $K$ of $\Sigma$-algebras is a *hidden quasivariety* if it there exists a set $Q \subseteq QEq_{\Sigma}(X)$ of quasi-equations such that $K$ contains exactly the $\Sigma$-algebras that behaviorally satisfy all quasi-equations in $Q$.

Given a class $K$ of $\Sigma$-algebras, the *hidden quasivariety generated by* $K$, denoted by $HQ(K)$, is the smallest hidden quasivariety containing $K$.

With respect to this hidden approach, a systematic study, in the spirit of universal algebra, is still to be done. There are already some interesting results, namely a Birkhoff-like result with respect to hidden varieties [Ros98]. Since this characterization, due to Rosu, uses closure operations that are not very common in universal algebra we will not present it here.

## 2.3 Standard abstract algebraic logic

In this section we will present some notions and results of the abstract theory of algebraization of logics, first introduced in a mathematical precise way by Blok and Pigozzi in [BP89]. There, they intended to generalize the so-called Lindenbaum-Tarski construction. In [Tar83] Tarski gives the precise connection between classical propositional logic and Boolean algebras. The technique consists of looking at the set of formulas as an algebra with operators induced by the connectives. Logical equivalence is a congruence in the formula algebra and the induced quotient algebra turns out to be a Boolean algebra.

The definition proposed by Blok and Pigozzi of *algebraizable logic*, is, in fact, what is now called a finitely algebraizable logic [FJP03]. Moreover they consider exclusively finitary logics, that is, logics that also satisfy the finitariness property.

In the sequel, as already stated, we will *not* restrict ourselves to finitary logics and will consider the wider notion of algebraizable logic as proposed, for example, in [FJP03]. We point to [Cze01, FJP03] for historical and technical details, and for the proofs of the results presented in this section.

### 2.3.1 Algebraization

In their seminal work, Blok and Pigozzi [BP89] consider only finitary logics. This is however too restrict and Herrmann [Her96] and Czelakowski [Cze92], independently, generalize the framework so that non-finitary logics can also be studied in the context of AAL.

The objects of study of standard tools of AAL are the *structural propositional logics*. As we have seen above, these are nothing but single-sorted logics, that is, many-sorted logics over a many-sorted signature with just the sort $\phi$ of formulas. In the remainder of this section, whenever we refer to a signature $\Sigma$ we are always assuming that $\Sigma$ is single-sorted.

We can now introduce the main notion of *algebraizable logic*.

**Definition 2.3.1.** A single-sorted logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ is *algebraizable* if there exists a class $K$ of $\Sigma$-algebras, a set $\Theta(\xi)$ of $\Sigma$-equations, and a set $\Delta(\xi_1, \xi_2)$ of $\mathcal{L}$-formulas such that the following conditions hold, for every $T \cup \{t\} \subseteq L_\Sigma(X)$ and $\Psi \cup \{\varphi \approx \psi\} \subseteq Eq_\Sigma(X)$:

*i)* $T \vdash t$ iff $\Theta[T] \vDash_\Sigma^K \Theta(t)$;

*ii)* $\Psi \vDash_\Sigma^K \varphi \approx \psi$ iff $\Delta[\Psi] \vdash \Delta(\varphi, \psi)$;

*iii)* $\xi \dashv\vdash \Delta[\Theta(\xi)]$;

*iv)* $\xi_1 \approx \xi_2 \;=\!\mid\mid\!\!=^{K}_{\Sigma}\; \Theta[\Delta(\xi_1, \xi_2)]$.

The set $\Theta$ of equations is called the set of *defining equations*, $\Delta$ is called the set of *equivalential formulas*, and $K$ is called an *equivalent algebraic semantics* for $\mathcal{L}$. When both $\Theta$ and $\Delta$ are finite, we say that $\mathcal{L}$ is *finitely algebraizable.*

The notion of algebraizable logic intuitively means that the consequence relation of a logic $\mathcal{L}$ can be captured by the equational consequence relation $\models^K_\Sigma$, and vice-versa, in a logically inverse way.

An enlightening characterization of algebraizable logic, as illustrated in Fig. 2.1 can be expressed using maps of logics [Cze01].

**Theorem 2.3.2.** *A structural single-sorted logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ is algebraizable iff there exists a class $K$ of $\Sigma$-algebras and a strong representation $(\theta, \tau)$ of $\mathcal{L}$ in $Eqn^K_\Sigma$ such that both $\theta$ and $\tau$ commute with substitutions.*

**Single-sorted logic $\mathcal{L}$** $\xLeftrightarrow{\text{Strong representation}}$ **Equational logic over $K$**

Figure 2.1: Algebraizable logic

Note that the fact that both maps commute with substitutions is essential to guarantee that each map can be given uniformly, respectively, by a set $\Theta(\xi)$ of equations, and a set $\Delta(\xi_1, \xi_2)$ of formulas.

The following definition of *equivalence set* generalizes the well-known phenomenon of classical propositional calculus, where the equivalence of formulas can be expressed by the equivalence symbol $\Leftrightarrow$, i.e., for each theory $T$, $\langle \xi_1, \xi_2 \rangle \in \Omega(T)$ iff $T \vdash \xi_1 \Leftrightarrow \xi_2$.

**Definition 2.3.3.** A single-sorted logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ is *equivalential* if there exists a set $\Delta(\xi_1, \xi_2) \subseteq L_\Sigma(\{\xi_1, \xi_2\})$ such that for every $\varphi, \psi, \delta, \varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_n \in L_\Sigma(X)$:

(R) $\vdash \Delta(\varphi, \varphi)$;

(S) $\Delta(\varphi, \psi) \vdash \Delta(\psi, \varphi)$;

(T) $\Delta(\varphi, \psi), \Delta(\psi, \delta) \vdash \Delta(\varphi, \delta)$;

(MP) $\Delta(\varphi, \psi), \varphi \vdash \psi$;

(RP) $\Delta(\varphi_1, \psi_1), \ldots, \Delta(\varphi_n, \psi_n) \vdash \Delta(f[\varphi_1, \ldots, \varphi_n], f[\psi_1, \ldots, \psi_n])$ for every $f : \phi^n \to \phi$.

In this case, $\Delta$ is called an *equivalence set* for $\mathcal{L}$. It is easy to see that $\Delta$ defines a congruence on $\mathbf{L_\Sigma}(\mathbf{X})$, that is, an equivalence relation that is compatible with all operations.

In the following proposition we present the first necessary condition for a logic to be algebraizable.

**Proposition 2.3.4.** *If a single-sorted logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ is algebraizable then it is equivalential.*

Indeed, when a logic $\mathcal{L}$ is algebraizable, the set of equivalence formulas constitutes an equivalence set for $\mathcal{L}$, thus showing that $\mathcal{L}$ is equivalential.

## 2.3.2 Equivalent algebraic semantics

In this section we present some standard results of AAL related to the properties of the algebraic counterpart of a given algebraizable logic. Issues such as unicity and axiomatization are also discussed.

We start with an important unicity result. It states that the algebraic counterpart of an algebraizable logic is, in some precise sense, unique. Moreover, it states that, although we can choose different sets of equivalence formulas and different sets of defining equations, these choices are equivalent.

**Theorem 2.3.5.** *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a single-sorted logic. Suppose that $\mathcal{L}$ is algebraizable and let $K$ and $K'$ be two equivalent algebraic semantics for $\mathcal{L}$ such that $\Delta(\xi_1, \xi_2)$ and $\Theta(\xi)$ are equivalence formulas and defining equations for $K$, and, similarly, $\Delta'(\xi_1, \xi_2)$ and $\Theta'(\xi)$ for $K'$. Then we have that:*

*i)* $\vDash_\Sigma^K = \vDash_\Sigma^{K'}$;

*ii)* $\Delta(\xi_1, \xi_2) \dashv\vdash \Delta'(\xi_1, \xi_2)$;

*iii)* $\Theta(\xi) =\!||\!\!=^K_\Sigma \Theta'(\xi)$.

Although a given algebraizable logic can have different equivalent algebraic semantics, there is one that can be canonically associated with $\mathcal{L}$: the largest one. So, we write *the* equivalent algebraic semantics when we want to refer to the largest equivalent algebraic semantics.

In [BP89] Blok and Pigozzi prove interesting results concerning the uniqueness and axiomatization of an equivalent algebraic semantics of a given finitary and finitely algebraizable propositional logic. They prove that a class $K$ of algebras is an equivalent algebraic semantics of a finitary and finitely algebraizable logic if and only if the quasivariety generated by $K$ is also an equivalent algebraic semantics. In terms of uniqueness they prove that there is an unique quasivariety equivalent to a given finitary and finitely algebraizable logic. This equivalent quasivariety semantics is precisely the largest equivalent algebraic semantics of $\mathcal{L}$. Moreover, the axiomatization of this quasivariety can be directly built from an axiomatization of the logic being algebraized, as stated in the following result.

**Theorem 2.3.6.** *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a finitary single-sorted logic obtained from a deductive system with a set of axioms $Ax$ and a set of inference rules $R$. Assume that $\mathcal{L}$ is finitely algebraizable with $\Theta$ the set of defining equations and $\Delta$ the set of equivalential formulas. Then, the equivalent quasivariety semantics is axiomatized by the following equations and conditional-equations:*

*i)* $\Theta(\varphi)$ *for each* $\varphi \in Ax$;

*ii)* $\Theta[\Delta(\xi, \xi)]$;

*iii)* $\Theta(\psi_0) \;\&\ldots\&\; \Theta(\psi_n) \to \Theta(\psi)$ *for each rule* $\frac{\psi_0 \ldots \psi_n}{\psi} \in R$;

*iv)* $\Theta[\Delta(\xi_1, \xi_2)] \to \xi_1 \approx \xi_2$.

### 2.3.3 The Leibniz hierarchy

There are several interesting and useful alternative characterizations of the notion of algebraizable notion. The most useful, namely to prove negative results, is perhaps the characterization that explores the properties of the so-called *Leibniz operator*. Indeed, the Leibniz operator is one of the most important standard tools in AAL. It can help to build a hierarchy of classes of logics using its properties. This hierarchy is depicted in Fig. 2.2. We use it as a roadmap for the remainder of the section.

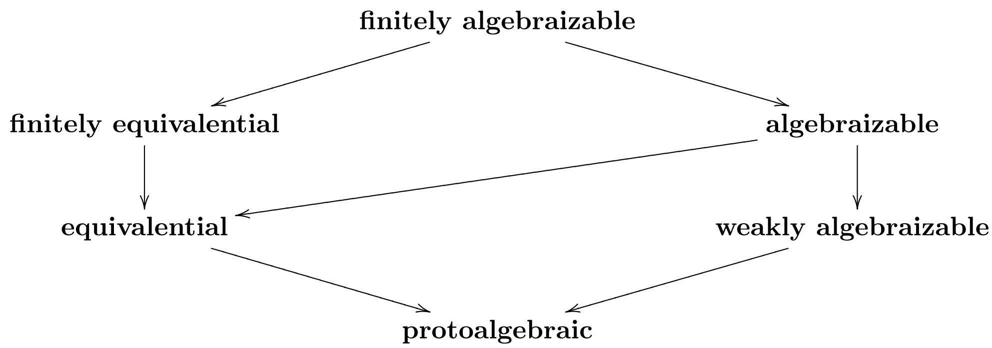



Figure 2.2: A view of the Leibniz hierarchy.

A congruence $\equiv$ on a $\Sigma$-algebra $\mathbf{A}$ is said to be *compatible* with a subset $D$ of $A_\phi$ if $b \in D$ whenever $a \in D$ and $a \equiv b$. In this case, $D$ is an union of equivalence classes of $\equiv$.

**Lemma 2.3.7.** *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a single-sorted logic. For each $T \in Th_\mathcal{L}$, there exists a largest congruence on $\mathbf{T_\Sigma}(\mathbf{X})$ compatible with $T$ and it is given by*

$$\{\langle \varphi, \psi \rangle : \textit{ for all } \delta \in L_\Sigma(\{\xi\}),\ \delta(\xi \backslash \varphi) \in T \textit{ iff } \delta(\xi \backslash \psi) \in T\}$$

**Definition 2.3.8.** Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a single-sorted logic. The *Leibniz operator* on the formula algebra is given by:

$$\begin{array}{rccl} \Omega: & Th_\mathcal{L} & \to & \mathrm{Cong}_{\mathbf{L_\Sigma(X)}} \\ & T & \mapsto & \text{largest congruence compatible with } T. \end{array}$$

When a pair of formulas $\langle \varphi, \psi \rangle$ is in $\Omega(T)$ for some $\mathcal{L}$-theory $T$, we say that $\varphi$ and $\psi$ are *indiscernible with respect to* $T$, or just $T$*-indiscernible.*

The term Leibniz congruence was introduced in [BP89] but the concept appears much early. The characterization of the Leibniz $\Gamma$-congruence given in the proof of Lemma 3.2.3 justifies the use of the term Leibniz. The famous Leibniz second order criterion says that two objects in the universe of discourse are equal if they share all the properties that can be expressed in the language of discourse.

The Leibniz operator plays a central role in AAL. As we will see, some important classes of logics can be characterized by its properties. These properties include monotonicity, injectivity and commutation with inverse substitutions, where the last one means that given a substitutions $\sigma$ over $\Sigma$ and a theory $T \in Th_{\mathcal{L}}$ we have that

$$\Omega(\sigma^{-1}(T)) = \sigma^{-1}(\Omega(T)).$$

An important class of logics, the *protoalgebraic logics*, can be introduced using the Leibniz operator.

**Definition 2.3.9.** A single-sorted logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ is *protoalgebraic* if, for every $\mathcal{L}$-theory $T$,

$$\text{if } \langle \varphi, \psi \rangle \in \Omega(T) \text{ then } T \cup \{\varphi\} \dashv\vdash T \cup \{\psi\}.$$

So, in a protoalgebraic logic, two $T$-indiscernible formulas are always interderivable with respect to $T$.

Although in general not algebraizable, the protoalgebraic logics constitute the main class of logics for which the advanced methods of algebraic logic can be applied. The next theorem gives us a first glimpse of the importance of the Leibniz operator. It shows that the protoalgebraicity of a logic $\mathcal{L}$ can be established just by looking at the behavior of $\Omega$ on the lattice of $\mathcal{L}$-theories. We also exhibit a syntactic characterization that will be useful in the sequel.

**Theorem 2.3.10.** *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a single-sorted logic. Then the following are equivalent:*

*i) is protoalgebraic;*

*ii) $\Omega$ is monotone;*

*iii) there exists a set $\Delta(\xi_1, \xi_2) \subseteq L_\Sigma(X)$ of formulas with two distinguished variables of sort $\phi$ and possibly parametric variables, satisfying*

**reflexivity:** $\vdash \Delta(\xi, \xi)$;

**detachment:** $\xi_1, \Delta(\xi_1, \xi_2) \vdash \xi_2$.

We now introduce another class of algebras in the Leibniz hierarchy, the *weakly algebraizable logics*.

**Definition 2.3.11.** A single-sorted logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ is *weakly algebraizable* if there exists a class $K$ of $\Sigma$-algebras, a set $\Theta(\xi, \underline{z}) \subseteq Eq_\Sigma(X)$ of $\phi$-equations and a set $\Delta(\xi_1, \xi_2, \underline{w}) \subseteq L_\Sigma(\{\xi_1, \xi_2, \underline{w}\})$ of formulas such that, for every $T \cup \{t\} \subseteq L_\Sigma(X)$ and for every $\Psi \cup \{t_1 \approx t_2\} \subseteq Eq_\Sigma(X)$ of $\phi$-equations:

*i)* $T \vdash t$ iff $\Theta[\langle T \rangle] \vDash^K_\Sigma \Theta(\langle t \rangle)$;

*ii)* $\Psi \vDash^K_\Sigma t_1 \approx t_2$ iff $\Delta[\langle \Psi \rangle] \vdash \Delta(\langle t_1, t_2 \rangle)$;

*iii)* $\xi \dashv\vdash \Delta[\langle \Theta(\langle \xi \rangle) \rangle]$;

*iv)* $\xi_1 \approx \xi_2 =\!\mid\mid\!\!=^K_\Sigma \Theta[\langle \Delta(\langle \xi_1, \xi_2 \rangle) \rangle]$;

The difference between the notion of weakly algebraizable logic and that of algebraizable logic is the fact that, in the former, both the equivalence set of formulas and the defining set of equations have parametric variables, that is, variables different from the distinguished one. Recall that, given a formula $\varphi(\xi, \underline{z})$, we write $\langle \varphi(\xi) \rangle$ to denote the set $\{\sigma\varphi \mid \sigma$ substitution such that $\sigma\xi = \xi\}$. So, these variables $\underline{z}$ are said to be parametric since they are suppose to represent every possible instantiation.

We can also characterize the notion of weakly algebraizable logic using the Leibniz operator.

**Theorem 2.3.12.** *A single-sorted logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ is weakly algebraizable iff $\Omega$ is monotone and injective.*

The notion of equivalential logic we have introduced in the last section can also have a characterization using the Leibniz operator.

**Theorem 2.3.13.** *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a single-sorted logic. Then the following are equivalent:*

*i) $\mathcal{L}$ is equivalential;*

*ii) $\Omega$ is monotone and commutes with inverse substitutions;*

*iii) $\Omega$ is monotone and $\sigma\Omega(T) \subseteq \Omega((\sigma T)^\vdash)$, for all substitutions $\sigma$ and $\mathcal{L}$-theories $T$.*

It is now clear why the hierarchy depicted in Fig. 2.2 is said to be the Leibniz hierarchy. In fact, each class of logics can be characterized by inspection of the properties of the corresponding Leibniz operator. In Fig. 2.3 we enrich the Leibniz hierarchy diagram presented before with the properties of each class of logics.

Concerning algebraizability, we have the following result.

**Theorem 2.3.14.** *A structural single-sorted logic* $\mathcal{L} = \langle \Sigma, \vdash \rangle$ *is algebraizable iff* $\Omega$ *is monotone, injective, and commutes with inverse substitutions.*

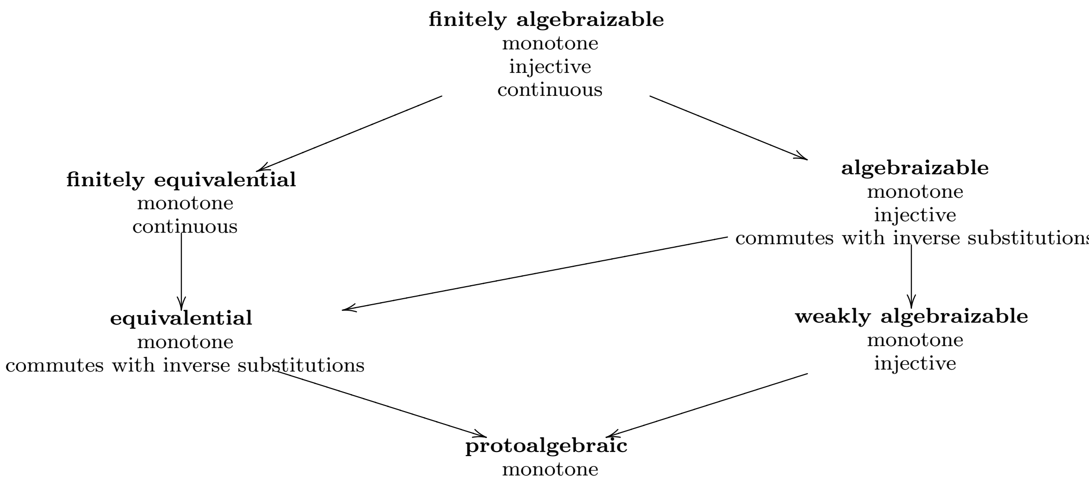



Figure 2.3: Leibniz hierarchy - properties of $\Omega$.

Note that using the Leibniz characterization of each class, the inclusions presented in Fig. 2.3 are immediate.

### 2.3.4 Intrinsic and sufficient characterizations

Although very important, the characterization involving the Leibniz operator is not very useful in practice to show algebraizability, since the concepts involved are hard to work with. The first theorem of this section gives an intrinsic characterization of algebraizable logic which involves conditions simpler to work with. We called it intrinsic since, as the characterization involving the Leibniz operator, it does not depend on an a priori existence of a class $K$ of algebras.

**Theorem 2.3.15.** *A single-sorted logic* $\mathcal{L} = \langle \Sigma, \vdash \rangle$ *is algebraizable iff it is equiv-*

*alential with equivalence set* $\Delta(\xi_1, \xi_2)$ *and there exists a set* $\Theta(\xi) \subseteq Eq_{\Sigma,\phi}(\{\xi\})$ *of* $\phi$*-equations such that*

$$\xi \dashv\vdash \Delta[\Theta(\xi)].$$

As a corollary of the Theorem 2.3.15 we have a sufficient condition for a logic to be algebraizable. It is a result of practical importance since it is rather simple to verify if a given logic satisfies the conditions there presented.

**Corollary 2.3.16.** *Let* $\mathcal{L} = \langle \Sigma, \vdash \rangle$ *be a single-sorted logic. A sufficient condition for* $\mathcal{L}$ *to be algebraizable is that it is equivalential with equivalence set* $\Delta(\xi_1, \xi_2)$ *that also satisfies:*

**(G-rule)** $\xi_1, \xi_2 \vdash \Delta(\xi_1, \xi_2)$.

In this case $\Delta$ and $\xi \approx \Delta(\xi, \xi)$ are, respectively, the set of equivalence formulas and the set of defining equations. If the sufficient conditions of the above corollary are satisfied the logic is said to be *regularly algebraizable*. We point to [Cze01] for a detailed study of this class of logics. Most of the usual examples studied in the literature of AAL are regularly algebraizable.

### 2.3.5 Matrix semantics

Matrix semantics is one of the important tools chosen for semantical investigation in standard AAL, with a lot of fruitful and enlightening results already established. In this section we present some definitions and results relevant for the remainder of the work. We refer the interested reader to Wójcicki's 1988 book [Wój88] and Czelakowski's book [Cze01] as sources for the large body of research on this topic.

**Definition 2.3.17.** A *(logical) matrix* over a single-sorted signature $\Sigma$ is a pair $\langle \mathbf{A}, D \rangle$ where $\mathbf{A}$ is a $\Sigma$-algebra and $D \subseteq A$, is the set of *designated values*.

Given a matrix $\mathcal{M} = \langle \mathbf{A}, D \rangle$ over $\Sigma$, we can define a consequence relation over $\Sigma$, denoted by $\vdash_{\mathcal{M}}$, such that, for every $T \cup \{\varphi\} \subseteq L_\Sigma(X)$, we have that $T \vdash_{\mathcal{M}} \varphi$ iff for every assignment $h$ over $\mathbf{A}$ we have that:

$$h(\varphi) \in D \text{ whenever } h(\psi) \in D \text{ for every } \psi \in T.$$

**Definition 2.3.18.** Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic and $\mathcal{M} = \langle \mathbf{A}, D \rangle$ a matrix over $\Sigma$. The matrix $\mathcal{M}$ is a *model* of $\mathcal{L}$ if $\vdash \subseteq \vdash_{\mathcal{M}}$, that is, for every $T \cup \{\varphi\} \subseteq L_\Sigma(X)$, we have that $T \vdash_{\mathcal{M}} \varphi$ whenever $T \vdash \varphi$. In this case, $D$ is called a *$\mathcal{L}$-filter* of $\mathbf{A}$.

Given a $\Sigma$-algebra $\mathbf{A}$, the set of all $\mathcal{L}$-filters of $\mathbf{A}$, which we denoted by $Fi_{\mathcal{L}}(\mathbf{A})$, is closed under intersections of arbitrary families and is thus a complete lattice. Therefore, given any set $C \subseteq A$, there is always the least $\mathcal{L}$-filter of $\mathbf{A}$ that contains $C$. It is the $\mathcal{L}$-filter of $\mathbf{A}$ generated by $C$ and is denoted by $Fi^{\mathbf{A}}_{\mathcal{L}}(C)$. The class of all matrix models of $\mathcal{L}$ is denoted by $Mod(\mathcal{L})$.

A *matrix congruence* $\theta$ over a matrix $\mathcal{M} = \langle \mathbf{A}, D \rangle$ is a congruence over $\mathbf{A}$ compatible with $D$, that is, $\theta$ is a congruence over $\mathbf{A}$ and for every $a, b \in A_\phi$, if $\langle a, b \rangle \in \theta_\phi$ and $a \in D$ then $b \in D$. It is easy to see that every matrix $\mathcal{M} = \langle \mathbf{A}, D \rangle$ has a largest matrix congruence. This is called *Leibniz congruence of* $\mathcal{M}$ and is denoted by $\Omega_{\mathbf{A}}(D)$.

**Definition 2.3.19.** Given a $\Sigma$-algebra $\mathbf{A}$ we can consider the *Leibniz operator on* $\mathbf{A}$

$$\Omega_{\mathbf{A}} : Fi_{\mathcal{L}}(\mathbf{A}) \to Cong_{\mathbf{A}}$$
$$D \mapsto \Omega_{\mathbf{A}}(D).$$

We have the following characterization result for $\Omega_{\mathbf{A}}(D)$.

**Proposition 2.3.20.** *Given a matrix $\mathcal{M} = \langle \mathbf{A}, D \rangle$ over $\Sigma$ we have that $\langle a, b \rangle \in \Omega_{\mathbf{A}}(D)$ iff for every $\varphi(\xi, \xi_1, \ldots, \xi_n) \in L_\Sigma(X)$ and every $\langle a_1, \ldots, a_n \rangle \in A^n_\phi$ we have that:*

$$\varphi_{\mathbf{A}}(a, a_1, \ldots, a_n) \in D \quad \textit{iff} \quad \varphi_{\mathbf{A}}(b, a_1, \ldots, a_n) \in D.$$

The standard class of algebras that AAL canonically associates to a logic $\mathcal{L}$ is a subclass of the class of all algebraic reducts of matrices in $Mod(\mathcal{L})$. This subclass is related to the so-called *Leibniz-reduced matrix models.* A matrix $\mathcal{M} = \langle \mathbf{A}, D \rangle$ is said to be *reduced* (or *Leibniz-reduced*) if its Leibniz congruence $\Omega_{\mathbf{A}}(D)$ is the identity. Thus, to each matrix $\mathcal{M} = \langle \mathbf{A}, D \rangle$ corresponds the reduced matrix $\langle \mathbf{A}/_{\Omega_{\mathbf{A}}(D)}, D/_{\Omega_{\mathbf{A}}(D)} \rangle$. This matrix is the *reduction* of $\mathcal{M}$. The class of all Leibniz-reduced matrix models of a logic $\mathcal{L}$ is denoted by $Mod^*(\mathcal{L})$.

We can consider the class of $\Sigma$-algebras $Alg^*(\mathcal{L})$ of algebraic reducts of matrix models of $\mathcal{L}$, that is,

$$Alg^*(\mathcal{L}) = \{\mathbf{A} : \text{ there exists } D \subseteq A_\phi \text{ such that } \langle \mathbf{A}, D\rangle \in Mod^*(\mathcal{L})\}.$$

Actually, the class of algebras that standard AAL canonically associates to a logic $\mathcal{L}$ is the class of algebraic reduct of the so-called Suszko-reduced matrix models of $\mathcal{L}$. This class, however, coincides with $Alg^*(\mathcal{L})$ for the protoalgebraic logics. So, since it is not our intention to study non-protoalgebraic logics, we will not go into details about Suszko congruences and Suszko-reduced matrices.

We now present a first result relating the algebraizability of a logic with the Leibniz congruence on its matrix models. It states that when a logic is algebraizable, its set of equivalence formulas defines the Leibniz congruence on every matrix model.

**Proposition 2.3.21.** *Let $\mathcal{L} = \langle \Sigma, \vdash\rangle$ be a single-sorted logic and suppose that $\mathcal{L}$ is algebraizable logic with $\Delta(\xi_1, \xi_2)$ the set of equivalence formulas. Let $\mathcal{M} = \langle \mathbf{A}, D\rangle \in Mod(\mathcal{L})$ be a matrix model of $\mathcal{L}$. Then,*

$$\langle a, b\rangle \in \Omega_{\mathbf{A}}(D) \quad \textit{iff} \quad \Delta_{\mathbf{A}}(a, b) \subseteq D.$$

We can now see the following theorem as the matrix version of the characterization Theorem 2.3.14. When the logics under consideration have small, finite matrix models this theorem gives a very useful tool for showing non-algebraizability.

**Theorem 2.3.22.** *Let $\mathcal{L} = \langle \Sigma, \vdash\rangle$ be a many-sorted logic and $K$ a class of $\Sigma$-algebras. Then:*

*1) The following are equivalent:*

- *i) $\mathcal{L}$ is algebraizable and $K$ is the equivalent algebraic semantics;*
- *ii) for every $\Sigma$-algebra $\mathbf{A}$ we have that $\Omega_{\mathbf{A}}$ is an isomorphism between the lattices of $\mathcal{L}$-filters of $\mathbf{A}$ and of $K$-congruences of $\mathbf{A}$, that commutes with inverse substitutions.*

*2) Assume $\mathcal{L}$ is algebraizable with equivalent algebraic semantics $K$. Let $\Theta(\xi)$ be the set of defining equations for $K$. For each $\Sigma$-algebra $\mathbf{A}$ and congruence $\theta$ of $\mathbf{A}$ define*

$$H_{\mathbf{A}}(\theta) = \{a \in A_\phi : \langle \gamma_{\mathbf{A}}(a), \delta_{\mathbf{A}}(a)\rangle \in \theta, \textit{ for every } \gamma \approx \delta \in \Theta\}.$$

*Then $H_{\mathbf{A}}$ restricted to the $K$-congruences of $\mathbf{A}$ is the inverse of $\Omega_{\mathbf{A}}$.*

The above theorem gives some insight on the precise connection between equivalent algebraic semantics and matrix semantics. The following corollary gives a particular way of describing this connection involving the class $Alg^*(\mathcal{L})$.

**Corollary 2.3.23.** *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a single-sorted logic and assume that $\mathcal{L}$ is algebraizable with $K$ the equivalent algebraic semantics. Then $K = Alg^*(\mathcal{L})$.*

### 2.3.6 Examples and limitations

The theory of AAL is fruitful in positive and interesting results. One of the purposes of this section is to illustrate some of these positive and interesting results with the help of some examples. A second and more important purpose of the section is to point out some limitations of the theory, paving the way for the generalization of the theory we will undertake in the subsequent chapters.

**Example 2.3.24** (Classical and intuitionistic propositional logics)**.**

The main paradigm of AAL is the well-established connection between classical propositional logic (CPL) and the variety of Boolean algebras. This was really the starting point to the idea of connecting logic with algebra, which evolved by trying to generalize this connection to other logics. Another important example is intuitionistic propositional logic (IPL). Its algebraization gives rise to the class of Heyting algebras. It is interesting to note that, in contrast to Boolean algebras, which were defined before the Lindenbaum-Tarski techniques were first applied to generate them from CPL, Heyting algebras seem to be the first algebras of logic that were identified precisely by applying these techniques (which are the ancient roots of the modern theory of AAL) to a given axiomatization of IPL.

∎

Despite the enormous success of the standard tools of AAL, not only in the generality of the results, but also in the large number of examples, we can point out some of their limitations. From our point of view, one major limitation is its inability to correctly deal with logics with a many-sorted language. Let us, first of all, discuss the paradigmatic example of first-order classical logic (FOL).

**Example 2.3.25** (First-Order Classical Logic)**.**

Research on the algebraization of FOL goes back to the seminal work initiated by Tarski in the 1940s, and published in collaboration with Henkin and Monk in [HMT71]. This line of research is known as the cylindric approach, and it is the one that we will follow in the present work. Nevertheless, we note that there is another important approach to the algebraization of FOL, known as the polyadic approach, that differs from the cylindric approach mainly because it deals with explicit substitutions.

In [BP89], Blok and Pigozzi, following the cylindric approach, present a single-sorted algebraization of FOL in the terms we have just introduced. This example introduces their ideas using our notation.

Their main idea is to massage the first-order language into a single-sorted language and then consider a structural single-sorted deductive system **PR**, as introduced by Németi, for first-order logic over this single-sorted language. The single-sorted language of **PR** is obtained from a first-order language that is more restricted than the usual FOL language.

Consider a first-order language $\langle \mathcal{C}, \mathcal{R}, \mathcal{F} \rangle$ with equality, where $\mathcal{C}$ is the set of constant symbols, $\mathcal{R}$ is the set of relation symbols and $\mathcal{F}$ is the set of function symbols. It is usually assumed, in the literature about the algebraization of first-order logic, that $\mathcal{F} = \emptyset$. Moreover, it is assumed that the individual variables are canonically ordered in a sequence $v_1, v_2, \ldots, v_n, \ldots$.

Recall that atomic formulas that do not involve equality are of the form $R(y_1, \ldots, y_n)$ where $R$ is a $n$-ary relation symbol and $y_1, \ldots, y_n$ range over the individual variables. An atomic formula of the form $R(v_1, \ldots, v_n)$ where $R$ is a $n$-ary relation symbol and the variables occur in the canonical order, is said to be in restricted form. If we consider just the first-order formulas whose atomic formulas that do not involve equality are in restricted form, we get the restricted first-order language over $\langle \mathcal{C}, \mathcal{R} \rangle$.

The single-sorted signature $\Sigma_{RFOL} = \langle S, F \rangle$ obtained from this restricted first-order language is such that:

- $S = \{\phi\}$;
- $c_R : \to \phi$ for every $R \in \mathcal{R}$;
- $\top, \bot : \to \phi$, $\neg : \phi \to \phi$ and $\wedge, \vee, \Rightarrow : \phi^2 \to \phi$;
- $\forall_i : \phi \to \phi$ for every $i \in \mathbb{N}$;
- $\exists_i : \phi \to \phi$ for every $i \in \mathbb{N}$;
- $=_{i,j} : \to \phi$ for every pair $i, j \in \mathbb{N}$.

Each nullary operation $c_R$ is intended to represent within the single-sorted language the restricted atomic formula $R(v_1, \ldots, v_n)$, while the nullary operation $=_{i,j}$ is intended to represent the equality atomic formula $v_i = v_j$. For each $i \in \mathbb{N}$, the unary operations $\forall_i$ and $\exists_i$ are intended to represent, respectively, the universal quantifier $\forall_{v_i}$ and the existential quantifier $\exists_{v_i}$. For practical reasons we denote $=_{i,j}$, $\forall_i$ and $\exists_i$, respectively by $v_i = v_j$, $\forall_{v_i}$ and $\exists_{v_i}$. Nevertheless, we reinforce the idea that, in the single-sorted signature, the symbols $v_i = v_j$, $\forall_{v_i}$ and $\exists_{v_i}$ are to be considered indivisible.

The structural single-sorted deductive system **PR** over this single-sorted language consists of the axioms, where $k, j, i$ range over $\mathbb{N}$:

A1. all classical tautologies;

A2. $\forall_{v_k}(\xi_1 \Rightarrow \xi_2) \Rightarrow (\forall_{v_k}\xi_1 \Rightarrow \forall_{v_k}\xi_2)$;

A3. $(\forall_{v_k}\xi) \Rightarrow \xi$;

A4. $(\forall_{v_k}\forall_{v_j}\xi) \Rightarrow (\forall_{v_j}\forall_{v_k}\xi)$;

A5. $(\forall_{v_k}\xi) \Rightarrow (\forall_{v_k}\forall_{v_k}\xi)$;

A6. $(\exists_{v_k}\xi) \Rightarrow (\forall_{v_k}\exists_{v_k}\xi)$;

A7. $v_k = v_k$;

A8. $\exists_{v_k}(v_k = v_j)$;

A9. $(v_k = v_j) \Rightarrow ((v_k = v_i) \Rightarrow (v_j = v_i)$;

A10. $(v_k = v_j) \Rightarrow (\xi \Rightarrow \forall_{v_k}((v_k = v_j) \Rightarrow \xi)$, if $k \neq j$;

A11. $(\exists_{v_k}\xi) \Leftrightarrow (\neg\forall_{v_k}\neg\xi)$;

A12. $c_R \Rightarrow \forall_{v_k}c_R$, if $k$ is greater than the rank of $R$;

and the rules:

R1. $\frac{\xi_1 \quad \xi_1 \Rightarrow \xi_2}{\xi_2}$ (modus ponens);

R2. $\frac{\xi}{\forall_{v_k}\xi}$ (generalization).

Since **PR** is a single-sorted logic we can now study its algebraization [BP89].

THEOREM. **PR** is algebraizable and **its** equivalent algebraic semantics is the variety of cylindric algebras.

By a cylindric algebra we mean a $\Sigma_{RFOL}$-algebra

$$\mathbf{A} = \langle A_\phi, \wedge_\mathbf{A}, \vee_\mathbf{A}, \Rightarrow_\mathbf{A}, \neg_\mathbf{A}, \top_\mathbf{A}, \bot_\mathbf{A}, (c_R)_\mathbf{A}, (\forall_{v_k})_\mathbf{A}, (\exists_{v_k})_\mathbf{A}, (v_k = v_j)_\mathbf{A}\rangle_{k,j\in\mathbb{N},R\in\mathcal{R}}$$

such that, for every $k, i, j \in \mathbb{N}$,

C0. $\langle A_\phi, \wedge_\mathbf{A}, \vee_\mathbf{A}, \Rightarrow_\mathbf{A}, \neg_\mathbf{A}, \top_\mathbf{A}, \bot_\mathbf{A}\rangle$ is a Boolean algebra;

and $\mathbf{A}$ also satisfies the following equations:

C1. $\exists_{v_k}\bot \approx \bot$;

C2. $\xi \wedge \exists_{v_k}\xi \approx \xi$;

C3. $\exists_{v_k}(\xi_1 \wedge \exists_{v_k}\xi_2) \approx (\exists_{v_k}\xi_1) \wedge (\exists_{v_k}\xi_2)$;

C4. $\exists_{v_k}\exists_{v_j}\xi \approx \exists_{v_j}\exists_{v_k}\xi$;

C5. $(v_k = v_k) \approx \top$;

C6. $(v_i = v_j) \approx \exists_{v_k}((v_i = v_k) \wedge (v_k = v_j))$, if $k \neq j$;

C7. $(\exists_{v_k}((v_i = v_k) \wedge \xi)) \wedge (\exists_{v_k}((v_i = v_k) \wedge \neg\xi)) \approx \bot$, if $k \neq j$;

C8. $(\exists_{v_k}c_R) \approx c_R$ for $k$ greater than the rank of $R$.

In the literature about cylindric algebras it is usual to see the following notation for the operations: $c_k = (\exists_{v_k})_\mathbf{A}$, $d_{i,j} = (v_k \approx v_j)_\mathbf{A}$, $\times = \wedge_\mathbf{A}$, $+ = \vee_\mathbf{A}$ , $- = \neg_\mathbf{A}$, $1 = \top_\mathbf{A}$ and $0 = \bot_\mathbf{A}$. The elements $d_{i,j}$ are called diagonal elements, and the operations $c_k$ are called cylindrifications.

Despite the success of the example of FOL within AAL, we can point out some drawbacks. A major one is related to the fact that the first-order language they start from differs in several important respects from standard FOL. In our opinion, there are two main reasons for this fact. The first, and more important one, is related to the fact that FOL is not a structural logic in the sense we have defined above. The second cause, responsible for some additional restrictions in the language is the fact that, since the standard tools of AAL only applies to propositional logics, the atomic formulas of FOL have to be represented, within the propositional language, as constant symbols.

Another important drawback is the fact that, given the many-sorted character of first-order logic, where we have at least syntactic categories for terms and formulas, and possibly also for variables, it would be desirable to have an algebraic counterpart that reflects this many-sorted character. This is clearly not the case with plain cylindric algebras.

∎

One of our motivations is precisely to extend the theory of AAL to cope with logics that, like first-order logic, have a many-sorted language. Given an algebraizable many-sorted logic, this will allow us to reflect its many-sorted character in its corresponding algebraic counterpart.

But it is not only at the purely many-sorted level that the limitations of standard AAL arise. Even at the propositional level there are interesting logics that fall out of the scope of the theory. It is the case of certain so-called *non-truth-functional logics*, such as the paraconsistent systems of da Costa [dC74]. The major problem with these logics is that they lack congruence for some connective(s). Roughly speaking, a logic is said to be paraconsistent if its consequence relation is not *explosive* [CM02]. We say that a logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ is an *explosive logic* with respect to a *negation* connective $\neg$ if, for all formulas $\varphi$ and $\psi$, it is true that $\{\varphi, \neg\varphi\} \vdash \psi$.

**Example 2.3.26** (Paraconsistent Logic $\mathcal{C}_1$ of da Costa)**.**

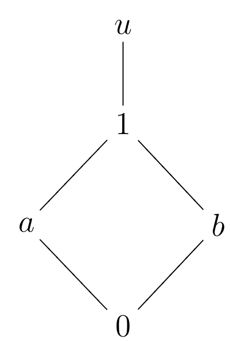
It was proved, first by Mortensen [Mor80], and afterwards by Lewin, Mikenberg and Schwarze [LMS91] that $\mathcal{C}_1$ is not algebraizable according with the standard notion of algebraizable logic. So, we can say that $\mathcal{C}_1$ is an example of a logic whose non-algebraizability is well-studied. Nevertheless, it is rather strange that a relatively well-behaved logic fails to have an algebraic counterpart.

In the proof of the next theorem we present here the argument used in [LMS91] for proving non-algebraizability of $\mathcal{C}_1$, since it is very interesting and it makes use of the tools of AAL we have just introduced.

Theorem. *$\mathcal{C}_1$ is not algebraizable.*

*Proof.* Recall from Theorem 2.3.22 that if a logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ is algebraizable with equivalent algebraic semantics $K$ then, for every $\Sigma$-algebra $\mathbf{A}$, $\Omega_{\mathbf{A}}$ is an isomorphism between the lattice of $\mathcal{L}$-filters and the lattice of $K$-congruences of $\mathbf{A}$. So, to show that a given logic is not algebraizable one as to present an algebra $\mathbf{A}$, such that $\Omega_{\mathbf{A}}$ is not an isomorphism. In the case of $\mathcal{C}_1$ consider the following $\Sigma_{\mathcal{C}_1}$-algebra $\mathbf{A} = \langle A_\phi, \mathbf{t}_\mathbf{A}, \mathbf{f}_\mathbf{A}, \neg_\mathbf{A}, \wedge_\mathbf{A}, \vee_\mathbf{A}, \Rightarrow_\mathbf{A}\rangle$ such that:

- $A_\phi = \{0, a, b, 1, u\}$
- $\mathbf{t}_\mathbf{A} = 1$ and $\mathbf{f}_\mathbf{A} = 0$
- $\langle A_\phi, \wedge_\mathbf{A}, \vee_\mathbf{A}\rangle$ is a lattice as depicted




- the operations $\neg_\mathbf{A}$ and $\Rightarrow_\mathbf{A}$ are given in the following tables

| $\Rightarrow_\mathbf{A}$ | $u$ | 1 | a | b | 0 |
|---|---|---|---|---|---|
| $u$ | $u$ | $u$ | $a$ | $b$ | 0 |
| 1 | $u$ | 1 | $a$ | $b$ | 0 |
| $a$ | $u$ | 1 | 1 | $b$ | $b$ |
| $b$ | $u$ | 1 | $a$ | 1 | $a$ |
| 0 | $u$ | 1 | 1 | 1 | 1 |

| | $\neg_\mathbf{A}$ |
|---|---|
| $u$ | 1 |
| 1 | 0 |
| $a$ | $b$ |
| $b$ | $a$ |
| 0 | 1 |

Notice that if we restrict the operations to the elements $\{a, b, 0, 1\}$ then we get a Boolean algebra. Consider the subsets $D_1 = \{u, 1, a\}$ and $D_2 = \{u, 1, b\}$ of $A_\phi$. It is an easy exercise to prove that $\langle \mathbf{A}, D_1\rangle$ and $\langle \mathbf{A}, D_2\rangle$ are matrix models of $\mathcal{C}_1$. So, both $D_1$ and $D_2$ are $\mathcal{C}_1$-filters of $\mathbf{A}$. It is also an easy exercise to see that there are no non-trivial congruences on $\mathbf{A}$. So, we have just the trivial congruences $\nabla = \{\langle x, x\rangle : x \in A_\phi\}$ and $\mathbf{\Delta} = A_\phi \times A_\phi$.

Finally, just note that the largest congruence on $\mathbf{A}$ compatible with both $D_1$ and $D_2$ is the trivial congruence $\mathbf{\Delta}$. So, $\Omega_\mathbf{A}$ cannot be an isomorphism. $\square$

Although it is defined as a logic weaker than Classical Propositional Logic (CPL), it happens that the defined connective $\sim$ indeed corresponds to classical negation.

Therefore, the fragment $\{\sim, \wedge, \vee, \Rightarrow, \mathbf{t}, \mathbf{f}\}$ corresponds to CPL. So, despite of its innocent aspect, $\mathcal{C}_1$ is a non-truth-functional logic, namely it lacks congruence for its paraconsistent negation connective with respect to the equivalence $\Leftrightarrow$ that algebraizes the CPL fragment. Exploring this fact, da Costa himself introduces in [dC66] a so-called class of Curry algebraic structures as a possible algebraic counterpart of $\mathcal{C}_1$. In fact, nowadays, these algebraic structures are known as da Costa algebras [CdA84]. However, their precise nature remains unknown, given the non-algebraizability results reported above. The problem of algebraizing $\mathcal{C}_1$ and its connection with da Costa algebras an valuation semantics will be the subject of a forthcoming section.

## 2.4 Remarks

We conclude with a brief summary of the achievements of this chapter. We have introduced the central notion of logic using two approaches: the consequence based approach and the operator based approach. These were shown to be equivalent ways of introducing a logic. We also studied the theory space of a logic, in particular we proved that it is always a complete partial order.

With the aim of studying not only the logics but also the relation between logics, we introduced the notion of map between logics. The existence of a map between two logics induces a strong connection between the corresponding deductive systems. By strengthening the conditions of map we obtained the notions of conservative map and of strong representation. Nonetheless to say that existence of notion of strong representation between two logics implies a tight connection between them. In fact, almost all of the proposals in the literature for the notion of equivalence between logics are particular cases of the notion of strong representation. They just differ on the way they map the formulas between the logics. Of course, our main motivation about strong representation is that it abstracts the tight connection between an algebraizable logic and the equational logic defined over the algebraic semantics of the logic. We extended the notion of map to the theory space and proved several results involving the notions of conservative map and strong representation. Namely, we proved that the existence of a strong representation between two logics implies the isomorphism of the correspondent theory spaces.

We introduced several relevant notions and tools of many-sorted behavioral logic, paving therefore the way to a many-sorted behavioral generalization of the tools of AAL that we will undertake in the subsequent chapters. We started by introducing the notion of many-sorted signature. The notion of many-sorted logic was then introduced as a logic whose language is built from a many-sorted signature. We then introduced the notion of many-sorted algebra along with several notions and constructions usual in the realm of universal algebra. Many-sorted equational logic associated with a class of many-sorted algebras was introduced. We presented the notions of variety and quasivariety of logics along with some important characterizations theorems. We introduced the notion of hidden many-sorted signature as a many-sorted signature which is divided in a visible and a hidden part. The key notion of behavioral equivalence was introduced. Two hidden elements of an algebra are behaviorally equivalent if they cannot be distinguished by any visible operation. Substituting, in the hidden part, the role of equality by behaviorally equivalence we obtained behavioral versions of the notions of standard universal algebra.

We ended with a review of some of the standard key notions and results of AAL. These include not only the central notion of algebraizable logic, but also a substantial part of the so-called Leibniz hierarchy, along with the respective characterization results. Some important semantic notions and results were present, namely those involving logical matrices. Finally, we presented some well-known examples in the area of AAL, along with a discussion on the limitations of the standard tools of AAL.

# Chapter 3

# Behavioral abstract algebraic logic

We have seen in the previous chapter that, despite of its success, the scope of application of the standard tools of AAL is relatively limited. Logics with a many-sorted language, even if well behaved, are good examples of logics that fall out of their scope. It goes without saying that rich logics, with many-sorted languages, are essential to specify and reason about complex systems, as also argued and justified by the theory of combined logics [SSC99]. However, even in the class of propositional based (single-sorted) logics many interesting examples simply fall out of the scope of the standard tools of AAL. In particular, there are well-known examples of logics that may be seen as resulting from the extension (by adding connectives and rules) of algebraizable logics that turn out not to be algebraizable. This is the case, for example, of certain non-truth-functional logics, herein understood as logics which are extensions of algebraizable logics by some new connectives not satisfying the congruence property with respect to the equivalence of the algebraizable fragment. With the proliferation of logical systems, with applications ranging from computer science, to mathematics and philosophy, the examples of non-algebraizable logics that, therefore, lack from a meaningful and insightful algebraic counterpart are expected to become more and more common.

Although the standard tools of AAL can associate a class of algebras to every logic, the connection between a non-algebraizable logic and the corresponding class of algebras is, of course, not very strong nor very interesting. This phenomenon is well known and may happen for several reasons, and in different degrees, depending on whether the Leibniz operator will lack the properties of injectivity, monotonicity or commutation with inverse substitutions. The particular issue of non-injectivity, staying within the realm of protoalgebraic and equivalential logics, has been carefully studied in [FJ01], where the authors restrict the models of the logic by considering just the matrices with a so-called Leibniz filter. Although this is a very interesting approach, the resulting logic is, of course, different from the original one. Contrarily to what is done in [FJ01], we do not want, at all, to change the logic we start from. Our strategy is rather to change a bit the algebraic perspective. This is achieved by considering behavioral equivalence rather then equality as the basic concept. Our aim in this chapter is precisely to propose and study an extension of the tools of AAL that may encompass some of these less orthodox logics while still associating to them meaningful and insightful algebraic counterparts. Contrarily to what is done in [FJ01], we do not want, at all, to change the logic we start from. Our strategy is rather to change a bit the algebraic perspective. This is achieved by considering behavioral equivalence rather then equality as the basic concept.

In more concrete terms, we introduce and study a generalization of the standard tools of AAL obtained by using many-sorted behavioral logic in the role traditionally played by unsorted equational logic. We start by setting up the framework for our many-sorted behavioral approach. We then introduce the central notion of $\Gamma$-behaviorally algebraizable logic, where $\Gamma$ is a subsignature of the original signature of the logic. The subsignature $\Gamma$ is a parameter and, once fixed, it means that the algebraic part of the behavioral algebraization process is built over the notion of $\Gamma$-behavioral equivalence. We then introduce the notion of $\Gamma$-behaviorally equivalential and use it in some necessary conditions for a logic to be behaviorally algebraizable. We prove that the novel notion of behaviorally algebraizable logic is not as broad as it becomes trivial, by proving that it is in the class of standard protoalgebraic logics, which is considered the largest class of logics amenable to the methods of AAL. We continue by introducing a behavioral version of the Leibniz operator and engage on a generalization of the Leibniz hierarchy. We introduce the behavioral versions of protoalgebraic logic and of weakly algebraizable logic along with several characterization results. Besides the results involving the Leibniz operator itself, we have also results involving the notion of set of behavioral equivalence formulas. Characterization results for the class of behaviorally algebraizable and behaviorally equivalential logics are also obtained. We end the chapter with some intrinsic and sufficient conditions that are very useful in practice to show that a given logic is behavioral algebraizable.

The chapter is organized as follows. In Section 1 we present the notion of behaviorally algebraizable logic. We also establish some necessary conditions for a logic to be behaviorally algebraizable. Section 2 is devoted to the study of a behavioral generalization of the so-called Leibniz hierarchy. In Section 3 we prove some intrinsic and sufficient condition that are very useful in practice to show that a given logic is behavioral algebraizable. We conclude, in Section 4, with some remarks.

## 3.1 Generalizing algebraization

In this section we propose a behavioral extension of the notion of algebraizable logic. The role that unsorted equational logic plays in the standard theory of algebraization is, in our work, played by many-sorted behavioral logic.

Along with our proposal, we present some necessary conditions for a logic to be behaviorally algebraizable. These are important to show that the generalized notion is not as broad that it becomes trivial.

Recall that our aim is to build a framework general enough to capture some logics that fall out of the scope of the standard tools of AAL. With respect to many-sortedness, some work has already been presented in [CG07]. Our aim here is to go further ahead and to capture also logics that are not algebraizable in the standard sense (although they still seem to be sufficiently well-behaved to be studied in algebraic terms). Namely the so-called non-truth-functional logics, which are extensions of algebraizable logics by some new connectives not satisfying the congruence property with respect to the equivalence of the algebraizable fragment. Many-sorted behavioral logic seems to be the correct tool for this enterprise since, besides providing a rich many-sorted framework, it allows the isolation of the fragment of the language that corresponds to the algebraizable part of the logic. In its more general form, as introduced for instance in [GM00], behavioral equivalence is an equivalence relation that is only required to be compatible with respect to the operations in a given subsignature of the original signature.

Consider given a many-sorted language generated from a many-sorted signature $\Sigma = \langle S, F \rangle$. Recall that we have a distinguished sort $\phi$ of formulas. In the many-sorted approach to AAL presented in [CG07] the theory was developed by replacing the role of unsorted equational logic by many-sorted behavioral logic over the same signature and taking the sort $\phi$ as the unique visible sort. Despite the success of this generalization to cope with many-sorted logics, a lot of non-algebraizable logics could still not be captured. This is due to the fact that, since the sort $\phi$ is considered visible, we have equational reasoning about formulas, which forces every connective to be congruent. To allow for non-congruent connectives, the sort $\phi$ must be a hidden sort too, so that one is forced to reason behaviorally about formulas as well. This can be achieved by considering behavioral logic over an extended signature.

**Definition 3.1.1.** Given a many-sorted signature $\Sigma = \langle S, F \rangle$ we define an *extended signature* $\Sigma^o = \langle S^o, F^o \rangle$ such that $S^o = S \biguplus \{v\}$, where $v$ is to be considered the sort of *observations* of formulas. The indexed set of operations $F^o = \{{F^o}_{ws}\}_{w \in (S^o)^*, s \in S^o}$ is such that:

- ${F^o}_{ws} = F_{ws}$ if $ws \in S^*$;
- $F^o_{\phi v} = \{o\}$;
- $F^o_{ws} = \emptyset$ otherwise.

Intuitively, we are just extending the signature with a new sort $v$ for the observations that we can perform on formulas using operation $o$. The extended hidden signature obtained from $\Sigma$, that we also denote by $\Sigma^o$, can then be defined as $\langle \Sigma^o, \{v\} \rangle$. The choice of $v$ as the name for the new sort is now clear. It is intended to represent the only visible sort of the extended hidden signature.

In the sequel, given a signature $\Sigma = \langle S, F \rangle$, a subsignature $\Gamma$ of $\Sigma$ and a class $K$ of $\Sigma^o$-algebras, we use $Bhv_\Sigma^{K,\Gamma}$ to refer to the logic $\langle Eq_{\Sigma^o}, \models\!\equiv_\Sigma^{K,\Gamma} \rangle$, where $\models\!\equiv_\Sigma^{K,\Gamma}$ is the behavioral consequence relation over $\Sigma^o$ associated with $K$ and $\Gamma$.

First of all, note that $K$ is a class of algebras over the extended signature and not just over the original one. Nevertheless, given a $\Sigma^o$-algebra, we can always consider its restriction to $\Sigma$ which is, of course, a $\Sigma$-algebra. Note also that in this behavioral consequence over the extended signature, and for each $s \in S$,

$$\mathcal{E}_\Sigma^\Gamma[x\!:\!s] = \{o(c) : c \in \mathcal{C}_{\Sigma,\phi}^\Gamma[x\!:\!s]\ \}$$

is the set of possible experiments of sort $s$.

From $Bhv_\Sigma^{K,\Gamma}$ we can define a logic $BEqn_\Sigma^{K,\Gamma} = \langle Eq_\Sigma, \vDash_{\Sigma,bhv}^{K,\Gamma} \rangle$ where $\vDash_{\Sigma,bhv}^{K,\Gamma}$ is just the restriction of $\models\!\equiv_\Sigma^{K,\Gamma}$ to $\Sigma$. The set of theories of $BEqn_\Sigma^{K,\Gamma}$ is denoted by $Th_\Sigma^{K,\Gamma}$. With this construction we obtain an important ingredient of our theory: a logic for behaviorally reason about equations over the original signature $\Sigma$.

The following lemma states a property of this behavioral consequence that will be often used in the sequel.

**Lemma 3.1.2.** *Let $\Sigma = \langle S, F \rangle$ be a many-sorted signature, $\Gamma$ a subsignature of $\Sigma$, $K$ a class of $\Sigma^o$-algebras, $t \approx t' \in Eq_{\Sigma,s}(X)$ an equation, $c(x\!:\!s, x_1\!:\!s_1, \ldots, x_m\!:\!s_m) \in \mathcal{C}_{\Sigma,s'}^\Gamma[x:s]$ a context, and $\langle t_1, \ldots, t_m \rangle \in T_{\Sigma,s_1}(X) \times \ldots \times T_{\Sigma,s_m}(X)$. Then, we have that*

$$t \approx t' \vDash_{\Sigma,bhv}^{K,\Gamma} c(t, t_1, \ldots, t_m) \approx c(t', t_1, \ldots, t_m).$$

*Proof.* Let $\mathbf{A} \in K$ and $h$ an assignment over $\Sigma$. Suppose that $h(t) \equiv_\Gamma h(t')$. We aim to prove that $h(c(t, t_1, \ldots, t_m)) \equiv_\Gamma h(c(t', t_1, \ldots, t_m))$. Let

$\epsilon(x':s', x'_1:s'_1, \ldots, x'_n:s'_n) \in \mathcal{E}^{\Gamma}_{\Sigma}[x':s']$ and $\langle a_1, \ldots, a_n\rangle \in A_{s'_1} \times \ldots \times A_{s'_n}$. Note that $\epsilon(c(x:s, x_1:s_1, \ldots, x_m:s_m), x'_1:s'_1, \ldots, x'_n:s'_n) \in \mathcal{E}^{\Gamma}_{\Sigma}[x:s]$. Since we are assuming that $h(t) \equiv_\Gamma h(t')$, we have that

$$\epsilon_{\mathbf{A}}(c_{\mathbf{A}}(h(t), h(t_1), \ldots, h(t_m)), a_1, \ldots, a_n) = \epsilon_{\mathbf{A}}(c_{\mathbf{A}}(h(t'), h(t_1), \ldots, h(t_m)), a_1, \ldots, a_n).$$

This is equivalent to the fact that

$$\epsilon_{\mathbf{A}}(h(c(t, t_1, \ldots, t_m)), a_1, \ldots, a_n) = \epsilon_{\mathbf{A}}(h(c(t', t_1, \ldots, t_m)), a_1, \ldots, a_n).$$

So, we can conclude that $h(c(t, t_1, \ldots, t_n)) \equiv_\Gamma h(c(t, t_1, \ldots, t_n))$.

□

Consider given a subsignature $\Gamma$ of $\Sigma$. We now introduce the main notion of $\Gamma$-behaviorally algebraizable logic.

**Definition 3.1.3.** ($\Gamma$-behaviorally algebraizable logic)
A many-sorted logic $\mathcal{L} = \langle \Sigma, \vdash\rangle$ is $\Gamma$-*behaviorally algebraizable* if there exists a class $K$ of $\Sigma^o$-algebras, a set $\Theta(\xi) \subseteq Comp^{K,\Gamma}_{\Sigma}(\{\xi\})$ of $\phi$-equations and a set $\Delta(\xi_1, \xi_2) \subseteq T_{\Gamma,\phi}(\{\xi_1, \xi_2\})$ of formulas such that, for every $T \cup \{t\} \subseteq L_\Sigma(X)$ and for every set $\Phi \cup \{t_1 \approx t_2\}$ of $\phi$-equations,

i) $T \vdash t$ iff $\Theta[T] \vDash^{K,\Gamma}_{\Sigma,bhv} \Theta(t)$;

ii) $\Phi \vDash^{K,\Gamma}_{\Sigma,bhv} t_1 \approx t_2$ iff $\Delta[\Phi] \vdash \Delta(t_1, t_2)$;

iii) $\xi \dashv\vdash \Delta[\Theta(\xi)]$;

iv) $\xi_1 \approx \xi_2 =\mid\mid=^{K,\Gamma}_{\Sigma,bhv} \Theta[\Delta(\xi_1, \xi_2)]$;

Following the standard notation of AAL, $\Theta$ is called the set of *defining equations*, $\Delta$ the set of *equivalence formulas*, and $K$ a *behaviorally equivalent algebraic semantics* for $\mathcal{L}$. Note that this definition is parameterized by the choice of the subsignature $\Gamma$ of $\Sigma$. We say that a logic $\mathcal{L} = \langle \Sigma, \vdash\rangle$ is behaviorally algebraizable if there exists a subsignature $\Gamma$ of $\Sigma$ such that $\mathcal{L}$ is $\Gamma$-behaviorally algebraizable.

If the set of defining equations and of equivalence formulas are finite we say that $\mathcal{L}$ is *finitely* $\Gamma$*-behaviorally algebraizable*. As in standard AAL, conditions *i*) and *iv*) jointly imply *ii*) and *iii*), and vice-versa.

The following theorem, depicted in Fig. 3.1, is a characterization of behaviorally algebraizable logic using maps of logics.

**Theorem 3.1.4.** *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic and $\Gamma$ a subsignature of $\Sigma$. Then, $\mathcal{L}$ is $\Gamma$-behaviorally algebraizable iff there exists a class $K$ of $\Sigma^o$-algebras and a strong representation $\langle \theta, \tau \rangle$ of $\mathcal{L}$ in $BEqn_{\Sigma}^{K,\Gamma}$, such that $\theta$ is given by a $\Theta(\xi) \subseteq Comp_{\Sigma}^{K,\Gamma}(\{\xi\})$ of $\phi$-equations and $\tau$ is given by a set $\Delta(\xi_1, \xi_2) \subseteq T_{\Gamma,\phi}(\{\xi_1, \xi_2\})$ of formulas.*

*Proof.* The result follows from the fact that conditions *i*), *ii*), *iii*) and *iv*) of the definition of $\Gamma$-behaviorally algebraizable logic are equivalent to the fact that the pair $\langle \theta, \tau \rangle$ defined, respectively, by $\Theta(\xi)$ and $\Delta(\xi_1, \xi_2)$, is a strong representation. □

**Many-sorted logic $\mathcal{L}$** $\xLeftrightarrow{\text{Strong representation}}$ **Many-sorted behavioral logic over $K$**

Figure 3.1: Behaviorally algebraizable logic

Consider given a $\Gamma$-behaviorally algebraizable logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ and let $\Delta(\xi_1, \xi_2)$ be the set of equivalence formulas. Consider the set $CC_{\Delta}[x:\phi] \subseteq \mathcal{C}_{\Sigma}^{\Sigma}[x:\phi]$ defined as follows: $c \in CC_{\Delta}[x:\phi]$ iff for every $\varphi, \psi \in L_{\Sigma}(X)$, we have that $\Delta(\varphi, \psi) \vdash \Delta(c[\varphi], c[\psi])$. We call $CC_{\Delta}[x:\phi]$ the set of *congruent contexts for* $\Delta$.

**Proposition 3.1.5.** *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic and $\Gamma$ a subsignature of $\Sigma$. Suppose that $\mathcal{L}$ is $\Gamma$-behaviorally algebraizable logic with $\Theta(\xi)$ a set of defining equations, $\Delta(\xi_1, \xi_2)$ a set of equivalence formulas and $K$ a $\Gamma$-behaviorally equivalent algebraic semantics. Then $\mathcal{C}_{\Gamma,\phi}[x:\phi] \subseteq CC_{\Delta}[x:\phi]$. Moreover, every $c \in CC_{\Delta}[x:\phi]$ is congruent with respect to $\vDash_{\Sigma,bhv}^{K,\Gamma}$.*

*Proof.* Using Lemma 3.1.2 we can conclude that, for every $c \in \mathcal{C}_{\Gamma,\phi}[x:\phi]$, we have that $\xi_1 \approx \xi_2 \vDash_{\Sigma,bhv}^{K,\Gamma} c[\xi_1] \approx c[\xi_2]$. Using now the properties of the set of equivalence formulas, we can easily conclude that $\Delta(\xi_1, \xi_2) \vdash \Delta(c[\xi_1], c[\xi_2])$. So, $\mathcal{C}_{\Gamma,\phi}[x:\phi] \subseteq CC_{\Delta}[x:\phi]$.

Now let $c \in CC_{\Delta}[x:\phi]$. So, we have that $\Delta(\xi_1, \xi_2) \vdash \Delta(c[\xi_1], c[\xi_2])$. Using properties *i*) and *iv*) of the set of defining equations we can conclude that $\xi_1 \approx \xi_2 \vDash_{\Sigma,bhv}^{K,\Gamma} c[\xi_1] \approx c[\xi_2]$. So, $c$ is congruent with respect to $\vDash_{\Sigma,bhv}^{K,\Gamma}$. □

It is well-known for behavioral logic [Ros00] that, when a context is behaviorally congruent, we can always add it to the set of admissible contexts without changing the behavioral consequence. Therefore, although we can have $\mathcal{C}_{\Gamma,\phi} \subset CC_{\Delta}$, the behavioral consequence is the same as if we had chosen the whole $CC_{\Delta}$ as the set of contexts.

We now focus on trying to answer a natural question that arises at this point: what are the limits of this new definition of algebraizability? We will see later on that the notion of behaviorally algebraizable logic extends the standard notion of algebraizable logic. Still, this is at least as important as knowing whether the notion is so broad that everything becomes behaviorally algebraizable with an appropriate choice of $\Gamma$. In this direction, we end this section by studying some necessary conditions for a logic to be behaviorally algebraizable. They will help us to show that the limits of applicability of the notion are very reasonable and not as broad as it might seem.

In [PW74] Prucnal and Wrónski introduce the standard notion of equivalential logic. Equivalence systems generalize the well-known phenomenon of classical propositional calculus where the equivalence of formulas can be expressed by the equivalence symbol $\Leftrightarrow$, i.e., for each theory $T$, $\langle \xi_1, \xi_2 \rangle \in \Omega(T)$ iff $T \vdash \xi_1 \Leftrightarrow \xi_2$. We extend the notion of equivalential logic to our behavioral setting.

**Definition 3.1.6.** A many-sorted logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ is $\Gamma$-behaviorally equivalential if there exists a set $\Delta(\xi_1, \xi_2) \subseteq T_{\Gamma,\phi}(\{\xi_1, \xi_2\})$ of formulas such that for every $\varphi, \psi, \delta, \varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_n \in L_\Sigma(X)$:

(R) $\vdash \Delta(\varphi, \varphi)$;

(S) $\Delta(\varphi, \psi) \vdash \Delta(\psi, \varphi)$;

(T) $\Delta(\varphi, \psi), \Delta(\psi, \delta) \vdash \Delta(\varphi, \delta)$;

(MP) $\Delta(\varphi, \psi), \varphi \vdash \psi$;

($\mathrm{RP}_\Gamma$) $\Delta(\varphi_1, \psi_1), \ldots, \Delta(\varphi_n, \psi_n) \vdash \Delta(c[\varphi_1, \ldots, \varphi_n], c[\psi_1, \ldots, \psi_n])$
for every $c : \phi^n \to \phi \in Der_{\Gamma,\phi}$.

In this case, $\Delta$ is called a $\Gamma$-*behavioral equivalence set* for $\mathcal{L}$. Recall that a congruence is an equivalence relation that is compatible with *all* operations. Note that the main difference between this behavioral version of equivalentiality and the standard notion is that in the former the set $\Delta$ is no longer assumed to define a congruence.

Instead, it is only assumed to preserve the operations of the subsignature $\Gamma$. We say that a logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ is behaviorally equuivalential if there exists a subsignature $\Gamma$ of $\Sigma$ such that $\mathcal{L}$ is $\Gamma$-behaviorally equivalential. In the following proposition we present a first necessary condition for behavioral algebraizability. The result extends a well-known standard result of AAL.

**Proposition 3.1.7.** *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic and $\Gamma$ a subsignature of $\Sigma$. If $\mathcal{L}$ is $\Gamma$-behaviorally algebraizable then it is $\Gamma$-behaviorally equivalential.*

*Proof.* Suppose $\mathcal{L}$ is behaviorally algebraizable with $\Theta(\xi)$, $\Delta(\xi_1, \xi_2)$, $\Gamma$ and $K$. Using the properties of $\Theta(\xi)$ and $\Delta(\xi_1, \xi_2)$ it is easy to prove that $\Delta(\xi_1, \xi_2)$ satisfies (R), (S) and (T). For (MP), note that, since $\mathcal{L}$ is algebraizable, $\Delta(\varphi, \psi), \varphi \vdash \psi$ is equivalent to $\varphi \approx \psi, \Theta(\varphi) \vDash^{K,\Gamma}_{\Sigma,bhv} \Theta(\psi)$. But this last condition follows from the fact that $\Theta(\xi) \in Comp^{K,\Gamma}_{\Sigma}(\{\xi\})$. Condition ($\mathrm{RP}_\Gamma$) follows easily from the the fact that, given $t_1 \approx t_2 \in Eq_{\Sigma,s}(X)$ and $c \in \mathcal{C}^{\Gamma}_{\Sigma}[x\!:\!s]$, we have that $t_1 \approx t_2 \vDash^{K,\Gamma}_{\Sigma,bhv} c[t_1] \approx c[t_2]$. □

From the notion of behaviorally equivalential logic we can isolate a simpler necessary condition for behavioral algebraization.

**Definition 3.1.8.** A many-sorted logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ *has an equivalence set (of formulas)* if there exists a set $\Delta \subseteq L_\Sigma(\xi_1, \xi_2)$ of formulas that satisfies the following conditions:

**(R)** $\vdash \Delta(\varphi, \varphi)$;

**(S)** $\Delta(\varphi, \psi) \vdash \Delta(\psi, \varphi)$;

**(T)** $\Delta(\varphi, \psi), \Delta(\psi, \delta) \vdash \Delta(\varphi, \delta)$;

**(MP)** $\Delta(\varphi, \psi), \varphi \vdash \psi$;

Note that we dropped the condition that $\Delta$ should be congruent for all operations of some subsignature $\Gamma$ of $\Sigma$.

The following result is an immediate consequence of Proposition 3.1.7, and it is also a necessary condition for a logic to be behaviorally algebraizable.

**Corollary 3.1.9.** *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic. If $\mathcal{L}$ is $\Gamma$-behaviorally algebraizable for some subsignature $\Gamma$ of $\Sigma$, then it has an equivalence set.*

In some sense, to be behaviorally algebraizable a logic must be at least expressive enough to enable the definition of an equivalence by means of a set of formulas in two variables. This is a natural requirement since a logic that does not have any equivalence set cannot represent within itself any kind of behavioral equivalence, and so, it must fail to be behaviorally algebraizable. One such example is the inf-sup fragment of classical propositional logic, where no equivalence set can be defined. This logic is a well-known example of a non-protoalgebraic logic.

We can give another necessary condition for a logic to be behaviorally algebraizable. Although it is a weaker condition, it is an important one since it is related with the notion of protoalgebraic logic.

Recall, from Theorem 2.3.10, that in AAL a standard characterization of protoalgebraic logic can be given by the existence of a set $\Delta(\xi_1, \xi_2) \subseteq L_\Sigma(X)$ of formulas with two distinguished variables of sort $\phi$, and possibly parametric variables, satisfying $\vdash \Delta(\xi, \xi)$ (reflexivity) and $\xi_1, \Delta(\xi_1, \xi_2) \vdash \xi_2$ (detachment). In [Mar04] this characterization of protoalgebraicity is proved for many-sorted logics. So, as an immediate consequence of Proposition 3.1.7, we have the following result.

**Corollary 3.1.10.** *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic. If $\mathcal{L}$ is behaviorally algebraizable then it is many-sorted protoalgebraic.*

This result let us conclude that our generalized notion of algebraizable logic is not too broad. A behaviorally algebraizable logic necessarily belongs to what is considered the largest class of logics amenable to the tools of AAL: the class of protoalgebraic logics.

## 3.2 The behavioral Leibniz hierarchy

One of the goals of AAL is to discover general criteria for a class of algebras (or for a class of mathematical objects closely related to algebra, such as logical matrices) to be the algebraic counterpart of a logic, and to develop the methods for obtaining it. Another important goal of AAL is a classification of logics based on the properties of their algebraic counterparts. Ideally, when it is known that a given logic belongs to a particular group in the classification, one has general theorems providing important information about its properties. Following these goals, we propose in this section a behavioral generalization of some of the standard notions and results of AAL. This is basically a systematic continuation of the effort that was already started in the previous section. Our main aim is to draw a behavioral Leibniz hierarchy that generalizes part of the standard Leibniz hierarchy.

Until now we have focused on generalizing the notion of algebraizable logic. To further support our methodology, we now show how to extend other standard notions and results of AAL to the behavioral setting. Recall that one of the main tools of AAL is the Leibniz operator. It can help to build a hierarchy of classes of logics using its properties. The behavioral hierarchy is depicted in Fig. 3.2. We use it as a roadmap for the remainder of the section.

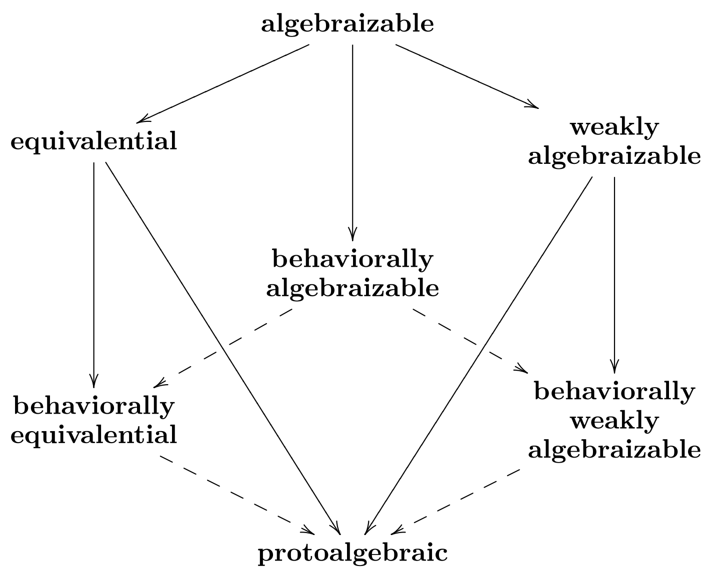



Figure 3.2: A view of the behavioral Leibniz hierarchy.

First of all, we need to introduce the behavioral variant of the notion of Leibniz operator. This behavioral version is based on a generalization of the notion of congruence: the notion of Γ-congruence, for a subsignature Γ of the original signature.

**Definition 3.2.1.** Consider given a signature $\Sigma = \langle S, F\rangle$ and a subsignature $\Gamma$ of $\Sigma$. A $\Gamma$-*congruence* over a $\Sigma$-algebra $\mathbf{A}$ is an equivalence relation $\theta$ over $\mathbf{A}$ such that, for every $\langle a_1, b_1\rangle \in \theta_{s_1}, \ldots, \langle a_n, b_n\rangle \in \theta_{s_n}$ and $f : s_1 \ldots s_n \to s \in \Gamma$, we have that:

$$\langle f_{\mathbf{A}}(a_1, \ldots, a_n), f_{\mathbf{A}}(b_1, \ldots, b_n)\rangle \in \theta_s.$$

We denote the set of all Γ-congruences over a Σ-algebra $\mathbf{A}$ by $Con^{\Sigma}_{\Gamma}(\mathbf{A})$. The difference between a Γ-congruence and a congruence over $\mathbf{A}$ is that a Γ-congruence is assumed to satisfy the congruence property just for contexts generated from the subsignature $\Gamma$. The contexts outside $\Gamma$ do not necessarily satisfy the congruence property. Clearly, a congruence is just a $\Sigma$-congruence in our setting, that is, we just have to take $\Gamma = \Sigma$.

A $\Gamma_\phi$-*congruence* over a $\Sigma$-algebra $\mathbf{A}$ is a $\phi-$reduct $\theta$ of a $\Gamma$-congruence over $\mathbf{A}$, that is, an equivalence relation $\theta$ over $A_\phi$ such that, if $\langle a_1, b_1\rangle \in \theta, \ldots, \langle a_n, b_n\rangle \in \theta$ and $f : \phi^n \to \phi \in Der_{\Gamma,\phi^n\phi}$, then

$$\langle f_{\mathbf{A}}(a_1, \ldots, a_n), f_{\mathbf{A}}(b_1, \ldots, b_n)\rangle \in \theta.$$

The set of all $\Gamma_\phi$-congruences of $\mathbf{A}$ is denoted by $Con^{\Sigma}_{\Gamma,\phi}(\mathbf{A})$. As we will see in the sequel, the importance of $\Gamma_\phi$-congruences reflects the distinguished role that the sort $\phi$ plays in our theory.

The next lemma is a generalization for $\Gamma$-congruences of a well-known result for congruences [BS81, MT92].

**Lemma 3.2.2.** *Given a signature $\Sigma$ and a subsignature $\Gamma$ of $\Sigma$, $Con^{\Sigma}_{\Gamma}(\mathbf{A})$ is a complete sublattice of $Eqv^{\Sigma}(\mathbf{A})$, the complete lattice of equivalences on $\mathbf{A}$.*

*Proof.* We only need to prove that $Con^{\Sigma}_{\Gamma}(\mathbf{A})$ is closed under the supremum (join) and the infimum (meet) of $Eqv^{\Sigma}(\mathbf{A})$. To verify that $Con^{\Sigma}_{\Gamma}(\mathbf{A})$ is closed under arbitrary intersections is straightforward. For arbitrary joins in $Con^{\Sigma}_{\Gamma}(\mathbf{A})$ suppose $\alpha_i \in Con^{\Sigma}_{\Gamma}(\mathbf{A})$ for $i \in I$. Then, if $f : s_1 \ldots s_n \to s \in \Gamma$ is a $\Gamma$-operation and

$$\langle a_1, b_1\rangle \in (\bigvee_{i\in I} \alpha_i)_{s_1}, \ldots, \langle a_n, b_n\rangle \in (\bigvee_{i\in I} \alpha_i))_{s_n},$$

where $\bigvee$ is the join of $Eqv^{\Sigma}(\mathbf{A})$, then it follows that one can find $i_0, \ldots, i_k \in I$ such that

$$\langle a_i, b_i\rangle \in (\alpha_{i_0} \circ \ldots \circ \alpha_{i_k})_{s_i}, \qquad 0 \leq i \leq n.$$

An easy argument then suffices to show that

$$\langle f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)\rangle \in (\alpha_{i_0} \circ \ldots \circ \alpha_{i_k})_s.$$

Therefore $\bigvee_{i\in I} \alpha_i$ is a $\Gamma$-congruence on $\mathbf{A}$.

□

It is easy to see that $Con^{\Sigma}_{\Gamma,\phi}(\mathbf{A})$ is a complete sublattice of $Eqv^{\Sigma_{|\phi}}(\mathbf{A}_{|\phi})$. The fact that every theory of $\vDash^{K,\Gamma}_{\Sigma,bhv}$ is a $\Gamma$-congruence over $\mathbf{T}_{\mathbf{\Sigma}}(\mathbf{X})$ is an easy exercise and generalizes the well-known relation between $\vDash_K$ and $Con^{\Sigma}_{\Sigma}(\mathbf{T_\Sigma}(\mathbf{X}))$. A $\Gamma$-congruence $\theta$ over a $\Sigma$-algebra $\mathbf{A}$ is compatible with a set $\Phi \subseteq A_\phi$ if for every $a_1, a_2 \in A_\phi$, if $\langle a_1, a_2\rangle \in \theta_\phi$ and $a_1 \in \Phi$ then $a_2 \in \Phi$.

Recall that the Leibniz congruence is the largest congruence compatible with a given $\mathcal{L}$-theory. The following lemma asserts the existence of the largest $\Gamma$-congruence over $\mathbf{T_\Sigma}(\mathbf{X})$ compatible with a given $\mathcal{L}$-theory $T$, thus generalizing the standard existence result.

**Lemma 3.2.3.** *Let $\mathcal{L} = \langle \Sigma, \vdash\rangle$ be a many-sorted logic and $\Gamma$ a subsignature of $\Sigma$. For each $T \in Th_{\mathcal{L}}$, there is a largest $\Gamma$-congruence compatible with $T$.*

*Proof.* Let $T \in Th_{\mathcal{L}}$ and consider the binary relation $\Phi_T$ over $\mathbf{T_\Sigma}(\mathbf{X})$ such that, for every $s \in S$, we have that $\langle t_1, t_2\rangle \in (\Phi_T)_s$ iff for every $c(x:s, x_1:s_1, \ldots, x_n:s_n) \in \mathcal{C}^{\Gamma}_{\Sigma,\phi}[x:s]$ and every $u_1 \in T_{\Sigma,s_1}(X), \ldots, u_n \in T_{\Sigma,s_n}(X)$ we have that

$$c[t_1, u_1, \ldots, u_n] \in T \quad \text{iff} \quad c[t_2, u_1, \ldots, u_n] \in T.$$

It is easy to conclude that $\Phi_T$ is a $\Gamma$-congruence compatible with $T$. We now prove that it is indeed the largest one.

Let $\alpha$ be a $\Gamma$-congruence over $\mathbf{T_\Sigma}(\mathbf{X})$ compatible with $T$. We aim to prove that $\alpha \subseteq \Phi_T$. Consider $\langle t_1, t_2\rangle \in \alpha_s$, for some $s \in S$. Since $\alpha$ is a $\Gamma$-congruence we can conclude that, for every $c(x:s, x_1:s_1, \ldots, x_n:s_n) \in \mathcal{C}^{\Gamma}_{\Sigma,\phi}[x:s]$ and every $u_1 \in T_{\Sigma,s_1}(X), \ldots, u_n \in T_{\Sigma,s_n}(X)$ we have that $\langle c[t_1, u_1, \ldots, u_n], c[t_2, u_1, \ldots, u_n]\rangle \in \alpha_\phi$. Using now the fact that $\alpha$ is compatible with $T$, we can conclude that $c[t_1, u_1, \ldots, u_n] \in T$ iff $c[t_2, u_1, \ldots, u_n] \in T$. So, we have that $\langle t_1, t_2\rangle \in \Phi_T$.

□

Now that we have proved that, given a $\mathcal{L}$-theory $T$, the largest $\Gamma$-congruence compatible with $T$ exists, we can use this result to extend the notion of Leibniz operator to this behavioral setting.

**Definition 3.2.4.** Let $\mathcal{L} = \langle \Sigma, \vdash\rangle$ be a many-sorted logic and $\Gamma$ a subsignature of $\Sigma$. The *behavioral Leibniz operator* on the term algebra, is

$$\begin{array}{rrcl} \Omega^{bhv}_{\Gamma}: & Th_{\mathcal{L}} & \to & Con^{\Sigma}_{\Gamma}(\mathbf{T_\Sigma}(\mathbf{X})) \\ & T & \mapsto & \text{largest } \Gamma\text{-congruence over } \mathbf{T_\Sigma}(\mathbf{X}) \text{ compatible with } T. \end{array}$$

As before, this definition is parametrized by the choice of $\Gamma$. Note that in the proof of Lemma 3.2.3 one can find an useful characterization of $\Omega^{bhv}_{\Gamma}$.

The behavioral Leibniz operator plays a central role in our approach. As we will see, some important classes of logics can be characterized by its properties. As a first example, we use the behavioral Leibniz operator to define a behavioral version of the notion of protoalgebraic logic. Consider given a subsignature $\Gamma$ of $\Sigma$. We now introduce the main notion of $\Gamma$-behaviorally protoalgebraic logic.

**Definition 3.2.5.** A many-sorted logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ is $\Gamma$-*behaviorally protoalgebraic* if, for every $T \in Th_{\mathcal{L}}$ and $\varphi, \psi \in L_\Sigma(X)$, we have that

$$\text{if } \langle \varphi, \psi \rangle \in \Omega^{bhv}_\Gamma(T) \text{ then } T, \varphi \vdash \psi \text{ and } T, \psi \vdash \varphi.$$

Again, this definition is parametrized by the choice of $\Gamma$. When this choice is important we say that a logic is $\Gamma$-behaviorally protoalgebraic. In what follows, if $\Gamma$ is clear from the context then it can be omitted. We say that a logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ is behaviorally protoalgebraic if there exists a subsignature $\Gamma$ of $\Sigma$ such that $\mathcal{L}$ is $\Gamma$-behaviorally protoalgebraic.

Our aim now is to prove some equivalent characterizations of the notion of behaviorally protoalgebraic logic. These equivalent characterizations are behavioral versions of the standard results for protoalgebraic logics. Some of them are useful to show that the standard and the behavioral notions of protoalgebraic logic coincide. Before the main characterization result, we need to introduce some preliminary notions and results.

Consider given a many-sorted logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ and a subsignature $\Gamma$ of $\Sigma$. Let $\sigma_{\xi_2 \to \xi_1}$ be the substitution over $\Sigma$ that substitutes $\xi_2$ with $\xi_1$, that is, $\sigma_{\xi_2 \to \xi_1, \phi}(\xi_2) = \xi_1$, and leaves the remaining variables unchanged. Using this substitution we can introduce the set

$$T^{\mathcal{L},\Gamma}_{\xi_1,\xi_2} = \{\varphi \in L_\Gamma(X) : \vdash \sigma_{\xi_2 \to \xi_1}(\varphi)\}.$$

Intuitively, $T^{\mathcal{L},\Gamma}_{\xi_1,\xi_2}$ is the set of all formulas $\varphi$ that become a theorem when every occurrence of $\xi_2$ in $\varphi$ is substituted by $\xi_1$. When the logic $\mathcal{L}$ is clear from the context, we write just $T^{\Gamma}_{\xi_1,\xi_2}$ instead of $T^{\mathcal{L},\Gamma}_{\xi_1,\xi_2}$.

The non-behavioral unsorted analogue of $T^{\Gamma}_{\xi_1,\xi_2}$ is used by Herrmann in [Her96] as a fundamental tool in the development of his theory. In our framework $T^{\Gamma}_{\xi_1,\xi_2}$ is also an important tool and, in particular, it can be used to give an alternative characterization of the notion of behavioral protoalgebraicity. Before we prove the following lemma that asserts some simple but very useful properties of $T^{\Gamma}_{\xi_1,\xi_2}$.

**Lemma 3.2.6.** *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic and $\Gamma$ a subsignature of $\Sigma$. Then,* *i)* *if* $\sigma$ *is a substitution over* $\Gamma$ *such that* $\sigma_{\xi_2\to\xi_1}(\sigma\xi_1) = \sigma_{\xi_2\to\xi_1}(\sigma\xi_2)$ *then* $T^{\Gamma}_{\xi_1,\xi_2}$ *is closed under* $\sigma$*, that is,* $\sigma[T^{\Gamma}_{\xi_1,\xi_2}] \subseteq T^{\Gamma}_{\xi_1,\xi_2}$*;*

*ii)* $\langle\xi_1,\xi_2\rangle \in \Omega^{bhv}_{\Gamma,\phi}((T^{\Gamma}_{\xi_1,\xi_2})^{\vdash})$*;*

*iii)* *assuming that* $\mathcal{L}$ *is* $\Gamma$*-behaviorally protoalgebraic, then* $\Delta(\xi_1,\xi_2) \subseteq L_{\Gamma}(\{\xi_1,\xi_2\})$ *is a* $\Gamma$*-behavioral equivalence for* $\mathcal{L}$ *iff* $\Delta \subseteq T^{\Gamma}_{\xi_1,\xi_2}$ *and* $\Delta^{\vdash} = (T^{\Gamma}_{\xi_1,\xi_2})^{\vdash}$*.*

*Proof.* *i)* Let $\sigma$ be a substitution such that $\sigma_{\xi_2\to\xi_1}(\sigma\xi_1) = \sigma_{\xi_2\to\xi_1}(\sigma\xi_2)$ and let $\varphi \in T^{\Gamma}_{\xi_1,\xi_2}$. By definition of $T^{\Gamma}_{\xi_1,\xi_2}$ we have that $\vdash \sigma_{\xi_2\to\xi_1}\varphi$. We note that

$$\sigma_{\xi_2\to\xi_1}(\sigma(\sigma_{\xi_2\to\xi_1}\varphi)) = \sigma_{\xi_2\to\xi_1}(\sigma\varphi)$$

which can be proved by an easy induction on the complexity of formulas. By structurality, $\vdash \sigma(\sigma_{\xi_2\to\xi_1}\varphi)$ and therefore we have also that $\vdash \sigma_{\xi_2\to\xi_1}(\sigma(\sigma_{\xi_2\to\xi_1}\varphi))$. So, we can conclude that $\vdash \sigma_{\xi_2\to\xi_1}(\sigma\varphi)$ by the above equality. This means that $\sigma\varphi \in T^{\Gamma}_{\xi_1,\xi_2}$.

*ii)* Let $c(x : \phi, x_1 : s_1, \ldots, x_n : s_n) \in \mathcal{C}^{\Gamma}_{\Sigma,\phi}[x : \phi]$ be a $\Gamma$-context, $u_1 \in T_{\Sigma,s_1}(X), \ldots, u_n \in T_{\Sigma,s_n}(X)$ and $\xi$ a variable. Note that $\sigma_{\xi_2\to\xi_1}(c(\xi_1,u_1,\ldots,u_n)) = \sigma_{\xi_2\to\xi_1}(c(\xi_2,u_1,\ldots,u_n))$. Therefore $c(\xi_1,u_1,\ldots,u_n) \in (T^{\Gamma}_{\xi_1,\xi_2})^{\vdash}$ iff $c(\xi_2,u_1,\ldots,u_n) \in (T^{\Gamma}_{\xi_1,\xi_2})^{\vdash}$. So, using Lemma 3.2.3 we can conclude that $\langle\xi_1,\xi_2\rangle \in \Omega^{bhv}_{\Gamma,\phi}((T^{\Gamma}_{\xi_1,\xi_2})^{\vdash})$.

*iii)* Assume first that $\Delta(\xi_1,\xi_2)$ is a $\Gamma$-behavioral equivalence for $\mathcal{L}$. Using (R) we can conclude that $\Delta \subseteq T^{\Gamma}_{\xi_1,\xi_2}$. So, $(\Delta)^{\vdash} \subseteq (T^{\Gamma}_{\xi_1,\xi_2})^{\vdash}$. For the other inclusion let us prove that $\Delta \vdash T^{\Gamma}_{\xi_1,\xi_2}$. Let $\varphi \in T^{\Gamma}_{\xi_1,\xi_2}$, that is, $\vdash \sigma_{\xi_2\to\xi_1}\varphi$. We have that $\Delta(\xi_1,\xi_2) \vdash \Delta(\sigma_{\xi_2\to\xi_1}\varphi,\varphi)$ using ($\mathrm{RP}_{\Gamma}$). Using now (MP) we can conclude that $\Delta(\xi_1,\xi_2) \vdash \varphi$.

Suppose now that $\Delta \subseteq T^{\Gamma}_{\xi_1,\xi_2}$ and $\Delta^{\vdash} = (T^{\Gamma}_{\xi_1,\xi_2})^{\vdash}$. We aim to prove that $\Delta(\xi_1,\xi_2)$ satisfies conditions (R), (S), (T) and ($\mathrm{RP}_{\Gamma}$) in Definition 3.1.6 of $\Gamma$-behavioral equivalence set. Property (R) follows from the fact that $\Delta \subseteq T^{\Gamma}_{\xi_1,\xi_2}$. We now prove the single replacement property,

($SRP_{\Gamma}$) $\quad \Delta(\xi_1,\xi_2) \vdash \Delta(\varphi(\xi_1),\varphi(\xi_2))$ for every $\varphi(\xi) \in L_{\Gamma}(X)$ and $\xi$ variable.

Consider the substitution $\sigma$ such that $\sigma_{\phi}(\xi_1) = \varphi(\xi_1)$ and $\sigma_{\phi}(\xi_2) = \varphi\xi_2)$. It is easy to see that $\sigma_{\xi_2\to\xi_1}(\sigma\xi_1) = \sigma_{\xi_2\to\xi_1}(\sigma\xi_2)$. So we can use *i)* to conclude that

$$\Delta(\varphi(\xi_1),\varphi(\xi_2)) = \sigma\Delta(\xi_1,\xi_2) \subseteq T^{\Gamma}_{\xi_1,\xi_2}.$$

Finally we have that $\Delta(\xi_1,\xi_2) \vdash \Delta(\varphi(\xi_1),\varphi(\xi_2))$ since $(\Delta(\xi_1,\xi_2))^{\vdash} = (T^{\Gamma}_{\xi_1,\xi_2})^{\vdash}$.

To prove (MP) recall that $\mathcal{L}$ is $\Gamma$-behaviorally protoalgebraic. This implies that $\xi_1, T^{\Gamma}_{\xi_1,\xi_2} \vdash \xi_2$, and so we have that $\xi_2 \in (\{\xi_1\} \cup T^{\Gamma}_{\xi_1,\xi_2})^{\vdash}$. Since $\Delta^{\vdash} = (T^{\Gamma}_{\xi_1,\xi_2})^{\vdash}$ we can conclude that $\xi_2 \in (\{\xi_1\} \cup \Delta(\xi_1,\xi_2))^{\vdash}$.

To prove (S) we use ($\mathrm{SRP}_\Gamma$) with $\varphi$ as $\Delta(\xi,\xi_1)$. So we have that

$$\Delta(\xi_1,\xi_2) \vdash \Delta(\Delta(\xi_1,\xi_1),\Delta(\xi_2,\xi_1)).$$

Using now (MP) and (R) we can conclude that $\Delta(\xi_1,\xi_2) \vdash \Delta(\xi_2,\xi_1)$.

Let us now prove (T). We use ($\mathrm{SRP}_\Gamma$) with $\varphi$ as $\delta(\xi_1,\xi)$, for every $\delta(\xi_1,\xi_2) \in \Delta(\xi_1,\xi_2)$. Then we have, for every $\delta(\xi_1,\xi_2) \in \Delta(\xi_1,\xi_2)$, that $\Delta(\xi_2,\xi_3) \vdash \Delta(\delta(\xi_1,\xi_2),\delta(\xi_1,\xi_3))$. So, using (MP) we can conclude that $\Delta(\xi_1,\xi_2),\Delta(\xi_2,\xi_3) \vdash \Delta(\xi_1,\xi_3)$

Finally, ($\mathrm{RP}_\Gamma$) follows in the usual way from (T) and ($\mathrm{SRP}_\Gamma$).

□

The following notion of behavioral protoequivalence system of formulas is the basis of a characterization of behavioral protoalgebraicity. It generalizes the concept of (many-sorted) protoequivalence system given in [Mar04]. In the single-sorted case it generalizes the notion of protoequivalence system [Cze01] where no parametric variables are assumed.

**Definition 3.2.7.** Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic and $\Gamma$ a subsignature of $\Sigma$. A a set $\Delta(\xi_1,\xi_2,\underline{z}) \subseteq L_\Gamma(\{\xi_1,\xi_2,\underline{z}\})$ where $\underline{z} = \langle z_1 : s_1, z_2 : s_2, \ldots \rangle$ is a set of *parametric variables* with sort different from $\phi$ and at most one variable of each sort is said a $\Gamma$-*protoequivalence system for* $\mathcal{L}$ if it satisfies the following conditions:

**(R)** $\vdash \Delta(\xi,\xi,\underline{z})$;

**(MP)** $\xi_1, \Delta(\langle \xi_1,\xi_2 \rangle) \vdash \xi_2$.

The following notion of parametrized equivalence system may seem, at first sight, very similar to the above notion of protoequivalence system. Besides assuming the replacement condition (RP), the major difference is the restriction on the parametric variables. This notion is also a basis of a characterization of behavioral protoalgebraicity.

**Definition 3.2.8.** Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic and $\Gamma$ a subsignature of $\Sigma$. A a set $\Delta(\xi_1, \xi_2, \underline{z}) \subseteq L_\Gamma(X)$ where $\underline{z} = \langle z_1 : s_1, z_2 : s_2, \ldots \rangle$ is a set of *parametric variables* is said a *parametrized* $\Gamma$*-equivalence system for* $\mathcal{L}$ if it satisfies the following conditions:

**(R)** $\vdash \Delta(\xi, \xi, \underline{z})$;

**(MP)** $\xi_1, \Delta(\langle \xi_1, \xi_2 \rangle) \vdash \xi_2$;

**(SRP$_\Gamma$)** $\Delta(\langle \xi_1, \xi_2 \rangle) \vdash \Delta(\langle c[\xi_1], c[\xi_2] \rangle)$, for every $c \in \mathcal{C}^\Gamma_{\Sigma,\phi}[\xi : \phi]$.

The following theorem is a behavioral version of well-known characterizations of the notion of protoalgebraic logic [Cze01].

**Theorem 3.2.9.** *Let* $\mathcal{L} = \langle \Sigma, \vdash \rangle$ *be a many-sorted logic and* $\Gamma$ *a subsignature of* $\Sigma$*. Then, the following conditions are equivalent:*

*i)* $\mathcal{L}$ *is* $\Gamma$*-behaviorally protoalgebraic;*

*ii)* $\Omega^{bhv}_{\Gamma,\phi}$ *is monotone;*

*iii)* $\xi_1, T^\Gamma_{\xi_1,\xi_2} \vdash \xi_2$*;*

*iv) there exists a* $\Gamma$*-protoequivalence system for* $\mathcal{L}$*;*

*v) there exists a parametrized* $\Gamma$*-equivalence system for* $\mathcal{L}$*.*

*Proof.* *i)* $\Rightarrow$ *ii)*: Assume $\mathcal{L}$ is $\Gamma$-protoalgebraic. Let $T_1, T_2 \subseteq Th_{\mathcal{L}}$ such that $T_1 \subseteq T_2$. We need to prove that $\Omega^{bhv}_\Gamma(T_1)$ is compatible with $T_2$. For this purpose, let $\varphi \in T_2$ and $\langle \varphi, \psi \rangle \in \Omega^{bhv}_{\Gamma,\phi}(T_1)$. Therefore $T_1, \varphi \dashv\vdash \psi, T_1$ by $\Gamma$-protoalgebraizability. Since $T_1 \subseteq T_2$ we have that $T_2, \varphi \dashv\vdash \psi, T_2$. So, since $\varphi \in T_2$ and $T_2$ is a theory, we can conclude that $\psi \in T_2$. Now that we have proved that $\Omega^{bhv}_\Gamma(T_1)$ is compatible with $T_2$, we can conclude that $\Omega^{bhv}_{\Gamma,\phi}(T_1) \subseteq \Omega^{bhv}_{\Gamma,\phi}(T_2)$ since $\Omega^{bhv}_\Gamma(T_2)$ is the largest $\Gamma$-congruence compatible with $T_2$.

*ii)* $\Rightarrow$ *iii)*: By Lemma 3.2.6 we have that $\langle \xi_1, \xi_2 \rangle \in \Omega^{bhv}_{\Gamma,\phi}((T^\Gamma_{\xi_1,\xi_2})^\vdash)$. Since $\Omega^{bhv}_{\Gamma,\phi}$ is monotone we have $\langle \xi_1, \xi_2 \rangle \in \Omega^{bhv}_{\Gamma,\phi}((\{\xi_1\} \cup T^\Gamma_{\xi_1,\xi_2})^\vdash)$, and by compatibility we can conclude that $\xi_2 \in (\{\xi_1\} \cup T^{\Gamma}_{\xi_1,\xi_2})^{\vdash}$, that is, $\xi_1, T^{\Gamma}_{\xi_1,\xi_2} \vdash \xi_2$.

*iii*) $\Rightarrow$ *iv*): Take $\Delta(\xi_1,\xi_2) := \sigma T^{\Gamma}_{\xi_1,\xi_2}$ where $\sigma$ is a substitution such that $\sigma_\phi(\xi_1) = \xi_1$ and $\sigma\phi(\xi) = \xi_2$ for every $\xi \neq \xi_1$ and, for every $s \neq \phi$, $\sigma_s(x) = x_0$ for every $x \in X_s$, where $x_0$ is a fixed variable of sort $s$. So, the conditions over the variables are verified. To verify (R) and (MP) note first that, since $\sigma$ is a substitution over $\Gamma$ and $\sigma_{\xi_2\to\xi_1}(\sigma\xi_1) = \sigma_{\xi_2\to\xi_1}(\sigma\xi_2)$ we have, using Lemma 3.2.6, that $\sigma T^{\Gamma}_{\xi_1,\xi_2} \subseteq T^{\Gamma}_{\xi_1,\xi_2}$. So, (R) is satisfied. In turn, (MP) follows from *iii*) and structurality.

*iv*) $\Rightarrow$ *i*): Suppose that there exists a $\Gamma$-protoequivalence set $\Delta(\xi_1,\xi_2,\underline{z})$ for $\mathcal{L}$. Let $\varphi,\psi \in L_\Sigma(X)$ and let $T$ be a theory of $\mathcal{L}$ such that $\langle\varphi,\psi\rangle \in \Omega^{bhv}_{\Gamma,\phi}(T)$. So, for every $\delta(\xi_1,\xi_2) \in \Delta(\xi_1,\xi_2)$, we have that $\langle\delta(\varphi,\psi),\delta(\varphi,\varphi)\rangle \in \Omega^{bhv}_{\Gamma,\phi}(T)$. So, by compatibility and using (R) we have that $\Delta(\varphi,\psi) \subseteq T$. So, using (MP) we have that $T,\varphi \vdash \psi$. In the same way we have that $T,\psi \vdash \varphi$. So $T,\varphi \dashv\vdash \psi, T$.

*iii*) $\Rightarrow$ *v*): Take $\Delta(\xi_1,\xi_2) := T^{\Gamma}_{\xi_1,\xi_2}$. Condition (R) is an immediate consequence of the definition of $T^{\Gamma}_{\xi_1,\xi_2}$ and (MP) is an immediate consequence of *iii*). For condition ($\mathrm{SRP}_\Gamma$), let $c \in \mathcal{C}^{\Gamma}_{\Sigma,\phi}[\xi:\phi]$. Now consider the substitution $\sigma$ over $\Sigma$ such that $\sigma_\phi(\xi_1) = c[\xi_1]$ and $\sigma_\phi(\xi_2) = c[\xi_2]$. It is easy to verify that $\sigma_{\xi_2\to\xi_1}(\sigma(\xi_1)) = \sigma_{\xi_2\to\xi_1}(\sigma(\xi_2))$, and using Lemma 3.2.6 we can conclude that $\Delta(c[\xi_1],c[\xi_2]) = \sigma T^{\Gamma}_{\xi_1,\xi_2} \subseteq T^{\Gamma}_{\xi_1,\xi_2} = \Delta(\xi_1,\xi_2)$. So, $\Delta(\langle\xi_1,\xi_2\rangle) \vdash \Delta(\langle c[\xi_1],c[\xi_2]\rangle)$.

*v*) $\Rightarrow$ *i*): Suppose that there exists a parametrized $\Gamma$-equivalence system $\Delta(\xi_1,\xi_2,\underline{z})$ for $\mathcal{L}$. Let $\varphi,\psi \in L_\Sigma(X)$ and let $T$ be a theory of $\mathcal{L}$ such that $\langle\varphi,\psi\rangle \in \Omega^{bhv}_{\Gamma,\phi}(T)$. So, for every $\delta(\xi_1,\xi_2) \in \Delta(\xi_1,\xi_2)$, we have that $\langle\delta(\varphi,\psi),\delta(\varphi,\varphi)\rangle \in \Omega^{bhv}_{\Gamma,\phi}(T)$. So, by compatibility and using (R) we have that $\Delta(\varphi,\psi) \subseteq T$. So, using (MP) we have that $T,\varphi \vdash \psi$. In the same way we have that $T,\psi \vdash \varphi$. So $T,\varphi \dashv\vdash \psi, T$.

$\square$

The standard notion of protoequivalence system for a protoalgebraic logic does not have parameter variables. The need for this extra assumption in our result arises from the use of many-sorted languages, as it was already observed in [Mar04]. Indeed, in the single-sorted case, the $\Gamma$-protoequivalence system of Theorem 3.2.9 can be taken without parameter variables.

The following result is an immediate consequence of Theorem 3.2.9.

**Corollary 3.2.10.** *Let $\mathcal{L} = \langle\Sigma,\vdash\rangle$ be a many-sorted logic and $\Gamma$ a subsignature of $\Sigma$. Then $\mathcal{L}$ is $\Gamma$-behaviorally protoalgebraic whenever it is $\Gamma$-behaviorally equivalential.*

More interestingly, condition *iv*) of Theorem 3.2.9 also allows us to conclude that if a logic is behaviorally protoalgebraic then it is also protoalgebraic in the standard sense. This is an important fact since it means that all our behavioral hierarchy is contained in the class of protoalgebraic logics, the class of logics that is widely considered to be largest class amenable to the tools of AAL.

After focusing on the notion of behaviorally protoalgebraic logic, we turn our attention to other classes of logics in the Leibniz hierarchy. One such example is the behavioral version of the notion of weakly algebraizable logic. Consider given a subsignature $\Gamma$ of $\Sigma$. We now introduce the main notion of $\Gamma$-behaviorally weakly algebraizable logic.

**Definition 3.2.11.** A many-sorted logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ is $\Gamma$-*behaviorally weakly algebraizable* if there exists a class $K$ of $\Sigma^o$-algebras, a set $\Theta(\xi, \underline{z}) \subseteq Comp_\Sigma^{K,\Gamma}(X)$ of $\phi$-equations and a set $\Delta(\xi_1, \xi_2, \underline{w}) \subseteq L_\Gamma(\{\xi_1, \xi_2, \underline{w}\})$ of formulas such that, for every $T \cup \{\varphi\} \subseteq L_\Sigma(X)$ and for every set $\Phi \cup \{\varphi_1 \approx \varphi_2\}$ of $\phi$-equations:

*i*) $T \vdash \varphi$ iff $\Theta[\langle T \rangle] \models_{\Sigma,bhv}^{K,\Gamma} \Theta(\langle \varphi \rangle)$;

*ii*) $\Phi \models_{\Sigma,bhv}^{K,\Gamma} \varphi_1 \approx \varphi_2$ iff $\Delta[\langle \Phi \rangle] \vdash \Delta(\langle \varphi_1, \varphi_2 \rangle)$;

*iii*) $\xi \dashv\vdash \Delta[\langle \Theta(\langle \xi \rangle) \rangle]$;

*iv*) $\xi_1 \approx \xi_2 \mathrel{=\!\!|\!\models}_{\Sigma,bhv}^{K,\Gamma} \Theta[\langle \Delta(\langle \xi_1, \xi_2 \rangle) \rangle]$;

The difference between the notion of behaviorally weakly algebraizable logic and the notion of behaviorally algebraizable logic is the fact that, in the former, both the set of equivalence formulas and the set of defining equations have parametric variables. Recall that if $\varphi(\xi, \underline{w}) \in L_\Sigma(\{\xi, \underline{w}\})$ is a formula then $\varphi(\langle \xi \rangle)$ denotes the set $\{\varphi(\xi, \underline{\gamma}) : \underline{\gamma}$ possible instantiation of $\underline{w}\}$. In what follows, if $\Gamma$ is clear from the context then it can be omitted. We say that a logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ is behaviorally weakly algebraizable if there exists a subsignature $\Gamma$ of $\Sigma$ such that $\mathcal{L}$ is $\Gamma$-behaviorally weakly algebraizable. The fact that conditions *i*) and *iii*) are jointly equivalent to conditions *ii*) and *iv*) is easily proved as in the case of behaviorally algebraizable logic.

**Proposition 3.2.12.** *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted signature and $\Gamma$ a subsig-*

*nature of* $\Sigma$. *Suppose that* $\mathcal{L}$ *is* $\Gamma$*-behaviorally weakly algebraizable with* $\Delta(\xi_1, \xi_2, \underline{w})$ *the set of equivalence formulas. Then* $\Delta(\xi_1, \xi_2, \underline{w})$ *is a parametrized* $\Gamma$*-equivalence system for* $\mathcal{L}$.

*Proof.* Let $K$ be the $\Gamma$-behavioral equivalent algebraic semantics and let $\Theta(\xi, \underline{z}) \subseteq Comp_\Sigma^{K,\Gamma}(X)$ be the respective set of defining equations.

Condition (R) follows easily from the fact that $\vDash_{\Sigma,bhv}^{K,\Gamma} \xi \approx \xi$. For (MP), note that, since $\mathcal{L}$ is $\Gamma$-behaviorally weakly algebraizable, $\Delta(\langle\varphi, \psi\rangle), \varphi \vdash \psi$ is equivalent to

$$\varphi \approx \psi, \Theta(\langle\varphi\rangle) \vDash_{\Sigma,bhv}^{K,\Gamma} \Theta(\langle\psi\rangle).$$

But the last condition follows from the fact that $\Theta(\langle\xi\rangle) \subseteq Comp_\Sigma^{K,\Gamma}(\{\xi\})$.

For condition ($\text{SRP}_\Gamma$) let $c \in \mathcal{C}_{\Sigma,\phi}^{\Gamma}[\xi : \phi]$. The condition $\Delta(\langle\xi_1, \xi_2\rangle) \vdash \Delta(\langle c[\xi_1], c[\xi_2]\rangle)$ follows easily from the the fact that $\xi_1 \approx \xi_2 \vDash_{\Sigma,bhv}^{K,\Gamma} c[\xi_1] \approx c[\xi_2]$ and from condition *ii*) in the definition of $\Gamma$-behaviorally weakly algebraizable.

□

As an immediate consequence of Proposition 3.2.12 we have the following result.

**Corollary 3.2.13.** *Let* $\mathcal{L} = \langle\Sigma, \vdash\rangle$ *be a many-sorted logic and* $\Gamma$ *a subsignature of* $\Sigma$. *If* $\mathcal{L}$ *is* $\Gamma$*-behaviorally weakly algebraizable then it is* $\Gamma$*-behaviorally protoalgebraic.*

The next result presents an important characterization of the Leibniz $\Gamma$-congruence in the cases where the logic is $\Gamma$-behaviorally weakly algebraizable.

**Theorem 3.2.14.** *Let* $\mathcal{L} = \langle\Sigma, \vdash\rangle$ *be a many-sorted logic and* $\Gamma$ *a subsignature of* $\Sigma$. *Suppose that* $\Delta(\xi_1, \xi_2, \underline{z}) \subseteq L_\Gamma(\{\xi_1, \xi_2, \underline{z}\})$ *is a parametrized* $\Gamma$*-equivalence system for* $\mathcal{L}$. *Then, for every* $\varphi, \psi \in L_\Sigma(X)$ *we have that*

$$\langle\varphi, \psi\rangle \in \Omega_\Gamma^{bhv}(T) \quad \textit{iff} \quad \Delta(\langle\varphi, \psi\rangle) \subseteq T.$$

*Proof.* First let $\langle\varphi, \psi\rangle \in \Omega_{\Gamma,\phi}^{bhv}(T)$. Then, by compatibility, we have that $\Delta(\langle\varphi, \psi\rangle) \subseteq T$ iff $\Delta(\langle\varphi, \varphi\rangle) \subseteq T$. Since $\Delta(\xi_1, \xi_2)$ satisfies (R) we can conclude that $\Delta(\langle\varphi, \psi\rangle) \subseteq T$.

On the other direction, suppose that $\Delta(\langle\varphi, \psi\rangle) \subseteq T$. So, using (RP), we have, for every $c \in \mathcal{C}_{\Sigma,\phi}^{\Gamma}[\xi]$ that $\Delta(\langle c[\varphi], c[\psi]\rangle) \subseteq T$. Using (MP) we can conclude that $c[\varphi] \in T$ iff $c[\psi] \in T$. So, we have that $\langle\varphi, \psi\rangle \in \Omega_{\Gamma,\phi}^{bhv}(T)$.

□

In our behavioral setting we can generalize the standard characterization of weakly algebraizable logics using the Leibniz operator.

**Theorem 3.2.15.** *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted signature and $\Gamma$ a subsignature of $\Sigma$. Then $\mathcal{L}$ is $\Gamma$-behaviorally weakly algebraizable iff $\Omega^{bhv}_{\Gamma,\phi}$ is monotone and injective.*

*Proof.* First assume that $\mathcal{L}$ is $\Gamma$-behaviorally weakly algebraizable with $\Theta(\xi, \underline{z}) \subseteq Comp^{K,\Gamma}_{\Sigma}(X)$ the set of defining equations and $\Delta(\xi_1, \xi_2, \underline{w}) \subseteq L_\Gamma(\{\xi_1, \xi_2, \underline{w}\})$ the set of equivalence formulas. Using Corollary 3.2.13 we have that $\mathcal{L}$ is $\Gamma$-behaviorally protoalgebraic and using Theorem 3.2.9 we can conclude that $\Omega^{bhv}_{\Gamma,\phi}$ is monotone.

To prove that it is also injective let $T_1, T_2 \in Th_{\mathcal{L}}$ such that $\Omega^{bhv}_{\Gamma,\phi}(T_1) = \Omega^{bhv}_{\Gamma,\phi}(T_2)$. Now consider the following sequence of equivalent sentences:

$$
\begin{array}{lll}
\varphi \in T_1 & \text{iff} \;\; \Delta[\langle \Theta(\langle \varphi \rangle) \rangle] \subseteq T_1 & \\
 & \text{iff} \;\; \Theta(\langle \varphi \rangle) \subseteq \Omega^{bhv}_{\Gamma,\phi}(T_1) & \text{using Theorem 3.2.14} \\
 & \text{iff} \;\; \Theta(\langle \varphi \rangle) \subseteq \Omega^{bhv}_{\Gamma,\phi}(T_2) & \text{using Theorem 3.2.14} \\
 & \text{iff} \;\; \Delta[\langle \Theta(\langle \varphi \rangle) \rangle] \subseteq T_2 & \\
 & \text{iff} \;\; \varphi \in T_2. &
\end{array}
$$

So, $T_1 = T_2$, showing that $\Omega^{bhv}_{\Gamma,\phi}$ is injective.

Let us now assume that $\Omega^{bhv}_{\Gamma,\phi}$ is monotone and injective. Using Theorem 3.2.9 we know that $\mathcal{L}$ is $\Gamma$-behaviorally protoalgebraic. So, there exists a parametrized $\Gamma$-equivalence system $\Delta(\xi_1, \xi_2, \underline{w}) \subseteq L_\Gamma(X)$.

Now take

$$K = \{\mathbf{T}_{\mathbf{\Sigma^o}}(\mathbf{X^o})_{/_{(\Omega^{bhv}_{\Gamma}(T))^o}} : T \in Th_{\mathcal{L}}\}$$

a class of $\Sigma^o$-algebras. We aim to prove that $\mathcal{L}$ is $\Gamma$-behaviorally weakly algebraizable with $K$ a $\Gamma$-behaviorally equivalent algebraic semantics.

Using Theorem 3.2.14 and taking into account that the definition of $K$, it is an easy exercise to prove that, for every set $\Phi$ of $\phi$-equations and $\varphi, \psi \in L_\Sigma(X)$, we have

$$\Phi \vDash^{K,\Gamma}_{\Sigma,bhv} \varphi \approx \psi \quad \text{iff} \quad \Delta[\langle \Phi \rangle] \vdash \Delta(\langle \varphi, \psi \rangle).$$

We now prove $\xi \dashv\vdash \Delta[\langle \Theta(\langle \xi \rangle) \rangle]$ for some set $\Theta(\xi, \underline{z})$ of $\phi$-equations with the variable $\xi$ and parametric variables $\underline{z}$.

Let $T_\xi = \{\xi\}^\vdash$ and take $\Theta(\xi) = \Omega^{bhv}_{\Gamma,\phi}(T_\xi)$. We show that $\xi \dashv\vdash \Delta[\langle\Omega^{bhv}_{\Gamma,\phi}(T_\xi)\rangle]$, or equivalently that $T_\xi = (\Delta[\langle\Omega^{bhv}_{\Gamma,\phi}(T_\xi)]\rangle)^\vdash$. For that, consider the following sequence of equivalent sentences:

$$\begin{aligned}
\langle\varphi,\psi\rangle \in \Omega^{bhv}_{\Gamma,\phi}((\Delta[\langle\Omega^{bhv}_{\Gamma,\phi}(T_\xi)\rangle])^\vdash) \quad &\text{iff} \quad \Delta(\langle\varphi,\psi\rangle) \subseteq (\Delta[\langle\Omega^{bhv}_{\Gamma,\phi}(T_\xi)\rangle])^\vdash \\
&\text{iff} \quad \Delta[\langle\Omega^{bhv}_{\Gamma,\phi}(T_\xi)\rangle] \vdash \Delta(\langle\varphi,\psi\rangle) \\
&\text{iff} \quad \Omega^{bhv}_{\Gamma,\phi}(T_\xi) \vDash^{K,\Gamma}_{\Sigma,bhv} \varphi \approx \psi \\
&\text{iff} \quad \langle\varphi,\psi\rangle \in \Omega^{bhv}_{\Gamma,\phi}(T_\xi).
\end{aligned}$$

So, $\Omega^{bhv}_{\Gamma,\phi}(T_\xi) = \Omega^{bhv}_{\Gamma}((\Delta[\langle\Omega^{bhv}_{\Gamma,\phi}(T_\xi)\rangle])^\vdash)$. By injectivity of $\Omega^{bhv}_{\Gamma,\phi}$ we have that

$$T_\xi = (\Delta[\langle\Omega^{bhv}_{\Gamma,\phi}(T_\xi)\rangle])^\vdash.$$

□

In Section 3.1 we have introduced the notion of behaviorally equivalential logic. In the next proposition we group two interesting properties regarding behaviorally equivalential logics and the behavioral Leibniz operator. The first one generalizes the close connection between a set of equivalence formulas and the Leibniz congruence. The second condition generalizes the well-known criterion for equivalentiality due to Herrmann [Her96].

**Proposition 3.2.16.** *Let $\mathcal{L} = \langle\Sigma,\vdash\rangle$ a many-sorted logic and $\Gamma$ a subsignature of $\Sigma$. Let $\Delta(\xi_1,\xi_2) \subseteq L_\Gamma(\{\xi_1,\xi_2\})$ a set of formulas. Then,*

*i) if $\Delta(\xi_1,\xi_2)$ is a $\Gamma$-behavioral equivalence set for $\mathcal{L}$ then, for every $T \in Th_\mathcal{L}$ and $\varphi,\psi \in L_\Sigma(X)$, we have that*

$$\langle\varphi,\psi\rangle \in \Omega^{bhv}_{\Gamma,\phi}(T) \quad \textit{iff} \quad \Delta(\varphi,\psi) \subseteq T.$$

*ii) Herrmann's Test: suppose $\mathcal{L}$ is $\Gamma$-behaviorally protoalgebraic. Then, $\Delta(\xi_1,\xi_2) \subseteq L_\Gamma(\{\xi_1,\xi_2\})$ is a $\Gamma$-behavioral equivalence set for $\mathcal{L}$ iff $\Delta(\xi_1,\xi_2) \subseteq T^\Gamma_{\xi_1,\xi_2}$ and it satisfies $\langle\xi_1,\xi_2\rangle \in \Omega^{bhv}_{\Gamma,\phi}(\Delta(\xi_1,\xi_2)^\vdash)$;*

*Proof.* *i)* First let $\langle\varphi,\psi\rangle \in \Omega^{bhv}_{\Gamma,\phi}(T)$. Then, by compatibility, we have that $\Delta(\varphi,\psi) \subseteq T$ iff $\Delta(\varphi,\varphi) \subseteq T$. Since $\Delta$ satisfies (R) we can conclude that $\Delta(\varphi,\psi) \subseteq T$.

On the other direction, suppose that $\Delta(\varphi,\psi) \subseteq T$. So, for every $c \in \mathcal{C}^{\Gamma}_{\Sigma,\phi}[\xi]$ we have that $\Delta(c[\varphi], c[\psi]) \subseteq T$. So, using (MP) we can conclude that $c[\varphi] \in T$ iff $c[\psi] \in T$. So, we have that $\langle \varphi,\psi\rangle \in \Omega^{bhv}_{\Gamma,\phi}(T)$.

*ii)* Suppose first that $\Delta(\xi_1,\xi_2)$ is a $\Gamma$-behavioral equivalence for $\mathcal{L}$. Since $\vdash \Delta(\xi_1,\xi_1)$ we have that $\Delta \subseteq T^{\Gamma}_{\xi_1,\xi_2}$. By Lemma 3.2.6 we have that $(\Delta(\xi_1,\xi_2))^{\vdash} = (T^{\Gamma}_{\xi_1,\xi_2})^{\vdash}$. Again by Lemma 3.2.6 we have that $\langle \xi_1,\xi_2\rangle \in \Omega^{bhv}_{\Gamma,\phi}((T^{\Gamma}_{\xi_1,\xi_2})^{\vdash}) = \Omega^{bhv}_{\Gamma,\phi}(\Delta(\xi_1,\xi_2)^{\vdash})$. Now suppose that $\Delta(\xi_1,\xi_2) \subseteq T^{\Gamma}_{\xi_1,\xi_2}$ and $\langle \xi_1,\xi_2\rangle \in \Omega^{bhv}_{\Gamma,\phi}(\Delta(\xi_1,\xi_2)^{\vdash})$. Then $\Delta(\xi_1,\xi_2)^{\vdash} \subseteq (T^{\Gamma}_{\xi_1,\xi_2})^{\vdash}$. To prove the reverse inclusion, let $\varphi \in T^{\Gamma}_{\xi_1,\xi_2}$. By definition of $T^{\Gamma}_{\xi_1,\xi_2}$ we have that $\varphi(\xi_1,\xi_1)$. So $\varphi(\xi_1,\xi_1) \in \Delta(\xi_1,\xi_2)^{\vdash}$. Since $\langle \xi_1,\xi_2\rangle \in \Omega^{bhv}_{\Gamma,\phi}(\Delta(\xi_1,\xi_2)^{\vdash})$ we have by compatibility that $\varphi(\xi_1,\xi_2) \in \Delta(\xi_1,\xi_2)^{\vdash}$ iff $\varphi(\xi_1,\xi_1) \in \Delta(\xi_1,\xi_2)^{\vdash}$. So, $\varphi(\xi_1,\xi_2) \in \Delta(\xi_1,\xi_2)^{\vdash}$. So, we have that $(T^{\Gamma}_{\xi_1,\xi_2})^{\vdash} \subseteq \Delta(\xi_1,\xi_2)^{\vdash}$ and we can conclude that $(T^{\Gamma}_{\xi_1,\xi_2})^{\vdash} = \Delta(\xi_1,\xi_2)^{\vdash}$. By Lemma 3.2.6 we have that $\Delta$ is a $\Gamma$-behavioral equivalence.

□

We now show that the notion of behavioral equivalentiality can also be characterized by properties of the behavioral Leibniz operator. This result also generalizes a well-known standard result of AAL.

**Theorem 3.2.17.** *Let $\mathcal{L} = \langle \Sigma, \vdash\rangle$ be a many-sorted logic and $\Gamma$ a subsignature of $\Sigma$. Assuming that $\mathcal{L}$ is $\Gamma$-standard, then the following are equivalent:*

*i)* *$\mathcal{L}$ is $\Gamma$-behaviorally equivalential;*

*ii)* *$\Omega^{bhv}_{\Gamma,\phi}$ is monotone and commutes with inverse substitutions;*

*iii)* *$\Omega^{bhv}_{\Gamma,\phi}$ is monotone and $\sigma\Omega^{bhv}_{\Gamma,\phi}(T) \subseteq \Omega^{bhv}_{\Gamma,\phi}((\sigma T)^{\vdash})$, for all substitutions and $\mathcal{L}$-theories $T$.*

*Proof.* *i)* $\Rightarrow$ *ii)*: Suppose that $\mathcal{L}$ is $\Gamma$-behaviorally equivalential and let $\Delta(\xi_1,\xi_2)$ be a $\Gamma$-behavioral equivalence set for $\mathcal{L}$. Using Corollary 3.2.10 we have that, since $\mathcal{L}$ is $\Gamma$-behaviorally equivalential, then it is also $\Gamma$-behaviorally protoalgebraic. By Theorem 3.2.9 we can conclude that $\Omega^{bhv}_{\Gamma,\phi}$ is monotone. To prove that $\Omega^{bhv}_{\Gamma,\phi}$ commutes with inverse substitutions, consider some $T \in Th_{\mathcal{L}}$ and a substitution $\sigma$. Now, we have the following sequence of equivalent sentences:

$$\begin{aligned}\langle t_1,t_2\rangle \in \sigma^{-1}\Omega^{bhv}_{\Gamma,\phi}(T) \quad &\text{iff} \quad \langle \sigma t_1,\sigma t_2\rangle \in \Omega^{bhv}_{\Gamma,\phi}(T)\\ &\text{iff} \quad \Delta(\sigma t_1,\sigma t_2)\subseteq T\\ &\text{iff} \quad \sigma\Delta(t_1,t_2)\subseteq T\\ &\text{iff} \quad \Delta(t_1,t_2)\subseteq \sigma^{-1}T\\ &\text{iff} \quad \langle t_1,t_2\rangle \in \Omega^{bhv}_{\Gamma,\phi}(\sigma^{-1}T).\end{aligned}$$

*ii*) $\Rightarrow$ *iii*): Let $T \in Th_{\mathcal{L}}$ and let $\sigma$ be a substitution over $\Sigma$. Let $T_0 = (\sigma T)^{\vdash}$. It is obvious that $T \subseteq \sigma^{-1}T_0$ and therefore $\Omega^{bhv}_{\Gamma,\phi}(T) \subseteq \Omega^{bhv}_{\Gamma,\phi}(\sigma^{-1}T_0)$. Since $\Omega^{bhv}_{\Gamma,\phi}$ commutes with inverse substitutions we have that $\Omega^{bhv}_{\Gamma,\phi}(\sigma^{-1}T_0) = \sigma^{-1}\Omega^{bhv}_{\Gamma,\phi}(T_0)$. Thus, $\Omega^{bhv}_{\Gamma,\phi}(T) \subseteq \sigma^{-1}\Omega^{bhv}_{\Gamma,\phi}(T_0)$. This yields $\sigma\Omega^{bhv}_{\Gamma,\phi}(T) \subseteq \Omega^{bhv}_{\Gamma,\phi}((\sigma T)^{\vdash})$.

*iii*) $\Rightarrow$ *i*): Assume condition *iii*). By Proposition 3.2.16, $\mathcal{L}$ is equivalential provided some $\Delta(\xi_1,\xi_2) \subseteq L_\Sigma(\{xi_1,\xi_2\})$ satisfies $\Delta \subseteq T^\Gamma_{\xi_1,\xi_2}$ and $\langle \xi_1,\xi_2\rangle \in \Omega^{bhv}_\Gamma((\Delta)^{\vdash}$. Recall that since $\mathcal{L}$ is $\Gamma$-standard there exists a closed term over $\Gamma$ for each sort $s \in S$. Let $\sigma$ be a substitution such that $\sigma_\phi(\xi_1) = \xi_1$ and $\sigma_\phi(\xi)) = \xi_2$ for every $\xi \in X_\phi$ and, for every $s \in S$ and every $x \in X_s$, $\sigma_s(x) = t_s$ where $t_s$ is a closed term of sort $s$. Now take $\Delta(\xi_1,\xi_2) = \sigma T^\Gamma_{\xi_1,\xi_2}$. So, $\Delta \subseteq L_\Gamma(\{\xi_1,\xi_2\})$. Since $\sigma_{\xi_2\to\xi_1}(\sigma\xi_1) = \sigma_{\xi_2\to\xi_1}(\sigma\xi_2)$, by Lemma 3.2.6, we have that $\Delta = \sigma T^\Gamma_{\xi_1,\xi_2} \subseteq T^\Gamma_{\xi_1,\xi_2}$. We know that $\langle \xi_1,\xi_2\rangle \in \Omega^{bhv}_{\Gamma,\phi}((T^\Gamma_{\xi_1,\xi_2})^{\vdash})$ by Lemma 3.2.6. So, $\langle \sigma\xi_1,\sigma\xi_2\rangle \in \sigma\Omega^{bhv}_{\Gamma,\phi}((T^\Gamma_{\xi_1,\xi_2})^{\vdash})$. By hypothesis, $\langle \xi_1,\xi_2\rangle \in \Omega^{bhv}_{\Gamma,\phi}((\sigma T^\Gamma_{\xi_1,\xi_2})^{\vdash})$. So, $\langle \xi_1,\xi_2\rangle \in \Omega^{bhv}_{\Gamma,\phi}((T^\Gamma_{\xi_1,\xi_2})^{\vdash})$. By Proposition 3.2.16 we can conclude that $\Delta$ is a $\Gamma$-behavioral equivalence.

□

At this point it is important to recall that quotient algebras are an essential ingredient of AAL. However, in our framework, we cannot perform quotients directly since we are now working with $\Gamma$-congruences instead of congruences. This is precisely where algebras over the extended signature $\Sigma^o$ play a key role. Towards the main theorem of this section, the characterization of behavioral algebraizability using properties of the behavioral Leibniz operator, we see how can we use algebras over the extended signature to simulate the quotient construction.

Given a $\Gamma$-congruence $\theta$ over $\mathbf{T_\Sigma(X)}$ we are able to construct from it a congruence over $\mathbf{T^o_\Sigma(X^o)}$ that keeps the relevant information of the original $\Gamma$-congruence. Consider the relation $\theta^o = \{\theta^o_s\}_{s\in S^o}$ over $\mathbf{T^o_\Sigma(X^o)}$ such that:

- $\theta^o_v = \{\langle o(\varphi), o(\psi)\rangle : \langle \varphi,\psi\rangle \in \theta_\phi\} \cup \{\langle t,t\rangle : t \in T_{\Sigma^o,v}(X^o)\}$;

- $\theta^o_s$ is the identity relation over $T_{\Sigma^o,s}(X^o)$ for every $s \neq v$.

It is an easy exercise to verify that $\theta^o$ is indeed a congruence on $\mathbf{T^o_\Sigma}(\mathbf{X^o})$.

It is now possible to establish the characterization of behavioral algebraizability using the behavioral Leibniz operator. The result generalizes the well-known standard result [Her96]. The proof closely follows the the one presented by Herrmann in [Her96]. The techniques used therein are easier to adapt to the behavioral setting than, for example, those used in the proof given by Blok and Pigozzi [BP89].

**Theorem 3.2.18.** *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a $\Gamma$-standard many-sorted logic, where $\Gamma$ is a subsignature of $\Sigma$. Then, $\mathcal{L}$ is $\Gamma$-behaviorally algebraizable iff $\Omega^{bhv}_{\Gamma,\phi}$ is injective, monotone and commutes with inverse substitutions.*

*Proof.* First assume that $\mathcal{L}$ is $\Gamma$-behaviorally algebraizable. So, it is equivalential, and therefore $\Omega^{bhv}_{\Gamma,\phi}$ is monotone and commutes with inverse substitutions. To prove that it is also injective let $T_1, T_2 \in Th_\mathcal{L}$ such that $\Omega^{bhv}_{\Gamma,\phi}(T_1) = \Omega^{bhv}_{\Gamma,\phi}(T_2)$. Now consider the following sequence of equivalent sentences:

$$\begin{aligned}
\varphi \in T_1 \quad & \text{iff} \quad \Delta[\Theta(\varphi)] \subseteq T_1 \\
& \text{iff} \quad \Theta(\varphi) \subseteq \Omega^{bhv}_{\Gamma,\phi}(T_1) \\
& \text{iff} \quad \Theta(\varphi) \subseteq \Omega^{bhv}_{\Gamma,\phi}(T_2) \\
& \text{iff} \quad \Delta[\Theta(\varphi)] \subseteq T_2 \\
& \text{iff} \quad \varphi \in T_2.
\end{aligned}$$

So, $T_1 = T_2$, showing that $\Omega^{bhv}_{\Gamma,\phi}$ is injective.

Assume now that $\Omega^{bhv}_{\Gamma,\phi}$ is injective, monotone and commutes with inverse substitutions. So, by Theorem 3.2.17 $\mathcal{L}$ is $\Gamma$-behaviorally equivalential. Let $\Delta(\xi_1, \xi_2)$ be an equivalence for $\mathcal{L}$.

Let $K = \{\mathbf{T_{\Sigma^o}}(\mathbf{X^o})_{/_{(\Omega^{bhv}_\Gamma(T))^o}} : T \in Th_\mathcal{L}\}$ be a class of $\Sigma^o$-algebras. Using Lemma 3.2.16 and taking into account the definition of $K$, it is an easy exercise to prove that, for every set $\Phi$ of $\phi$-equations and $\varphi, \psi \in L_\Sigma(X)$, we have

$$\Phi \vDash^{K,\Gamma}_{\Sigma,bhv} \varphi \approx \psi \quad \text{iff} \quad \Delta[\Phi] \vdash \Delta(t_1, t_2).$$

Let us now prove that $\xi \dashv\vdash \Delta[\Theta(\xi)]$ for some set $\Theta(\xi) \subseteq Eq_{\Sigma,\phi}(\{\xi\})$ of $\phi$-equations.

Let $T_\xi = \{\xi\}^{\vdash}$ and take a substitution $\sigma$ such that $\sigma_\phi(\xi') = \xi$ for every $\xi' \in X_\phi$ and, for every $s \in S$ and $s \neq \phi$, we have that $\sigma_s(x) = t_s$ where $t_s$ is a closed term of sort $s$. Take $\Theta(\xi) = \sigma\Omega^{bhv}_{\Gamma,\phi}(T_\xi)$. So, $\Theta(\xi) \subseteq Eq_\Sigma(\{\xi\})$. Since $\sigma(\Delta(\Omega^{bhv}_{\Gamma,\phi}(T_\xi))) = \Delta(\sigma\Omega^{bhv}_{\Gamma,\phi}(T_\xi)) = \Delta(\Theta(\xi))$, it suffices to show that $\xi \dashv\vdash \Delta[\Omega^{bhv}_{\Gamma,\phi}(T_\xi)]$, or equivalently that $T_\xi = (\Delta[\Omega^{bhv}_{\Gamma,\phi}(T_\xi)])^{\vdash}$. For that, consider the following sequence of equivalent sentences:

$$\begin{aligned}
\langle\varphi,\psi\rangle \in \Omega^{bhv}_{\Gamma,\phi}((\Delta[\Omega^{bhv}_{\Gamma,\phi}(T_\xi)])^{\vdash}) \quad &\text{iff} \quad \Delta(\varphi,\psi) \subseteq (\Delta[\Omega^{bhv}_{\Gamma,\phi}(T_\xi)])^{\vdash}\\
&\text{iff} \quad (\Delta[\Omega^{bhv}_{\Gamma,\phi}(T_\xi)]) \vdash \Delta(\varphi,\psi)\\
&\text{iff} \quad \Omega^{bhv}_{\Gamma,\phi}(T_\xi) \vDash^{K,\Gamma}_{\Sigma,bhv} \varphi \approx \psi\\
&\text{iff} \quad \langle\varphi,\psi\rangle \in \Omega^{bhv}_{\Gamma,\phi}(T_\xi).
\end{aligned}$$

So, $\Omega^{bhv}_{\Gamma,\phi}(T_\xi) = \Omega^{bhv}_{\Gamma}((\Delta[\Omega^{bhv}_{\Gamma,\phi}(T_\xi)])^{\vdash})$. By injectivity of $\Omega^{bhv}_{\Gamma,\phi}$ we have that

$$T_\xi = (\Delta[\Omega^{bhv}_{\Gamma,\phi}(T_\xi)])^{\vdash}.$$

□

## 3.3 Intrinsic and sufficient characterizations

At a first glance, the definition of behaviorally algebraizable logic may seem impure, since it depends on the *a priori* existence of a behavioral equivalent algebraic semantics. The characterization of behavioral algebraizability using the behavioral Leibniz operator already shows that this is, in fact, an intrinsic property of a logic. We now provide a second intrinsic characterization of behavioral algebraizability and, as a corollary, we are able to obtain an useful sufficient condition.

We have seen that $\Gamma$-behavioral equivalentiality is a necessary condition for a many-sorted logic to be $\Gamma$-behaviorally algebraizable. The following theorem shows that we get a necessary and sufficient condition for $\Gamma$-behavioral algebraizability, just by adding some natural assumption.

**Theorem 3.3.1.** *Let $\mathcal{L} = \langle\Sigma,\vdash\rangle$ be a many-sorted logic and $\Gamma$ a subsignature of $\Sigma$. Then we have that $\mathcal{L}$ is $\Gamma$-behaviorally algebraizable iff it is $\Gamma$-behaviorally equivalential with $\Gamma$-behavioral equivalence set $\Delta(\xi_1,\xi_2)$ and there exists a set $\Theta(\xi) \subseteq Eq_{\Sigma,\phi}(\{\xi\})$ of $\phi$-equations such that $\xi \dashv\vdash \Delta[\Theta(\xi)]$.*

*Proof.* Suppose first that $\mathcal{L}$ is $\Gamma$-behaviorally algebraizable. Then, using Proposition 3.1.7, we have that $\mathcal{L}$ is $\Gamma$-behaviorally equivalential. The existence of the set $\Theta(\xi)$ of $\phi$-equations such that $\xi \dashv\vdash \Delta[\Theta(\xi)]$ is immediate from the definition of behaviorally algebraizable.

On the other direction, suppose that $\mathcal{L}$ is $\Gamma$-behaviorally equivalential and that there exists a set $\Theta(\xi) \subseteq Eq_{\Sigma,\phi}(\xi)$ of $\phi$-equations such that $\xi \dashv\vdash \Delta[\Theta(\xi)]$. For each theory $T \in Th_{\mathcal{L}}$ we define a binary relation $\Omega_\Delta(T)$ over $\mathbf{L_\Sigma}(\mathbf{X})$ such that

$$(\Omega_\Delta(T)) = \{\langle \varphi_1, \varphi_2 \rangle : \Delta(\varphi_1, \varphi_2) \subseteq T\}.$$

By Proposition 3.2.16 we have that $\Omega_\Delta(T) = \Omega^{bhv}_{\Gamma,\phi}(T)$ for every $T \in Th_{\mathcal{L}}$.

We now prove that $\Omega_\Delta : Th_{\mathcal{L}} \to Con_\phi(\mathbf{T_\Sigma}(\mathbf{X}))$ is monotone, injective and commutes with inverse substitutions.

Let $T_1, T_2 \in Th_{\mathcal{L}}$ such that $T_1 \subseteq T_2$. Suppose that $\langle \varphi_1, \varphi_2 \rangle \in \Omega_\Delta(T_1)$. Then $\Delta(\varphi_1, \varphi_2) \subseteq T_1$. Since $T_1 \subseteq T_2$ we have that $\Delta(\varphi_1, \varphi_2) \subseteq T_2$ and so $\langle \varphi_1, \varphi_2 \rangle \in \Omega_\Delta(T_2)$. Thus $\Omega_\Delta$ is monotone.

Suppose that $\Omega_\Delta(T_2) = \Omega_\Delta(T_1)$ and let $\varphi \in T_1$. Then, using the fact that $\varphi \dashv\vdash \Delta[\Theta(\varphi)]$, we have that $\Delta[\Theta(\varphi)] \subseteq T_1$ and therefore $\langle \delta(\varphi), \epsilon(\varphi) \rangle \in \Omega_\Delta(T_1)$ for every $\delta \approx \epsilon \in \Theta$. Thus $\langle \delta(\varphi), \epsilon(\varphi) \rangle \in \Omega_\Delta(T_2)$ for every $\delta \approx \epsilon \in \Theta$ and so $\Delta[\Theta(\varphi)] \subseteq T_2$ and $\varphi \in T_2$ using the fact that $\varphi \dashv\vdash \Delta[\Theta(\varphi)]$. This shows that $T_1 \subseteq T_2$, and by symmetry we have that $T_1 = T_2$. Thus $\Omega_\Delta$ is injective.

Let $\sigma$ be a substitution over $\Sigma$. Then we have the following sequence of equivalent sentences:

$$\begin{aligned}
\langle \varphi_1, \varphi_2 \rangle \in (\Omega_\Delta(\sigma^{-1}T)) \quad &\text{iff} \quad \Delta(\varphi_1, \varphi_2) \subseteq \sigma^{-1}T \\
&\text{iff} \quad \sigma\Delta(\varphi_1, \varphi_2) \subseteq T \\
&\text{iff} \quad \Delta(\sigma\varphi_1, \sigma\varphi_2) \subseteq T \\
&\text{iff} \quad \langle \sigma\varphi_1, \sigma\varphi_2 \rangle \in (\Omega_\Delta(T)) \\
&\text{iff} \quad \langle \varphi_1, \varphi_2 \rangle \in \sigma^{-1}(\Omega_\Delta(T)).
\end{aligned}$$

So $\Omega_\Delta(\sigma^{-1}T) = \sigma^{-1}\Omega_\Delta(T)$, that is, $\Omega_\Delta$ commutes with inverse substitutions. Since $\Omega_\Delta = \Omega^{bhv}_{\Gamma,\phi}$ we can apply Theorem 3.2.18 to conclude that $\mathcal{L}$ is $\Gamma$-behaviorally algebraizable. Note that Theorem 3.2.18 has the assumption that $\mathcal{L}$ is $\Gamma$-standard. This assumption is only used in the construction of the equivalence set $\Delta$, to guarantee that $\Delta$ has no parametric variables of sorts different from $\phi$. In this case, since we are assuming the existence of a set $\Delta$ with no parametric variables, we do not need to assume that $\mathcal{L}$ is $\Gamma$-standard.

□

As a corollary, we can give a useful sufficient condition for a logic to be behaviorally algebraizable. The result extends a well-known standard sufficient condition of AAL [BP89].

**Corollary 3.3.2.** *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic and $\Gamma$ a subsignature of $\Sigma$. A sufficient condition for $\mathcal{L}$ to be $\Gamma$-behaviorally algebraizable is that it is $\Gamma$-behaviorally equivalential with $\Gamma$-behavioral equivalence set $\Delta(\xi_1, \xi_2)$ satisfying also:*

**(G)** $\xi_1, \xi_2 \vdash \Delta(\xi_1, \xi_2)$.

*In this case $\Delta(\xi_1, \xi_2)$ and $\Theta(\xi) = \{\xi \approx e(\xi, \xi) : e \in \Delta\}$ are, respectively, the equivalence formulas and defining equations for $\mathcal{L}$.*

*Proof.* Since $\Delta[\Theta(\xi)] = \Delta[\xi, \Delta(\xi, \xi)]$, and using (G) we have that $\xi, \Delta(\xi, \xi) \vdash \Delta[\Theta(\xi)]$. Since $\Delta(\xi, \xi)$ is a $\mathcal{L}$-theorem we can conclude that $\xi \vdash \Delta[\Theta(\xi)]$. Moreover, $\Delta[\Theta(\xi)] \vdash \xi$ is a consequence of (MP), using again the fact that $\Delta(\xi, \xi)$ is a $\mathcal{L}$-theorem. Since all conditions of Theorem 3.3.1 hold, we can conclude that $\mathcal{L}$ is $\Gamma$-behaviorally algebraizable.

□

## 3.4 Remarks

We conclude with a brief summary of the achievements of this chapter. We have introduced and studied a novel generalization of the standard tools of AAL obtained by using many-sorted behavioral logic in the role traditionally played by unsorted equational logic. Continuing the effort done in the previous chapter, we set up the framework for our many-sorted behavioral approach. The extension of the signature allowed to define a behavioral consequence relation over formulas. We then introduced the central notion of $\Gamma$-behaviorally algebraizable logic, where $\Gamma$ is a subsignature of the original signature of the logic. As we referred several times, the approach is parametrized by the choice of the subsignature $\Gamma$ of the original signature. Some necessary conditions for a logic to be behaviorally algebraizable were introduced, namely some involving the notion of behaviorally equivalential logic and the notion of set of equivalence formulas. When introducing a novel notion one as to study its limits. Indeed, we proved that the novel notion of behaviorally algebraizable logic is not as broad as it becomes trivial by proving that it is in the class of standard protoalgebraic logics. This class is considered the largest class of logics amenable to the standard methods of AAL. We then introduced a behavioral notion of the Leibniz operator. This was obtained by substitution the role of congruences by $\Gamma$-congruences, where $\Gamma$ is a subsignature of the original signature. A $\Gamma$-congruence is an equivalence relation compatible with the operations in the subsignature $\Gamma$. We then engaged on a generalization of Leibniz hierarchy. We introduced behavioral versions of the notion of protoalgebraic logic and of weakly algebraizable logic, along with several of their characterization results. Characterization results for the class of behaviorally algebraizable and behaviorally equivalential logics were also obtained. We ended the chapter with some intrinsic and sufficient conditions for a logic to be behaviorally algebraizable. These are very useful in practice to show that a given logic is behavioral algebraizable.

# Chapter 4

# BAAL - Semantical considerations

We now continue the effort towards the generalization of the standard notions and results of AAL to the behavioral setting, now in a semantical perspective. We start by characterizing the class of algebras that our behavioral approach canonically associates with a given behaviorally algebraizable logic. We prove that a unicity result with respect to the algebraic counterpart of a behaviorally algebraizable logic can be obtained. We prove also a result that allows to produce the axiomatization of the algebraic counterpart of a behaviorally algebraizable logic $\mathcal{L}$ from the deductive system of $\mathcal{L}$. Matrix semantics is the standard tool for semantical investigations in AAL [Cze01]. The generalization of this tool to the behavioral setting is not straightforward and can lead to two different approaches. We start by exploring the most natural approach, the one centered on the standard notion of logical matrix. We generalize some of the results of the theory of logical matrices, ultimately aiming at bridging results, relating metalogical properties of a logic with algebraic properties of its associated class of algebras. We introduce a class $Alg_\Gamma$ of algebras generalizing the standard class $Alg$ of algebraic reducts of reduced matrices. Moreover, we prove that, in the case of a behaviorally algebraizable logic $\mathcal{L}$, the class $Alg_\Gamma(\mathcal{L})$ coincides with the largest behaviorally equivalent algebraic semantics. Given a logic $\mathcal{L}$ which is algebraizable in the standard sense and it also $\Gamma$-behaviorally algebraizable for some subsignature $\Gamma$ of the original signature, we study then the relationship between the classes $Alg_\Gamma(\mathcal{L})$ and $Alg(\mathcal{L})$. We establish relations between the classes of equations and quasi-equations satisfied by these two classes of algebras. We then develop the second approach to the generalization of the standard notion of logical matrix. This approach is strongly connected with the theory of valuation semantics [dCB94]. We introduce an algebraic version of valuation, the notion of $\Gamma$-valuation, and we prove a completeness theorem with respect to the class $Mod_\Gamma(\mathcal{L})$ of all $\Gamma$-valuation models. We prove also a result relating a metalogical property of a logic $\mathcal{L}$ and an algebraic property of $Mod_\Gamma(\mathcal{L})$. We end by showing how to extract a class $\mathcal{M}_K$ of $\Gamma$-valuations that is complete with respect to $\mathcal{L}$, from the algebraic counterpart $K$ of a $\Gamma$-behaviorally algebraizable logic $\mathcal{L}$.

The chapter is organized as follows. In Section 1 we study the class of algebras canonically associated with a given behaviorally algebraizable logic. Section 2 then focuses on a first generalization of the standard notion of logical matrix, that closely follows the standard theory. In Section 3 we study the relationship between the standard class of algebras associated with a logic and the class of algebras that our framework associates with the logic. Then, in section 4, we develop a second generalization of the standard notion of logical matrix, strongly connected with valuation semantics. We conclude, in Section 5, with some remarks.

## 4.1 Behaviorally equivalent algebraic semantics

An important goal when AAL is applied to the study of a particular logic, is to discover the class of algebras that are canonically associated with that logic. The strong connection between a logic and its associated class of algebras can be very useful for metalogical investigation and also helps to give a more clear insight about the logic. Two good examples are the strong connections between IPL (Intuitionistic Propositional Logic) and CPL (Classical Propositional Logic) with, respectively, the class of Heyting algebras and the class of Boolean algebras.

In this section we study the class of algebras canonically associated with a given behaviorally algebraizable logic. Issues like uniqueness and axiomatization of the algebraic counterpart are also discussed. At the end we show that under some mild conditions it is possible to define operations in the new sort $v$. These can be seen as the representation in $v$ of the congruent operations of sort $\phi$, thus promoting to some extent the behavioral reasoning to plain-old equational reasoning on the visible sort.

Recall that in AAL a logic is algebraizable in the standard sense, or equivalential in the standard sense, in an essentially unique way. This property derives from the fact that the equivalence set of an algebraizable logic $\mathcal{L}$ represents within $\mathcal{L}$ the relation of equality in the algebraic models of $\mathcal{L}$. The distinct feature of the equality relation is that it is a congruence relation, that is, an equivalence relation preserved by *all* primitive operations. Therefore, it should be clear that we cannot expect this kind of uniqueness within our behavioral framework. In fact, there is no guarantee that a logic cannot be behaviorally algebraizable with a different (an possibly non-comparable) choice of equivalence sets, giving rise to different behavioral algebraizations. Since uniqueness fails, we start this section by studying the relationship between existing equivalence sets within the same logic.

In the sequel, consider fixed a many-sorted signature $\Sigma$ and a many-sorted logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$. Recall that an equivalence set for $\mathcal{L}$ is a set $\Delta(\xi_1, \xi_2) \subseteq L_\Sigma(\{\xi_1, \xi_2\})$ of formulas that satisfies the following conditions:

**(R)** $\vdash \Delta(\varphi, \varphi)$;

**(S)** $\Delta(\varphi, \psi) \vdash \Delta(\psi, \varphi)$;

**(T)** $\Delta(\varphi, \psi), \Delta(\psi, \delta) \vdash \Delta(\varphi, \delta)$;

**(MP)** $\Delta(\varphi, \psi), \varphi \vdash \psi$;

Given an equivalence set $\Delta(\xi_1, \xi_2)$ and a theory $T \in Th_\mathcal{L}$, we can define a binary relation $C_\Delta(T)$ over $L_\Sigma(X)$ as follows:

$$\langle \varphi, \psi \rangle \in C_\Delta(T) \quad \text{iff} \quad T \vdash \Delta(\varphi, \psi).$$

Clearly, since $\Delta$ satisfies conditions (R), (S) and (T), we have that $C_\Delta(T)$ is indeed an equivalence relation over $L_\Sigma(X)$. Condition (MP) implies that, additionally, $C_\Delta(T)$ is compatible with $T$.

For every $T \in Th_\mathcal{L}$, let $Eqv_\mathcal{L}(T) \subseteq Eqv_{L_\Sigma(X)}$ be a set of equivalences over $L_\Sigma(X)$ defined by:

$$Eqv_\mathcal{L}(T) = \{C_\Delta(T) : \Delta(\xi_1, \xi_2) \text{ is an equivalence set of } \mathcal{L}\}.$$

Intuitively, $Eqv_\mathcal{L}(T)$ can be seen as the set of equivalences over $L_\Sigma(X)$ that can be defined by a set of formulas with two variables over the deductive consequence of $\mathcal{L}$. Clearly, inclusion defines a partial order on $Eqv_\mathcal{L}(T)$. We can see that

$$C_{\Delta_1}(T) \subseteq C_{\Delta_2}(T) \quad \text{iff} \quad T, \Delta_2(\xi_1, \xi_2) \vdash \Delta_1(\xi_1, \xi_2)$$

and that, in particular,

$$C_{\Delta_1}(T) = C_{\Delta_2}(T) \quad \text{iff} \quad T, \Delta_1(\xi_1, \xi_2) \dashv\vdash T, \Delta_2(\xi_1, \xi_2).$$

**Proposition 4.1.1.** *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic. For every $T \in Th_\mathcal{L}$ we have that $\langle Eqv_\mathcal{L}(T), \subseteq \rangle$ constitutes a complete lattice.*

*Proof.* Let $T \in Th_{\mathcal{L}}$. Let us verify that the infimum of a set $\{C_{\Delta_i}(T) : i \in I\}$ is $C_{\Delta^*}(T)$ where $\Delta^*(\xi_1, \xi_2) = \bigcup_{i \in I} \Delta_i(\xi_1, \xi_2)$. First we need to verify that $C_{\Delta^*}(T)$ is a lower bound of $\{C_{\Delta_i}(T) : i \in I\}$ and then we have to verify that it is the greatest one.

Let $\langle \varphi, \psi \rangle \in C_{\Delta^*}(T)$. Then we have that $T \vdash \bigcup_{i \in I} \Delta_i(\varphi, \psi)$. In particular, for each $i \in I$, we have that $T \vdash \Delta_i(\varphi, \psi)$. So, we have that $\langle \varphi, \psi \rangle \in C_{\Delta_i}(T)$. We can conclude that $C_{\Delta^*}(T) \subseteq C_{\Delta_i}(T)$ for every $i \in I$, and so $C_{\Delta^*}(T)$ is a lower bound of $\{C_{\Delta_i}(T) : i \in I\}$.

To prove that $C_{\Delta^*}(T)$ is indeed the greatest lower bound of $\{C_{\Delta_i}(T) : i \in I\}$, let $C_{\Delta}(T) \in Eqv_{\mathcal{L}}(T)$ such that $C_{\Delta}(T) \subseteq C_{\Delta_i}(T)$ for every $i \in I$. Then we have that $\Delta_i(\xi_1, \xi_2) \vdash \Delta(\xi_1, \xi_2)$ for every $i \in I$, and so $\bigcup_{i \in I} \Delta_i(\xi_1, \xi_2) \vdash \Delta(\xi_1, \xi_2)$. We can then conclude that $C_{\Delta}(T) \subseteq C_{\Delta^*}(T)$.

□

We already gave some clues why we cannot aim at full uniqueness in the behavioral algebraization process. One of the main reasons is that the behavioral algebraization process is parametrized by the choice of the subsignature $\Gamma$. Nevertheless, it is interesting that, once $\Gamma$ is fixed, we can prove a uniqueness result with the same flavor as the standard one proved in [BP89].

**Theorem 4.1.2.** *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic and $\Gamma$ a subsignature of $\Sigma$. Suppose that $\mathcal{L}$ is $\Gamma$-behaviorally algebraizable logic and let $K$ and $K'$ be two $\Gamma$-behaviorally equivalent algebraic semantics for $\mathcal{L}$ such that $\Delta(\xi_1, \xi_2)$ and $\Theta(\xi)$ are the equivalence formulas and defining equations for $K$, and similarly $\Delta'(\xi_1, \xi_2)$ and $\Theta'(\xi)$ for $K'$. Then we have that:*

*i)* $\vDash^{K,\Gamma}_{\Sigma,bhv} = \vDash^{K',\Gamma}_{\Sigma,bhv}$;

*ii)* $\Delta(\xi_1, \xi_2) \dashv\vdash \Delta'(\xi_1, \xi_2)$;

*iii)* $\Theta(\xi) =\!\mid\mid\!=^{K,\Gamma}_{\Sigma,bhv} \Theta'(\xi)$.

*Proof.* We first prove condition *ii)*, that is, we prove that $\Delta(\xi_1, \xi_2) \dashv\vdash \Delta'(\xi_1, \xi_2)$. Note that $\Delta'(\xi_1, \xi_2) : \phi^2 \to \phi \in Der_{\Gamma,\phi}$. So, since $\Delta$ is a $\Gamma$-behavioral equivalence set for $\mathcal{L}$, we have that $\Delta(\xi_1, \xi_2) \vdash \Delta(\Delta'(\xi_1, \xi_1), \Delta'(\xi_1, \xi_2))$. Using $(MP)$ and the fact that $\vdash \Delta'(\xi_1, \xi_1)$, we can conclude that $\Delta(\xi_1, \xi_2) \vdash \Delta'(\xi_1, \xi_2)$. Moreover, using a symmetric argument we obtain $\Delta(\xi_1, \xi_2) \dashv\vdash \Delta'(\xi_1, \xi_2)$.

To prove condition *i)* we use the fact that $\Delta(\xi_1, \xi_2) \dashv\vdash \Delta'(\xi_1, \xi_2)$ and that $K$ and $K'$ are two $\Gamma$-behaviorally equivalent algebraic semantics for $\mathcal{L}$. In fact, for any

$\Psi \cup \{\varphi \approx \psi\} \subseteq Eq_\Sigma(X)$, we have the following equivalent conditions:

$$\begin{aligned}\Psi \vDash^{K,\Gamma}_{\Sigma,bhv} \varphi \approx \psi \quad &\text{iff} \quad \{\Delta(\delta_1,\delta_2) : \delta_1 \approx \delta_2 \in \Psi\} \vdash \Delta(\varphi,\psi)\\ &\text{iff} \quad \{\Delta'(\delta_1,\delta_2) : \delta_1 \approx \delta_2 \in \Psi\} \vdash \Delta'(\varphi,\psi)\\ &\text{iff} \quad \Psi \vDash^{K',\Gamma}_{\Sigma,bhv} \varphi \approx \psi.\end{aligned}$$

To establish condition *iii)* we use again the fact that $\Delta(\xi_1,\xi_2) \dashv\vdash \Delta'(\xi_1,\xi_2)$ and that $K$ and $K'$ are two $\Gamma$-behaviorally equivalent algebraic semantics for $\mathcal{L}$. In fact, we have the following equivalent conditions:

$$\begin{aligned}\Theta(\xi) \mathrel{=\!\!||\!\!=}^{K,\Gamma}_{\Sigma,bhv} \Theta'(\xi) \quad &\text{iff} \quad \Delta[\Theta(\xi)] \dashv\vdash \Delta[\Theta'(\xi)]\\ &\text{iff} \quad \Delta[\Theta(\xi)] \dashv\vdash \Delta'[\Theta'(\xi)]\\ &\text{iff} \quad \xi \dashv\vdash \xi.\end{aligned}$$

Since this last condition is true in every logic, we can conclude that

$$\Theta(\xi) \mathrel{=\!\!||\!\!=}^{K,\Gamma}_{\Sigma,bhv} \Theta'(\xi).$$

□

Theorem 4.1.2 allows us to conclude that, as in the standard result, given a $\Gamma$-behaviorally algebraizable logic $\mathcal{L}$ we can consider *the* largest $\Gamma$-behaviorally equivalent algebraic semantics, denoted by $K^\Gamma_\mathcal{L}$. But contrarily to the case of standard AAL, in our approach $K^\Gamma_\mathcal{L}$ is not the class of algebras that should be canonically associated with $\mathcal{L}$. Indeed, as we will see, it is a subclass of $K^\Gamma_\mathcal{L}$ that will allow us to generalize the stnadard results of AAL.

Consider now the particular case where a many-sorted logic $\mathcal{L} = \langle \Sigma, \vdash\rangle$ is finitary and finitely $\Gamma$-behaviorally algebraizable for some subsignature $\Gamma$ of $\Sigma$. An immediate consequence of Theorem 4.1.2 is that, if $K$ and $K'$ are two $\Gamma$-behaviorally equivalent algebraic semantics for $\mathcal{L}$, then $K$ and $K'$ must $\Gamma$-behaviorally satisfy the same quasi-equations. So, $K$ and $K'$ generate the same $\Gamma$-hidden quasivariety and this $\Gamma$-hidden quasivariety is also a $\Gamma$-behaviorally equivalent algebraic semantics for $\mathcal{L}$. Therefore, we can talk about *the* equivalent $\Gamma$-hidden quasivariety semantics of a finitary and finitely $\Gamma$-behaviorally algebraizable logic. It is interesting to note that, similarly to what Blok and Pigozzi propose for finitary and finitely algebraizable propositional logics [BP89], we can construct a basis for the quasi-equations of the unique equivalent $\Gamma$-hidden quasivariety semantics given an axiomatization of $\mathcal{L}$.

**Theorem 4.1.3.** *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a finitary many-sorted logic given by a deductive system composed of a set $Ax$ of axioms and a set $Ir$ of inference rules, and consider $\Gamma$ a subsignature of $\Sigma$. Assume that $\mathcal{L}$ is finitely $\Gamma$-behaviorally algebraizable with defining equations $\Theta(\xi)$ and equivalence formulas $\Delta(\xi_1, \xi_2)$. Then, the unique equivalent $\Gamma$-hidden quasivariety semantics for $\mathcal{L}$ is axiomatized by the following equations and quasi-equations:*

*i)* $\Theta(\varphi)$*, for every theorem* $\varphi$ *of* $\mathcal{L}$*;*

*ii)* $\Theta[\Delta(\xi, \xi)]$*;*

*iii)* $\Theta(\psi_1) \wedge \ldots \wedge \Theta(\psi_n) \rightarrow \Theta(\varphi)$ *for every* $\langle \psi_1, \ldots, \psi_n, \varphi \rangle \in Ir$*;*

*iv)* $\Theta[\Delta(\xi_1, \xi_2)] \rightarrow \xi_1 \approx \xi_2$*.*

*Proof.* Let $H$ be the $\Gamma$-hidden quasivariety defined by conditions *i)* - *iv)*. We show that $H$ is the equivalent $\Gamma$-hidden quasivariety semantics of $\mathcal{L}$.

Equation *ii)* and quasi-equation *iv)* together are equivalent to

$$\xi_1 \approx \xi_2 =\mid\models^{H,\Gamma}_{\Sigma,bhv} \Theta[\Delta(\xi_1, \xi_2)].$$

To prove that

$$T \vdash \varphi \text{ iff } \Theta[T] \models^{H,\Gamma}_{\Sigma,bhv} \Theta(\varphi)$$

let us rewrite this condition in a more useful way:

$$T \vdash \varphi \text{ iff } \varphi \in \Psi$$

where

$$\Psi = \{\psi \in L_\Sigma(X) : \Theta[T] \models^{H,\Gamma}_{\Sigma,bhv} \Theta(\psi)\}.$$

From condition *i)* we have that $Ax \subseteq \Psi$ and by condition *iii)* we can conclude that $\Psi$ is closed under the inference rules of $\mathcal{L}$. So, $\Psi$ is a theory of $\mathcal{L}$. Since $T \subseteq \Phi$ we can conclude that if $T \vdash \varphi$ then $\varphi \in \Psi$.

To show the converse, assume that $\varphi \in \Psi$. Let $K$ be the $\Gamma$-hidden equivalent quasivariety semantics for $\mathcal{L}$, which exists by hypothesis. Then $K$ necessarily satisfies conditions *i)*-*iv)* and so $K \subseteq H$. Therefore $\Theta[T] \models^{K,\Gamma}_{\Sigma,bhv} \Theta(\varphi)$ and, using the fact that $K$ is a behaviorally equivalent algebraic semantics for $\mathcal{L}$, we can conclude that $T \vdash \varphi$.

□

It might seem unnatural that, although the language of a many-sorted logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ is over the signature $\Sigma$, we associate to $\mathcal{L}$ a class of algebras over the extended signature $\Sigma^o$. This is, nevertheless, a key point of our approach since it is this precise technical detail that allows us to have behavioral reasoning over the whole $\Sigma$ and, in particular, over the formulas.

Since the logic $\mathcal{L}$ is over the signature $\Sigma$ we do not have full control on the new sort $v$, in the sense that, given a $\Sigma^o$-algebra $\mathbf{A}$, the set $A_v$ might contain more information than what is needed for defining the consequence $\models^{K,\Gamma}_{\Sigma,bhv}$. Therefore, in some sense, the largest $\Gamma$-behaviorally equivalent algebraic semantics of a $\Gamma$-behaviorally algebraizable logic $\mathcal{L}$, $K^{\Gamma}_{\mathcal{L}}$, contains more algebras than the ones we would like to canonically associate with $\mathcal{L}$. We will see how we can extract from $K^{\Gamma}_{\mathcal{L}}$ the class of algebras we are interested in canonically associate with the logic $\mathcal{L}$.

First of all, recall that in the construction of an extended signature $\Sigma^o$ from a many-sorted signature $\Sigma$, we just added a new sort $v$ and an operation $o : \phi \to v$. No operation on the new sort $v$ was defined . We now see that, given a $\Sigma^o$-algebra $\mathbf{A}$ and under some mild conditions, some connectives in the visible sort $v$ arise naturally from the connectives in the sort $\phi$.

Let $\mathbf{A}$ be a $\Sigma^o$-algebra such that $o_{\mathbf{A}}$ is surjective and let $f : \phi^n \to \phi \in Der_{\Sigma,\phi}$. Assume that $\mathbf{A}$ satisfies the visible quasi-equation

$$o(\xi_1^1) \approx o(\xi_1^2) \& \ldots \& o(\xi_n^1) \approx o(\xi_n^2) \to o(f(\xi_1^1, \ldots, \xi_n^1)) \approx o(f(\xi_1^2, \ldots, \xi_n^2)).$$

This quasi-equation expresses the fact that $f_{\mathbf{A}}$ behaves well with respect to operation $o$, and in this case we say that $f$ is a congruent connective on $\mathbf{A}$. We can define a $n$-ary operation $f^v : v^n \to v$ over $\mathbf{A}$ such that, for every $a_1, \ldots, a_n \in A_\phi$,

$$f^v_{\mathbf{A}}(o_{\mathbf{A}}(a_1), \ldots, o_{\mathbf{A}}(a_n)) = o_{\mathbf{A}}(f_{\mathbf{A}}(a_1, \ldots, a_n)).$$

It is easy to see that this operation is well-defined since we are assuming that $o_{\mathbf{A}}$ is surjective and that $\mathbf{A}$ satisfies the above visible quasi-equation.

Let $\Sigma$ be a many-sorted signature and $\Gamma$ a subsignature of $\Sigma$. Given an $\Sigma^o$-algebra $\mathbf{A}$ recall that $\equiv_{\Gamma}$ denotes the $\Gamma$-behavioral equivalence relation over $\mathbf{A}$. From $\mathbf{A}$ we can define a $\Sigma^o$-algebra $\mathbf{A}^*$ in the following way:

- $\mathbf{A}^*_{|\Sigma} = \mathbf{A}_{|\Sigma}$,
- $A^*_v = \{[a]_{\equiv_\Gamma} : a \in A_\phi\}$;

- $o_{\mathbf{A}^*}(a) = [a]_{\equiv_\Gamma}$.

**Lemma 4.1.4.** *Given a $\Sigma^o$-algebra $\mathbf{A}$, an assignment $h$ over $\mathbf{A}$ and $t_1, t_2 \in T_{\Sigma,s}(X)$, we have that*

$$\mathbf{A}, h \Vdash_\Gamma t_1 \approx t_2 \quad \textit{iff} \quad \mathbf{A}^*, h \Vdash_\Gamma t_1 \approx t_2.$$

*Proof.* First of all, recall that every experiment $\epsilon \in \mathcal{E}^\Gamma_\Sigma[x\!:\!s]$ is of the form $o(c(x\!:\!s))$ where $c(x\!:\!s) \in \mathcal{C}^\Gamma_{\Sigma,\phi}[x\!:\!s]$. Consider now the following sequence of equivalent conditions:

$$\begin{array}{lll}
\mathbf{A}, h \Vdash_\Gamma t_1 \approx t_2 & \text{iff} & h(t_1) \equiv_\Gamma h(t_2) \quad \text{on } \mathbf{A} \\[2ex]
& \text{iff} & \begin{array}{c} \text{for every } c(x\!:\!s, x_1\!:\!s_1, \dots, x_n\!:\!s_n) \in \mathcal{C}^\Gamma_{\Sigma,\phi}[x\!:\!s] \\ \text{and } \langle a_1, \dots, a_n \rangle \in A_{s_1} \times \dots \times A_{s_n} \\ o_{\mathbf{A}}(c_{\mathbf{A}}(h(t_1), a_1, \dots, a_n)) = o_{\mathbf{A}}(c_{\mathbf{A}}(h(t_2), a_1, \dots, a_n)) \end{array} \\[6ex]
& \text{iff} & \begin{array}{c} \text{for every } c(x\!:\!s, x_1\!:\!s_1, \dots, x_n\!:\!s_n) \in \mathcal{C}^\Gamma_{\Sigma,\phi}[x\!:\!s] \\ \text{and } \langle a_1, \dots, a_n \rangle \in A_{s_1} \times \dots \times A_{s_n} \\ c_{\mathbf{A}}(h(t_1), a_1, \dots, a_n) \equiv_\Gamma c_{\mathbf{A}}(h(t_2), a_1, \dots, a_n) \end{array} \\[6ex]
& \text{iff} & \begin{array}{c} \text{for every } c(x\!:\!s, x_1\!:\!s_1, \dots, x_n\!:\!s_n) \in \mathcal{C}^\Gamma_{\Sigma,\phi}[x\!:\!s] \\ \text{and } \langle a_1, \dots, a_n \rangle \in A_{s_1} \times \dots \times A_{s_n} \\ [c_{\mathbf{A}}(h(t_1), a_1, \dots, a_n)]_{\equiv_\Gamma} = [c_{\mathbf{A}}(h(t_2), a_1, \dots, a_n)]_{\equiv_\Gamma} \end{array} \\[6ex]
& \text{iff} & \begin{array}{c} \text{for every } c(x\!:\!s, x_1\!:\!s_1, \dots, x_n\!:\!s_n) \in \mathcal{C}^\Gamma_{\Sigma,\phi}[x\!:\!s] \\ \text{and } \langle a_1, \dots, a_n \rangle \in A_{s_1} \times \dots \times A_{s_n} \\ o_{\mathbf{A}^*}(c_{\mathbf{A}^*}(h(t_1), a_1, \dots, a_n)) = o_{\mathbf{A}^*}(c_{\mathbf{A}^*}(h(t_2), a_1, \dots, a_n)) \end{array} \\[6ex]
& \text{iff} & h(t_1) \equiv_\Gamma h(t_2) \quad \text{on } \mathbf{A}^* \\[2ex]
& \text{iff} & \mathbf{A}^*, h \Vdash_\Gamma t_1 \approx t_2
\end{array}$$

□

Given a class $K$ of $\Sigma^o$-algebras we can apply the above construction to every algebra in $K$, thus obtaining

$$K^* = \{\mathbf{A}^* : \mathbf{A} \in K\}.$$

**Proposition 4.1.5.** *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic and $\Gamma$ a subsignature of $\Sigma$. Suppose that $\mathcal{L}$ is $\Gamma$-behaviorally algebraizable and $K$ is a $\Gamma$-behaviorally equivalent algebraic semantics for $\mathcal{L}$. Then we have that*

*i)* $\vDash^{K,\Gamma}_{\Sigma,bhv} = \vDash^{K^*,\Gamma}_{\Sigma,bhv}$ *;*

*ii)* $K^*$ *is also a $\Gamma$-behaviorally equivalent algebraic semantics for $\mathcal{L}$.*

*Proof.* First of all, we note that condition *ii*) is an immediate consequence of condition *i*). To prove condition *i*) consider the following sequence of equivalent sentences, that is a consequence of the definition of $K^*$ and Lemma 4.1.4:

$$\Psi \vDash^{K,\Gamma}_{\Sigma,bhv} t_1 \approx t_2 \quad \text{iff} \quad \begin{array}{c}\text{for every } \mathbf{A} \in K \text{ and } h \text{ assignment over } \mathbf{A} \text{ we have that} \\ \mathbf{A}, h \Vvdash_\Gamma t_1 \approx t_2 \text{ whenever } \mathbf{A}, h \Vvdash_\Gamma r_1 \approx r_2 \ \text{ for every } r_1 \approx r_2 \in \Psi\end{array}$$

$$\text{iff} \quad \begin{array}{c}\text{for every } \mathbf{A} \in K \text{ and } h \text{ assignment over } \mathbf{A} \text{ we have that} \\ \mathbf{A}^*, h \Vvdash_\Gamma \varphi \approx \psi \text{ whenever } \mathbf{A}^*, h \Vvdash_\Gamma \gamma_1 \approx \gamma_2 \ \text{ for every } \gamma_1 \approx \gamma_2 \in \Psi\end{array}$$

$$\text{iff} \quad \Psi \vDash^{K^*,\Gamma}_{\Sigma,bhv} t_1 \approx t_2$$

□

Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic and $\Gamma$ a subsignature of $\Sigma$. Suppose that $\mathcal{L}$ is $\Gamma$-behaviorally algebraizable and let $K^\Gamma_\mathcal{L}$ be its largest $\Gamma$-behaviorally equivalent algebraic semantics. We can apply the above $*$-construction to $K^\Gamma_\mathcal{L}$ and obtain a class $(K^\Gamma_\mathcal{L})^*$ of $\Sigma^o$-algebras. By definition of $K^\Gamma_\mathcal{L}$ and by Proposition 4.1.5 we have that $(K^\Gamma_\mathcal{L})^*$ is a subclass of $K^\Gamma_\mathcal{L}$. The class $(K^\Gamma_\mathcal{L})^*$ is the class of $\Sigma^o$-algebras we canonically associate to $\mathcal{L}$.

The following lemma presents an important property of the class $(K^\Gamma_\mathcal{L})^*$. It states that, using operation $o$, the behavioral reasoning over formulas can be reduced to pure equational reasoning.

**Lemma 4.1.6.** *For every $\mathbf{A} \in (K^\Gamma_\mathcal{L})^*$, $h$ an assignment over $\mathbf{A}$ and $\varphi_1, \varphi_2 \in L_\Sigma(X)$, we have that*

$$\mathbf{A}, h \Vvdash_\Gamma \varphi_1 \approx \varphi_2 \quad \textit{iff} \quad \mathbf{A}, h \Vdash o(\varphi_1) \approx o(\varphi_2).$$

*Proof.* First of all note that, since $\mathbf{A} \in (K^\Gamma_\mathcal{L})^*$, there exists $\mathbf{B} \in K^\Gamma_\mathcal{L}$ such that $\mathbf{A} = \mathbf{B}^*$.

The fact that $\mathbf{A}, h \Vdash\!\vdash_\Gamma \varphi_1 \approx \varphi_2$ implies $\mathbf{A}, h \Vdash o(\varphi_1) \approx o(\varphi_2)$ is a trivial consequence of $o(\xi)$ being an experiment.

Suppose now that $\mathbf{A}, h \Vdash o(\varphi_1) \approx o(\varphi_2)$. This means that $o_\mathbf{A}(h(\varphi_1)) = o_\mathbf{A}(h(\varphi_2))$, and by definition of $\mathbf{B}^*$ we have that $h(\varphi_1) \equiv_\Gamma h(\varphi_2)$ in $\mathbf{B}$. So, we have that $\mathbf{B}, h \Vdash\!\vdash_\Gamma \varphi_1 \approx \varphi_2$, and using Lemma 4.1.4 we can conclude that $\mathbf{A}, h \Vdash\!\vdash_\Gamma \varphi_1 \approx \varphi_2$.

□

The following lemma asserts that, for every algebra belonging to $(K^\Gamma_\mathcal{L})^*$, the connectives of $\Gamma$ are all congruent.

**Lemma 4.1.7.** *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic and $\Gamma$ a subsignature of $\Sigma$. Suppose that $\mathcal{L}$ is $\Gamma$-behaviorally algebraizable. Then, every operation $f : \phi^n \to \phi \in \Gamma$ is congruent in every member of $(K^\Gamma_\mathcal{L})^*$.*

*Proof.* Just for sake of notation take $K = (K^\Gamma_\mathcal{L})^*$ in this proof. We have to show that, for every $f : \phi^n \to \phi \in \Gamma$, the quasi-equation

$$o(\xi_1^1) \approx o(\xi_1^2) \& \ldots \& o(\xi_n^1) \approx o(\xi_n^2) \to o(f(\xi_1^1, \ldots, \xi_n^1)) \approx o(f(\xi_1^2, \ldots, \xi_n^2))$$

is satisfied in every algebra of $K$. By Lemma 4.1.6 this is equivalent to prove that

$$\xi_1^1 \approx \xi_1^2, \ldots, \xi_n^1 \approx \xi_n^2 \vDash^{K,\Gamma}_{\Sigma,bhv} f(\xi_1^1, \ldots, \xi_n^1) \approx f(\xi_1^2, \ldots, \xi_n^2).$$

From Lemma 3.1.2 we have, for every $c(x{:}s, x_1{:}s_1, \ldots, x_m{:}s_m) \in \mathcal{C}^\Gamma_{\Sigma,\phi}[\xi]$ and $\langle \varphi_1, \ldots, \varphi_m \rangle \in (L_\Sigma(X))^m$, that

$$\varphi \approx \psi \vDash^{K,\Gamma}_{\Sigma,bhv} c(\varphi, \varphi_1, \ldots, \varphi_m) \approx c(\psi, \varphi_1, \ldots, \varphi_m).$$

So, in particular, we have that

$$\xi_1^1 \approx \xi_1^2 \vDash^{K,\Gamma}_{\Sigma,bhv} f(\xi_1^1, \xi_2^1, \ldots, \xi_n^1) \approx f(\xi_1^2, \xi_2^1 \ldots, \xi_n^1),$$

and we have also that

$$\xi_2^1 \approx \xi_2^2 \vDash^{K,\Gamma}_{\Sigma,bhv} f(\xi_1^2, \xi_2^1, \ldots, \xi_n^1) \approx f(\xi_1^2, \xi_2^2 \ldots, \xi_n^1).$$

Applying this idea for every $1 \leq i \leq n$, we obtain

$$\xi_i^1 \approx \xi_i^2 \vDash^{K,\Gamma}_{\Sigma,bhv} f(\xi_1^2, \xi_2^2, \ldots, \xi_{i-1}^2, \xi_i^1, \xi_{i+1}^1 \ldots, \xi_n^1) \approx f(\xi_1^2, \xi_2^2 \ldots, \xi_{i-1}^2, \xi_i^2, \xi_{i+1}^1, \ldots, \xi_n^1).$$

We then apply Transitivity (T) and obtain

$$\xi_1^1 \approx \xi_1^2, \ldots, \xi_n^1 \approx \xi_n^2 \vDash^{K,\Gamma}_{\Sigma,bhv} f(\xi_1^1, \ldots, \xi_n^1) \approx f(\xi_1^2, \ldots, \xi_n^2).$$

□

Lemma 4.1.7 implies that we can define, for every algebra $\mathbf{A}$ in $(K_{\mathcal{L}}^{\Gamma})^*$ and for every operation $f : \phi^n \to \phi$ in $\Gamma$, its visible counterpart $f_{\mathbf{A}}^v : A_v^n \to A_v$ on $\mathbf{A}$. Thus, for every $\Sigma^o$-algebra in $(K_{\mathcal{L}}^{\Gamma})^*$, we can consider, without loss of generality, that the algebra $\mathbf{A}$ is endowed with the operations on the sort $v$ that arise in this way from congruent operations on the sort $\phi$.

## 4.2 Matrix semantics

Matrix semantics is one of the important standard tools used for semantical investigation in AAL and a lot of fruitful and enlightening results already established. We point to Wójcicki's 1988 book [Wój88] and Czelakowski's book [Cze01] as sources for the large body of research on this topic.

In this section we introduce and study a possible generalization of the notion of matrix semantics. This generalization is the natural one, closely following the path of research of the theory of logical matrices. We show that some important results from standard theory of AAL regarding matrix semantics generalize smoothly to our behavioral setting.

In the sequel we consider fixed a many-sorted signature $\Sigma$ and a subsignature $\Gamma$ of $\Sigma$.

**Definition 4.2.1.** A *(many-sorted logical) matrix* over $\Sigma$ is a tuple $\mathcal{M} = \langle \mathbf{A}, D \rangle$ where $\mathbf{A}$ is a $\Sigma$-algebra and $D \subseteq A_\phi$ is the set of *designated values.*

An *assignment* over $\mathcal{M}$ is, as usual, a homomorphism $h : \mathbf{T}_{\mathbf{\Sigma}}(\mathbf{X}) \to \mathbf{A}$. Given a matrix $\mathcal{M} = \langle \mathbf{A}, D \rangle$ over $\Sigma$, we can define a consequence relation over $\Sigma$, denoted by $\vdash_{\mathcal{M}}$, such that, for every $T \cup \{\varphi\} \subseteq L_\Sigma(X)$, we have that $T \vdash_{\mathcal{M}} \varphi$ iff for every assignment $h$ over $\mathcal{M}$ we have that

$$h(\varphi) \in D \quad \text{whenever} \quad h(\psi) \in D \text{ for every } \psi \in T.$$

**Definition 4.2.2.** Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic and $\mathcal{M} = \langle \mathbf{A}, D \rangle$ a matrix over $\Sigma$. The matrix $\mathcal{M}$ is a *model* of $\mathcal{L}$ if $\vdash \subseteq \vdash_{\mathcal{M}}$, that is, for every $T \cup \{\varphi\} \subseteq L_\Sigma(X)$, we have that $T \vdash_{\mathcal{M}} \varphi$ whenever $T \vdash \varphi$. In this case, $D$ is called a $\mathcal{L}$-filter of $\mathbf{A}$.

Given a $\Sigma$-algebra $\mathbf{A}$, the set of all $\mathcal{L}$-filters of $\mathbf{A}$, is denoted by $Fi_{\mathcal{L}}(\mathbf{A})$. We endow $Fi_{\mathcal{L}}(\mathbf{A})$ with set-theoretical inclusion and since it is closed under intersections of arbitrary families it is a complete lattice. Therefore, given any set $C \subseteq A$, there is always the least $\mathcal{L}$-filter of $\mathbf{A}$ that contains $C$. This least $\mathcal{L}$-filter is called the $\mathcal{L}$-filter of $\mathbf{A}$ generated by $C$ and is denoted by $Fi^{\mathbf{A}}_{\mathcal{L}}(C)$. The class of all matrix models of $\mathcal{L}$ is denoted by $Mod(\mathcal{L})$.

As we have seen up to now, all the definitions we have presented are immediate generalizations of the standard ones. The first differences begin to appear when studying congruences on a matrix, since now the $\Gamma$-congruences play a key role.

**Definition 4.2.3.** A *matrix* $\Gamma$*-congruence* over a matrix $\mathcal{M} = \langle \mathbf{A}, D\rangle$ is a $\Gamma$-congruence $\theta$ over $\mathbf{A}$ compatible with $D$, that is, $\theta$ is a $\Gamma$-congruence over $\mathbf{A}$ and for every $a, b \in A_\phi$, we have that $b \in D$ whenever $\langle a, b\rangle \in \theta_\phi$ and $a \in D$.

A matrix $\Gamma_\phi$-congruence over a matrix $\mathcal{M} = \langle \mathbf{A}, D\rangle$ is the $\phi$-restriction of a matrix $\Gamma$-congruence.

**Proposition 4.2.4.** *Let* $\mathcal{M} = \langle \mathbf{A}, D\rangle$ *be a matrix over* $\Sigma$*. Then, there is a largest matrix* $\Gamma$*-congruence over* $\mathcal{M}$*.*

*Proof.* Consider the binary relation $\Phi_{\mathcal{M}}$ over $\mathbf{A}$ such that, for every $s \in S$, we have that $\langle a, b\rangle \in (\Phi_{\mathcal{M}})_s$ iff for every $c(x : s, x_1 : s_1, \ldots, x_n : s_n) \in \mathcal{C}^{\Gamma}_{\Sigma,\phi}[x : s]$ and every $\langle a_1, \ldots, a_n\rangle \in A_{s_1} \times \ldots \times A_{s_n}$ we have that

$$c_{\mathbf{A}}(a, a_1, \ldots, a_n) \in D \quad \text{iff} \quad c_{\mathbf{A}}(b, a_1, \ldots, a_n) \in D.$$

It is easy to conclude that $\Phi_{\mathcal{M}}$ is a matrix $\Gamma$-congruence. We now prove that it is indeed the largest one.

Let $\alpha$ be a matrix $\Gamma$-congruence over $\mathcal{M}$. We aim to prove that $\alpha \subseteq \Phi_{\mathcal{M}}$. Consider $\langle a, b\rangle \in \alpha_s$, for some $s \in S$. Since $\alpha$ is a $\Gamma$-congruence we can conclude that, for every $c(x : s, x_1 : s_1, \ldots, x_n : s_n) \in \mathcal{C}^{\Gamma}_{\Sigma,\phi}[x : s]$ and every $\langle a_1, \ldots, a_n\rangle \in A_{s_1} \times \ldots \times A_{s_n}$ we have that

$$\langle c_{\mathbf{A}}(a, a_1, \ldots, a_n), c_{\mathbf{A}}(b, a_1, \ldots, a_n)\rangle \in \alpha_\phi.$$

Using now the fact that $\alpha$ is compatible with $T$, we can conclude that

$$c_{\mathbf{A}}(a, a_1, \ldots, a_n) \in D \quad \text{iff} \quad c_{\mathbf{A}}(b, a_1, \ldots, a_n) \in D.$$

So, we have that $\langle a, b\rangle \in \Phi_{\mathcal{M}}$.

□

The fact that the largest matrix $\Gamma$-congruence exists, allows us to define the matrix version of the behavioral Leibniz congruence.

**Definition 4.2.5.** Let $\mathcal{M} = \langle \mathbf{A}, D\rangle$ be a matrix over $\Sigma$. Then, the $\Gamma$-*behavioral Leibniz congruence of* $\mathcal{M}$, denoted by $\Omega^{bhv}_{\Gamma,\mathbf{A}}(D)$, is the largest matrix $\Gamma$-congruence over $\mathcal{M}$.

We denote by $\Omega^{bhv}_{\Gamma,\mathbf{A},\phi}(D)$ the restriction of $\Omega^{bhv}_{\Gamma,\mathbf{A}}(D)$ to the sort $\phi$. As we will see, due to the fundamental role that the sort $\phi$ plays in our approach, the restriction $\Omega^{bhv}_{\Gamma,\mathbf{A},\phi}(D)$ is very useful.

The following characterization result is an immediate consequence of the proof of Proposition 4.2.4 and it generalizes the standard result.

**Corollary 4.2.6.** *Given a matrix* $\mathcal{M} = \langle \mathbf{A}, D\rangle$ *over* $\Sigma$, *we have that, for every* $s \in S$, $\langle a, b\rangle \in (\Omega^{bhv}_{\Gamma,\mathbf{A}}(D))_s$ *iff for every* $c(x{:}s, x_1{:}s_1, \ldots, x_n{:}s_n) \in \mathcal{C}^{\Gamma}_{\Sigma,\phi}[x : s]$ *and every* $\langle a_1, \ldots, a_n\rangle \in A_{s_1} \times \ldots \times A_{s_n}$ *we have that*

$$c_{\mathbf{A}}(a, a_1, \ldots, a_n) \in D \quad \textit{iff} \quad c_{\mathbf{A}}(b, a_1, \ldots, a_n) \in D.$$

Recall that the standard class of algebras that AAL canonically associates to a logic is the class of algebraic reducts of the Leibniz reduced matrices. This class of algebras is precisely the class of algebras of the form $\mathbf{A}/_{\Omega_{\mathbf{A}}(D)}$ with $D$ a $\mathcal{L}$-filter of $\mathbf{A}$. In our behavioral approach we cannot perform quotients since the behavioral Leibniz congruence is not, in general, a congruence. The operations outside $\Gamma$ would not be necessarily well-defined in the quotient construction. To overcome this difficulty, the extension of the signature and the use of algebras over the extended signature are a key ingredients.

We now see how we can obtain an algebra over the extended signature that, in some sense, represents this quotient.

Given a matrix $\Gamma_\phi$-congruence $\theta$ over a matrix $\mathcal{M} = \langle \mathbf{A}, D\rangle$, consider the $\Sigma^o$-algebra $\mathbf{A}^o_\theta$ such that:

- $\mathbf{A}^o_{\theta|_\Sigma} = \mathbf{A}$;
- $(A^o_\theta)_v = A_\phi/\theta = \{[a]_\theta : a \in A_\phi\}$;
- $o_{\mathbf{A}^o_\theta}(a) = [a]_\theta$.

The idea, as we will explain better below, is to use the visible part $(A^o_\theta)_v$ to simulate the quotient.

Consider given a $\Sigma$-algebra $\mathbf{A}$ and a class $K$ of $\Sigma^o$-algebras. A $\Gamma_\phi$-congruence $\theta$ over $\mathbf{A}$ is said to be a $K$-$\Gamma_\phi$-congruence if $\mathbf{A}^o_\theta \in K$. We denote by $Con^K_\Gamma(\mathbf{A})$ the set of all $K$-$\Gamma_\phi$-congruences of $\mathbf{A}$. It is easy to see that $Con^K_\Gamma(\mathbf{A})$ is a sublattice of $Eqv^{\Sigma_{|\phi}}(\mathbf{A}_{|\phi})$, the lattice of equivalences of $\mathbf{A}_\phi$.

In our approach we canonically associate to a given logic a class of algebras over the extended signature that corresponds, in this more general approach, to the algebraic reducts of Leibniz reduced matrices. Given a many-sorted logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ and a subsignature $\Gamma$ of $\Sigma$, consider the class

$$Alg^*_\Gamma(\mathcal{L}) = \{\mathbf{A}^o_{\Omega^{bhv}_{\Gamma,\mathbf{A},\phi}(D)} : \langle \mathbf{A}, D\rangle \in Mod(\mathcal{L})\}.$$

The class $Alg^*_\Gamma(\mathcal{L})$ is the class of algebras that, in this semantical approach, we canonically associate with $\mathcal{L}$. Note that $Alg^*_\Gamma(\mathcal{L})$ is also parameterized by the choice of the subsignature $\Gamma$ of $\Sigma$.

We say that $\mathbf{A}^o_{\Omega^{bhv}_{\Gamma,\mathbf{A},\phi}(D)}$ simulates the quotient of $\mathbf{A}$ by $\Omega^{bhv}_{\Gamma,\mathbf{A}}(D)$ because when $\langle a, b\rangle \in (\Omega^{bhv}_{\Gamma,\mathbf{A}}(D))_s$, even when $a$ and $b$ do not collapse in $\mathbf{A}^o_{\Omega^{bhv}_{\Gamma,\mathbf{A},\phi}(D)}$, we have nevertheless that $a \equiv_\Gamma b$ in $\mathbf{A}^o_{\Omega^{bhv}_{\Gamma,\mathbf{A},\phi}(D)}$.

The first connection between behavioral algebraization and the semantic version of the behavioral Leibniz congruence is given in the following proposition. It asserts that, when a logic is behaviorally algebraizable, the equivalence set of formulas defines the behavioral Leibniz congruence on the matrix models of the logic. This result is a generalization of a well-known standard result of AAL [Cze01].

**Proposition 4.2.7.** *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be many-sorted logic and $\Gamma$ a subsignature of $\Sigma$. Suppose that $\mathcal{L}$ is $\Gamma$-behaviorally algebraizable logic with $\Delta(\xi_1, \xi_2)$ the set of equivalence formulas. Let $\mathcal{M} = \langle \mathbf{A}, D\rangle$ be a matrix model of $\mathcal{L}$. Then,*

$$\langle a, b\rangle \in \Omega^{bhv}_{\Gamma,\mathbf{A},\phi}(D) \quad \textit{iff} \quad \Delta_\mathbf{A}(a, b) \subseteq D.$$

*Proof.* First let $\langle a, b\rangle \in \Omega^{bhv}_{\Gamma,\mathbf{A},\phi}(D)$. Then, by compatibility, we have that $\Delta_\mathbf{A}(a, b) \subseteq D$ iff $\Delta_\mathbf{A}(a, a) \subseteq D$. Since $\Delta$ satisfies (R) we can conclude that $\Delta_\mathbf{A}(a, b) \subseteq D$.

On the other direction, suppose that $\Delta_\mathbf{A}(a, b) \subseteq D$. So, for every $c \in \mathcal{C}^\Gamma_{\Sigma,\phi}[\xi]$ we have that $\Delta_\mathbf{A}(c_\mathbf{A}(a), c_\mathbf{A}(b)) \subseteq D$. Then, using (MP) we can conclude that

$$c_\mathbf{A}(a) \in D \quad \text{iff} \quad c_\mathbf{A}(b) \in D.$$

Hence, we conclude that $\langle a, b\rangle \in \Omega^{bhv}_{\Gamma,\mathbf{A},\phi}(D)$.

□

The following theorem is the major result of this section. It gives a semantic characterization of behavioral algebraizability and can be viewed as the matrix version of Theorem 3.2.18. This result is well-known for finitary and finitely algebraizable logics. We extend it herein for our behavioral approach also dropping the finitariness condition.

**Theorem 4.2.8.** *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic, $\Gamma$ a subsignature of $\Sigma$ and $K$ a class of $\Sigma^o$-algebras.*

*1) The following are equivalent:*

- *i) $\mathcal{L}$ is $\Gamma$-behaviorally algebraizable and $K$ is the $\Gamma$-behaviorally equivalent algebraic semantics;*
- *ii) for every $\Sigma$-algebra $\mathbf{A}$ we have that $\Omega^{bhv}_{\Gamma,\mathbf{A},\phi}$ is an isomorphism between the lattices of $\mathcal{L}$-filters and $K$-$\Gamma_\phi$-congruences of $\mathbf{A}$, that commutes with inverse substitutions.*

*2) Assume $\mathcal{L}$ is $\Gamma$-behaviorally algebraizable with $K$ the $\Gamma$-behaviorally equivalent algebraic semantics. Let $\Theta(\xi)$ be the set of defining equations for $K$. For each $\Sigma$-algebra $\mathbf{A}$ and $\Gamma_\phi$-congruence $\theta$ of $\mathbf{A}$ define:*

$$H_{\mathbf{A}}(\theta) = \{a \in A_\phi : \langle \gamma_{\mathbf{A}}(a), \delta_{\mathbf{A}}(a)\rangle \in \theta, \text{ for every } \gamma \approx \delta \in \Theta\}.$$

*Then $H_{\mathbf{A}}$ restricted to the $K$-$\Gamma_\phi$-congruences of $\mathbf{A}$ is the inverse of $\Omega^{bhv}_{\Gamma,\mathbf{A},\phi}$.*

*Proof.* Let us begin by proving the equivalence between $i)$ and $ii)$.

$i) \Rightarrow ii)$: Assume first that $i)$ holds and let $\mathbf{A}$ be a $\Sigma$-algebra. The fact that $\Omega^{bhv}_{\Gamma,\mathbf{A},\phi}$ commutes with inverse substitutions is an easy consequence of Proposition 4.2.7. Now let $D$ be a $\mathcal{L}$-filter on $\mathbf{A}$. We start by showing that $\Omega^{bhv}_{\Gamma,\mathbf{A},\phi}(D)$ is a $K$-$\Gamma_\phi$-congruence. Suppose that $\{\varphi_1^i \approx \varphi_2^i : i \in I\} \vDash^{K,\Gamma}_{\Sigma,bhv} \varphi_1 \approx \varphi_2$ and consider $h : \mathbf{T_\Sigma(X)} \to \mathbf{A}$ such that $\langle h_\phi(\varphi_1^i), h_\phi(\varphi_2^i)\rangle \in \Omega^{bhv}_{\Gamma,\mathbf{A},\phi}(D)$ for every $i \in I$. Then, by Proposition 4.2.7, we have that, for every $i \in I$, $\Delta_{\mathbf{A}}(h_\phi(\varphi_1^i), h_\phi(\varphi_2^i)) \subseteq D$. Since $K$ is assumed to be the $\Gamma$-behaviorally equivalent algebraic semantics for $\mathcal{L}$ we have, by condition $ii)$ of Definition 3.1.3, that $\{\varphi_1^i \approx \varphi_2^i : i \in I\} \vDash^{K,\Gamma}_{\Sigma,bhv} \varphi_1 \approx \varphi_2$ is equivalent to

$$\{\Delta(\varphi_1^i, \varphi_2^i) : i \in I\} \vdash \Delta(\varphi_1, \varphi_2).$$

So, since $D$ is a $\mathcal{L}$-filter, we have that $\Delta_{\mathbf{A}}(h_\phi(\varphi_1), h_\phi(\varphi_2)) \subseteq D$. Again using Proposition 4.2.7 we have that $\langle h_\phi(\varphi_1), h_\phi(\varphi_2)\rangle \in \Omega^{bhv}_{\Gamma,\mathbf{A},\phi}(D)$. Hence, $\Omega^{bhv}_{\Gamma,\mathbf{A},\phi}(D)$ is closed under $\models^{K,\Gamma}_{\Sigma,bhv}$ consequence, and so it is a $K$-$\Gamma_\phi$-congruence. So, $\Omega^{bhv}_{\Gamma,\mathbf{A},\phi}$ indeed maps the $Fi_{\mathcal{L}}(\mathbf{A})$ into $Con^{\Sigma}_{\Gamma,K,\phi}(\mathbf{A})$.

Now let $\theta$ be an arbitrary $K$-$\Gamma_\phi$-congruence of $\mathbf{A}$ and let $H_{\mathbf{A}}(\theta)$ be the subset of $A_\phi$ defined in 2). By the dual of the above argument, with condition $i)$ of Definition 3.1.3 in place of condition $ii)$, we get that $H_{\mathbf{A}}(\theta)$ is a $\mathcal{L}$-filter.

Next we show that indeed $\Omega^{bhv}_{\Gamma,\mathbf{A},\phi}(H_{\mathbf{A}}(\theta)) = \theta$. For all $a, b \in A_\phi$ we have that $\langle a, b\rangle \in \Omega^{bhv}_{\Gamma,\mathbf{A},\phi}(H_{\mathbf{A}}(\theta))$ iff we have that $\Theta_{\mathbf{A}}(\Delta_{\mathbf{A}}(a, b)) \subseteq \theta$. Recall condition $iv)$ of Definition 3.1.3: $\xi_1 \approx \xi_2 =\!\mid\models^{K,\Gamma}_{\Sigma,bhv} \Theta[\Delta(\xi_1, \xi_2)]$. So, we have that $\Theta_{\mathbf{A}}[\Delta_{\mathbf{A}}(a, b)] \in \theta$ iff $\langle a, b\rangle \in \theta$. Thus we have that $\Omega^{bhv}_{\Gamma,\mathbf{A},\phi}(H_{\mathbf{A}}(\theta)) = \theta$ proving that $\Omega^{bhv}_{\Gamma,\mathbf{A},\phi}$ is a mapping of the $\mathcal{L}$-filters onto the set of all $K$-$\Gamma_\phi$-congruences.

To prove that condition $ii)$ holds it remains to prove that $\Omega^{bhv}_{\Gamma,\mathbf{A},\phi}$ is monotone and injective. Recall condition $iii)$ of Definition 3.1.3: $\xi \dashv\vdash \Delta[\Theta(\xi)]$. So, $a \in D$ iff $\Delta_{\mathbf{A}}[\Theta_{\mathbf{A}}(a)] \in D$ iff $\{\langle\gamma(a), \delta(a)\rangle : \delta \approx \gamma \in \Theta(\xi)\} \subseteq \Omega^{bhv}_{\Gamma,\mathbf{A},\phi}(D)$. Therefore, for any $\mathcal{L}$-filters $D_1$ and $D_2$ we have that $D_1 \subseteq D_2$ iff $\Omega^{bhv}_{\Gamma,\mathbf{A},\phi}(D_1) \subseteq \Omega^{bhv}_{\Gamma,\mathbf{A},\phi}(D_2)$. Thus, $D_1 = D_2$ iff $\Omega^{bhv}_{\Gamma,\mathbf{A},\phi}(D_1) = \Omega^{bhv}_{\Gamma,\mathbf{A},\phi}(D_2)$.

$ii) \Rightarrow i)$: By taking $\mathbf{A} = \mathbf{T}_{\mathbf{\Sigma}}(\mathbf{X})$ and using Theorem 3.2.18 we can conclude that $\mathcal{L}$ is $\Gamma$-behaviorally algebraizable.

We have already seen that $\Omega^{bhv}_{\Gamma,\mathbf{A},\phi}(H_{\mathbf{A}}(\theta)) = \theta$ for every $K$-$\Gamma_\phi$-congruence $\theta$. The dual result, $H_{\mathbf{A}}(\Omega^{bhv}_{\Gamma,\mathbf{A},\phi}(D)) = D$ for every $\mathcal{L}$-filter $D$, is similarly established but with condition $iv)$ in Definition 3.1.3 instead of condition $iii)$.

□

Note that, when a logic has small, finite matrix models, Theorem 4.2.8 gives us a very useful tool for showing non $\Gamma$-behavioral algebraizability. This theorem is also important since it gives some insight about the precise connection between $\Gamma$-behaviorally equivalent algebraic semantics and matrix semantics. This connection is described in the following corollary and it is a generalization of the standard result of AAL [Cze01, BP89].

**Corollary 4.2.9.** *Let $\mathcal{L} = \langle\Sigma, \vdash\rangle$ be a many-sorted logic and $\Gamma$ a subsignature of $\Sigma$. Assume that $\mathcal{L}$ is $\Gamma$-behaviorally algebraizable and let $K^{\Gamma}_{\mathcal{L}}$ be the $\Gamma$-behaviorally equivalent algebraic semantics. Then $(K^{\Gamma}_{\mathcal{L}})^* = Alg^*_{\Gamma}(\mathcal{L})$.*

*Proof.* Consider $\mathbf{B} \in Alg^*_{\Gamma}(\mathcal{L})$. So $\mathbf{B} = \mathbf{A}^{o}_{\Omega^{bhv}_{\Gamma,\mathbf{A},\phi}(D)}$ for some $\langle\mathbf{A}, D\rangle \in Mod(\mathcal{L})$. By Theorem 4.2.8 we have that $\Omega^{bhv}_{\Gamma,\mathbf{A},\phi}(D)$ is a $K^{\Gamma}_{\mathcal{L}}$-$\Gamma_\phi$-congruence and so $\mathbf{B} \in (K^{\Gamma}_{\mathcal{L}})^*$.

Consider now $\mathbf{B} \in (K_{\mathcal{L}}^{\Gamma})^*$. So $o_{\mathbf{B}} : B_\phi \to B_v$ is surjective. It is easy to see that $Ker(o_{\mathbf{B}}) = \{\langle a, b\rangle : o_{\mathbf{B}}(a) = o_{\mathbf{B}}(b)\}$ is a $\Gamma_\phi$-congruence over $\mathbf{B}_{|\Sigma}$. It is also easy to see that $(\mathbf{B}_{|\Sigma})^o_{Ker(o_{\mathbf{B}})} \cong \mathbf{B}$. So, $Ker(o_{\mathbf{B}})$ is a $K_{\mathcal{L}}^{\Gamma}$-$\Gamma_\phi$-congruence. Using Theorem 4.2.8 we have that $Ker(o_{\mathbf{B}}) = \Omega^{bhv}_{\Gamma,\mathbf{B}_{|\Sigma},\phi}(D)$ for some $\mathcal{L}$-filter $D$. So, $\langle \mathbf{B}_{|\Sigma}, D\rangle$ is a matrix model of $\mathcal{L}$ and we can conclude that $\mathbf{B} \in Alg^*_\Gamma(\mathcal{L})$.

□

## 4.3 $Alg(\mathcal{L})$ versus $Alg_\Gamma(\mathcal{L})$

In this section we aim to study the relationship between two classes of algebras that can be canonically associated with a logic $\mathcal{L}$, $Alg(\mathcal{L})$ and $Alg_\Gamma(\mathcal{L})$. Recall that, for every protoalgebraic logic $\mathcal{L} = \langle \Sigma, \vdash\rangle$, there is a standard class of algebras canonically associated with it. This is the class $Alg(\mathcal{L})$ of the algebraic reducts of the reduced matrix models. Moreover, when $\mathcal{L}$ is algebraizable then it has a strong connection with $Alg(\mathcal{L})$.

In the same way, when $\mathcal{L}$ is $\Gamma$-behaviorally algebraizable, for some subsignature $\Gamma$ of $\Sigma$, we have that the class $Alg_\Gamma(\mathcal{L})$ has a strong connection with $\mathcal{L}$.

Therefore, in the cases where $\mathcal{L}$ is algebraizable and $\Gamma$-behaviorally algebraizable, it is interesting to study the relationship between the classes $Alg(\mathcal{L})$ and $Alg_\Gamma(\mathcal{L})$.

Suppose that a logic $\mathcal{L} = \langle \Sigma, \vdash\rangle$ is algebraizable with equivalence set $\Delta$ and defining equations $\Theta$. Suppose also that there exists a subsignature $\Gamma$ of $\Sigma$ such that $\mathcal{L}$ is $\Gamma$-behaviorally algebraizable with $\Gamma$-behavioral equivalence set $\Delta'$ and $\Gamma$-behavioral defining equations $\Theta'$.

One natural and very interesting question that arises immediately is about the relationship between the respective algebraic counterparts.

Recall that, since $\mathcal{L}$ is algebraizable, we can canonically associate to $\mathcal{L}$ the class of $\Sigma$-algebras

$$Alg(\mathcal{L}) = \{\mathbf{A}_{/\Omega_{\mathbf{A}}(D)} : \langle \mathbf{A}, D\rangle \text{ is a matrix model of } \mathcal{L}\}.$$

Moreover, since $\mathcal{L}$ is $\Gamma$-behaviorally algebraizable, we can also canonically associate to $\mathcal{L}$ the class of $\Sigma^o$-algebras

$$Alg_\Gamma(\mathcal{L}) = \{\mathbf{A}^o_{\Omega^{bhv}_{\Gamma,\mathbf{A}}(D)} : \langle \mathbf{A}, D\rangle \text{ is a matrix model of } \mathcal{L}\}.$$

First of all, note that classes $Alg(\mathcal{L})$ and $Alg_\Gamma(\mathcal{L})$ are not directly comparable since they are classes of algebras over different signatures. Of course, although different, the signatures $\Sigma$ and $\Sigma^o$ are closely related and, therefore, we can engage on a study of the relationship between these two classes of algebras.

The crucial point in the study of the relationship between $Alg(\mathcal{L})$ and $Alg_\Gamma(\mathcal{L})$ is the strong relationship both share with the logic $\mathcal{L}$.

Recall from Theorem 2.3.22 that, since $\mathcal{L}$ is algebraizable, for each $\Sigma$-algebra $\mathbf{A}$ (not necessarily in $Alg(\mathcal{L})$), the Leibniz operator on $\mathbf{A}$, $\Omega_\mathbf{A}$, is an isomorphism between the lattice of $\mathcal{L}$-filters of $\mathbf{A}$ and the lattice of $Alg(\mathcal{L})$-congruences of $\mathbf{A}$.

Similarly, now following Theorem 4.2.8 and since $\mathcal{L}$ is $\Gamma$-behaviorally algebraizable, we have that the $\Gamma$-behavioral Leibniz operator on $\mathbf{A}$, $\Omega^{bhv}_{\Gamma,\mathbf{A}}$, is an isomorphism between the lattice of $\mathcal{L}$-filters of $\mathbf{A}$ and the lattice of $Alg_\Gamma(\mathcal{L})$-$\Gamma$-congruences of $\mathbf{A}$, for each $\Sigma$-algebra $\mathbf{A}$ (not necessarily in $Alg(\mathcal{L})$),.

Therefore, for every $\Sigma$-algebra $\mathbf{A}$, we have an isomorphism $\pi$ between the lattice of $Alg(\mathcal{L})$-congruences of $\mathbf{A}$ and the lattice of $Alg_\Gamma(\mathcal{L})$-$\Gamma$-congruences of $\mathbf{A}$. Of course, the isomorphism $\pi$ is such that we have $\Omega_\mathbf{A}(D) \overset{\pi}{\mapsto} \Omega^{bhv}_{\Gamma,\mathbf{A}}(D)$, for each $\mathcal{L}$-filter $D$ of $\mathbf{A}$. Recalling from Theorems 2.3.22 and 4.2.8 the isomorphisms $\Omega_\mathbf{A}$ and $\Omega^{bhv}_{\Gamma,\mathbf{A}}$ we have the following more accurate characterization:

$$\pi(\theta) = \Omega^{bhv}_{\Gamma,\mathbf{A}}(\{a \in A : \langle \delta_\mathbf{A}(a), \epsilon_\mathbf{A}(a)\rangle \in \theta \text{ for every } \delta \approx \epsilon \in \Theta\})$$

$$\pi^{-1}(\theta) = \Omega_\mathbf{A}(\{a \in A : \langle \delta_\mathbf{A}(a), \epsilon_\mathbf{A}(a)\rangle \in \theta \text{ for every } \delta \approx \epsilon \in \Theta'\})$$

With this characterization we can obtain an algebra in $Alg(\mathcal{L})$ from a algebra in $Alg_\Gamma(\mathcal{L})$ and vice-versa.

Consider given a $\Sigma^o$-algebra $\mathbf{B} \in Alg_\Gamma(\mathcal{L})$. Then we can consider the subset $D$ of $B_\phi$ defined as $D = \{a \in B_\phi : \delta_\mathbf{B}(a) \equiv_\Gamma \epsilon_\mathbf{B}(a) \text{ for every } \delta \approx \epsilon \in \Theta'\}$. So we can consider the $\Sigma$-algebra $\mathbf{A}$ obtained from $\mathbf{B}$ and defined as $\mathbf{A} = \mathbf{B}_\phi/_{\Omega_{\mathbf{B}_\phi}(D)}$. Since $\langle \mathbf{B}_\phi, D\rangle$ is a matrix model of $\mathcal{L}$ we have that $\mathbf{A} \in Alg(\mathcal{L})$.

Furthermore, given a $\Sigma$-algebra $\mathbf{A} \in Alg(\mathcal{L})$, we can consider the set $D = \{a \in A : \delta_\mathbf{A}(a) = \epsilon_\mathbf{A}(a) \text{ for every } \delta \approx \epsilon \in \Theta\}$. We can then consider the $\Sigma^o$-algebra $\mathbf{B}$ obtained from $\mathbf{A}$ and defined as $\mathbf{B} = \mathbf{A}_{\Omega^{bhv}_{\Gamma,\mathbf{A}}(D)}$. Since $\langle \mathbf{A}, D\rangle$ is a matrix model of $\mathcal{L}$ we have that $\mathbf{B} \in Alg_\Gamma(\mathcal{L})$.

Recall that, given a class $K$ of $\Sigma$-algebras, the set $QEq(K)$ is the set of all quasi-equations over $\Sigma$ satisfied by all the algebras in $K$ and the set $Eq(K)$ is the set of all equations over $\Sigma$ satisfied by all the algebras in $K$.

Similarly, given a class $K$ of $\Sigma^o$-algebras, we can define the the set $QEq_\Gamma(K)$ as the set of all quasi-equations over $\Sigma$ that are $\Gamma$-behaviorally satisfied by all the algebras in $K$ and the set $Eq_\Gamma(K)$ is the set of equations over $\Sigma$ that are $\Gamma$-behaviorally satisfied by all the algebras in $K$.

We can now prove some results relating these sets of equations and quasi-equations.

**Proposition 4.3.1.** *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic and $\Gamma$ a subsignature of $\Sigma$. Suppose that $\mathcal{L}$ is algebraizable with equivalence set $\Delta$ and defining equations $\Theta$. Suppose also that $\mathcal{L}$ is $\Gamma$-behaviorally algebraizable with $\Gamma$-behavioral equivalence set $\Delta'$ and $\Gamma$-behavioral defining equations $\Theta'$. Then we have that*

$$i)\ Eq(Alg(\mathcal{L})) \subseteq Eq_\Gamma(Alg_\Gamma(\mathcal{L})).$$

*If $\Delta = \{\mu_1, \dots, \mu_l\}$ and $\Theta' = \{\delta_1 \approx \epsilon_1, \dots, \delta_m \approx \epsilon_m\}$ are finite sets, we have*

$$ii)\ ((\varphi_1 \approx \psi_1)\& \dots \&(\varphi_n \approx \psi_n)) \to (\varphi \approx \psi) \in QEq(Alg(\mathcal{L}))$$

*iff*

*for every $\mu \in \Delta$ and every $(\delta \approx \epsilon) \in \Theta'$ we have that*

$$(\&_{k=1}^n(\&_{i=1}^l(\&_{j=1}^m\ \delta_j(\mu_i(\varphi_k,\psi_k)) \approx \epsilon_j(\mu_i(\varphi_k,\psi_k))))) \to (\delta(\mu(\varphi,\psi)) \approx \epsilon(\mu(\varphi,\psi))) \in QEq_\Gamma(Alg_\Gamma(\mathcal{L})).$$

*If $\Delta' = \{\mu'_1, \dots, \mu'_l\}$ and $\Theta = \{\delta'_1 \approx \epsilon'_1, \dots, \delta'_m \approx \epsilon'_m\}$ are finite sets, we have*

$$iii)\ ((\varphi_1 \approx \psi_1)\& \dots \&(\varphi_n \approx \psi_n)) \to (\varphi \approx \psi) \in QEq_\Gamma(Alg_\Gamma(\mathcal{L}))$$

*iff*

*for every $\mu \in \Delta'$ and every $(\delta \approx \epsilon) \in \Theta$ we have that*

$$(\&_{k=1}^n(\&_{i=1}^l(\&_{j=1}^m\ \delta'_j(\mu'_i(\varphi_k,\psi_k)) \approx \epsilon'_j(\mu'_i(\varphi_k,\psi_k))))) \to (\delta(\mu(\varphi,\psi)) \approx \epsilon(\mu(\varphi,\psi))) \in QEq(Alg(\mathcal{L})).$$

*Proof.* For statement $i)$ suppose that $\varphi \approx \psi \in Eq(Alg(\mathcal{L}))$. Then, we have that $\vDash_\Sigma^{Alg(\mathcal{L})} \varphi \approx \psi$, and since $Alg(\mathcal{L})$ is an equivalent algebraic semantics we can conclude that $\vdash \Delta(\varphi,\psi)$. Therefore, we have that $\vdash \Delta'(\varphi,\psi)$ and, since $Alg_\Gamma(\mathcal{L})$ is a $\Gamma$-behaviorally equivalent algebraic semantics, we can conclude that $\vDash_{\Sigma,bhv}^{Alg_\Gamma(\mathcal{L}),\Gamma} \varphi \approx \psi$. So, $\varphi \approx \psi \in Eq_\Gamma(Alg(\mathcal{L}))$.

For statement *ii*) just notice that $\{\varphi_i \approx \psi_i : i \in I\} \vDash_{\Sigma}^{Alg(\mathcal{L})} \varphi \approx \psi$ iff $\{\Delta(\varphi_i, \psi_i) : i \in I\} \vdash \Delta(\varphi, \psi)$ iff $\{\Theta'[\Delta(\varphi_i, \psi_i)] : i \in I\} \vDash_{\Sigma,bhv}^{Alg_\Gamma(\mathcal{L}),\Gamma} \Theta'[\Delta(\varphi, \psi)]$.

In the same way, for condition *iii*), just notice that $\{\varphi_i \approx \psi_i : i \in I\} \vDash_{\Sigma,bhv}^{Alg_\Gamma(\mathcal{L}),\Gamma} \varphi \approx \psi$ iff $\{\Delta'(\varphi_i, \psi_i) : i \in I\} \vdash \Delta'(\varphi, \psi)$ iff $\{\Theta[\Delta'(\varphi_i, \psi_i)] : i \in I\} \vDash_{\Sigma}^{Alg(\mathcal{L})} \Theta[\Delta'(\varphi, \psi)]$.

□

Let us now draw some conclusions of the above results. First of all observe that although different and over different signatures, we can conclude that the classes $Alg(\mathcal{L})$ and $Alg_\Gamma(\mathcal{L})$ have a strong connection.

The major conclusion that we can draw is that the class $Alg_\Gamma(\mathcal{L})$ allows us to see $Alg(\mathcal{L})$ from a different perspective. This change of perspective can, in some cases, provide some new insight about the logic $\mathcal{L}$. The gains depend, of course, on the concrete example we have on hand. Note that we are not saying at all that one should prefer $Alg_\Gamma(\mathcal{L})$ to $Alg(\mathcal{L})$. On the contrary, they represent different perspectives of the algebraic counterpart of the same logic and should be studied together for a better understanding of the logic.

## 4.4 Valuation semantics

Valuation semantics [dCB94] has been proposed as an effort to generalize the notion of matrix semantics, in order to give a semantical tool to study a larger class of logics. Contrarily to the theory matrix semantics, the original theory of valuation semantics does not have any algebraic flavor. The aim of our proposal of valuation semantics is to generalize both standard matrix semantics and valuation semantics in order to have a notion that combines the important features of both.

Recall that a many-sorted matrix over a signature $\Sigma$ is a tuple $\langle \mathbf{A}, D \rangle$ where $\mathbf{A}$ is a $\Sigma$-algebra and $D \subseteq A_\phi$ is the set of designated elements. In this setting, formulas are homomorphically interpreted into the algebra $\mathbf{A}$. The key idea of the valuation semantics is to drop this homomorphism condition, in the sense that there are operations that are always interpreted homomorphically, but there are also some that are not.

Our idea for an algebraic version of valuation semantics is that a valuation should be constituted by a map from the set of formulas to an algebra over a chosen subsignature of the original signature. Then, the homomorphism condition is only imposed for the operations of that subsignature.

In the sequel we consider fixed a many-sorted signature $\Sigma$ and a subsignature $\Gamma$ of $\Sigma$.

**Definition 4.4.1.** A $\Gamma$-*valuation* is a triple $v = \langle \mathbf{A}, D, h\rangle$ where $\mathbf{A}$ is a $\Gamma$-algebra, $D \subseteq A_\phi$ is the set of *designated elements of* $\mathbf{A}$ and $h$ is a sorted function $h : T_\Sigma(X) \to A$ that satisfies, for every $f : s_1 \dots s_n \to s \in \Gamma$ and $\langle t_1, \dots, t_n\rangle \in T_{\Sigma,s_1}(X) \times \dots \times T_{\Sigma,s_n}(X)$, the following condition:

$$h(f(t_1, \dots, t_n)) = f_{\mathbf{A}}(h(t_1), \dots, h(t_n)).$$

Given a $\Gamma$-valuation $v = \langle \mathbf{A}, D, h\rangle$ and a formula $\varphi \in L_\Sigma(X)$, we say that $v$ satisfies $\varphi$, denoted by $v \Vdash \varphi$, if $h(\varphi) \in D$.

**Definition 4.4.2.** A $\Gamma$-*valuation semantics* over $\Sigma$ is a collection $\mathcal{M}$ of $\Gamma$-valuations.

It is important to note that the notion of matrix semantics and the notion of valuation semantics are particular cases of the notion of $\Gamma$-valuation semantics.

The first one can be obtained by taking $\Gamma = \Sigma$ and considering that for each $\Sigma$-algebra $\mathbf{A}$, if $\langle \mathbf{A}, D, h\rangle \in \mathcal{M}$ then also $\langle \mathbf{A}, D, h'\rangle \in \mathcal{M}$ for every homomorphism $h' : \mathbf{T}_{\mathbf{\Sigma}}(\mathbf{X}) \to \mathbf{A}$.

The standard notion of valuation semantics can be obtained from our notion, by observing that valuation semantics assume fixed a set of truth-values. So, by taking $\Gamma = \emptyset$ and considering that all $\Gamma$-valuations share the same algebraic reduct, we have a traditional valuation semantics.

Given a $\Gamma$-valuation semantics $\mathcal{M} = \{\langle \mathbf{A}_i, D_i, h_i\rangle : i \in I\}$ over $\Sigma$, the consequence relation over $\Sigma$ associated with $\mathcal{M}$, denoted by $\vdash_{\mathcal{M}}$, is such that, for every $\Phi \cup \{\varphi\} \subseteq L_\Sigma(X)$, we have that $\Phi \vdash_{\mathcal{M}} \varphi$ iff for every $\Gamma$-valuation $v \in \mathcal{M}$

$$v \Vdash \varphi \ \text{ whenever } \ v \Vdash \psi \text{ for every } \psi \in \Phi.$$

Before we continue, let us make two remarks. First of all, note that $\vdash_{\mathcal{M}}$ is a consequence relation defined over the whole signature $\Sigma$. Note also that, with our approach, we are now able to algebraically specify not only the class of algebras associated with a given logic, but also the admissible ways that a formula can be interpreted in these algebras.

**Definition 4.4.3.** Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic. A $\Gamma$-valuation $v = \langle \mathbf{A}, D, h \rangle$ is said to be a *model of* $\mathcal{L}$ if, for every $\Phi \cup \{\varphi\} \subseteq L_\Sigma(X)$ we have that

$$v \Vdash \varphi \text{ whenever } \; v \Vdash \Phi \text{ and } \; \Phi \vdash \varphi.$$

In this case $D$ is called a $\Gamma$-*deductive filter of* $\mathcal{L}$ or just a $\mathcal{L}$-$\Gamma$-*filter*. The set of all models of $\mathcal{L}$ is denoted by $Mod_\Gamma(\mathcal{L})$.

**Definition 4.4.4.** Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic and $\mathcal{M}$ a $\Gamma$-valuation semantics. Then,

- $\mathcal{M}$ is *sound* for $\mathcal{L}$ if $\vdash \subseteq \vdash_{\mathcal{M}}$, that is, for every $\Phi \cup \{\varphi\} \subseteq L_\Sigma(X)$,
$$\Phi \vdash_{\mathcal{M}} \varphi \quad \text{whenever} \quad \Phi \vdash \varphi.$$
- $\mathcal{M}$ is *adequate* for $\mathcal{L}$ if $\vdash_{\mathcal{M}} \subseteq \vdash$, that is, for every $\Phi \cup \{\varphi\} \subseteq L_\Sigma(X)$,
$$\Phi \vdash \varphi \quad \text{whenever} \quad \Phi \vdash_{\mathcal{M}} \varphi.$$
- $\mathcal{M}$ is *complete* with respect to $\mathcal{L}$ if it is both sound and adequate.

For each subset $\Phi$ of $L_\Sigma(X)$ we can define the $\Gamma$-valuation $v_\Gamma^\Phi = \langle \mathbf{T_\Sigma(X)}_{|_\Gamma}, \Phi, id \rangle$ where $id$ is the identity function from $T_\Sigma(X)$ to $T_\Sigma(X)$. The $\Gamma$-valuations of the form $v_\Gamma^\Phi$ are called *Lindenbaum* $\Gamma$-*valuations* for the signature $\Sigma$. The family of $\Gamma$-valuations

$$\mathbf{V}_\Gamma(\mathcal{L}) = \{v_\Gamma^\Phi : \Phi \in Th(\mathcal{L})\}$$

is called the *Lindenbaum* $\Gamma$-*bundle* for $\mathcal{L}$. As usual, these syntactical models are very useful when proving completeness results.

**Theorem 4.4.5.** *Every many-sorted logic* $\mathcal{L}$ *is complete with respect to the Lindenbaum* $\Gamma$-*bundle* $\boldsymbol{V}_\Gamma(\mathcal{L})$.

*Proof.* We first prove that $\mathbf{V}_\Gamma(\mathcal{L})$ is sound for $\mathcal{L}$, that is, for every $T \in Th(\mathcal{L})$, we have that $v_\Gamma^T \in Mod_\Gamma(\mathcal{L})$. Suppose that $\Phi \vdash \varphi$ for some $\Phi \cup \{\varphi\} \subseteq L_\Sigma(X)$ and that $v_\Gamma^T \Vdash \gamma$ for every $\gamma \in \Phi$. So, we have that $\Phi \subseteq T$ and since $T$ is a theory we can conclude that $\Phi^\vdash \subseteq T$. Since $\varphi \in \Phi^\vdash$, we can conclude that $\varphi \in T$ and so $v_\Gamma^T \Vdash \varphi$.

Now let us prove that $\mathbf{V}_\Gamma(\mathcal{L})$ is an adequate $\Gamma$-valuation semantics for $\mathcal{L}$. Suppose that $\Phi \nvdash \varphi$ for some $\Phi \cup \{\varphi\} \subseteq L_\Sigma(X)$. Then $v_\Gamma^{\Phi^\vdash} \Vdash \gamma$ for every $\gamma \in \Phi$ and, since $\varphi \notin \Phi^\vdash$, we have that $v_\Gamma^{\Phi^\vdash} \nVdash \varphi$ . Therefore, we can conclude that $\Phi \nvdash_{\mathbf{V}_\Gamma(\mathcal{L})} \varphi$.

□

As an immediate corollary we have the following completeness result with respect to the class $Mod_\Gamma(\mathcal{L})$ .

**Corollary 4.4.6.** *Every many-sorted logic $\mathcal{L}$ is complete with respect to $Mod_\Gamma(\mathcal{L})$.*

*Proof.* This result follows from the very general and well-known fact that, if we add a model of a logic to a complete semantics for that logic, we still obtain a complete semantics.

The class $Mod_\Gamma(\mathcal{L})$ is obtained precisely by enriching the complete $\Gamma$-valuation semantics $\mathbf{V}_\Gamma(\mathcal{L})$ with all models of $\mathcal{L}$. So $Mod_\Gamma(\mathcal{L})$ is still a complete $\Gamma$-valuation semantics for $\mathcal{L}$.

□

Recall that the class of all matrix models of $\mathcal{L}$ plays an important role in the standard approach, since it algebraically captures some of the metalogical properties of $\mathcal{L}$. In our approach, this role is played by $Mod_\Gamma(\mathcal{L})$.

To see that some important results of the fruitful theory of logical matrices generalize to valuation semantics, we end this section by presenting an example of such a result. We start by recalling some notation concerning the notion of ultraproduct.

Given a set of $\Gamma$-valuations $\{v_i : i \in I\}$ and an ultrafilter $\mathcal{U}$ over $I$ we can define an equivalence relation $\sim_\mathcal{U}$ on the cartesian product $\Pi_{i\in I}v_i$ as follows:

$$a \sim_\mathcal{U} b \quad \text{iff} \quad \{i \in I : a_i = b_i\} \in \mathcal{U}.$$

The *ultraproduct* of the $\Gamma$-valuations $v_i$, $i \in I$, modulo an ultrafilter $\mathcal{U}$, is the quotient of $\Pi_{i\in I}v_i$ by the equivalence $\sim_\mathcal{U}$ and it is denoted by $\Pi_\mathcal{U}v_i$.

The following result generalizes the so-called Bloom's Theorem [Blo75] to our setting.

**Theorem 4.4.7.** *A many-sorted logic $\mathcal{L} = \langle\Sigma, \vdash\rangle$ is finitary iff the class $Mod_\Gamma(\mathcal{L})$ is closed under ultraproducts.*

*Proof.* Suppose first that $\mathcal{L}$ is finitary. Let $\{v_i : i \in I\} \subseteq Mod_\Gamma(\mathcal{L})$ be a class of $\Gamma$-valuations and $\mathcal{U}$ an ultrafilter over $I$. We aim to prove that $\Pi_\mathcal{U}v_i \in Mod_\Gamma(\mathcal{L})$. So, suppose that $\Phi \vdash \varphi$ and that $\Pi_\mathcal{U}v_i \Vdash \gamma$ for every $\gamma \in \Phi$. Since $\mathcal{L}$ is finitary we have that there exists $\Phi' = \{\gamma_1, \ldots, \gamma_n\} \subseteq \Phi$ such that $\Phi' \vdash \varphi$. For each $j \in \{1, \ldots, n\}$ and since $\Pi_\mathcal{U}v_i \Vdash \gamma_j$, we can conclude that $I_j = \{i \in I : v_i \Vdash \gamma_j\} \in \mathcal{U}$. Since $\mathcal{U}$ is an ultrafilter we have that $I_1 \cap \ldots \cap I_n \in \mathcal{U}$, and therefore $I_1 \cap \ldots \cap I_n \neq \emptyset$. Recall that $v_i \in Mod_\Gamma(\mathcal{L})$ for every $i \in I_1 \cap \ldots \cap I_n$ and since $\Phi' \vdash \varphi$ we have that $I_1 \cap \ldots \cap I_n \subseteq \{i \in I : v_i \Vdash \varphi\}$. Since $\mathcal{U}$ is an ultrafilter we have that $\{i \in I : v_i \Vdash \varphi\} \in \mathcal{U}$ and so $\Pi_{\mathcal{U}} v_i \Vdash \varphi$.

Suppose now that $Mod_\Gamma(\mathcal{L})$ is closed under ultraproducts. To prove that $\mathcal{L}$ is finitary let $\Phi$ be an infinite set of formulas and assume that $\Phi' \nvdash \varphi$ for every finite $\Phi' \subseteq \Phi$. Let $I$ denote the set of all finite subsets of $\Phi$. For each $i \in I$ define $i^* = \{j \in I : i \subseteq j\}$. Using well-known results on ultrafilters [BS81] we can conclude that there exists an ultrafilter $\mathcal{U}$ over $I$ that contains the family $\{i^* : i \in I\}$. For every $i \in I$, let $v_i = \langle \mathbf{L}_{\mathbf{\Sigma}_{|\mathbf{\Gamma}}}(\mathbf{X}), D_i, h_i \rangle$ where $D_i = i^{\vdash}$ and $h_i$ is the identity function from $L_\Sigma(X)$ to $L_\Sigma(X)$. Clearly, $v_i \in Mod_\Gamma(\mathcal{L})$ for every $i \in I$. Let $\Pi_{\mathcal{U}} v_i$ be the ultraproduct of the family $v_i$ by the ultrafilter $\mathcal{U}$. Then, for every $\delta \in \Phi$ we have that $\{\delta\}^* \subseteq \{i \in I : h_i(\delta) \in D_i\}$. So, $\{i \in I : h_i(\delta) \in D_i\} \in \mathcal{U}$ for every $\delta \in \Phi$ and consequently we have that $\Pi_{\mathcal{U}} v_i \Vdash \delta$ for every $\delta \in \Phi$. But $\{i \in I : h_i(\varphi) \in D_i\} = \emptyset$ and so $\Pi_{\mathcal{U}} v_i \nVdash \varphi$. Since $\Pi_{\mathcal{U}} v_i \in Mod_\Gamma(\mathcal{L})$, we have that $\Phi \nvdash \varphi$. So we can conclude that $\mathcal{L}$ is finitary.

□

Recall that our aim is to show the precise connection between valuation semantics and behavioral algebraization. In the sequel, we consider fixed a $\Gamma$-behaviorally algebraizable many-sorted logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ with equivalent algebraic semantics $K$ and defining equations $\Theta(\xi) = \{\delta_i(\xi) \approx \epsilon_i(\xi) : i \in I\}$. In the main result of this subsection we show how to extract from $K$ a complete $\Gamma$-valuation semantics for $\mathcal{L}$.

Let $\mathbf{A}$ be an $\Sigma^o$-algebra and recall that $\equiv_\Gamma$ denotes the $\Gamma$-behavioral equivalence over $\mathbf{A}$. Since $\equiv_\Gamma$ is not a congruence over $\mathbf{A}$, we cannot perform the quotient of $\mathbf{A}$ by $\equiv_\Gamma$. We can, nevertheless, obtain from $\mathbf{A}$ a $\Gamma$-algebra $\mathbf{A}_\Gamma = \mathbf{A}_{|\Gamma}/_{\equiv_\Gamma}$ by performing the quotient of the $\Gamma$-reduct of $\mathbf{A}$ by the $\Gamma$-congruence $\equiv_\Gamma$. The $\Gamma$-algebra $\mathbf{A}_\Gamma$ is well-defined since $\equiv_\Gamma$ is a congruence over $\mathbf{A}_{|\Gamma}$.

Given a $\Sigma^o$-algebra $\mathbf{A}$ and an assignment $h$ over $\mathbf{A}$, we can build a $\Gamma$-valuation $v_{\mathbf{A},h}$ as follows:

- $v_{\mathbf{A},h} = \langle B_\mathbf{A}, D_\mathbf{A}, h_\mathbf{A} \rangle$;
- $B_\mathbf{A} = \mathbf{A}_\Gamma$;
- $D_\mathbf{A} = \{[a]_{\equiv_\Gamma} \in B_{\mathbf{A},\phi} : \delta_{i\mathbf{A}}(a) \equiv_\Gamma \epsilon_{i\mathbf{A}}(a) \text{ for every } i \in I\}$;
- $h_\mathbf{A} = \iota_{\equiv_\Gamma} \circ h$.

Note that $\iota_{\equiv_\Gamma}$ denotes the natural map from $\mathbf{A}_{|\Gamma}$ to $\mathbf{A}_\Gamma$, that associates to each $a \in \mathbf{A}_{|\Gamma}$ the correspondent equivalence class $[a]_{\equiv_\Gamma}$.

Note also that the fact that $D_{\mathbf{A}}$ is well-defined, that is, it does not depend on the particular choice of a representative for the equivalence class, is an immediate consequence of the fact that $\Theta(\xi) \subseteq Comp_{\Sigma}^{K,\Gamma}(\{\xi\})$, since this implies that

$$\xi_1 \approx \xi_2, \{\delta_i(\xi_1) \approx \epsilon_i(\xi_1) : i \in I\} \vDash_{\Sigma,bhv}^{K,\Gamma} \{\delta_i(\xi_2) \approx \epsilon_i(\xi_2) : i \in I\}.$$

Applying the above construction to every $\mathbf{A} \in K$ and every assignment $h$ over $\mathbf{A}$, we obtain the following $\Gamma$-valuation semantics:

$$\mathcal{M}_K = \{v_{\mathbf{A},h} : \mathbf{A} \in K \text{ and } h \text{ assignment over } \mathbf{A}\}.$$

The following lemma states a first connection between the pair $\langle \mathbf{A}, h\rangle$, where $\mathbf{A} \in K$ and $h$ is an assignment over $\mathbf{A}$, and the corresponding $\Gamma$-valuation $v_{\mathbf{A},h}$.

**Lemma 4.4.8.** *Given an algebra $\mathbf{A} \in K$, an assignment $h$ over $\mathbf{A}$ and a formula $\varphi \in L_\Sigma(X)$, we have that:*

$$\mathbf{A}, h \Vdash\!\vdash \delta_i(\varphi) \approx \epsilon_i(\varphi) \textit{ for every } i \in I \quad \textit{iff} \quad v_{\mathbf{A},h} \Vdash \varphi.$$

*Proof.* The result follows from the sequence of equivalent conditions:

$$\begin{aligned}
\mathbf{A}, h \Vdash\!\vdash \delta_i(\varphi) \approx \epsilon_i(\varphi) \text{ for every } i \in I \quad & \text{iff} \quad \delta_{i\mathbf{A}}(h(\varphi)) \equiv_\Gamma \epsilon_{i\mathbf{A}}(h(\varphi)) \text{ for every } i \in I \\
& \text{iff} \quad [h(\varphi)]_{\equiv_\Gamma} \in D_{\mathbf{A}} \\
& \text{iff} \quad v_{\mathbf{A},h} \Vdash \varphi
\end{aligned}$$

□

We can now prove a result that relates the behavioral consequence associated with $K$ and the corresponding valuation semantics, $\mathcal{M}_K$. This is a generalization of a standard result linking matrix semantics and AAL [FJP03].

**Theorem 4.4.9.** *Let $\mathcal{L} = \langle \Sigma, \vdash\rangle$ be a many-sorted $\Gamma$-behaviorally algebraizable logic with equivalent algebraic semantics $K$ and defining equations $\Theta(\xi) = \{\delta_i \approx \epsilon_i : i \in I\}$. Then*

$$\Phi \vdash_{\mathcal{M}_K} \varphi \quad \textit{iff} \quad \{\Theta(\gamma) : \gamma \in \Phi\} \vDash_{\Sigma,bhv}^{K,\Gamma} \Theta(\varphi).$$

*Proof.* First, assume that $\Phi \vdash_{\mathcal{M}_K} \varphi$ and consider $\mathbf{A} \in K$ and $h$ an assignment over $\mathbf{A}$ such that $\mathbf{A}, h \Vvdash \delta_i(\gamma) \approx \epsilon_i(\gamma)$ for every $i \in I$ and $\gamma \in \Phi$. Using Lemma 4.4.8 this is equivalent to $v_{\mathbf{A},h} \Vdash \gamma$ for every $\gamma \in \Phi$. Since $v_{\mathbf{A},h} \in \mathcal{M}_K$ and since we are assuming that $\Phi \vdash_{\mathcal{M}_K} \varphi$, we can conclude that $v_{\mathbf{A},h} \Vdash \varphi$. Using again Lemma 4.4.8 we can conclude that $\mathbf{A}, h \Vvdash \delta_i(\varphi) \approx \epsilon_i(\varphi)$ for every $i \in I$.

Assume now that $\{\Theta(\gamma) : \gamma \in \Phi\} \vDash^{K,\Gamma}_{\Sigma,bhv} \Theta(\varphi)$ and consider a $\Gamma$-valuation $v = \langle \mathbf{A}, D, h\rangle \in \mathcal{M}_K$ such that $v \Vdash \gamma$ for every $\gamma \in \Phi$. Using Lemma 4.4.8 we can conclude that $\mathbf{A}, h \Vvdash \delta_i(\gamma) \approx \epsilon_i(\gamma)$ for every $i \in I$ and $\gamma \in \Phi$. Since we are assuming that $\mathbf{A} \in K$ and that $\{\Theta(\gamma) : \gamma \in \Phi\} \vDash^{K,\Gamma}_{\Sigma,bhv} \Theta(\varphi)$, we can conclude that $\mathbf{A}, h \Vvdash \delta_i(\varphi) \approx \epsilon_i(\varphi)$ for every $i \in I$, which, by Lemma 4.4.8, is equivalent to $v \Vdash \varphi$.

□

As a consequence of the above theorem we have the following result that asserts that the construction of $\mathcal{M}_K$ from $K$ indeed yields a complete $\Gamma$-valuation semantics for $\mathcal{L}$.

**Corollary 4.4.10.** *Let $\mathcal{L} = \langle \Sigma, \vdash\rangle$ be a many-sorted $\Gamma$–behaviorally algebraizable logic with equivalent algebraic semantics $K$ and defining equations $\Theta(\xi)$. Then, $\mathcal{M}_K$ is a complete $\Gamma$-valuation semantics for $\mathcal{L}$.*

*Proof.* Recall that, since $\mathcal{L}$ is $\Gamma$-behaviorally algebraizable with equivalent algebraic semantics $K$ and defining equations $\Theta(\xi)$, we have, for every $\Phi \cup \{\varphi\} \subseteq L_\Sigma(X)$, that

$$\Phi \vdash \varphi \quad \text{iff} \quad \{\Theta(\gamma) : \gamma \in \Phi\} \vDash^{K,\Gamma}_{\Sigma,bhv} \Theta(\varphi).$$

Using Theorem 4.4.9 we can conclude that

$$\Phi \vdash \varphi \quad \text{iff} \quad \{\Theta(\gamma) : \gamma \in \Phi\} \vDash^{K,\Gamma}_{\Sigma,bhv} \Theta(\varphi) \quad \text{iff} \quad \Phi \vdash_{\mathcal{M}_K} \varphi.$$

□

## 4.5 Remarks

We conclude with a brief summary of the achievements of this chapter. We have continued the effort towards the generalization of the standard notions and results of AAL to the behavioral setting, now in a semantical perspective. We started by characterizing the class of algebras that our behavioral approach canonically associates with a given behaviorally algebraizable logic. We proved a unicity result with respect to the algebraic counterpart of a behaviorally algebraizable logic. This unicity result could be established once the parameter $\Gamma$ id fixed. We also proved a result that allows us to produce the axiomatization of the algebraic counterpart of a behaviorally algebraizable logic $\mathcal{L}$ from the deductive system of $\mathcal{L}$. Matrix semantics is the standard tool for semantical investigations in AAL. The generalization of this tool to the behavioral setting is not, however, straightforward and can lead to two different approaches. We started by exploring the most natural approach, the one centered on the standard notion of logical matrix. We generalized some of results of the theory of logical matrix, ultimately aiming at bridge results relating metalogical properties of a logic with algebraic properties of its associated class of algebras. We introduced a class $Alg_\Gamma$ of algebras generalizing the standard class $Alg$ of algebraic reducts of reduced matrices. Moreover, we proved that, in the case of a behaviorally algebraizable logic $\mathcal{L}$, the class $Alg_\Gamma(\mathcal{L})$ coincides with the largest behaviorally equivalent algebraic semantics. Given an algebraizable and $\Gamma$-behaviorally logic $\mathcal{L}$, we studied then the relationship between the class $Alg_\Gamma(\mathcal{L})$ and $Alg(\mathcal{L})$. We established relations between the classes of equations and quasi-equations satisfied by these two classes of algebras. We then developed the second approach to the generalization of the standard notion of logical matrix. This approach was strongly connected with the theory of valuation semantics. We introduces an algebraic version of valuation, the notion of $\Gamma$-valuation, and proved a completeness theorem with respect to the class $Mod_\Gamma(\mathcal{L})$ of all $\Gamma$-valuation models. We proved also a result relating a metalogical property of a logic $\mathcal{L}$ and an algebraic property of $Mod_\Gamma(\mathcal{L})$. We ended by showing how to extract, from the algebraic counterpart $K$ of a $\Gamma$-behaviorally algebraizable logic $\mathcal{L}$, a class $\mathcal{M}_K$ of $\Gamma$-valuations that is complete with respect to $\mathcal{L}$.

# Chapter 5

# Worked examples

We now present some examples to further illustrate the relevance of our new approach to the algebraization of logics. In the first example, we show that our behavioral approach is indeed an extension of the existing tools of AAL [FJP03, CG07]. In the many-sorted case we also present some non-behavioral many-sorted definitions and results that can be useful when applying the theory to particular examples of logics. We proceed with the example of paraconsistent logic $\mathcal{C}_1$ of da Costa, whose non-algebraizability in the standard sense is well-known [dC74, Mor80, LMS91]. We show that $\mathcal{C}_1$ is behaviorally algebraizable and, moreover, we give an algebraic counterpart for it. Recall that, although the standard non-algebraizability of $\mathcal{C}_1$ is well-known, there have been some proposals of algebraic counterparts of $\mathcal{C}_1$, namely the class of so-called da Costa algebras and the non-truth-functional bivaluation semantics. Of course, since $\mathcal{C}_1$ is not algebraizable, their precise connection with $\mathcal{C}_1$ could never be established at the light of the standard tools of AAL. We prove that both the class of da Costa algebras and the class of bivaluations can now be obtained from the class of algebras that our approach canonically associates with $\mathcal{C}_1$, thus explaining their precise connection with $\mathcal{C}_1$. We also study the example of the Carnap-style presentation of modal logic $S5$, whose non-algebraizability in the standard sense is again well-known [BP89]. We prove that $S5$ is behaviorally algebraizable and we propose an algebraic counterpart for it. We continue by briefly analyzing the example of first order logic $FOL$, whose standard algebraization is well-studied [BP89, ANS01]. Our approach can be useful to shed light on the essential distinction between terms and formulas. Next, in the example of global logic we follow the exogenous semantic approach for enriching a logic [MSS05] and present a sound a complete deductive system for global logic $GL(\mathcal{L})$ over a given local logic $\mathcal{L}$. We also prove that $GL(\mathcal{L})$ is behaviorally algebraizable independently of $\mathcal{L}$. Moreover, we prove that in the cases where $\mathcal{L}$ is algebraizable we are able to recover the algebraic counterpart of $\mathcal{L}$ from the algebraic counterpart of $GL(\mathcal{L})$. Still following the exogenous semantic approach for enriching a logic we present the example of exogenous propositional probability logic EPPL. We prove that EPPL is behaviorally algebraizable and provide an algebraic counterpart for it. We proceed by exemplifying the power of our approach, by showing that it can be directly applied to study the algebraization of $k$-deductive systems [BP98, Mar04]. Finally, we study the example of Nelson's logic $N$, which is algebraizable according to the standard definition [Ras81], but its behavioral algebraization can help to give an extra insight on the role of Heyting algebras in the algebraic counterpart of $N$.

## 5.1 Important particular cases

In this section we show that standard and also many-sorted algebraization are particular cases of the behavioral notion of algebraizable logic we have proposed herein. Our goal is also to present some particular notions and results of the behavioral theory specifically tailored for the many-sorted case. This is of practical importance since this particularized notions and results assume a simpler presentation and can be applied to logics where there is no need to assume non-congruent operations.

### 5.1.1 Standard algebraization

In this example we hint on how to prove that a logic that is algebraizable in the standard sense is also algebraizable in our more general setting. Recall that the objects of study of the standard tools of AAL are structural propositional logics, that correspond, in our setting, to single-sorted logics.

Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a single-sorted logic. Recall that $\phi$ is the unique sort of $\Sigma$. Let $\mathbf{A}$ be a $\Sigma$-algebra and consider $\mathbf{A}^o$ a $\Sigma^o$-algebra obtained from $\mathbf{A}$ in the following way:

- $A^o_v = A_\phi$;
- $A^o_\phi = A_\phi$;
- $o_{\mathbf{A}^o}(a) = a$ for every $a \in A^o_\phi$;
- $f_{\mathbf{A}^o}(a_1, \ldots, a_n) = f_{\mathbf{A}}(a_1, \ldots, a_n)$ for every connective $f$ over $\Sigma$.

Intuitively, by taking the visible sort of $\mathbf{A}^o$ to be a copy of $\mathbf{A}$ and $o_{\mathbf{A}^o}$ to be the identity function, we are aiming at a collapse between behavioral satisfaction in $\mathbf{A}^o$ and equational satisfaction in $\mathbf{A}$. In fact, we can obtain the following result.

**Lemma 5.1.1.** *Let* $\mathbf{A}$ *be a* $\Sigma$*-algebra and* $\mathbf{A}^o$ *the* $\Sigma^o$*-algebra obtained from* $\mathbf{A}$ *using the above construction. Then, for every* $a, b \in A^o_\phi$, *it holds in* $\mathbf{A}^o$ *that:*

$$a \equiv_\Sigma b \quad \textit{iff} \quad a = b.$$

*Proof.* The fact that $a = b$ immediately implies that $a \equiv_\Sigma b$. Assume now that $a \equiv_\Sigma b$. Then, $a$ and $b$ are indistinguishable under every experiment. Therefore, in particular, we have that $a = o_{\mathbf{A}^o}(a) = o_{\mathbf{A}^o}(b) = b$.

□

Note that in the above lemma the behavioral equivalence is obtained by considering that $\Gamma = \Sigma$, that is, we choose the whole signature $\Sigma$ to produce the experiments.

For every class $K$ of $\Sigma$-algebras we can consider the following class of $\Sigma^o$-algebras

$$K^o = \{\mathbf{A}^o : \mathbf{A} \in K\}.$$

**Proposition 5.1.2.** *Given a single-sorted signature* $\Sigma$ *and a class* $K$ *of* $\Sigma$*-algebras, then*

$$\vDash^K_\Sigma \;=\; \vDash^{K^o,\Sigma}_{\Sigma,bhv}.$$

*Proof.* The result follows from the fact that, given an equation $t_1 \approx t_2$ and an assignment $h$ over $\mathbf{A}$, we have that $\mathbf{A}, h \Vdash t_1 \approx t_2$ iff $\mathbf{A}^o, h \Vvdash_\Sigma t_1 \approx t_2$. But this last condition is an immediate consequence of the previous lemma.

□

If we apply this construction to an equivalent algebraic semantics of an algebraizable logic we obtain the following immediate Corollary.

**Corollary 5.1.3.** *Let* $\mathcal{L} = \langle \Sigma, \vdash \rangle$ *be a single-sorted algebraizable logic with* $K$ *an equivalent algebraic semantics. Then* $\mathcal{L}$ *is* $\Sigma$*-behaviorally algebraizable with* $K^o$ *a* $\Sigma$*-behaviorally equivalent algebraic semantics.*

### 5.1.2 Many-sorted algebraization

We have two main aims in this example. The first is to recall the many-sorted algebraization, as introduced in [CG07]. This is particularly helpful when applying our approach to many-sorted logics and, moreover, we are not interested on the isolation of an algebraizable fragment of it. In those logics, the use of many-sorted approach can help to ease the notation very significantly.

The second aim of the example is to prove that a logic is behaviorally algebraizable whenever it is many-sorted algebraizable, that is, to prove that the many-sorted algebraization is a particular case of the behavioral algebraization by taking $\Gamma = \Sigma$.

We start by presenting some notions and results of BAAL particularized to the (non-behavioral) many-sorted case. We do not present the proof of the results since they are just particular cases of the results of BAAL. For more details on many-sorted algebraization we point to [CG07].

**Definition 5.1.4.** A many-sorted logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ is *algebraizable* if there exists a class $K$ of $\Sigma$-algebras, a set $\Theta(\xi) \subseteq Eq_\Sigma(\{\xi\})$ of $\phi$-equations and a set $\Delta(\xi_1, \xi_2) \subseteq L_\Sigma(\{\xi_1, \xi_2\})$ of formulas such that, for every $T \cup \{\varphi\} \subseteq L_\Sigma(X)$ and for every set $\Phi \cup \{t_1 \approx t_2\}$ of $\phi$-equations:

*i)* $T \vdash \varphi$ iff $\Theta[T] \vDash^K_\Sigma \Theta(\varphi)$;

*ii)* $\Phi \vDash^K_\Sigma t_1 \approx t_2$ iff $\Delta[\Phi] \vdash \Delta(t_1, t_2)$;

*iii)* $\xi \dashv\vdash \Delta[\Theta(\xi)]$;

*iv)* $\xi_1 \approx \xi_2 =\mid\mid\!\!=^K_\Sigma \Theta[\Delta(\xi_1, \xi_2)]$.

Following the standard notation of AAL, $\Theta$ is called the set of *defining equations*, $\Delta$ the set of *equivalence formulas*, and $K$ an *equivalent algebraic semantics* for $\mathcal{L}$. If the set of defining equations and of equivalence formulas are finite we say that $\mathcal{L}$ is *finitely* $\Gamma$*-behaviorally algebraizable*. Similarly to standard AAL, conditions *i)* and *iv)* jointly imply *ii)* and *iii)*, and vice-versa.

A necessary condition for a many-sorted logic to be algebraizable is that is it *equivalential*.

**Definition 5.1.5.** A many-sorted logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ is *equivalential* if there exists a set $\Delta(\xi_1, \xi_2) \subseteq L_\Sigma(\{\xi_1, \xi_2\})$ of formulas such that, for every $\varphi, \psi, \delta, \varphi_1, \dots, \varphi_n, \psi_1, \dots, \psi_n \in L_\Sigma(X)$ and every operation $c : \phi^n \to \phi$ of $\Sigma$:

(R) $\vdash \Delta(\varphi, \varphi)$;

(S) $\Delta(\varphi, \psi) \vdash \Delta(\psi, \varphi)$;

(T) $\Delta(\varphi, \psi), \Delta(\psi, \delta) \vdash \Delta(\varphi, \delta)$;

(MP) $\Delta(\varphi, \psi), \varphi \vdash \psi$;

(RP) $\Delta(\varphi_1, \psi_1), \dots, \Delta(\varphi_n, \psi_n) \vdash \Delta(c[\varphi_1, \dots, \varphi_n], c[\psi_1, \dots, \psi_n])$.

The following intrinsic characterization is very useful when one wants to prove that a given many-sorted logic is algebraizable.

**Theorem 5.1.6.** *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic. We have that $\mathcal{L}$ is algebraizable iff it is equivalential with equivalence set $\Delta(\xi_1, \xi_2)$ and there exists a set $\Theta(\xi) \subseteq Eq_{\Sigma,\phi}(\{\xi\})$ of $\phi$-equations such that*

$$\xi \dashv\vdash \Delta[\Theta(\xi)].$$

As a corollary, we can give a useful sufficient condition for a logic to be algebraizable. It provides easy to check conditions to prove that a logic is algebraizable.

**Corollary 5.1.7.** *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic. A sufficient condition for $\mathcal{L}$ to be algebraizable is that it is equivalential with equivalence set $\Delta(\xi_1, \xi_2)$ satisfying also:*

**(G)** $\xi_1, \xi_2 \vdash \Delta(\xi_1, \xi_2)$.

*In this case $\Delta(\xi_1, \xi_2)$ and $\Theta(\xi) = \{\xi \approx e(\xi, \xi) : e \in \Delta\}$ are, respectively, the equivalence formulas and defining equations for $\mathcal{L}$.*

In this (non-behavioral) many-sorted particular case, and since our notions are no longer parametrized by a subsignature $\Gamma$ of $\Sigma$, we get the standard unicity results with respect to equivalent algebraic semantics.

**Theorem 5.1.8.** *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted logic. Suppose that $\mathcal{L}$ is an algebraizable logic and let $K$ and $K'$ be two equivalent algebraic semantics for $\mathcal{L}$ such that $\Delta(\xi_1, \xi_2)$ and $\Theta(\xi)$ are equivalence formulas and defining equations for $K$, and, similarly, $\Delta'(\xi_1, \xi_2)$ and $\Theta'(\xi)$ for $K'$. Then we have that:*

*i)* $\vDash_\Sigma^K = \vDash_\Sigma^{K'}$;

*ii)* $\Delta(\xi_1, \xi_2) \dashv\vdash \Delta'(\xi_1, \xi_2)$;

*iii)* $\Theta(\xi) =\!\mid\mid\!\!=_\Sigma^K \Theta'(\xi)$.

Theorem 5.1.8 allows us to conclude that, as in standard AAL, given an algebraizable logic $\mathcal{L}$ we can consider the largest equivalent algebraic semantics, that we denote by $K_\mathcal{L}$.

Consider now the particular case where a many-sorted logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ is finitary and finitely algebraizable. An immediate consequence of the above theorem is that, if $K$ and $K'$ are two equivalent algebraic semantics for $\mathcal{L}$, then $K$ and $K'$ must satisfy the same quasi-equations. Then, $K$ and $K'$ generate the same quasivariety and this quasivariety is also an equivalent algebraic semantics for $\mathcal{L}$. Therefore, we can talk about *the* equivalent quasivariety semantics of a finitary and finitely algebraizable logic. It is interesting to note that, similarly to what Blok and Pigozzi [BP89] do for finitary and finitely algebraizable propositional logics,given an axiomatization of $\mathcal{L}$ we can construct a basis for the quasi-equations of the unique equivalent quasivariety semantics.

**Theorem 5.1.9.** *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a finitary many-sorted logic presented by a Hilbert-style deductive system composed of a set $Ax$ of axioms and a set $Ir$ of inference rules. Assume that $\mathcal{L}$ is finitely algebraizable with defining equation $\Theta(\xi)$ and equivalence formulas $\Delta(\xi_1, \xi_2)$. Then the unique equivalent quasivariety semantics for $\mathcal{L}$ is axiomatized by the following equations and quasi-equations:*

*i)* $\Theta(\varphi)$*, for every theorem* $\varphi$ *of* $\mathcal{L}$*;*

*ii)* $\Theta(\Delta(\xi, \xi))$*;*

*iii)* $\Theta(\psi_1) \wedge \ldots \wedge \Theta(\psi_n) \to \Theta(\varphi)$ *for every* $\langle \psi_1, \ldots, \psi_n, \varphi \rangle \in Ir$*;*

*iv)* $\Theta(\Delta(\xi_1, \xi_2)) \rightarrow \xi_1 \approx \xi_2$.

We end this example by showing that an algebraizable many-sorted logic is also behaviorally algebraizable. We use a construction similar to the one we did in Section 5.1.1 in the case of standard algebraization.

Let $\mathbf{A}$ be a $\Sigma$-algebra and consider $\mathbf{A}^o$ a $\Sigma^o$-algebra obtained from $\mathbf{A}$ such that:

- $(\mathbf{A}^o)_{|\Sigma} = \mathbf{A}$;
- $A^o_v = A_\phi$;
- $o_{\mathbf{A}^o}(a) = a$ for every $a \in A^o_\phi$.

Intuitively, by taking the visible sort of $\mathbf{A}^o$ to be a copy of $A_\phi$ and $o_{\mathbf{A}^o}$ to be the identity function, we are aiming at a collapsing between behavioral satisfaction in $\mathbf{A}^o$ and equational satisfaction in $\mathbf{A}$. In fact, we can prove the following result.

**Lemma 5.1.10.** *Let $\mathbf{A}$ be a $\Sigma$-algebra and $\mathbf{A}^o$ the $\Sigma^o$-algebra obtained from $\mathbf{A}$ using the above construction. Then, for every $a, b \in A^o_s$, we have that:*

$$a \equiv_\Sigma b \quad \textit{iff} \quad a = b.$$

*Proof.* The fact that $a = b$ immediately implies that $a \equiv_\Sigma b$. Assume now that $a \equiv_\Sigma b$. Then $a$ and $b$ are indistinguishable under every experiment. Therefore, in particular, we have that $a = o_{\mathbf{A}^o}(a) = o_{\mathbf{A}^o}(b) = b$.

□

For every class $K$ of $\Sigma$-algebras we can consider the following class of $\Sigma^o$-algebras

$$K^o = \{\mathbf{A}^o : \mathbf{A} \in K\}.$$

**Proposition 5.1.11.** *Given a many-sorted signature $\Sigma$, a class $K$ of $\Sigma$-algebras and a set $\Phi \cup \{\varphi \approx \psi\} \subseteq Eq_{\Sigma,\phi}(X)$ of $\phi$-equations we have that:*

$$\Phi \vDash^K_\Sigma \varphi \approx \psi \qquad \textit{iff} \qquad \Phi \vDash^{K^o,\Sigma}_{\Sigma,bhv} \varphi \approx \psi.$$

*Proof.* The result follows from the fact that, given a $\phi$-equation $\varphi \approx \psi$ and an assignment $h$ over $\mathbf{A}$, we have that $\mathbf{A}, h \Vdash \varphi \approx \psi$ iff $\mathbf{A}^o, h \Vvdash_\Sigma \varphi \approx \psi$. This last condition follows easily from Lemma 5.1.10.

□

If we apply this construction to an equivalent algebraic semantics of an algebraizable logic we obtain the following immediate corollary.

**Corollary 5.1.12.** *Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a many-sorted algebraizable logic with $K$ an equivalent algebraic semantics. Then $\mathcal{L}$ is $\Sigma$-behaviorally algebraizable with $K^o$ a $\Sigma$-behaviorally equivalent algebraic semantics.*

## 5.2 da Costa's paraconsistent logic $\mathcal{C}_1$

We now analyze the behavioral algebraization of the paraconsistent logic $\mathcal{C}_1$ of da Costa [dC74, dC63]. This is one of the motivating examples of our approach and it was inspired by the work in [CCC$^+$03]. It was proved, first by Mortensen [Mor80], and later by Lewin, Mikenberg and Schwarze [LMS91], that $\mathcal{C}_1$ is not algebraizable according to the standard notion. So, we can say that $\mathcal{C}_1$ is an example of a logic whose non-algebraizability is well studied. Nevertheless, it is rather strange that a relatively well-behaved logic fails to have an algebraic counterpart. Of course, one could argue that a class of algebras, namely $Alg(\mathcal{C}_1)$, is always associated with $\mathcal{C}_1$. This class is however not very interesting and, therefore, no work has been devoted to it in the literature.

First of all let us recalling the logic $\mathcal{C}_1 = \langle \Sigma_{\mathcal{C}_1}, \vdash_{\mathcal{C}_1} \rangle$. The single-sorted signature of $\mathcal{C}_1$, $\Sigma_{\mathcal{C}_1} = \langle \{\phi\}, F \rangle$, is such that $F_\phi = \{\mathbf{t}, \mathbf{f}\}$, $F_{\phi\phi} = \{\neg\}$, $F_{\phi\phi\phi} = \{\wedge, \vee, \Rightarrow\}$ and $F_{ws} = \emptyset$ otherwise. We can define an unary derived connective $\sim$ over $\Sigma_{\mathcal{C}_1}$ such that $\sim \xi = (\xi^\circ \wedge (\neg \xi))$, where $\varphi^\circ$ is just an abbreviation of $\neg(\varphi \wedge (\neg \varphi))$. This derived connective is intended to correspond to classical negation. The fact that we can define classical negation within $\mathcal{C}_1$ is indeed an essential feature of its forthcoming behavioral algebraization.

The consequence relation of $\mathcal{C}_1$ can be defined in a Hilbert-style way from the following axioms:

A1) $\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_1)$;

A2) $(\xi_1 \wedge \xi_2) \Rightarrow \xi_1$;

A3) $(\xi_1 \wedge \xi_2) \Rightarrow \xi_2$;

A4) $\xi_1 \Rightarrow (\xi_2 \Rightarrow (\xi_1 \wedge \xi_2))$;

A5) $\xi_1 \Rightarrow (\xi_1 \vee \xi_2)$;

A6) $\xi_2 \Rightarrow (\xi_1 \vee \xi_2)$;

A7) $\neg\neg\xi_1 \Rightarrow \xi_1$;

A8) $\xi_1 \vee \neg\xi_1$;

A9) $\xi_1^\circ \Rightarrow (\xi_1 \Rightarrow (\neg\xi_1 \Rightarrow \xi_2))$;

A10) $(\xi_1^\circ \wedge \xi_2^\circ) \Rightarrow (\xi_1 \wedge \xi_2)^\circ$;

A11) $(\xi_1^\circ \wedge \xi_2^\circ) \Rightarrow (\xi_1 \vee \xi_2)^\circ$;

A12) $(\xi_1^\circ \wedge \xi_2^\circ) \Rightarrow (\xi_1 \Rightarrow \xi_2)^\circ$;

A13) $\mathbf{t} \Leftrightarrow (\xi_1 \Rightarrow \xi_1)$;

A14) $\mathbf{f} \Leftrightarrow (\xi_1^\circ \wedge (\xi_1 \wedge \neg\xi_1))$;

A15) $(\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_3)) \Rightarrow ((\xi_1 \Rightarrow \xi_2) \Rightarrow (\xi_1 \Rightarrow \xi_3))$;

A16) $(\xi_1 \Rightarrow \xi_3) \Rightarrow ((\xi_2 \Rightarrow \xi_3) \Rightarrow ((\xi_1 \vee \xi_2) \Rightarrow \xi_3))$;

and the rule of inference:

(MP) $\dfrac{\xi_1 \quad \xi_1 \Rightarrow \xi_2}{\xi_2}$

Although it is defined as a logic weaker than Classical Propositional Logic (CPL), it happens that the defined connective $\sim$ indeed corresponds to classical negation. Therefore, the fragment $\{\sim, \wedge, \vee, \Rightarrow, \mathbf{t}, \mathbf{f}\}$ corresponds to CPL. So, despite of its innocent aspect, $\mathcal{C}_1$ is a non-truth-functional logic, namely it lacks congruence for its paraconsistent negation connective with respect to the equivalence $\Leftrightarrow$ that algebraizes the CPL fragment. In general, it may happen that $\vdash_{\mathcal{C}_1} (\varphi \Leftrightarrow \psi)$ but $\nvdash_{\mathcal{C}_1} (\neg\varphi \Leftrightarrow \neg\psi)$. Although $\mathcal{C}_1$ is not algebraizable, in [CCC$^+$03] the authors have introduced a class of two-sorted algebras as a possible algebraic counterpart for $\mathcal{C}_1$, exploring the fact that CPL is a fragment of $\mathcal{C}_1$. However, their precise nature remains unknown, given the non-algebraizability results reported above. One of the objectives of this example is to capture, using our approach, the precise connection between $\mathcal{C}_1$ and this class of two-sorted algebras.

As we have pointed out several times before, behavioral algebraization depends on the choice of the subsignature $\Gamma$. It is well-known that the non-algebraizability of $\mathcal{C}_1$ is due to the fact that its paraconsistent negation is non-truth-functional.

Hence, the key idea of this example is to leave paraconsistent negation out of the chosen subsignature. Consider now the subsignature $\Gamma = \langle\{\phi\}, F^\Gamma\rangle$ of $\Sigma_{\mathcal{C}_1}$ such that $F^\Gamma_{\phi\phi} = \{\sim\}$ and $F^\Gamma_{ws} = F_{ws}$ for every $ws \neq \phi\phi$. Note that, since paraconsistent negation $\neg$ is used in the definition of classical negation $\sim$, the subsignature $\Gamma$ is not just the reduct of $\Sigma_{\mathcal{C}_1}$ obtained by excluding $\neg$.

Let $K_{\mathcal{C}_1}$ be a class of $\Sigma^o$-algebras that $\Gamma$-behaviorally satisfy the following set of hidden equations:

*i)* $\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_1) \approx \mathbf{t}$;

*ii)* $(\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_3)) \Rightarrow ((\xi_1 \Rightarrow \xi_2) \Rightarrow (\xi_1 \Rightarrow \xi_3)) \approx \mathbf{t}$;

*iii)* $(\xi_1 \wedge \xi_2) \Rightarrow \xi_1 \approx \mathbf{t}$;

*iv)* $(\xi_1 \wedge \xi_2) \Rightarrow \xi_2 \approx \mathbf{t}$;

*v)* $\xi_1 \Rightarrow (\xi_2 \Rightarrow (\xi_1 \wedge \xi_2)) \approx \mathbf{t}$;

*vi)* $\xi_1 \Rightarrow (\xi_1 \vee \xi_2) \approx \mathbf{t}$;

*vii)* $\xi_2 \Rightarrow (\xi_1 \vee \xi_2) \approx \mathbf{t}$;

*viii)* $(\xi_1 \Rightarrow \xi_3) \Rightarrow ((\xi_2 \Rightarrow \xi_3) \Rightarrow ((\xi_1 \vee \xi_2) \Rightarrow \xi_3)) \approx \mathbf{t}$;

*ix)* $\neg\neg\xi_1 \Rightarrow \xi_1 \approx \mathbf{t}$;

*x)* $\xi_1 \vee \neg\xi_1 \approx \mathbf{t}$;

*xi)* $(\sim \xi_1) \Rightarrow (\neg\xi_1) \approx \mathbf{t}$;

*xii)* $\xi_1^\circ \wedge (\xi_1 \wedge \neg\xi_1) \approx \mathbf{f}$;

*xiii)* $(\xi_1^\circ \wedge \xi_2^\circ) \Rightarrow (\xi_1 \wedge \xi_2)^\circ \approx \mathbf{t}$;

*xiv)* $(\xi_1^\circ \wedge \xi_2^\circ) \Rightarrow (\xi_1 \vee \xi_2)^\circ \approx \mathbf{t}$;

*xv)* $(\xi_1^\circ \wedge \xi_2^\circ) \Rightarrow (\xi_1 \Rightarrow \xi_2)^\circ \approx \mathbf{t}$;

and $\Gamma$-behaviorally satisfy the following hidden quasi-equations:

*xvi)* $(\xi_1 \approx \mathbf{t})\ \ \&\ \ ((\xi_1 \Rightarrow \xi_2) \approx \mathbf{t})\ \rightarrow\ (\xi_2 \approx \mathbf{t})$;

*xvii)* $((\xi_1 \Rightarrow \xi_2) \approx \mathbf{t})\ \ \&\ \ ((\xi_2 \Rightarrow \xi_1) \approx \mathbf{t})\ \rightarrow\ (\xi_1 \approx \xi_2)$.

We are interested in the class $K^*_{\mathcal{C}_1} = \{\mathbf{A}^* : \mathbf{A} \in K_{\mathcal{C}_1}\}$. Note that, by Lemma 4.1.7, we have that $K^*_{\mathcal{C}_1}$ satisfies the following visible quasi-equations:

*xviii)* $(o(\xi_1) \approx o(\xi_2)) \rightarrow (o(\sim \xi_1) \approx o(\sim \xi_2))$;

*xix)* $(o(\xi_1) \approx o(\xi_2))\&(o(\xi_3) \approx o(\xi_4)) \rightarrow (o(\xi_1 \vee \xi_3) \approx o(\xi_2 \vee \xi_4))$;

*xx)* $(o(\xi_1) \approx o(\xi_2))\&(o(\xi_3) \approx o(\xi_4)) \rightarrow (o(\xi_1 \wedge \xi_3) \approx o(\xi_2 \wedge \xi_4))$;

*xxi)* $(o(\xi_1) \approx o(\xi_2))\&(o(\xi_3) \approx o(\xi_4)) \rightarrow (o(\xi_1 \Rightarrow \xi_3) \approx o(\xi_2 \Rightarrow \xi_4))$.

Since $K^*_{\mathcal{C}_1}$ satisfies the above quasi-equations *xviii)* - *xxi)*, we can define over every member $\mathbf{A} \in K^*_{\mathcal{C}_1}$ the operations:

- $\sim^v : v \rightarrow v$ such that $\sim^v_{\mathbf{A}} (o_{\mathbf{A}}(a)) = o_{\mathbf{A}}(\sim_{\mathbf{A}} a)$;
- $\vee^v : vv \rightarrow v$ such that $o_{\mathbf{A}}(a) \vee^v o_{\mathbf{A}}(b) = o_{\mathbf{A}}(a \vee_{\mathbf{A}} b)$;
- $\wedge^v : vv \rightarrow v$ such that $o_{\mathbf{A}}(a) \wedge^v o_{\mathbf{A}}(b) = o_{\mathbf{A}}(a \wedge_{\mathbf{A}} b)$;
- $\Rightarrow^v : vv \rightarrow v$ such that $o_{\mathbf{A}}(a) \Rightarrow^v o_{\mathbf{A}}(b) = o_{\mathbf{A}}(a \Rightarrow_{\mathbf{A}} b)$.

To simplify the presentation consider the following abbreviations:

$$o(\mathbf{f}) = \bot \quad o(\mathbf{t}) = \top \quad \sim^v = - \quad \wedge^v = \sqcap \quad \vee^v = \sqcup \quad \Rightarrow^v = \sqsupset.$$

Due to the careful choice of the subsignature $\Gamma$, and since $K^*_{\mathcal{C}_1}$ satisfies the above quasi-equations *xviii)* - *xxi)*, we can obtain the following useful lemma.

**Lemma 5.2.1.** *Given* $\mathbf{A} \in K^*_{\mathcal{C}_1}$*, an equation* $\varphi \approx \psi$ *and* $h$ *an assignment then*

$$\mathbf{A}, h \Vvdash_\Gamma \varphi \approx \psi \textit{ iff } \mathbf{A}, h \Vdash o(\varphi) \approx o(\psi).$$

*Proof.* The fact that $\mathbf{A}, h \Vvdash_\Gamma \varphi \approx \psi$ implies $\mathbf{A}, h \Vdash o(\varphi) \approx o(\psi)$ follows from $\xi \in C^\Gamma_{\Sigma_{\mathcal{C}_1},\phi}[\xi]$. The other direction follows from an easy induction on the structure of contexts, recalling that $\mathbf{A}$ satisfies the quasi-equations *xviii)* - *xxi)*.

□

The class $K^*_{\mathcal{C}_1}$ was proposed in [CCC$^+$03] as a possible algebraic counterpart of $\mathcal{C}_1$, but the connection between $\mathcal{C}_1$ and $K^*_{\mathcal{C}_1}$ was never established at the light of the theory of algebraization. In fact, they introduced this class of algebras over a richer signature that contained, a priori, the visible connectives $\sqcup, \sqcap, \top, \bot, \sim$ and assumed that the visible part of every algebra in this class is a Boolean algebra. It is interesting to note that, although we define herein the class $K^*_{\mathcal{C}_1}$ over a poorest signature, we are able to define the same visible connectives as abbreviations and further prove the following result.

**Proposition 5.2.2.** *For every algebra* $\mathbf{A} \in K^*_{\mathcal{C}_1}$, $\langle A_v, \sqcup_{\mathbf{A}}, \sqcap_{\mathbf{A}}, \top_{\mathbf{A}}, \bot_{\mathbf{A}}, -_{\mathbf{A}} \rangle$ *is a Boolean algebra.*

*Proof.* This result is a consequence of Lemma 5.2.1, the fact that $\mathcal{C}_1$ satisfies the usual axioms for positive Boolean connectives and the fact that $\sim$ defines classical negation within $\mathcal{C}_1$.

□

We are now in conditions to prove that $\mathcal{C}_1$ is behaviorally algebraizable with respect to the subsignature $\Gamma$ of $\Sigma_{\mathcal{C}_1}$ introduced above.

**Theorem 5.2.3.** $\mathcal{C}_1$ *is* $\Gamma$*-behaviorally algebraizable, with* $K^*_{\mathcal{C}_1}$ *a* $\Gamma$*-behaviorally equivalent algebraic semantics with* $\Theta(\xi) = \{\xi \approx \mathbf{t}\}$ *a set of defining equations and* $\Delta(\xi_1, \xi_2) = \{\xi_1 \Rightarrow \xi_2,\ \xi_2 \Rightarrow \xi_1\}$ *a set of equivalence formulas.*

*Proof.* First of all, note that $\Theta(\xi) \subseteq Eq_{\Gamma,\phi}(\xi)$ and $\Delta(\xi_1, \xi_2) \subseteq T_{\Gamma,\phi(\xi_1,\xi_2)}$. Now we have to prove that for every $T \cup \{\varphi\} \subseteq L_\Sigma(X)$:

*i)* $T \vdash_{\mathcal{C}_1} \varphi \quad \text{iff} \quad \{\gamma \approx \mathbf{t} : \gamma \in T\} \vDash^{K_{\mathcal{C}_1},\Gamma}_{\Sigma,bhv} \varphi \approx \mathbf{t}$;

*ii)* $\xi_1 \approx \xi_2 \;=\!|\!|\!\!=^{K_{\mathcal{C}_1},\Gamma}_{\Sigma,bhv} (\xi_1 \equiv \xi_2) \approx \mathbf{t}$.

Recall that in the visible sorts, behavioral logic coincides with equational logic. Using this fact together with Lemma 5.2.1, condition *i)* becomes equivalent to prove that for every $T \cup \{\varphi\} \subseteq L_\Sigma(X)$:

*i′)* $T \vdash_{\mathcal{C}_1} \varphi \quad \text{iff} \quad \{o(\gamma) \approx \top : \gamma \in T\} \vDash_{K^*_{\mathcal{C}_1}} o(\varphi) \approx \top$.

Note that we have now equational consequence instead of behavioral consequence. The fact that this condition holds was already proved in [CCC$^+$03].

Turning our attention to condition $ii$), and using Lemma 5.2.1, all we have to prove is that $o(\xi_1) \approx o(\xi_2) \vDash_{K_{\mathcal{C}_1}} o(\xi_1 \equiv \xi_2) \approx \top$ and that $o(\xi_1 \equiv \xi_2) \approx \top \vDash_{K_{\mathcal{C}_1}} o(\xi_1) \approx o(\xi_2)$. Both conditions follow from the fact that $\langle A_v, \sqcup_{\mathbf{A}}, \sqcap_{\mathbf{A}}, \top_{\mathbf{A}}, \bot_{\mathbf{A}}, -_{\mathbf{A}}\rangle$ is a Boolean algebra, for every $\Sigma^o$-algebra $\mathbf{A} \in K^*_{\mathcal{C}_1}$.

□

There are in the literature other proposals of algebraic counterparts for $\mathcal{C}_1$. One important example is the case of the so-called da Costa algebras proposed by da Costa [dC63, dC74]. Still, the precise connection between $\mathcal{C}_1$ and this class of algebraic structures has never been established.

In what follows we propose to establish the connection between $\mathcal{C}_1$ and the class of da Costa algebras at the light of our behavioral approach.

Let us begin by presenting the class of da Costa algebras. To be precise, the structure we will introduce is not an algebra in the technical sense of the word. This is due to the fact that, more than a set and operations over this set, these structure have also a relation over the set. Nevertheless, we will follow the notation as used in the literature and call these algebraic structures da Costa algebras. The definition we give here is a refinement of both the works of da Costa [dC66] and of Carnielli and Alcantara [CdA84], taking into account particular features of $\mathcal{C}_1$.

By a *da Costa algebra* we mean a structure

$$\mathcal{U} = \langle S, 0, 1, \leq, \wedge, \vee, \Rightarrow, \sim\rangle,$$

such that $0, 1 \in S$ and, for every $a, b, c \in S$, the following conditions hold:

1) $\leq \subseteq S \times S$ is a quasi-order, that is,

   Reflexivity: $a \leq a$;

   Transitivity: if $a \leq b$ and $b \leq c$ then $a \leq c$;

2) $a \wedge b \leq a$ and $a \wedge b \leq b$ ;

3) $a \wedge a \simeq a$ and $a \vee a \simeq a$, where $a \simeq b$ iff $a \leq b$ and $b \leq a$;

4) $a \wedge (b \vee c) \simeq (a \wedge b) \vee (a \wedge c)$;

5) $a \leq a \vee b$, $b \leq a \vee b$;

6) if $a \leq c$ and $b \leq c$ then $a \vee b \leq c$;

7) $a \wedge (a \Rightarrow b) \leq b$;

8) $a \wedge c \leq b$ then $c \leq (a \Rightarrow b)$;

9) $0 \leq a, \quad a \leq 1$;

10) $a^\circ \leq (\sim a)^\circ$, where $a^\circ = \sim (a \wedge \sim a)$;

11) $a \vee \sim a \simeq 1$;

12) $\sim (\sim x) \leq x$;

13) $a^\circ \leq (b \Rightarrow a) \Rightarrow ((b \Rightarrow \sim a) \Rightarrow \sim b)$;

14) $a^\circ \wedge \sim (a^\circ) \simeq 0$;

15) $a^\circ \wedge b^\circ \leq (a \wedge b)^\circ$;

16) $a^\circ \wedge b^\circ \leq (a \vee b)^\circ$;

17) $a^\circ \wedge b^\circ \leq (a \Rightarrow b)^\circ$.

Recall that the class $K_{\mathcal{C}_1}$ of algebras is the class of algebras that our behavioral approach canonically associates to $\mathcal{C}_1$. Therefore, in order to draw the relationship between da Costa algebras and $\mathcal{C}_1$, we focus on the precise connection between the class of da Costa algebras and the class $K_{\mathcal{C}_1}$ of two-sorted algebras.

The idea is to show that, given an algebra $\mathbf{A} \in K_{\mathcal{C}_1}$, we can obtain from it a da Costa algebra $\mathcal{U}_\mathbf{A}$ and also, given a da Costa algebra $\mathcal{U}$, we can obtain from it an algebra $\mathbf{A}_\mathcal{U} \in K_{\mathcal{C}_1}$. Moreover, these constructions are inverse of each other in the sense that $\mathbf{A}_{\mathcal{U}_\mathbf{A}}$ is isomorphic to $\mathbf{A}$ and, moreover, $\mathcal{U}_{\mathbf{A}_\mathcal{U}}$ is isomorphic to $\mathcal{U}$.

First, let $\mathbf{A} = \langle A_\phi, A_v, o_\mathbf{A}, \wedge_\mathbf{A}, \vee_\mathbf{A}, \Rightarrow_\mathbf{A}, \sim_\mathbf{A} \rangle \in K_{\mathcal{C}_1}$ and consider the following structure,

$$\mathcal{U}_\mathbf{A} = \langle S_\mathbf{A}, 0, 1, \leq_\mathbf{A}, \wedge_\mathbf{A}, \vee_\mathbf{A}, \Rightarrow_\mathbf{A}, \sim_\mathbf{A} \rangle,$$

obtained from $\mathbf{A}$ in the following way:

- $S_\mathbf{A} = A_\phi$
- $1 = \mathbf{t}_\mathbf{A}$ and $0 = \mathbf{f}_\mathbf{A}$
- $\leq_\mathbf{A} \subseteq S_\mathbf{A} \times S_\mathbf{A}$ is such that $a \leq_\mathbf{A} b$ iff $(a \Rightarrow_\mathbf{A} b) \equiv \mathbf{t}_\mathbf{A}$

Recall that we can consider a binary relation $\simeq_{\mathcal{U}_\mathbf{A}}$ on $S_\mathbf{A}$ defined by: $a \simeq_{\mathcal{U}_\mathbf{A}} b$ if $a \leq_{\mathcal{U}_\mathbf{A}} b$ and $b \leq_{\mathcal{U}_\mathbf{A}} a$. Using the fact that $\mathbf{A}$ satisfies quasi-equation $xvii)$ on the definition of $K_{\mathcal{C}_1}$, we can conclude that $\simeq_{\mathcal{U}_\mathbf{A}}$ coincides with behavioral equivalence $\equiv_\mathbf{A}$.

To see that $\mathcal{U}_\mathbf{A}$ is a da Costa algebra we just have to prove that it satisfies the conditions on the definition of a da Costa algebra. And here is where behavioral reasoning comes into play. Verifying that $\mathcal{U}_\mathbf{A}$ satisfies a condition of the form $a \leq_\mathbf{A} b$ is, by definition, the same as proving that in $\mathbf{A}$ we have that $(a \Rightarrow_\mathbf{A} b) \equiv \mathbf{t}_\mathbf{A}$. Now, using the fact that $K_{\mathcal{C}_1}$ is the behaviorally equivalent algebraic semantics of $\mathcal{C}_1$, it can be easily proved that $\mathcal{U}_\mathbf{A}$ satisfies all conditions in the definition of a da Costa algebra, since they amount to well-known properties of the consequence in $\mathcal{C}_1$.

Now let $\mathcal{U} = \langle S, 0, 1, \leq_\mathcal{U}, \wedge_\mathcal{U}, \vee_\mathcal{U}, \Rightarrow_\mathcal{U}, \sim_\mathcal{U}\rangle$ be a da Costa algebra. From $\mathcal{U}$ we will build a two-sorted algebra $\mathbf{A}_\mathcal{U}$ such that $\mathbf{A}_\mathcal{U} \in K_{\mathcal{C}_1}$. Recall that we can consider a binary relation $\simeq_\mathcal{U}$ on $S$ defined by: $a \simeq_\mathcal{U} b$ if $a \leq_\mathcal{U} b$ and $b \leq_\mathcal{U} a$. It can be easily shown that $\simeq_\mathcal{U}$ is an equivalence relation. As usual, when we have an equivalence relation on a set we can group equivalent elements thus making a partition of the set. Each group of equivalent elements form an equivalence class, and thus we obtain the set $S_{/\simeq_\mathcal{U}}$ of all equivalence classes.

Consider now the two-sorted algebra,

$$\mathbf{A}_\mathcal{U} = \langle A_\phi, A_v, o_\mathbf{A}, \wedge_\mathbf{A}, \vee_\mathbf{A}, \Rightarrow_\mathbf{A}, \sim_\mathbf{A}\rangle$$

obtained from $\mathcal{U}$ in the following way:

- $A_\phi = S$
- $A_v = S_{/\simeq_\mathcal{U}}$
- $o_\mathbf{A}(a) = [a]_{\simeq_\mathcal{U}}$
- $\wedge_\mathbf{A} = \wedge_\mathcal{U}$, $\vee_\mathbf{A} = \vee_\mathcal{U}$, $\Rightarrow_\mathbf{A} = \Rightarrow_\mathcal{U}$, $\sim_\mathbf{A} = \sim_\mathcal{U}$

Before we prove that $\mathbf{A}_\mathcal{U}$ is an algebra in $K_{\mathcal{C}_1}$, we will make some remarks. First of all, note that in a da Costa algebra $\mathcal{U}$ the equivalence relation $\simeq_\mathcal{U}$ is not in general a congruence. So, the usual quotient construction cannot be done because the non-congruent operations are not well-defined on the set $S_{/\simeq_\mathcal{U}}$. Using our two-sorted approach we can, nevertheless, simulate a quotient construction. This is the idea of the construction used to obtain $\mathbf{A}_{\mathcal{U}}$ from $\mathcal{U}$. Moreover, it is an easy exercise to prove that $\simeq_{\mathcal{U}}$ is an equivalence relation compatible with the positive connectives, that is, if $a, b, x, y \in S$ such that $a \simeq_{\mathcal{U}} b$ and $x \simeq_{\mathcal{U}} y$ then we have that $(a \wedge_{\mathcal{U}} x) \simeq_{\mathcal{U}} (b \wedge_{\mathcal{U}} y)$ and $(a \vee_{\mathcal{U}} x) \simeq_{\mathcal{U}} (b \vee_{\mathcal{U}} y)$ and $(a \Rightarrow_{\mathcal{U}} x) \simeq_{\mathcal{U}} (b \Rightarrow_{\mathcal{U}} y)$. This observation implies that we can prove for $\mathbf{A}_{\mathcal{U}}$ a result similar to lemma 5.2.1.

**Lemma 5.2.4.** *Given an equation $\varphi \approx \psi$ and $h$ an assignment then*

$$\mathbf{A}_{\mathcal{U}}, h \Vdash\!\vdash \varphi \approx \psi \quad \textit{iff} \quad \mathbf{A}_{\mathcal{U}}, h \Vdash o(\varphi) \approx o(\psi).$$

We will now prove that $\mathbf{A}_{\mathcal{U}}$ is an algebra of $K_{\mathcal{C}_1}$. First of all note that $o_{\mathbf{A}}$ is surjective by construction. So, we are left to prove that $\mathbf{A}_{\mathcal{U}}$ behaviorally satisfies all the equations and quasi-equations $i)$ to $xvii)$ and satisfies all quasi-equations $xviii)$ to $xxi)$ in the definition of $K_{\mathcal{C}_1}$.

Using lemma 5.2.4 and the definition of $\mathbf{A}_{\mathcal{U}}$ it is not hard to conclude that, given an equation $(\varphi \approx \psi)$ and an assignment $h$, we have that $\mathbf{A}_{\mathcal{U}}, h \Vdash\!\vdash (\varphi \approx \psi)$ iff $h(\varphi) \simeq_{\mathcal{U}} h(\psi)$. Moreover, in every da Costa algebra $\mathcal{U}$ we have that $(a \Rightarrow_{\mathcal{U}} b) \simeq_{\mathcal{U}} 1$ is equivalent to $a \leq_{\mathcal{U}} b$, for every $a$ and $b$. Using these observations it is easy to see that the verification that $\mathbf{A}_{\mathcal{U}}$ satisfies conditions $i)$ up to $xvii)$ of the definition of $K_{\mathcal{C}_1}$ amounts to well-known properties of every da Costa algebra. Almost all of these properties are proved, for example, in proposition 1 of [CdA84]. The fact that $\mathbf{A}_{\mathcal{U}}$ satisfies conditions $xviii)$ up to $xxi)$ is an immediate consequence of the already mentioned fact that $\simeq_{\mathcal{U}}$ is compatible with the positive connectives and classical negation.

**Theorem 5.2.5.** *More than isomorphic, $\mathcal{U}_{\mathbf{A}_{\mathcal{U}}}$ and $\mathcal{U}$ are in fact equal.*

*Proof.* Recall that given $\mathcal{U} = \langle S, 0, 1, \leq_{\mathcal{U}}, \wedge_{\mathcal{U}}, \vee_{\mathcal{U}}, \Rightarrow_{\mathcal{U}}, \sim_{\mathcal{U}} \rangle$ and applying the constructions we obtain $\mathcal{U}_{\mathbf{A}_{\mathcal{U}}} = \langle S, 0, 1, \leq_{\mathcal{U}_{\mathbf{A}_{\mathcal{U}}}}, \wedge_{\mathcal{U}}, \vee_{\mathcal{U}}, \Rightarrow_{\mathcal{U}}, \sim_{\mathcal{U}} \rangle$. So, all that remains to prove is that $\leq_{\mathcal{U}}$ and $\leq_{\mathcal{U}_{\mathbf{A}_{\mathcal{U}}}}$ coincide. For, observe that given $a, b \in S$ we have the following sequence of equivalent conditions: $a \leq_{\mathcal{U}_{\mathbf{A}_{\mathcal{U}}}} b$ iff $(a \Rightarrow_{\mathbf{A}_{\mathcal{U}}} b) \equiv_{\mathbf{A}_{\mathcal{U}}} \mathbf{t}_{\mathbf{A}_{\mathcal{U}}}$ iff $o_{\mathbf{A}_{\mathcal{U}}}(a \Rightarrow_{\mathbf{A}_{\mathcal{U}}} b) = o_{\mathbf{A}_{\mathcal{U}}}(\mathbf{t}_{\mathbf{A}_{\mathcal{U}}})$ iff $[a \Rightarrow_{\mathcal{U}} b]_{\simeq_{\mathcal{U}}} = [1]_{\simeq_{\mathcal{U}}}$ iff $(a \Rightarrow_{\mathcal{U}} b) \simeq_{\mathcal{U}} 1$ iff $a \leq_{\mathcal{U}} b$.

□

**Theorem 5.2.6.** *Let us now see that $\mathbf{A}_{\mathcal{U}_{\mathbf{A}}}$ and $\mathbf{A}$ are isomorphic as structures.*

*Proof.* Recall that given $\mathbf{A} = \langle A_\phi, A_v, o_{\mathbf{A}}, \wedge_{\mathbf{A}}, \vee_{\mathbf{A}}, \Rightarrow_{\mathbf{A}}, \sim_{\mathbf{A}} \rangle \in K_{\mathcal{C}_1}$ and applying the constructions we obtain $\mathbf{A}_{\mathcal{U}_{\mathbf{A}}} = \langle A_\phi, (A_\phi)_{/\simeq_{\mathcal{U}_{\mathbf{A}}}}, o_{\mathbf{A}_{\mathcal{U}_{\mathbf{A}}}}, \wedge_{\mathbf{A}}, \vee_{\mathbf{A}}, \Rightarrow_{\mathbf{A}}, \sim_{\mathbf{A}} \rangle$. Recall also that $\simeq_{\mathcal{U}_{\mathbf{A}}}$ coincides with behavioral equivalence $\equiv_{\mathbf{A}}$. So, all we have to prove is that $(A_\phi)_{/\equiv_{\mathbf{A}}}$ is isomorphic to $A_v$ and that both $o_{\mathbf{A}}$ and $o_{\mathbf{A}_{\mathcal{U}_{\mathbf{A}}}}$ respect this isomorphism, in the sense that, if $\pi : (A_\phi)_{/\equiv_{\mathbf{A}}} \to A_v$ is the isomorphism and $a \in A_\phi$ we have that $\pi(o_{\mathbf{A}_{\mathcal{U}_{\mathbf{A}}}}(a)) = o_{\mathbf{A}}(a)$.

Consider the function $\pi : (A_\phi)_{/\equiv_{\mathbf{A}}} \to A_v$ such that $[a]_{\equiv_{\mathbf{A}}} \stackrel{\pi}{\mapsto} o_{\mathbf{A}}(a)$. First of all we have to prove that $\pi$ is well-defined, that is, its definition does not depend on the particular choice of representative of each class of equivalence. For, consider $a, b \in A_\phi$ such that $a \equiv_{\mathbf{A}} b$ and let us prove that $\pi([a]_{\equiv_{\mathbf{A}}}) = \pi([b]_{\equiv_{\mathbf{A}}})$. This is immediate since by lemma 5.2.4 we have that $a \equiv_{\mathbf{A}} b$ is equivalent to $o_{\mathbf{A}}(a) = o_{\mathbf{A}}(b)$ and this is, by definition, $\pi([a]_{\equiv_{\mathbf{A}}}) = \pi([b]_{\equiv_{\mathbf{A}}})$.

Let us now prove that $\pi$ is indeed a bijection. Injective: let $a, b \in A_\phi$ such that $\pi([a]_{\equiv_{\mathbf{A}}}) = \pi([b]_{\equiv_{\mathbf{A}}})$. So, we have that $o_{\mathbf{A}}(a) = o_{\mathbf{A}}(b)$ and again by lemma 5.2.4 we can conclude that $[a]_{\equiv_{\mathbf{A}}} = [b]_{\equiv_{\mathbf{A}}}$. Surjective: follows immediately from the fact that $o_{\mathbf{A}}$ is surjective.

To conclude, we just have to prove that both $o_{\mathbf{A}}$ and $o_{\mathbf{A}_{\mathcal{U}_{\mathbf{A}}}}$ respect $\pi$ in the sense that $\pi(o_{\mathbf{A}_{\mathcal{U}_{\mathbf{A}}}}(a)) = o_{\mathbf{A}}(a)$. But this immediate since $o_{\mathbf{A}_{\mathcal{U}_{\mathbf{A}}}}(a) = [a]_{\equiv_{\mathbf{A}}}$.

□

The first major semantical analysis of the logics $\mathcal{C}_n$ was developed in [DCA77] by da Costa and Alves. There it was proposed a bivaluation semantics for each logic in the $\mathcal{C}_n$ hierarchy, in particular for $\mathcal{C}_1$.

We end this example with the study of the bivaluation semantics for $\mathcal{C}_1$ within our behavioral approach. Our aim is to obtain, as a by-product of our behavioral approach, the usual bivaluation semantics for $\mathcal{C}_1$ from the class $K_{\mathcal{C}_1}$ of two-sorted algebras. This reenforces the idea that our behavioral approach captures the semantical aspects of $\mathcal{C}_1$ in an unifying way.

First let us start by presenting the bivaluation semantics for $\mathcal{C}_1$ as introduced in [DCA77]. A bivaluation for $\mathcal{C}_1$ is a function $\nu : L_{\mathcal{C}_1} \to \{0, 1\}$ that satisfies the following conditions:

| | |
|---|---|
| val[i] | $\nu(\xi_1 \wedge \xi_2) = 1$ iff $\nu(\xi_1) = 1$ and $\nu(\xi_2) = 1$; |
| val[ii] | $\nu(\xi_1 \vee \xi_2) = 1$ iff $\nu(\xi_1) = 1$ or $\nu(\xi_2) = 1$; |
| val[iii] | $\nu(\xi_1 \Rightarrow \xi_2) = 1$ iff $\nu(\xi_1) = 0$ or $\nu(\xi_2) = 1$; |
| val[iv] | if $\nu(\xi_2^\circ) = \nu(\xi_1 \Rightarrow \xi_2) = \nu(\xi_1 \Rightarrow \neg\xi_2) = 1$ then $\nu(\xi_1) = 0$; |
| val[v] | if $\nu(\xi_1^\circ) = \nu(\xi_2^\circ) = 1$ then $\nu((\xi_1 * \xi_2)^\circ) = 1$ where $* \in \{\wedge, \vee, \Rightarrow\}$; |
| val[vi] | if $\nu(\xi) = 0$ then $\nu(\neg\xi) = 1$; |
| val[vii] | if $\nu(\neg\neg\xi) = 1$ then $\nu(\xi) = 1$. |

Let $\mathcal{V}_{\mathcal{C}_1}$ denote the valuation semantics for $\mathcal{C}_1$, that is, the class of all bivaluations satisfying the above conditions. In the remainder of the section we will concentrate on the intimately connection between $\mathcal{V}_{\mathcal{C}_1}$ and the class $K_{\mathcal{C}_1}$.

Consider given an algebra $\mathbf{A} = \langle A_\phi, A_v, o_\mathbf{A}, \wedge_\mathbf{A}, \vee_\mathbf{A}, \Rightarrow_\mathbf{A}, \sim_\mathbf{A}, 0_\mathbf{A}, 1_\mathbf{A}\rangle \in K_{\mathcal{C}_1}$. Recall that $\mathbf{A}_v = \langle A_v, \sqcup_\mathbf{A}, \sqcap_\mathbf{A}, \top_\mathbf{A}, \bot_\mathbf{A}, -_\mathbf{A}\rangle$ is a Boolean algebra. We will now make use of the well known result that every Boolean algebra is isomorphic to a subdirect power of $\mathbf{2}$ (the two elements Boolean algebra). It is not our intention to dwell on details about what this result means and we point to [BS81] for details. For our purposes it suffices to know that this implies the existence of a set $I$ and an injective homomorphism $\alpha : \mathbf{A}_v \to \mathbf{2}^I$, called a subdirect embedding, such that for each $i \in I$, we have that $\alpha_i : \mathbf{A}_v \to \mathbf{2}_i$ is surjective.

Consider the following set of functions

$$V_\mathbf{A} = \{\nu_{\mathbf{A},i,h} = \alpha_i \circ (o_\mathbf{A} \circ h) \mid \ i \in I \text{ and } h : L_{\mathcal{C}_1} \to A_\phi \text{ homomorphism}\}.$$

Note that for every $i \in I$ and every $h : L_{\mathcal{C}_1} \to A_\phi$ homomorphism we have that $\nu_{\mathbf{A},i,h}$ is a function from $L_{\mathcal{C}_1}$ to $\{0,1\}$. We can collect all function of this form in the set $\mathcal{V} = \bigcup_{\mathbf{A}\in K_{\mathcal{C}_1}} V_\mathbf{A}$.

In what follows we will sketch the proof that $\mathcal{V}$ is precisely the bivaluation semantics for $\mathcal{C}_1$, $\mathcal{V}_{\mathcal{C}_1}$. This proof can be divided in two parts. The first part is nothing but the verification that every element of $\mathcal{V}$ is a bivaluation for $\mathcal{C}_1$. This is an exercise and can be easily checked. To what concerns the second part, we have to prove that every bivaluation for $\mathcal{C}_1$ is in $\mathcal{V}$, that is, it can be obtained from an algebra in $K_{\mathcal{C}_1}$. To see this let $\nu$ be a bivaluation for $\mathcal{C}_1$ and consider the two-sorted algebra $\mathbf{A}_\nu = \langle A_\phi, A_v, o_\mathbf{A}, \wedge, \vee, \Rightarrow, \sim\rangle$ such that $A_\phi = L_{\mathcal{C}_1}$, $A_v = \{0,1\}$ and $o_\mathbf{A} = \nu$. First we have to prove is that $\mathbf{A}_\nu$ is in fact an algebra of $K_{\mathcal{C}_1}$ which is a laborious but in any case easy exercise. In this case, the algebra $\mathbf{A}_v$ is already a (trivial) subdirect product of $\mathbf{2}$ and so the homomorphism $\alpha$ is the identity. Taking the homomorphism $h : L_{\mathcal{C}_1} \to A_\phi$ as the identity, it becomes obvious that the $\nu$ is the bivaluation obtained from $\mathbf{A}_\nu$, $h$ and $\alpha$.

## 5.3 Lewis's modal logic $S5$

Various logics have appeared in the literature whose theorems coincide with those of Lewis's original system for $S5$. Here, we study a Carnap style presentation of $S5$ which is well known not to be algebraizable according to the standard definition [BP89]. Recall that $S5$ can be seen as an extension of CPL with the modality $\Box$. So, although $S5$ is not algebraizable, we can identify an algebraizable fragment of it, CPL. Therefore, using our approach we can build up an algebraic semantics for $S5$ based on Boolean algebras, the algebraic counterpart of CPL. This is in the realm of Boolean algebras with operators, which is the traditional algebraic semantics of normal modal logics. In this example we study the behavioral algebraization of $S5$.

We start by introducing the logic $S5$. The logic $S5 = \langle \Sigma_{S5}, \vdash_{S5} \rangle$ includes a one-sorted signature $\Sigma_{S5} = \langle \{\phi\}, F \rangle$ such that:

- $F_{\phi\phi} = \{\neg, \Box\}$;
- $F_{\phi\phi\phi} = \{\wedge, \vee, \Rightarrow\}$;
- $F_{ws} = \emptyset$ otherwise.

The possibility modality $\Diamond$ is obtained as usual through the abbreviation $\Diamond = \neg\Box\neg$. We also consider a constant $\mathbf{t}$ as abbreviation of $(\varphi \Rightarrow \varphi)$ for some arbitrary but fixed formula $\varphi$ of $S5$.

The consequence relation is obtained from a deductive system with the following axioms:

*i)* $\Box\varphi$ for every $\varphi$ classical tautology;

*ii)* $\Box\xi \Rightarrow \xi$;

*iii)* $\Box(\Box(\xi_1 \Rightarrow \xi_2) \Rightarrow (\Box\xi_1 \Rightarrow \Box\xi_2))$;

*iv)* $\Box(\Box\xi \Rightarrow \xi)$;

*v)* $\Box(\Diamond\xi \Rightarrow \Box\Diamond\xi)$;

and the inference rule:

**(MP)** $$\frac{\xi_1 \quad \xi_1 \Rightarrow \xi_2}{\xi_2}.$$

Now that we have introduced $S5$, we study its behavioral algebraization. Recall that the key point for the negative result concerning the algebraization of $S5$ is the lack of congruence of its modal operator $\Box$. Therefore, in the study of the behavioral algebraization of $S5$ it is natural to consider a subsignature $\Gamma$ of $\Sigma_{S5}$ that does not contain the modal operator $\Box$. Therefore, consider the subsignature $\Gamma = \langle\{\phi\}, F^{\Gamma}\rangle$ of $\Sigma_{S5}$ such that

- $F^{\Gamma}_{\phi\phi} = \{\neg\}$;
- $F^{\Gamma}_{ws} = F_{ws}$ for every $ws \neq \phi\phi$.

Note that $\Gamma$ is indeed obtained from $\Sigma_{S5}$ by ruling out $\Box$. We can now prove that $S5$ is $\Gamma$-behaviorally algebraizable.

Recall that Theorem 3.3.2 gives a sufficient and easy to check condition for a logic to be $\Gamma$-behaviorally algebraizable. In order to prove that $S5$ is $\Gamma$-behaviorally algebraizable, we just need to check that $S5$ is $\Gamma$-behaviorally equivalential and that the $\Gamma$-behavioral equivalence $\Delta(\xi_1, \xi_2)$ satisfies also the so-called (G)-rule: $\xi_1, \xi_2 \vdash \Delta(\xi_1, \xi_2)$.

**Theorem 5.3.1.** *$S5$ is $\Gamma$-behaviorally algebraizable.*

*Proof.* Consider the set of formulas $\Delta(\xi_1, \xi_2) = \{\xi_1 \Rightarrow \xi_2,\ \xi_2 \Rightarrow \xi_1\}$. Using well-known properties of $S5$ it can be easily proved that $\Delta$ is a $\Gamma$-behavioral equivalence set. The fact that $S5$ satisfies the (G)-rule is also well-known. Therefore, using Theorem 3.3.2 we can conclude that $S5$ is $\Gamma$-behaviorally algebraizable.

$\Box$

As a consequence of the behavioral algebraization of $S5$ we can now study its algebraic counterpart. To study the algebraic counterpart that our behavioral approach associates with $S5$ we use Theorem 4.1.3. This theorem gives an axiomatization of the largest $\Gamma$-behaviorally equivalent algebraic semantics of $S5$. Consider the class $K_{S5}$ of $\Sigma^{o}_{S5}$-algebras that $\Gamma$-behaviorally satisfy the following hidden equations:

*i)* $\Box\varphi \approx \mathbf{t}$ for every $\varphi$ classical tautology;

*ii)* $(\Box\xi \Rightarrow \xi) \approx \mathbf{t}$;

*iii)* $(\Box(\Box(\xi_1 \Rightarrow \xi_2) \Rightarrow (\Box\xi_1 \Rightarrow \Box\xi_2))) \approx \mathbf{t}$;

*iv)* $(\Box(\Box\xi \Rightarrow \xi)) \approx \mathbf{t}$;

$v)$ $(\Box(\Diamond\xi \Rightarrow \Box\Diamond\xi)) \approx \mathbf{t}$;

and Γ-behaviorally satisfy the hidden quasi-equations:

$i)$ $(\xi_1 \approx \mathbf{t})$ & $((\xi_1 \Rightarrow \xi_2) \approx \mathbf{t}) \rightarrow (\xi_2 \approx \mathbf{t})$;

$ii)$ $((\xi_1 \Rightarrow \xi_2) \approx \mathbf{t})$ & $((\xi_2 \Rightarrow \xi_1) \approx \mathbf{t}) \rightarrow (\xi_1 \approx \xi_2)$.

Recall that the class of algebras canonically associated with $S5$ is not $K_{S5}$ but its subclass $K^*_{S5} = \{\mathbf{A}^* : \mathbf{A} \in K_{S5}\}$.

An important feature of $K^*_{S5}$, given by Lemma 4.1.7, is that every algebra in $K^*_{S5}$ satisfies the following visible quasi-equations:

$i)$ $(o(\xi_1) \approx o(\xi_2)) \rightarrow (o(\neg\xi_1) \approx o(\neg\xi_2))$;

$ii)$ $(o(\xi_1) \approx o(\xi_2))\&(o(\xi_3) \approx o(\xi_4)) \rightarrow (o(\xi_1 \vee \xi_3) \approx o(\xi_2 \vee \xi_4))$;

$iii)$ $(o(\xi_1) \approx o(\xi_2))\&(o(\xi_3) \approx o(\xi_4)) \rightarrow (o(\xi_1 \wedge \xi_3) \approx o(\xi_2 \wedge \xi_4))$;

$iv)$ $(o(\xi_1) \approx o(\xi_2))\&(o(\xi_3) \approx o(\xi_4)) \rightarrow (o(\xi_1 \Rightarrow \xi_3) \approx o(\xi_2 \Rightarrow \xi_4))$.

Given a member $\mathbf{A} \in K^*_{S5}$, and since $\mathbf{A}$ satisfies the above quasi-equations $(i)$-$(iv)$, we can define the following operations over $\mathbf{A}$:

- $\neg^v : v \rightarrow v$ such that $\neg^v_{\mathbf{A}}(o_{\mathbf{A}}(a)) = o_{\mathbf{A}}(\neg_{\mathbf{A}} a)$;
- $\vee^v : vv \rightarrow v$ such that $o_{\mathbf{A}}(a) \vee^v o_{\mathbf{A}}(b) = o_{\mathbf{A}}(a \vee_{\mathbf{A}} b)$;
- $\wedge^v : vv \rightarrow v$ such that $o_{\mathbf{A}}(a) \wedge^v o_{\mathbf{A}}(b) = o_{\mathbf{A}}(a \wedge_{\mathbf{A}} b)$;
- $\Rightarrow^v : vv \rightarrow v$ such that $o_{\mathbf{A}}(a) \Rightarrow^v o_{\mathbf{A}}(b) = o_{\mathbf{A}}(a \Rightarrow_{\mathbf{A}} b)$.

For simplicity, consider the abbreviations:

$$o(\mathbf{f}) = \bot \qquad o(\mathbf{t}) = \top \qquad \neg^v = - \qquad \wedge^v = \sqcap \qquad \vee^v = \sqcup \qquad \Rightarrow^v = \sqsupset.$$

Due to the careful choice of the subsignature Γ, and since $K^*_{S5}$ satisfies the above quasi-equations $(i)$-$(iv)$, we can obtain the following useful lemma. It relates behavioral satisfaction with equational satisfaction on every algebra $\mathbf{A} \in K^*_{S5}$.

**Lemma 5.3.2.** *Given* $\mathbf{A} \in K^*_{S5}$*, an equation* $\varphi \approx \psi$ *and an assignment* $h$ *over* $\mathbf{A}$ *we have that:*

$$\mathbf{A}, h \Vdash\!\vdash_\Gamma \varphi \approx \psi \qquad \textit{iff} \qquad \mathbf{A}, h \Vdash o(\varphi) \approx o(\psi).$$

*Proof.* We can conclude that $\mathbf{A}, h \Vdash\!\vdash_\Gamma \varphi \approx \psi$ implies $\mathbf{A}, h \Vdash o(\varphi) \approx o(\psi)$ since $\xi \in C^{\Gamma}_{\Sigma_{\mathcal{C}_1},\phi}[\xi]$. The other direction follows from an easy induction on the structure of contexts, recalling that $\mathbf{A}$ satisfies the quasi-equations $i$)- $iv$).

□

**Proposition 5.3.3.** *If* $\mathbf{A} \in K^*_{S5}$ *then* $\langle A_v, \sqcup_{\mathbf{A}}, \sqcap_{\mathbf{A}}, \top_{\mathbf{A}}, \bot_{\mathbf{A}}, -_{\mathbf{A}} \rangle$ *is a Boolean algebra.*

*Proof.* This result is a consequence of the above lemma and the fact that $S5$ satisfies the usual axioms for Boolean connectives.

□

## 5.4 First-order classical logic

In Example 2.3.25, we have discussed the problems with the single-sorted algebraization of $FOL$ as developed in [BP89]. With our many-sorted framework we can now handle first-order logic as a two-sorted logic, with a sort for terms and a sort for formulas. This perspective seems to be much more convenient, and we no longer need to consider restricted $FOL$ formulas. Our main contributions with this example is the presentation of a two-sorted version of **PR** and also of a two-sorted version of cylindric algebras.

Since we do not need to assume non-congruent operations, we can use the notation of the many-sorted approach as introduced in Example 5.1.2.

Consider given a first-order language $\langle \mathcal{C}, \mathcal{R}, \mathcal{F} \rangle$ with equality, where $\mathcal{C}$ is the set of constant symbols, $\mathcal{R}$ is the set of relation symbols and $\mathcal{F}$ is the set of function symbols. Consider also a set $V$ of individual variables. We assume, as it is usually assumed in the algebraization of FOL, that $\mathcal{F} = \emptyset$. This assumption is just to simplify the notation.

The two-sorted signature $\Sigma_{FOL} = \langle S, F \rangle$ obtained from the restricted first-order language is such that:

- $S = \{\phi, t\}$;

- $v : \to t$ for every $v \in V$;
- $= :\ t^2 \to \phi$;
- $R\ : t^n \to \phi$ for every $R \in \mathcal{R}$;
- $\top, \bot : \to \phi$;
- $\neg : \phi \to \phi$;
- $\wedge, \vee, \Rightarrow : \phi^2 \to \phi$;
- $\forall_v : \phi \to \phi$ for every $v \in V$;
- $\exists_v : \phi \to \phi$ for every $v \in V$.

The structural two-sorted deductive system **PR** over this two-sorted language consists of the following axioms, where $v_k, v_j$ range over elements of $V$:

A1. $\varphi$ where $\varphi$ is a classical tautology;

A2. $\forall_{v_k}(\xi_1 \Rightarrow \xi_2) \Rightarrow (\forall_{v_k}\xi_1 \Rightarrow \forall_{v_k}\xi_2)$;

A3. $(\forall_{v_k}\xi) \Rightarrow \xi$;

A4. $(\forall_{v_k}\forall_{v_j}\xi) \Rightarrow (\forall_{v_j}\forall_{v_k}\xi)$;

A5. $(\forall_{v_k}\xi) \Rightarrow (\forall_{v_k}\forall_{v_k}\xi)$;

A6. $(\exists_{v_k}\xi) \Rightarrow (\forall_{v_k}\exists_{v_k}\xi)$;

A7. $x = x$;

A8. $\exists_{v_k}(v_k = x)$;

A9. $(x_k = x_j) \Rightarrow ((x_k = x_i) \Rightarrow (x_j = x_i)$;

A10. $(v_k = v_j) \Rightarrow (\xi \Rightarrow \forall_{v_k}((v_k = v_j) \Rightarrow \xi)$, if $v_k \neq v_j$;

A11. $(\exists_{v_k}\xi) \Leftrightarrow (\neg\forall_{v_k}\neg\xi)$;

A12. $R(v_1,\ldots,v_n) \Rightarrow \forall_v R(v_1,\ldots,v_n), \quad$ if $v \notin \{v_1,\ldots,v_n\}$;

and the rules:

R1. $\dfrac{\xi_1 \quad \xi_1 \Rightarrow \xi_2}{\xi_2}$ (modus ponens);

R2. $\dfrac{\xi}{\forall_{v_k}\xi}$ (generalization).

Since $FOL$ is a two-sorted logic we can now study its many-sorted algebraization. The fact that $FOL$ is many-sorted algebraizable follows from the well-known fact [Sho67] that $FOL$ satisfies all the conditions of Theorem 5.1.7.

**Theorem 5.4.1.** *The two-sorted logic $FOL$ is algebraizable and its equivalent algebraic semantics is the variety of two-sorted cylindric algebras.*

By a two-sorted cylindric algebra we mean a $\Sigma_{FOL}$-algebra

$$\mathbf{A} = \langle A_\phi, A_t, \wedge_\mathbf{A}, \vee_\mathbf{A}, \Rightarrow_\mathbf{A}, \neg_\mathbf{A}, \top_\mathbf{A}, \bot_\mathbf{A}, R_\mathbf{A}, (\forall_v)_\mathbf{A}, (\exists_v)_\mathbf{A}, =_\mathbf{A}, v_\mathbf{A}\rangle_{v\in V,\ R\in\mathcal{R}}$$

such that, for every $v_k, v_j, v_i \in V$,

C0. $\langle A_\phi, \wedge_\mathbf{A}, \vee_\mathbf{A}, \Rightarrow_\mathbf{A}, \neg_\mathbf{A}, \top_\mathbf{A}, \bot_\mathbf{A}\rangle$ is a Boolean algebra;

and $\mathbf{A}$ also satisfies the following equations:

C1. $\exists_{v_k}\bot \approx \bot$;

C2. $\xi \wedge \exists_{v_k}\xi \approx \xi$;

C3. $\exists_{v_k}(\xi_1 \wedge \exists_{v_k}\xi_2) \approx (\exists_{v_k}\xi_1) \wedge (\exists_{v_k}\xi_2)$;

C4. $\exists_{v_k}\exists_{v_j}\xi \approx \exists_{v_j}\exists_{v_k}\xi$;

C5. $(x = x) \approx \top$;

C6. $(v_i = v_j) \approx \exists_{v_k}((v_i = v_k) \land (v_k = v_j))$, if $v_k \neq v_j$;

C7. $(\exists_{v_k}((v_i = v_k) \land \xi)) \land (\exists_{v_k}((v_i = v_k) \land \neg\xi)) \approx \bot$ if $v_k \neq v_j$;

C8. $\exists_{v_k} R(v_1, \ldots, v_n) \approx R(v_1, \ldots, v_n)$ if $v_k \notin \{v_1, \ldots, v_n\}$.

The class of two-sorted cylindric algebras is a more natural class of algebras to be associated with $FOL$. Its restriction to the sort $\phi$ is a plain old cylindric algebra and it corresponds to a regular first-order interpretation structure on the sort of terms.

The aim of this example, more than showing the details, is to stress the potentiality of our approach in the algebraic treatment of extensions of FOL, namely admitting more sorts and the existence of non-congruent operations.

## 5.5 Exogenous global propositional logic

The exogenous semantics approach to enriching a logic consists in defining each model in the enrichment as a set of models in the original logic plus some relevant structure [MSS05]. The first step in the enrichment process envisage in [MSS05] is *globalization*. The idea is to start from a logic $\mathcal{L}$, called the *local logic*, and then obtain an enriched logic, called the *global logic obtained from* $\mathcal{L}$ and denoted by $GL(\mathcal{L})$. In Example 2.1.33, a particular case of globalization, with CPL as local logic, was illustrated.

This example as two main aims. The first one is to introduce a mechanism for constructing a sound and complete deductive system for $GL(\mathcal{L})$ from a deductive system for $\mathcal{L}$. In [MSS05] a sound and complete deductive system is presented just for the particular case of $GL(CPL)$. However, the design of such deduction system does not allow a straightforward generalization to other local logics.

The second main aim of this example is to study the algebraization of $GL(\mathcal{L})$ based on the properties of $\mathcal{L}$.

Consider fixed a local logic $\mathcal{L}_\ell = \langle L_\ell, \vDash_\ell \rangle$ with a set $L_\ell$ of *local formulas* and such that the consequence relation, $\vDash_\ell$, is semantically obtained from a class $\mathcal{M}_\ell$ of *local models* and a satisfaction relation $\Vdash_\ell \subseteq \mathcal{M}_\ell \times L_\ell$ in the usual way: for every

$\Phi \cup \{\varphi\} \subseteq L_\ell$ we have:

$$\Phi \vDash_\ell \varphi \quad \text{iff} \quad \begin{array}{c}\text{for every } m \in \mathcal{M}_\ell \text{ we have that} \\ m \Vdash_\ell \varphi \quad \text{whenever} \quad m \Vdash_\ell \gamma \text{ for every } \gamma \in \Phi.\end{array}$$

From $\mathcal{L}_\ell$ we can obtain a two-sorted logic $GL(\mathcal{L}_\ell) = \langle \Sigma_g, \vDash_g \rangle$ where the two-sorted signature $\Sigma_g$ is such that:

- $\Sigma_g = \{S, F\}$;
- $S = \{\ell, \phi\}$;
- $F_{\epsilon l} = L_\ell$;
- $F_{l\phi} = \{\Box\}$;
- $F_{\phi\phi} = \{\boxminus\}$;
- $F_{\phi\phi\phi} = \{\sqsupset\}$.

The sort $\ell$ is the sort of local formulas and $\phi$ is the sort of global formulas. Note that the formulas of the local logic are, in this two-sorted signature, constants of sort $\ell$.

We consider the usual abbreviations for the remainder (global) boolean connectives.

Recall that the key idea of globalization is to take global models as sets of local models. We inductively define the satisfaction of a global formula $\delta$ by a global model $M \subseteq \mathcal{M}_\ell$, denoted by $M \Vdash_g \delta$, as follows:

- $M \Vdash_g \Box\varphi$ iff for every $m \in M$, $m \Vdash_\ell \varphi$;
- $M \Vdash_g \boxminus\delta$ iff $M \nVdash_g \delta$;
- $M \Vdash_g \delta_1 \sqsupset \delta_2$ iff $M \nVdash_g \delta_1$ or $M \Vdash_g \delta_2$.

The global consequence relation, $\vDash_g$, is semantically obtained from the class $2^{\mathcal{M}_\ell}$ of *global models* and from the satisfaction relation $\Vdash_g \subseteq 2^{\mathcal{M}_\ell} \times L_\ell$ in the usual way: for every $\Psi \cup \{\delta\} \subseteq L_{\Sigma_g}(X)$ we have:

$$\Psi \vDash_\ell \delta \quad \text{iff} \quad \text{for every } M \in 2^{\mathcal{M}_\ell} \text{ we have that } M \Vdash_g \delta \text{ whenever } M \Vdash_g \gamma \text{ for every } \gamma \in \Psi.$$

An immediate and important consequence of the construction of $GL(\mathcal{L}_\ell)$ is that it is a conservative extension of $\mathcal{L}_\ell$.

**Theorem 5.5.1.** *Given $\Phi \cup \{\varphi\} \subseteq L_\ell$ we have that:*

$$\Phi \vDash_\ell \varphi \quad \textit{iff} \quad \{\Box\gamma : \gamma \in \Phi\} \vDash_g \Box\varphi.$$

*Proof.* Suppose first that $\Phi \vDash_\ell \varphi$. Let $M \subseteq 2^{\mathcal{M}_\ell}$ be a global model such that $M \Vdash_g \Box\gamma$ for every $\gamma \in \Phi$. Our aim is to prove that $M \Vdash_g \Box\varphi$. Since $M \Vdash_g \Box\gamma$ for every $\gamma \in \Phi$, we have that $m \Vdash_\ell \gamma$ for every $\gamma \in \Phi$ and for every $m \in M$. Therefore, using the fact that $\Phi \vDash_\ell \varphi$ we have that, for every $m \in M$, we have that $m \Vdash_\ell \varphi$. Therefore, we can conclude that $M \Vdash_g \Box\varphi$.

Suppose now that $\{\Box\gamma : \gamma \in \Phi\} \vDash_g \Box\varphi$. Let $m \in \mathcal{M}_\ell$ be a local model such that $m \Vdash_\ell \gamma$ for every $\gamma \in \Phi$. Our aim is to prove that $m \Vdash_\ell \varphi$. Since $\{m\} \Vdash_g \Box\gamma$ for every $\gamma \in \Phi$ and since $\{\Box\gamma : \gamma \in \Phi\} \vDash_g \Box\varphi$, we have that $\{m\} \Vdash_g \Box\varphi$. Therefore, we can conclude that $m \Vdash_\ell \varphi$.

□

The above construction of $GL(\mathcal{L}_\ell)$ from $\mathcal{L}_\ell$ was a semantical construction. We built the global models from the models of the local logic. Our aim now is to engage in a more syntactical construction. Let us assume that $\mathcal{L}_\ell$ can also be introduced as a deductive system $\mathcal{D}_\ell$ with a set $A_\ell$ of *(local) axioms* and a set $R_\ell$ of *(local) rules*. That is, $\mathcal{L}_\ell$ is sound and complete with respect to the logic obtained from the deductive system $\mathcal{D}_\ell$. Our aim is to construct a deductive system for $GL(\mathcal{L}_\ell)$ from the deductive system $\mathcal{D}_\ell$, and to study its soundness and completeness.

Using the deductive system $\mathcal{D}_\ell$ we can define a two-sorted deductive system $\mathcal{D}_g$ over $\Sigma_g$ that is constituted by the following axioms:

$I_\varphi$ $\quad \Box\varphi$ for every instance of an axiom $\varphi \in A_\ell$;

$I_r$ $\quad (\Box\psi_1 \sqsupset (\Box\psi_2 \sqsupset (\ldots(\Box\psi_n \sqsupset \Box\psi))\ldots))$ for every $r = \langle\{\psi_1, \ldots, \psi_n\}, \psi\rangle \in R_\ell$;

$C_1$ $\xi_1 \sqsupset (\xi_2 \sqsupset \xi_1)$;

$C_2$ $(\xi_1 \sqsupset (\xi_2 \sqsupset \xi_3)) \sqsupset ((\xi_1 \sqsupset \xi_2) \sqsupset (\xi_1 \sqsupset \xi_3))$;

$C_3$ $((\boxminus \xi_1) \sqsupset (\boxminus \xi_2)) \sqsupset (\xi_2 \sqsupset \xi_1)$;

and rule:

$$gMP \qquad \frac{\xi_1 \quad \xi_1 \sqsupset \xi_2}{\xi_2}.$$

The consequence relation obtained from the deductive system $\mathcal{D}_g$ over $\Sigma_g$ is denoted by $\vdash_g$. Note that for every classical tautological formula $\delta$ written with $\boxminus$ and $\sqsupset$, we have that $\vdash_g \delta$, since we have in the global deductive system $\mathcal{D}_g$ the three axioms of classical propositional logic plus the Modus Ponens rule.

The following theorem is a consequence of the construction of $\mathcal{D}_g$ from $\mathcal{D}_\ell$.

**Theorem 5.5.2.** *Given $\Phi \cup \{\varphi\} \subseteq L_\ell$ we have that:*

$$\textit{if} \quad \Phi \vdash_\ell \varphi \quad \textit{then} \quad \{\Box\gamma : \gamma \in \Phi\} \vdash_g \Box\varphi.$$

*Proof.* We prove this result by induction on the length of a derivation of $\varphi$ from $\Phi$ in $\mathcal{L}_\ell$.

Base:

- $\varphi \in \Phi$.

  Then $\Box\varphi \in \{\Box\gamma : \gamma \in \Phi\}$ and since $\vdash_g$ is a consequence relation, we can use Reflexivity to conclude that $\{\Box\gamma : \gamma \in \Phi\} \vdash_g \Box\varphi$.

- $\varphi$ is an axiom of $\mathcal{L}_\ell$.

  Then, using axiom $I_\varphi$, we have that $\Box\varphi$ is a global axiom, and so $\vdash_g \Box\varphi$. Using Weakening we get that $\{\Box\gamma : \gamma \in \Phi\} \vdash_g \Box\varphi$.

Induction Step:

- $\varphi$ is obtained from $\varphi_1, \ldots, \varphi_n$ using the rule instance $r = \langle\{\varphi_1, \ldots, \varphi_n\}, \varphi\rangle$.

  Therefore, $\Phi \vdash_l \varphi_i$ for every $i \in \{1, \ldots, n\}$. By induction hypothesis, $\{\Box\gamma : \gamma \in \Phi\} \vdash_g \Box\varphi_i$, for every $i \in \{1, \ldots, n\}$. Since $r$ is a rule instance of $\mathcal{L}_\ell$, we have that $(\Box\varphi_1 \sqsupset (\Box\varphi_2 \sqsupset \ldots (\Box\varphi_n \sqsupset \Box\varphi) \ldots))$ is an axiom instance of $GL(\mathcal{L}_\ell)$ and using the rule $gMP$ we can conclude that $\{\Box\gamma : \gamma \in \Phi\} \vdash_g \Box\varphi$.

□

The following theorem states that, independently of the chosen local logic $\mathcal{L}_\ell$, the Deduction theorem always holds for $GL(\mathcal{L}_\ell)$.

**Theorem 5.5.3.** *Given $\Phi \cup \{\delta_1, \delta_2\} \subseteq L_{\Sigma_g}(X)$ we have that*

$$\Phi, \delta_1 \vdash_g \delta_2 \quad \textit{iff} \quad \Phi \vdash_g \delta_1 \sqsupset \delta_2.$$

*Proof.* Recall that a sufficient condition for a logic to have the Deduction theorem is to have among its axioms the classical axioms $C_1$ and $C_2$, and have $MP$ as its only rule. This is clearly the case with $GL(\mathcal{L}_\ell)$ and so the Deduction theorem holds in $GL(\mathcal{L}_\ell)$.

□

In order to prove soundness and completeness theorems for global logic, recall that we are assuming that $\vdash_\ell$ is sound and complete with respect to $\vDash_\ell$.

We start by proving the easiest part: the soundness of $\vdash_g$ with respect to $\vDash_g$. The soundness result is an immediate consequence of the following two lemmas. The first one states that the axioms of $GL(\mathcal{L}_\ell)$ are all sound, that is, are all valid formulas.

**Lemma 5.5.4.** *The axioms of the global deductive system $\mathcal{D}_g$ are all valid formulas.*

*Proof.* Since the soundness of the global classical axioms $C_1$-$C_3$ is well-known, we restrict our attention to the local axioms and the interaction axioms.

- $\delta$ is an axiom of type $I_\varphi$, for some $\varphi \in L_\ell$ :

  In this case, $\delta$ is $\Box\varphi$. Let $M \subseteq \mathcal{M}_\ell$. Our aim is to prove that $M \Vdash_g \delta$. Recall that $M \Vdash_g \delta$ iff $m \Vdash_l \varphi$ for every $m \in M$. Since we are assuming that $\vdash_\ell$ is sound, then $\varphi$ is a valid formula of $\mathcal{L}_\ell$, that is, for every $m \in \mathcal{M}_\ell$ we have that $m \Vdash_l \varphi$. In particular, for every $m \in M$, we have that $m \Vdash_l \varphi$. Therefore, we can conclude that $M \Vdash_g \delta$.

- $\delta$ is an axiom of type $I_r$:

  Suppose that $\delta$ is $\Box\varphi_1 \sqsupset (\Box\varphi_2 \sqsupset (\ldots \sqsupset (\Box\varphi_n \sqsupset \Box\varphi)\ldots))$, where $\langle\{\varphi_1,\ldots,\varphi_n\},\varphi\rangle$ is a rule of $\mathcal{L}_\ell$. Let $M \subseteq \mathcal{M}_\ell$ be a global model. Our aim is to prove that $M \Vdash_g \delta$. Recall that $M \Vdash_g \Box\varphi_1 \sqsupset (\Box\varphi_2 \sqsupset (\ldots \sqsupset (\Box\varphi_n \sqsupset \Box\varphi)\ldots))$ iff $M \Vdash_g \Box\varphi$ whenever $M \Vdash_g \Box\varphi_i$ for every $i \in \{1,\ldots,n\}$.

  Assume that $M \Vdash_g \Box\varphi_i$ for every $i \in \{1,\ldots,n\}$. Then we have that $m \Vdash_\ell \varphi_i$ for every $i \in \{1,\ldots,n\}$ and for every $m \in M$.

  Since we are assuming that the rules of inference of $\mathcal{L}_\ell$ are sound, we have that $m \Vdash_l \varphi$ for every $m \in M$ whenever $m \Vdash_l \varphi_i$ for every $i \in \{1,\ldots,n\}$. Therefore we can conclude that $m \Vdash_\ell \varphi$, for every $m \in M$. This is equivalent to $M \Vdash_g \Box\varphi$.

□

The following lemma states the soundness of $gMP$, the global Modus Ponens rule.

**Lemma 5.5.5.** *For every* $\{\delta_1,\delta_2\} \subseteq L_{\Sigma_g}(X)$ *we have that*

$$\{\delta_1,(\delta_1 \sqsupset \delta_2)\} \vDash_g \delta_2.$$

*Proof.* Let $M \subseteq \mathcal{M}_\ell$ be a global model such that $M \Vdash_g \delta_1$ and $M \Vdash_g (\delta_1 \sqsupset \delta_2)$. Recall that $M \Vdash_g (\delta_1 \sqsupset \delta_2)$ iff $M \nVdash_g \delta_1$ or $M \Vdash_g \delta_2$. Since we are assuming that $M \Vdash_g \delta_1$ we can conclude that $M \Vdash_g \delta_2$.

□

The above lemma indicates that global Modus Ponens is sound, in the sense that the conclusion is a semantic consequence of the set of premises.

The following soundness theorem is a straightforward consequence of the soundness of the axioms and of the deduction rule.

**Theorem 5.5.6.** *Global logic* $GL(\mathcal{L}_\ell)$ *is sound, that is, for every* $\Phi \cup \{\delta\} \subseteq L_{\Sigma_g}(X)$ *we have that*

$$\textit{if} \quad \Phi \vdash_g \delta \quad \textit{then} \quad \Phi \vDash_g \delta.$$

*Proof.* The result follows immediately from lemmas 5.5.4 and 5.5.5 by an easy induction on the length of a derivation of $\delta$ from $\Phi$.

□

Our next goal is to prove a completeness theorem. We first need to establish some auxiliary results. As in the case of CPL, the key ingredient of the completeness result are the consistent sets and, in particular, maximal consistent sets.

**Definition 5.5.7.** A set $\Phi \subseteq L_{\Sigma_g}(X)$ is *(globally) consistent* if there exists a global formula $\delta \in L_{\Sigma_g}(X)$, such that $\Phi \nvdash_g \delta$. Otherwise $\Phi$ is called *inconsistent*.

The following lemma shows how to extend a consistent set in order to preserve consistency.

**Lemma 5.5.8.** *Let $\Phi \cup \{\delta\} \subseteq L_{\Sigma_g}(X)$. If $\Phi$ is consistent and $\Phi \nvdash_g \delta$, then $\Phi \cup \{\boxminus\delta\}$ is consistent.*

*Proof.* Suppose $\Phi \cup \{\boxminus\delta\}$ is inconsistent. Then, in particular, $\Phi \cup \{\boxminus\delta\} \vdash_g \delta$. Since $((\boxminus\delta) \sqsupset \delta) \sqsupset \delta$ is a theorem of $GL$, then $\Phi \vdash_g ((\boxminus\delta) \sqsupset \delta) \sqsupset \delta$. Using the Deduction theorem and $gMP$ we get that $\Phi \vdash_g \delta$, which contradicts the hypothesis.

□

**Definition 5.5.9.** A consistent set $\Phi \subseteq L_{\Sigma_g}(X)$ is *maximal* if it is not strictly contained in some consistent set, that is, there is no consistent set $\Phi'$, such that $\Phi \subsetneq \Phi'$.

The following lemma states an important property of maximal consistent sets.

**Lemma 5.5.10.** *Let $\Phi \subseteq L_{\Sigma_g}(X)$ be a maximal consistent set. Then, for every $\delta \in L_{\Sigma_g}(X)$, we have that $\delta \in \Phi$ or $\boxminus\delta \in \Phi$.*

*Proof.* Suppose that $\delta \notin \Phi$ and $\boxminus\delta \notin \Phi$. Since $\Phi$ is consistent, $\Phi \nvdash \delta$ or $\Phi \nvdash \boxminus\delta$. Suppose, without loss of generality, that $\Phi \nvdash \delta$. Using Lemma 5.5.8 we can conclude that $\Phi \cup \{\boxminus\delta\}$ is consistent. But this contradicts the maximality of $\Phi$, since $\boxminus\delta \notin \Phi$.
□

**Lemma 5.5.11.** *Let $\Phi \subseteq L_{\Sigma_g}(X)$ be a maximal consistent set. For every $\delta \in L_{\Sigma_g}(X)$ we have that $\delta \in \Phi$ whenever $\Phi \vdash_g \delta$.*

*Proof.* Suppose $\Phi \vdash_g \delta$ and $\delta \notin \Phi$. Then, using Lemma 5.5.10, we have that $\boxminus\delta \in \Phi$. Therefore, $\Phi \vdash_g \boxminus\delta$. Since $\delta \sqsupset ((\boxminus\delta) \sqsupset \gamma)$ is a theorem of $GL$ for every formula $\gamma \in L_{\Sigma_g}(X)$, we can conclude that $\Phi \vdash_g \delta \sqsupset ((\boxminus\delta) \sqsupset \gamma)$. Using $gMP$ we have that $\Phi \vdash_g \gamma$ for every $\gamma \in L_{\Sigma_g}(X)$. But this contradicts the consistency of $\Phi$.

□

The following proposition generalizes to $GL(\mathcal{L}_\ell)$ the so-called Lindenbaum's lemma.

**Proposition 5.5.12.** *Let* $\Phi \subseteq L_{\Sigma_g}(X)$ *be a consistent set. Then there exists a maximal consistent set* $\Phi^*$ *such that* $\Phi \subseteq \Phi^*$.

*Proof.* Consider $\delta_0, \delta_1, \ldots, \delta_n, \ldots$ an enumeration of the formulas of $L_{\Sigma_g}(X)$. Consider the sequence of sets of global formulas $\{\Phi_i\}_{i\in\mathbb{N}}$ such that:

- $\Phi_0 = \Phi$;
- $\Phi_{n+1} = \begin{cases} \Phi_n \cup \{\boxminus\delta_n\}, & \text{if } \Phi_n \nvdash_g \delta_n \\ \Phi_n, & \text{otherwise} \end{cases}$

Take $\Phi^* = (\bigcup_{n\in\mathbb{N}} \Phi_n)^{\vdash_g}$.

We now prove the following:

- For every $i \in \mathbb{N}$, $\Phi_i$ is consistent:

  Let us prove this by induction on $i \in \mathbb{N}$.

  Base.
  Suppose $i = 0$. Since $\Phi_0 = \Phi$, we have that $\Phi_0$ is consistent.

  Induction Step:
  If $\Phi_n \nvdash_g \delta_n$ then $\Phi_{n+1} = \Phi_n \cup \{\boxminus\delta_n\}$ is consistent by Lemma 5.5.8. Otherwise $\Phi_{n+1} = \Phi_n$ and therefore, by induction hypothesis, $\Phi_{n+1}$ is consistent.

- $\Phi^*$ is consistent:
  Suppose $\Phi^*$ is not consistent. Then $\Phi^* \vdash_g \delta$ for every $\delta \in L_{\Sigma_g}(X)$. In particular, given $\delta_0 \in L_{\Sigma_g}(X)$, we have that $\Phi^* \vdash_g \delta_0$ and $\Phi^* \vdash_g \boxminus\delta_0$. Clearly, $\bigcup_{n\in\mathbb{N}} \Phi_n \vdash_g \delta_0$ and $\bigcup_{n\in\mathbb{N}} \Phi_n \vdash_g \boxminus\delta_0$. Since $\vdash_g$ is finitary, there exists $n \in \mathbb{N}$ such that $\Phi_n \vdash_g \delta_0$ and $\Phi_n \vdash_g \boxminus\delta_0$. But this contradicts the consistency of $\Phi_n$.

- $\Phi^*$ is maximal:
  Suppose $\Phi^*$ is not maximal. Then there exists a consistent set $\Phi'$ such that $\Phi^* \subsetneq \Phi'$. Therefore, there exists a formula $\delta$ such that $\delta \in \Phi'$ and $\delta \notin$

$\Phi^*$. We know that $\delta$ is $\delta_n$ for some $n \in \mathbb{N}$. Therefore, either $\Phi_n \vdash_g \delta_n$ or $\Phi_n \nvdash_g \delta$. Suppose first that $\Phi_n \vdash_g \delta$. Then $\bigcup_{n\in\mathbb{N}} \Phi_n \vdash_g \delta$, and therefore $\delta \in \Phi^*$, contradicting the hypothesis. Suppose now that $\Phi_n \nvdash_g \delta$. Then $\Phi_{n+1} = \Phi_n \cup \{\boxminus\delta\}$. Since $\Phi_{n+1} \subseteq \Phi^* \subseteq \Phi'$, we have that $\Phi' \vdash_g \boxminus\delta$, which contradicts the consistency of $\Phi'$.

□

Given a set of formulas $\Phi \subseteq L_{\Sigma_g}(X)$, we can consider the set $\Phi_\ell$ of all local formulas in $\Phi$ as

$$\Phi_\ell = \{\varphi \in L_\ell : \Box\varphi \in \Phi\}.$$

Given a maximal consistent set $\Phi \subseteq L_{\Sigma_g}(X)$ we can consider the following global model

$$M_\Phi = \{m \in \mathcal{M}_\ell : m \Vdash \Phi_l\}.$$

**Lemma 5.5.13.** *Let $\Phi \subseteq L_{\Sigma_g}(X)$ be a maximal consistent set. Then, for every $\varphi \in L_\ell$,*

$$M_\Phi \Vdash_g \Box\varphi \qquad \textit{iff} \qquad \Box\varphi \in \Phi$$

*Proof.* First suppose that $M_\Phi \Vdash_g \Box\varphi$. Then $m \Vdash_l \varphi$, for every $m \in M_\Phi$ and by definition of $M_\Phi$, this implies that $\Phi_l \vDash_l \varphi$. Using the completeness theorem of the local logic, we get that $\Phi_l \vdash_l \varphi$. Then, using Weakening and Theorem 5.5.2, we have that $\Phi \vdash_g \Box\varphi$. Since $\Phi$ is maximal we can use Lemma 5.5.11 to conclude that $\varphi \in \Phi$.

On the other direction, suppose that $\Box\varphi \in \Phi$. By definition of $\Phi_\ell$ we have that $\varphi \in \Phi_\ell$. Therefore, by definition of $M_\Phi$, we can conclude that $M_\Phi \Vdash_g \Box\varphi$.

□

Note that the above lemma just applies to local formulas. We can extend it to every global formula by induction on the structure of a global formula.

**Proposition 5.5.14.** *Let $\Phi \subseteq L_{\Sigma_g}(X)$ be a maximal consistent set. Then, for every $\delta \in L_{\Sigma_g}(X)$,*

$$M_\Phi \Vdash_g \delta \qquad \textit{iff} \qquad \delta \in \Phi$$

*Proof.* Let us use induction on the structure of the formula $\delta$.

Base: $\delta$ is $\Box\varphi$ for some local formula $\varphi$.

This case is an immediate consequence of the Lemma 5.5.13.

Induction Step:

- $\delta$ is $\boxminus\delta_1$.

  Then $M_\Phi \Vdash_g \boxminus\delta_1$ iff $M_\Phi \nVdash_g \delta_1$ iff (by induction hypothesis) $\delta_1 \notin \Phi$ iff (by Lemma 5.5.10) $\boxminus\delta_1 \in \Phi$.

- $\delta$ is $(\delta_1 \sqsupset \delta_2)$.

  Suppose first that $M_\Phi \Vdash_g (\delta_1 \sqsupset \delta_2)$. Then we have two cases to consider.

  1. $M_\Phi \nVdash_g \delta_1$:

     Then, by induction hypothesis, $\delta_1 \notin \Phi$. Therefore, by Lemma 5.5.10, we have that $\boxminus\delta_1 \in \Phi$. Using axiom $C_1$, we have that $(\boxminus\delta_1) \sqsupset ((\boxminus\delta_2) \sqsupset (\boxminus\delta_1))$ and using gMP, we get that $(\boxminus\delta_2) \sqsupset (\boxminus\delta_1) \in \Phi$. Now we can use axiom $C_3$ to conclude that $((\boxminus\delta_2) \sqsupset (\boxminus\delta_1)) \sqsupset (\delta_1 \sqsupset \delta_2)$ and again by gMP, we get that $(\delta_1 \sqsupset \delta_2) \in \Phi$.

  2. $M_\Phi \Vdash_g \delta_2$:

     Then, by induction hypothesis, $\delta_2 \in \Phi$. Using axiom $C_1$ and using gMP, we can conclude that $(\delta_1 \sqsupset \delta_2) \in \Phi$.

  We prove the other direction by contraposition. Suppose that $M_\Phi \nVdash_g (\delta_1 \sqsupset \delta_2)$. Then $M_\Phi \Vdash_g \delta_1$ and $M_\Phi \nVdash_g \delta_2$. By induction hypothesis we have that $\delta_1 \in \Phi$ and $\delta_2 \notin \Phi$. Then by Lemma 5.5.10, $\boxminus\delta_2 \in \Phi$. Since $\delta_1 \sqsupset ((\boxminus\delta_2) \sqsupset \boxminus(\delta_1 \sqsupset \delta_2))$ is a theorem of $GL(\mathcal{L}_\ell)$ and using $gMP$, we get that $\boxminus(\delta_1 \sqsupset \delta_2) \in \Phi$. Since $\Phi$ is consistent, we conclude that $(\delta_1 \sqsupset \delta_2) \notin \Phi$.

□

We can now prove the result of strong completeness. Recall once more that we are assuming that the deductive system of the local logic is sound and complete with respect to the class $\mathcal{M}_\ell$ of models.

**Theorem 5.5.15.** *Global logic $GL(\mathcal{L}_\ell)$ is strongly complete, that is, for every $\Phi \cup \{\delta\} \subseteq L_{\Sigma_g}(X)$ we have that*

$$\Phi \vdash_g \delta \quad \textit{whenever} \quad \Phi \vDash_g \delta$$

*Proof.* We prove the result by contraposition. Suppose that $\Phi \nvdash_g \delta$. Then $\Phi$ is consistent, and using Lemma 5.5.8 we know that $\Phi \cup \{\boxminus\delta\}$ is also consistent. Therefore, by Proposition 5.5.12, there exists a maximal consistent set $\Phi^*$, such that $\Phi \cup \{\boxminus\delta\} \subseteq \Phi^*$. Using Proposition 5.5.14 we get that $M_{\Phi^*} \Vdash_g \Phi^*$, and since $\Phi \cup \{\boxminus\delta\} \subseteq \Phi^*$, we have that $M_{\Phi^*} \Vdash_g \Phi$ and $M_{\Phi^*} \Vdash_g (\boxminus\delta)$. Therefore, we have found a global model $M_{\Phi^*}$ such that, $M_{\Phi^*} \Vdash_g \Phi$ and $M_{\Phi^*} \nVdash_g \delta$. We can then conclude that $\Phi \nvDash_g \delta$.

□

In the remainder of this example we study the algebraization of $GL(\mathcal{L}_\ell)$. Since we are not assuming any non-congruent operation, we can use the particular case of the (non-behavioral) many-sorted algebraization, as illustrated in Example 5.1.2.

We begin by proving that $GL(\mathcal{L}_\ell)$ is algebraizable independently of the algebraizability of $\mathcal{L}_\ell$.

**Theorem 5.5.16.** *Global logic $GL(\mathcal{L}_\ell)$ is algebraizable.*

*Proof.* Recall that Theorem 5.1.7 establishes a sufficient condition for a logic to be algebraizable.

In this proof we use $\Delta = \{(\xi_1 \sqsupset \xi_2),\ (\xi_2 \sqsupset \xi_1)\}$ and just observe that the following conditions are all verified in $GL(\mathcal{L}_\ell)$ due to its classical flavor:

*(i)* $\vdash_g \delta_1 \Delta \delta_1$;

*(ii)* $\delta_1 \Delta \delta_2 \vdash_g \delta_2 \Delta \delta_1$;

*(iii)* $\delta_1 \Delta \delta_2, \delta_2 \Delta \delta_3 \vdash_g \delta_1 \Delta \delta_3$;

*(iv)* $\delta_1 \Delta \delta_2 \vdash_g (\boxminus\delta_1)\Delta(\boxminus\delta_2)$;

*(v)* $\delta_1 \Delta \delta_2, \delta_3 \Delta \delta_4 \vdash_g (\delta_1 \sqsupset \delta_3)\Delta(\delta_2 \sqsupset \delta_4)$;

*(vi)* $\delta_1, \delta_1 \Delta \delta_2 \vdash_g \delta_2$;

*(vii)* $\delta_1, \delta_2 \vdash_g \delta_1 \Delta \delta_2$.

Recall that, in this case, the set of defining equations is

$$\Theta(\xi) = \{\xi \approx \delta(\xi, \xi) : \delta(\xi_1, \xi_2) \in \Delta(\xi_1, \xi_2)\}.$$

□

Now that we have proved that $GL(\mathcal{L}_\ell)$ is algebraizable, we can study its algebraic counterpart. In this study we use Theorem 5.1.9. Recall that this theorem extracts an axiomatization of the equivalent quasivariety semantics of $GL(\mathcal{L}_\ell)$ from the deductive system of $GL(\mathcal{L}_\ell)$. The resulting quasivariety, that we will denote by $\mathcal{B}_g$, is constituted by the two-sorted algebras $\mathbf{B}$ such that $\mathbf{B}_\phi = \langle B_\phi, \sqcup^{\mathbf{B}}, \sqcap^{\mathbf{B}}, \boxminus^{\mathbf{B}}, \top^{\mathbf{B}}, \bot^{\mathbf{B}} \rangle$ is a Boolean algebra and also satisfies the following equations:

- $\Box\varphi \approx \top$ for every theorem $\varphi$ of $\mathcal{L}_\ell$;
- $((\sqcap_{1\leq i\leq n}\Box\psi_i) \sqsupset \Box\psi) \approx \top$ for every instance $\langle\{\psi_1, \ldots, \psi_n\}, \psi\rangle$ of a rule of $\mathcal{L}_\ell$.

Note that the above conditions do not say much about the $\ell$-reduct of an algebra in $\mathcal{B}_g$. To study it in more detail let us suppose that $\mathcal{L}_\ell$ is a single-sorted logic over a single sorted signature $\Sigma_\ell$. Moreover, suppose that $\mathcal{L}_\ell$ is finitely algebraizable with $\mathcal{Q}_{\mathcal{L}_\ell}$ the equivalent quasivariety, $\Theta_\ell$ the set of defining equations and $\Delta_\ell$ the set of equivalence formulas. An important question that now arises is the relationship between the two-sorted algebraic counterpart of $GL(\mathcal{L}_\ell)$ and the algebraic counterpart of $\mathcal{L}_\ell$. In Fig. 5.1 we have a view of the structure of an algebra in $\mathcal{B}_g$.

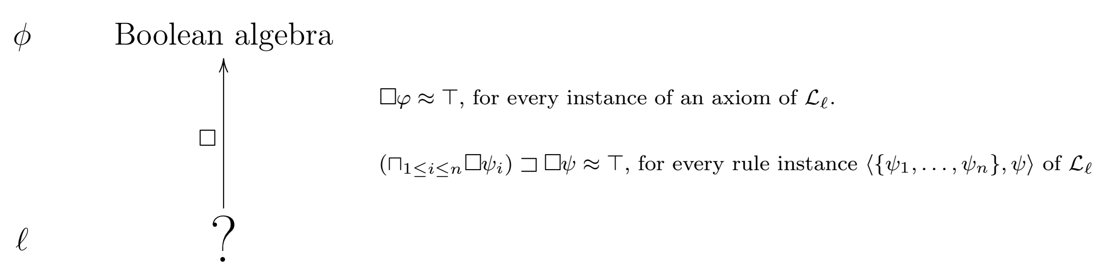



Figure 5.1: Algebraic counterpart of $GL(\mathcal{L}_\ell)$

The following lemma states an immediate relationship between the equivalence set $\Delta_\ell$ of $\mathcal{L}_\ell$ and the global equivalence set $\Delta = \{(\xi_1 \sqsupset \xi_2),\ (\xi_2 \sqsupset \xi_1)\}$.

**Lemma 5.5.17.** *Let $\Delta_\ell$ be the equivalence set of $\mathcal{L}_\ell$ and recall that*

$\Delta = \{(\xi_1 \sqsupset \xi_2),\ (\xi_2 \sqsupset \xi_1)\}$ *is the global equivalence set. Then, for every* $\varphi_1, \varphi_2 \in L_\ell$ *we have that*

$$\Box\Delta_\ell(\varphi_1, \varphi_2) \vdash_g \Delta(\Box\varphi_1, \Box\varphi_2).$$

*Proof.* Note that, since $\Delta_\ell$ is an equivalence set, the conditions $\Delta_\ell(\varphi_1, \varphi_2), \varphi_1 \vdash \varphi_2$ and $\Delta_\ell(\varphi_1, \varphi_2), \varphi_2 \vdash \varphi_1$ hold. So, using Theorem 5.5.1 we can conclude that $\Box\Delta_\ell(\varphi_1, \varphi_2), \Box\varphi_1 \vdash \Box\varphi_2$ and $\Box\Delta_\ell(\varphi_1, \varphi_2), \Box\varphi_2 \vdash \Box\varphi_1$. We can now use the Deduction Theorem 5.5.3 to conclude that $\Box\Delta_\ell(\varphi_1, \varphi_2) \vdash (\Box\varphi_1 \sqsupset \Box\varphi_2)$ and also that $\Box\Delta_\ell(\varphi_1, \varphi_2) \vdash (\Box\varphi_2 \sqsupset \Box\varphi_1)$.

□

Let $C_\ell$ be a basis of quasi-equations that define $\mathcal{Q}_{\mathcal{L}_\ell}$. We consider $\Gamma = \langle\{\ell, \phi\}, F\rangle$ a subsignature of $\Sigma_g$ such that $F_{\ell\phi} = \{\Box\}$ and $F_{s_1 \dots s_n s} = \emptyset$ otherwise. The next theorem states that, for every two-sorted algebra $\mathbf{B}$, although it is not always the case that $\mathbf{B}$ satisfies every quasi-equation in $C_\ell$, $\mathbf{B}$ always behaviorally satisfies every quasi-equation of $C$.

**Theorem 5.5.18.** *Let* $\boldsymbol{B} \in \mathcal{B}_g$*. Then we have that* $\boldsymbol{B} \Vvdash_\Gamma c$ *for every* $c \in C_\ell$*.*

*Proof.* Let $((\psi_1 \approx \varphi_1)\& \dots \&(\psi_n \approx \varphi_n)) \to (\psi \approx \varphi) \in C_\ell$. Then we have that $\{(\psi_1 \approx \varphi_1), \dots, (\psi_n \approx \varphi_n)\} \vDash_{\Sigma_\ell}^{\mathcal{Q}_{\mathcal{L}_\ell}} \psi \approx \varphi$. Using the fact that $\mathcal{L}_\ell$ is algebraizable and that $\Delta_\ell$ is a equivalence set of formulas for $\mathcal{L}_\ell$ we can conclude that $\{\Delta_\ell(\psi_1, \varphi_1), \dots, \Delta_\ell(\psi_n, \varphi_n)\} \vdash_\ell \Delta_\ell(\psi, \varphi)$. So, using Theorem 5.5.1 we can conclude that $\{\Box\Delta_\ell(\psi_1, \varphi_1), \dots, \Box\Delta_\ell(\psi_n, \varphi_n)\} \vdash_g \Box\Delta_\ell(\psi, \varphi)$. Using now Lemma 5.5.17 we have that $\{\Delta(\Box\psi_1, \Box\varphi_1), \dots, \Delta(\Box\psi_n, \Box\varphi_n)\} \vdash_g \Delta(\Box\psi, \Box\varphi)$. As a consequence, we have that $\{(\Box\psi_1 \approx \Box\varphi_1), \dots, (\Box\psi_n \approx \Box\varphi_n)\} \vDash_{\Sigma_g}^{\mathcal{B}_g} \Box\psi \approx \Box\varphi$. Recall that, by definition of the subsignature $\Gamma$, the only experiment is $\Box\xi$. Therefore, for every algebra $\mathbf{B} \in \mathcal{B}_g$, it immediately follows that $\mathbf{B} \Vvdash_\Gamma ((\psi_1 \approx \varphi_1)\& \dots \&(\psi_n \approx \varphi_n)) \to (\psi \approx \varphi)$.

□

Roughly speaking, the above theorem states that the $\ell$-reduct of every $\mathbf{B} \in \mathcal{B}_g$ is behaviorally equivalent to an algebra in $\mathcal{Q}_{\mathcal{L}_\ell}$, the quasivariety equivalent to $\mathcal{L}_\ell$.

For each $\mathbf{A} \in \mathcal{Q}_{\mathcal{L}_\ell}$ consider the set

$$D_{\mathbf{A}} = \{a \in A : \delta_{\mathbf{A}}(a) = \epsilon_{\mathbf{A}}(a) \text{ for every } \delta \approx \epsilon \in \Theta_\ell\}.$$

Intuitively, this set can be seen as the set of elements that can be considered designated elements in the algebra $\mathbf{A}$.

The following theorem describes how to canonically build a two-sorted algebra $B_{\mathbf{A}}$ in $\mathcal{B}_g$ whose $\ell$-reduct is precisely $\mathbf{A}$, given a single-sorted algebra $\mathbf{A}$ in $\mathcal{Q}_{\mathcal{L}_\ell}$.

**Theorem 5.5.19.** *For every* $\mathbf{A} \in \mathcal{Q}_{\mathcal{L}_\ell}$ *consider the* $\Sigma_g$*-algebra* $B_{\mathbf{A}}$ *such that:*

- $B_{\mathbf{A},\phi} = \mathbf{2}$;
- $B_{\mathbf{A},\ell} = \mathbf{A}$;
- $\Box_{B_{\mathbf{A}}}$ *is such that* $\Box_{B_{\mathbf{A}}}(a) = \top$ *iff* $a \in D_{\mathbf{A}}$.

*Then, we have that* $B_{\mathbf{A}} \in \mathcal{B}_g$.

*Proof.* Recall that a two-sorted algebra $\mathbf{B}$ belongs to $\mathcal{B}_g$ if and only if its $\phi$-reduct is a Boolean algebra and it also satisfies the equations $\Box\varphi \approx \top$ for every theorem $\varphi$ of $\mathcal{L}_\ell$ and $((\sqcap_{1\leq i\leq n}\Box\psi_i) \sqsupset \Box\psi) \approx \top$ for every instance $\langle\{\psi_1, \ldots, \psi_n\}, \psi\rangle$ of a rule of $\mathcal{L}_\ell$.

The fact that the $\phi$-reduct of $B_{\mathbf{A}}$ is a Boolean algebra is an immediate consequence of the construction of $B_{\mathbf{A}}$ from $\mathbf{A}$.

Assume now that $\langle\{\psi_1, \ldots, \psi_n\}, \psi\rangle$ is an instance of a rule of $\mathcal{L}_\ell$ and let $h$ be an assignment over $B_{\mathbf{A}}$. Our aim is to prove that $B_{\mathbf{A}}, h \Vdash ((\sqcap_{1\leq i\leq n}\Box\psi_i) \sqsupset \Box\psi) \approx \top$. Since $\{\psi_1, \ldots, \psi_n\} \vdash_\ell \psi$ and $\mathcal{Q}_{\mathcal{L}_\ell}$ is the equivalent algebraic semantics of $\mathcal{L}_\ell$, we have that $\{\Theta_\ell(\psi_1), \ldots, \Theta_\ell(\psi_n)\} \vDash_{\Sigma_\ell}^{\mathcal{Q}_{\mathcal{L}_\ell}} \Theta_\ell(\psi)$. So, in particular, we have that $\mathbf{A}, h_\ell \Vdash \Theta_\ell(\psi)$ whenever $\mathbf{A}, h_\ell \Vdash \Theta_\ell(\psi_i)$ for every $i \in \{1, \ldots, n\}$. Recall that, for every $\varphi \in L_\ell$, we have that $\mathbf{A}, h_\ell \Vdash \Theta_\ell(\varphi)$ if and only if $h_\ell(\varphi) \in D_{\mathbf{A}}$ if and only if $\Box_{B_{\mathbf{A}}}(h(\varphi)) = \top_{B_{\mathbf{A}}}$. Therefore, we can conclude that $B_{\mathbf{A}}, h \Vdash ((\sqcap_{1\leq i\leq n}\Box\psi_i) \sqsupset \Box\psi) \approx \top$.

We can prove that $B_{\mathbf{A}} \Vdash \Box\varphi \approx \top$ for every theorem $\varphi$ of $\mathcal{L}_\ell$ using a particular case of the argument used above for instances of rules.

$\Box$

## 5.6 Exogenous probabilistic propositional logic

In this example we study the algebraization of Exogenous Probabilistic Propositional Logic (EPPL) as introduced in [MSS05]. Therein, the authors introduce a probability logic built over $GL(CPL)$, the Global Logic over Classical Propositional Logic (CPL). The interest in probability logic has recently increased due to the growing importance of probability in security and in quantum logic [MS06].

The construction of this probability logic is a step further in the exogenous approach already presented in Section 5.5 in the example of global logic. In the EPPL case, the key idea is that a model of the probability logic is a probability space where the outcomes are classical valuations. Herein we just study the case of

EPPL with CPL as its local logic.

Let us start by introducing the language. The language of EPPL is an expansion of the language of $GL(CPL)$. It is obtained from the three-sorted signature $\Sigma_{\mathrm{P}} = \langle S, O \rangle$ where $S = \{g, \ell, t\}$ and $O = \{O_{ws}\}_{w \in S^*, s \in S}$ is such that:

- $O_{\epsilon\ell} = P \cup \{\mathbf{t}, \mathbf{f}\}$;
- $O_{\ell\ell} = \{\neg\}$;
- $O_{\ell\ell\ell} = \{\Rightarrow\}$;
- $O_{\epsilon t} = \{r : \text{ for every computable real number } r\}$;
- $O_{\ell t} = \{\int\}$;
- $O_{ttt} = \{+, \times\}$;
- $O_{\ell g} = \{\Box\}$;
- $O_{ttg} = \{\leq\}$;
- $O_{gg} = \{\boxminus\}$;
- $O_{ggg} = \{\sqsupset\}$.

We will assume fixed a three-sorted set $X = \{X\}_{s \in S}$ of variables. As usual, we can consider the algebra of terms $\mathbf{T}_{\boldsymbol{\Sigma}_{\mathrm{P}}}(X)$, the free algebra obtained from $\Sigma_{\mathrm{P}}$ over the set $X$. Note that, since we are assuming that the local logic is CPL, the operations of sort $\ell$ are the usual classical connectives and $P$ is the set of propositional variables.

In [MSS05] it is proved that EPPL is an infinitary logic. Therefore, EPPL can not have a strongly complete deductive system. So, in the sequel, we introduce EPPL just by semantic means and study its algebraization.

We start by introducing the models. These are extensions of the global models with an additional probability structure. Recall that a global model is a set of models of the local logic. In this particular case, the global models are sets of classical valuations and are called global valuations.

The denotation of terms and satisfaction of global formulas require a probability structure. Each interpretation structure has a probability space whose outcome space is a set of valuations. Moreover, there is an assignment for interpreting real variables.

Recall that a probability space is a triple $\langle \Omega, \mathfrak{B}, P\rangle$ where $\Omega$ is a non empty set, $\mathfrak{B} \subseteq 2^\Omega$ is a Borel field (that is, $\mathfrak{B}$ includes $\Omega$ and is closed for complements and countable unions) and $P : \mathfrak{B} \to [0,1]$ is a function such that:

- $P(\bigcup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} P(B_i)$ whenever $B_i \cap B_j = \emptyset$ for every $i \neq j$;
- $P(\Omega) = 1$

The elements of $\Omega$ are the *outcomes*, the elements of $\mathfrak{B}$ are the *events* and $P(B)$ is the probability of event $B$. In short, $P$ is a measure (additive map) with mass 1. For example, given a countable $\Omega$, it is usual to adopt $2^\Omega$ for $\mathfrak{B}$. Observe that, in this case, the probability $P$ is determined by the probability assigned to the singletons.

An interpretation structure is a pair $I = \langle V, P\rangle$ where $V$ is global valuation (a set of local valuations) and $P = \langle V, \mathfrak{B}, \mu\rangle$ is a probability space such that $\mathfrak{B}$ includes, for every $\varphi \in T_{\Sigma,\ell}(X)$, the set $mod(\varphi) \cap V = \{v \in V : v \Vdash \varphi\}$ . An assignment $h$ is a map such that $h(x) \in \mathbb{R}$ for each $x \in X_t$. The denotation of probability terms over the interpretation structure $I$ and an assignment $h$ is the map

$$[\![.]\!]_{I,h} : T_{\Sigma_{\mathrm{P}},t}(X) \to \mathbb{R}$$

inductively defined as follows:

- $[\![x]\!]_{I,h} = h(x)$;
- $[\![r]\!]_{I,h} = r$ for every computable real term;
- $[\![\int\varphi]\!]_{I,h} = \mu(mod(\varphi) \cap V)$;
- $[\![t_1 + t_2]\!]_{I,h} = [\![t_1]\!]_{I,h} + [\![t_2]\!]_{I,h}$;
- $[\![t_1 \times t_2]\!]_{I,h} = [\![t_1]\!]_{I,h} \times [\![t_2]\!]_{I,h}$.

The denotation of $(\int \varphi)$ is the probability, given by $\mu$, of the subset of $V$ that includes exactly all the models of $\varphi$. If $mod(\varphi) \cap V = V$ then the probability of $\varphi$ over $I$ is 1 even if $V \neq \mathcal{V}$. Moreover, the probability of $\varphi$ in a particular structure can be 0 even if the formula is a possible one. The satisfaction of probability formulas given an interpretation structure $I = \langle V, P\rangle$ and an assignment $h$ is inductively as follows:

- $I, h \Vdash_{\mathrm{P}} \Box\varphi$ iff $V \Vdash_g \Box\varphi$;

- $I, h \Vdash_{\mathrm{P}} (t_1 \leq t_2)$ iff $[\![t_1]\!]_{I,h} \leq [\![t_2]\!]_{I,h}$;
- $I, h \Vdash_{\mathrm{P}} \boxminus\delta$ iff $I, h \nVdash_{\mathrm{P}} \delta$;
- $I, h \Vdash_{\mathrm{P}} (\delta_1 \sqsupset \delta_2)$ iff $I, h \nVdash_{\mathrm{P}} \delta_1$ or $I, h \Vdash_{\mathrm{P}} \delta_2$;

The EPPL consequence relation can now be obtained, in the usual way, from the satisfaction of formulas. Given $T \cup \{\delta\} \subseteq L_{\Sigma_{\mathrm{P}}}(X)$ we can define $T \vDash_{\mathrm{P}} \delta$ iff for every interpretation structure $I = \langle V, P \rangle$ and every assignment $h$ we have that $I, h \Vdash_{\mathrm{P}} \delta$ whenever $I, h \Vdash_{\mathrm{P}} \gamma$ for every $\gamma \in T$.

We now focus on the problem of algebraizing EPPL. First of all, we prove that EPPL is algebraizable. For that we use the intrinsic characterization of algebraizable logic given in Theorem 5.1.7.

**Theorem 5.6.1.** *EPPL is algebraizable.*

*Proof.* Recall that Theorem 5.1.7 gives a sufficient condition for a logic to be algebraizable.

In this proof we use $\Delta = \{(\xi_1 \sqsupset \xi_2), (\xi_2 \sqsupset \xi_1)\}$ and just observe that the following conditions are all verified in EPPL due to its classical flavor:

*(i)* $\vDash_{\mathrm{P}} \delta_1 \Delta \delta_1$;

*(ii)* $\delta_1 \Delta \delta_2 \vDash_{\mathrm{P}} \delta_2 \Delta \delta_1$;

*(iii)* $\delta_1 \Delta \delta_2, \delta_2 \Delta \delta_3 \vDash_{\mathrm{P}} \delta_1 \Delta \delta_3$;

*(iv)* $\delta_1 \Delta \delta_2 \vDash_{\mathrm{P}} (\boxminus\delta_1)\Delta(\boxminus\delta_2)$;

*(v)* $\delta_1 \Delta \delta_2, \delta_3 \Delta \delta_4 \vDash_{\mathrm{P}} (\delta_1 \sqsupset \delta_3)\Delta(\delta_2 \sqsupset \delta_4)$;

*(vi)* $\delta_1, \delta_1 \Delta \delta_2 \vDash_{\mathrm{P}} \delta_2$;

*(vii)* $\delta_1, \delta_2 \vDash_{\mathrm{P}} \delta_1 \Delta \delta_2$.

Recall that, in this case, the set of defining equations is

$$\Theta(\xi) = \{\xi \approx \delta(\xi, \xi) : \delta(\xi_1, \xi_2) \in \Delta(\xi_1, \xi_2)\}.$$

□

More than proving that EPPL is algebraizable, our aim is to associate an algebraic counterpart to EPPL. Recall that, when a logic $\mathcal{L}$ is algebraizable, every equivalent algebraic semantics can be considered an algebraic counterpart for $\mathcal{L}$. Of course, among these equivalent algebraic semantics, one is considered canonical: the largest equivalent algebraic semantics. This largest equivalent algebraic semantics is not, however, easy to characterize when the logic is not finitary.

In non-finitary logics, equivalent algebraic algebraic with an easy characterization but that are only strictly contained in the largest one can, nevertheless, help to give a better insight about $\mathcal{L}$.

In what follows we present a class $\mathcal{B}_{\text{P}}$ of three-sorted algebras and prove that $\mathcal{B}_{\text{P}}$ is an equivalent algebraic semantics for EPPL. Although not the largest equivalent algebraic semantics $\mathcal{B}_{\text{P}}$ can give some insight about the algebraic counterpart of EPPL.

Consider the class $\mathcal{B}_{\text{P}}$ of algebras over $\Sigma_{\text{P}}$ such that $\mathbf{A} \in \mathcal{B}_{\text{P}}$ if there exists a set $I$ such that:

- $A_\ell = \otimes_{i \in I} \mathbf{B}_i$ where $\mathbf{B}_i$ is a Boolean algebra, for each $i \in I$;
- $A_t = \otimes_{i \in I} \mathbb{R}$;
- $A_g = \otimes_{i \in I} \mathbf{2}_i$ where each $\mathbf{2}_i$ is a two-value Boolean algebra;
- $\Box_{\mathbf{A}} : A_l \to A_g$ is the unique function obtained from $\{\Box_i\}_{i \in I}$ using the universal property of the product, where, for each $j \in I$,

$$\Box_j : \otimes_{i \in I} B_i \to 2 \text{ is such that } \Box_j((a)_{i \in I}) = 1 \text{ iff } a_j = \top_{\mathbf{B}_j};$$

- $\int_{\mathbf{A}} : A_l \to A_t$ is the unique function obtained from $\{\int_i\}_{i \in I}$ using the universal property of the product, where for each $j \in I$,

$$\int_j : \otimes_{i \in I} B_i \to 2 \text{ is such that } \int_j((a_i)_{i \in I}) = \mu_j(a_j), \text{ where } \mu_j : B_j \to \mathbb{R} \text{ is a finite additive probability measure over } \mathbf{B}_j;$$

- $\leq_{\mathbf{A}}: A_t \times A_t \to A_g$ is obtained from $\{\leq_i\}_{i \in I}$ using the universal property of the product, where, for each $j \in I$;

$$\leq_j: \otimes_{i \in I} \mathbb{R} \times \otimes_{i \in I} \mathbb{R} \to 2 \text{ is such that } \leq_j ((a_i)_{i \in I}, (b_i)_{i \in I}) = 1 \text{ iff } a_j \leq b_j.$$

We end this example with a result stating that $\mathcal{B}_{\rm P}$ is indeed an equivalent algebraic semantics for EPPL.

**Theorem 5.6.2.** *Let $T \cup \{\delta, \delta_1, \delta_2\} \subseteq L_{\Sigma_P}(X)$. Then we have that:*

$$T \vDash_{P} \delta \qquad \textit{iff} \qquad \{\gamma \approx \top : \gamma \in T\} \vDash^{\mathcal{B}_P}_{\Sigma_P} \delta \approx \top$$

*and*

$$\delta_1 \approx \delta_2 =\!\mid\mid\!=^{\mathcal{B}_P}_{\Sigma_P} (\delta_1 \equiv \delta_2) \approx \top.$$

*Proof.* The fact that $\delta_1 \approx \delta_2 =\!\mid\mid\!=^{\mathcal{B}_{\rm P}}_{\Sigma_{\rm P}} (\delta_1 \equiv \delta_2) \approx \top$ holds is an immediate consequence of the fact that, for every $\mathbf{B} \in \mathcal{B}_{\rm P}$, we have that $\mathbf{B}_g$ is a Boolean algebra.

Let us now prove that $T \vDash_{\rm P} \delta$ iff $\{\gamma \approx \top : \gamma \in T\} \vDash^{\mathcal{B}_{\rm P}}_{\Sigma_{\rm P}} \delta \approx \top$. We will prove both directions by contraposition.

Suppose first that $\{\gamma \approx \top : \gamma \in T\} \nvDash^{\mathcal{B}_{\rm P}}_{\Sigma_{\rm P}} \delta \approx \top$. Then, there exists an algebra $\mathbf{B} \in \mathcal{B}_{\rm P}$ and an assignment $h$ such that $\mathbf{B}, h \Vdash_{\rm P} \gamma \approx \top$ for every $\gamma \in T$ and $\mathbf{B}, h \nVdash_{\rm P} \delta \approx \top$. We know that there exists a set $I$ such that $\mathbf{B} = \otimes_{i\in I} \mathbf{2}_i$. So, we can conclude that there exists $j \in I$ such that $\mathbf{2}_j, h_j \Vdash_{\rm P} \gamma \approx \top$ for every $\gamma \in T$ and $\mathbf{2}_j, h_j \nVdash_{\rm P} \delta \approx \top$. Our aim is to build an interpretation structure that satisfies every element of $T$ and does not satisfy $\delta$.

A well-known Stones's theorem for Boolean algebra states that every Boolean algebra is isomorphic to a subdirect product of $\mathbf{2}$. In our case, this result allows us to conclude that there exists a set $K$ such that $\mathbf{B}_j \overset{\iota}{\hookrightarrow} \prod_{k\in K} \mathbf{2}_k$ is an embedding.

For each $k \in K$ consider the valuation $v_k : X_\ell \to \mathbf{2}_k$ defined as

$$v_k(x) = (\iota(h_j(x)))_k.$$

Consider the set $V^h_j = \{v_k : k \in K\}$. For each $b \in B_j$ define the subset $V_b$ of $V^h_j$ as $V_b = \{v_k : (\iota(b))_k = 1\}$. Then, we can consider the set $\mathfrak{B} = \{V_b : b \in B_j\}$. It is easy to see that $\mathfrak{B}$ is a Borel field. We can define a probability measure $\mu$ on $\mathfrak{B}$ such that, for every $V_b \in \mathfrak{B}$, we have that $\mu(V_b) = \int_{\mathbf{B}_j}(b)$.

We have, for every $\varphi \in T_{\Sigma,\ell}(X)$, that

$$\begin{aligned}
\langle V_j^h, \mathfrak{B}, \mu\rangle \Vdash \Box\varphi \quad & \text{iff} \quad v_k \Vdash \varphi \text{ for every } k \in K \\
& \text{iff} \quad (\iota(h(\varphi))_k = 1 \text{ for every } k \in K \\
& \text{iff} \quad h(\varphi) = \top_{\mathbf{B}_j} \\
& \text{iff} \quad \mathbf{2}_j, h \Vdash \Box\varphi \approx \top
\end{aligned}$$

With respect to the atomic formulas involving the probability constructor we can prove that $[\![\int\varphi]\!]_{h_j}^{\langle V_j^h,\mathfrak{B},\mu\rangle} = \int_{\mathbf{B}_j}(h_j(\varphi))$. Therefore it can be easily seen that $\langle V_j^h, \mathfrak{B}, \mu\rangle \Vdash t_1 \leq t_2$ iff $\mathbf{2}_j, h \Vdash (t_1 \leq t_2) \approx \top$.

With an easy induction on the structure of a global formula, we can conclude that

$$\langle V_j^h, \mathfrak{B}, \mu\rangle \Vdash \delta \quad \text{iff} \quad \mathbf{2}_j, h \Vdash \delta \approx \top.$$

In the other direction suppose that $T \not\vDash_{\mathrm{P}} \delta$. Then there exists an interpretation structure $I = \langle V, \mathfrak{B}, \mu\rangle$ and a assignment $h$ such that $I, h \Vdash_{\mathrm{P}} \gamma$ for every $\gamma \in T$ and $I, h \not\Vdash_{\mathrm{P}} \delta$. Our aim is to find a three-sorted algebra $\mathbf{B} \in \mathcal{B}_{\mathrm{P}}$ such that $\mathbf{B}, h \Vdash_{\mathrm{P}} (\gamma \approx \top)$ for every $\gamma \in T$ and $\mathbf{B}, h \not\Vdash_{\mathrm{P}} (\delta \approx \top)$.

Consider the $\Sigma_{\mathrm{P}}$-algebra $\mathbf{B}_I$ such that:

- $(\mathbf{B}_I)_g = \mathbf{2}$;
- $(\mathbf{B}_I)_\ell = \langle \mathfrak{B}, \cap, \cup, -, \emptyset, V\rangle$;
- $(\mathbf{B}_I)_\ell = \mathbb{R}$;
- $\Box_{\mathbf{B}_I}(a) = 1$ iff $a = V$;
- $\int_{\mathbf{B}_I}(a) = \mu(a)$.

Consider the assignment $h$ over $\mathbf{B}_I$ obtained from $h$ and defined as $h_g = h_g$, $h_t = h_t$ and $h_\ell(x) = \{v \in V : v \Vdash x\}$. An easy induction gives, for every $\alpha \in L_{\Sigma_{\mathrm{P}}}(X)$, that

$$I, h \Vdash \alpha \quad \text{iff} \quad \mathbf{B}_I, h \Vdash (\alpha \approx \top)\ .$$

So, it follows that $\mathbf{B}_I, h \Vdash_{\mathrm{P}} (\gamma \approx \top)$ for every $\gamma \in T$ and $\mathbf{B}_I, h \not\Vdash_{\mathrm{P}} (\delta \approx \top)$.

$\Box$

## 5.7 $k$-deductive systems

The higher dimensional systems, called $k$-deductive systems, constitute a natural generalization of deductive systems that encompass several other logical systems, namely equational and inequational logics. They were introduced by Blok and Pigozzi in [BP92] (see also [CP99, Mar04]) to provide a context to deal with logics which are assertional and equational. The algebraic theory of these higher dimensional systems, as in the deductive system setting, is supported by properties of the Leibniz congruence. In this example we show that our approach is general and expressive enough to capture the framework of $k$-deductive systems as a particular case. Our aim is to prove that a $k$-deductive system can be seen as a two-sorted logic and, moreover, that if it is algebraizable according to the standard notion then it is also behaviorally algebraizable. Therefore, we just need to work in a many-sorted setting without extending the signature. Example 5.1.1 shows that this is equivalent to working with an extended signature, and moreover we gain in simplicity of notation.

Consider given a propositional signature $P$. A $k$-deductive system is a logic for reasoning about tuples of formulas rather than formulas individually. A tuple of formulas can be naturally captured using a two-sorted signature. Therefore, a $k$-deductive system over $P$ can be introduced as a two-sorted logic.

From $P$ we can consider the two-sorted signature $\Sigma_P^k = \langle\{t, \phi\}, F\rangle$ such that:

- $F_{t^k\phi} = \{p\}$ ($k$-formulas);
- $F_{t^n t} = \{c : c \in P_n \text{ and } n \in \mathbb{N}\}$ ($k$-connectives);
- $F_{\phi t} = \{p_i : 1 \leq i \leq k\}$ (projections).

Given a $k$-deductive system $S = \langle P, \vdash_S\rangle$ we can consider a many-sorted logic $\mathcal{L}_S = \langle \Sigma_P^k, \vdash\rangle$ obtained from $S$ as follows:

$$\Phi \vdash p(\varphi_1, \ldots, \varphi_k) \quad \text{iff} \quad \{\langle\psi_1, \ldots, \psi_k\rangle : p(\psi_1, \ldots, \psi_k) \in \Phi\} \vdash_S \langle\varphi_1, \ldots, \varphi_k\rangle.$$

Given a $P$-algebra $\mathbf{A}$ we can consider an induced $\Sigma_P^k$-algebra $\mathbf{A}^*$ such that:

- $(A^*)_t = A$;
- $(A^*)_\phi = A^k$;

- $p_{\mathbf{A}^*}(a_1, \ldots, a_k) = \langle a_1, \ldots, a_k \rangle$;
- $(p_i)_{\mathbf{A}^*}(\langle a_1, \ldots, a_k \rangle) = a_i$, for every $1 \leq i \leq k$.

Given a class $K$ of $P$-algebras, we can apply this construction to the algebras of $K$ and obtain the class $K^* = \{\mathbf{A}^* : \mathbf{A} \in K\}$ of $\Sigma_P^k$-algebras.

We now show how can we use our framework to reason about the algebraization of a $k$-deductive system. The algebraization of a $k$-deductive systems in our many-sorted framework does not seem, at first sight, straightforward. This is due to the fact that, in $k$-deductive systems the equational consequence is defined over the propositional formulas, while in our approach it is defined over tuples of propositional formulas. Nevertheless, the following lemma asserts that the expressive power is the same in both approaches. We omit the proof since it is an easy exercise.

**Lemma 5.7.1.** *Let* $\mathbf{A}$ *be a* $P$*-algebra. Then we have that*

$$\mathbf{A}^* \Vdash p(\varphi_1, \ldots, \varphi_k) \approx p(\psi_1, \ldots, \psi_k) \quad \textit{iff} \quad \mathbf{A} \Vdash \varphi_i \approx \psi_i \textit{ for every } 1 \leq i \leq k.$$

*In particular,*

$$\mathbf{A} \Vdash \varphi \approx \psi \quad \textit{iff} \quad \mathbf{A}^* \Vdash p(\varphi, \ldots, \varphi) \approx p(\psi, \ldots, \psi).$$

Before we prove the main result we need to fix some notation. First of all, note that a $\phi$-equation without $\phi$-variables always has the form $p(\varphi_1, \ldots, \varphi_k) \approx p(\psi_1, \ldots, \psi_k)$. Given a set $\Phi$ of $\phi$-equations without $\phi$-variables, we can consider, for each $1 \leq i \leq k$, the set
$\Phi_i = \{\varphi_i \approx \psi_i : p(\varphi_1, \ldots, \varphi_i, \ldots, \varphi_k) \approx p(\psi_1, \ldots, \psi_i, \ldots, \psi_k) \in \Phi\}$.

**Proposition 5.7.2.** *A* $k$*-deductive system* $S = \langle P, \vdash_S \rangle$ *is algebraizable with equivalent algebraic semantics* $K$ *iff* $\mathcal{L}_S$ *is behaviorally algebraizable with* $\Gamma$*-behaviorally equivalent algebraic semantics* $K^*$.

*Proof.* Suppose first that $S$ is algebraizable and let $K$ be an equivalent algebraic semantics. Then, there exists a set $\Theta(x_1 : t, \ldots, x_k : t)$ of $k$-equations and a set $\Delta(x_1 : t, x_2 : t)$ of $k$-formulas such that $T \vdash_S \langle \varphi_1, \ldots, \varphi_k \rangle$ iff $\Theta[T] \vDash_K \Theta(\varphi_1, \ldots, \varphi_k)$ and $\varphi_1 \approx \varphi_2 =\!\mid\mid\!\!=_K \Theta[\Delta(\varphi_1, \varphi_2)]$. Given a $t$-term $\varphi(x_1 : t, \ldots, x_k : t)$ we consider the formula $\varphi^* = p(\varphi(p_1(\xi), \ldots, p_k(\xi)), \ldots, \varphi(p_1(\xi), \ldots, p_k(\xi)))$.

We can now consider the sets $\Theta^*(\xi) = \{\lambda^* \approx \epsilon* : \lambda \approx \epsilon \in \Theta\}$ and $\Delta^*(\xi_1, \xi_2) = \Delta(p_1(\xi_1), p_1(\xi_2)) \cup \ldots \cup \Delta(p_k(\xi_1), p_k(\xi_2))$. It is easy to check that $\mathcal{L}_S$ is algebraizable with $\Theta^*(\xi)$, $\Delta^*(\xi_1, \xi_2)$ and $K^*$.

Suppose now that $\mathcal{L}_S$ is algebraizable with $K^*$ an equivalent algebraic semantics. Then there exists a set $\Theta^*(\xi)$ of $\phi$-equations and a set $\Delta^*(\xi_1, \xi_2)$ of formulas such that $T \vdash_S p(\varphi_1, \ldots, \varphi_k)$ iff $\Theta^*[T] \vDash_{K^*} \Theta^*(p(\varphi_1, \ldots, \varphi_k))$ and $p(\varphi_1, \ldots, \varphi_k) \approx p(\psi_1, \ldots, \psi_k) =\!||\!=_{K^*} \Theta^*[\Delta^*(p(\varphi_1, \ldots, \varphi_k), p(\psi_1, \ldots, \psi_k))]$.

Now take $\Theta(x_1, \ldots, x_k) = \Theta^*_1(p(x_1, \ldots, x_k)) \cup \ldots \cup \Theta^*_k(p(x_1, \ldots, x_k))$ and $\Delta(x_1, x_2) = \Delta^*(p(x_1, \ldots, x_1), p(x_2, \ldots, x_2))$. It is now straightforward to prove that $S$ is algebraizable with $K$ an equivalent algebraic semantics.

□

## 5.8 Constructive logic with strong negation

Constructive logic with strong negation was formulated by Nelson [Nel49] in order to overcome some non-constructive properties of intuitionistic negation. The main criticism of intuitionistic negation is the fact that in Intuitionistic Propositional Logic (IPL), from the derivability of $\neg(\varphi \wedge \psi)$, it does not follows that at least one of the formulas $\neg\varphi$ or $\neg\psi$ is derivable in IPL. So, in order to obtain a constructive logic with this property, IPL was extended with an unary connective for strong negation satisfying the desired property. We closely follow the notation of Kracht [Kra98] and denote by $N$ the constructive logic with strong negation.

It is well-known that $N$ is algebraizable and that the equivalent algebraic semantics is the class of so-called $N$-lattices [Vak77, Sen84] (also Nelson algebras [SV08] or quasi-pseudo-Boolean algebras [Ras81]). The variety of $N$-lattices has been extensively studied [Ras81, Vak77, Sen84, Kra98]. One of the important results is the characterization of $N$-lattices trough Heyting algebras. We use this result extensively in this example.

Herein, our goal is to show that our framework can be useful even when applied to logics that are already algebraizable in the standard sense. The change of perspective can help to give a better insight about the algebraic counterpart of a given logic.

In more concrete terms, we show that $N$ can be behaviorally algebraized by choosing a subsignature $\Gamma$ of the original signature. This subsignature is obtained by excluding strong negation from the original signature, thus maintaining just the intuitionistic connectives. We then study the behavioral algebraic counterpart of $N$ and show that the characterization of $N$-lattices through Heyting algebras explicitly emerges, thus reinforcing the central role of Heyting algebras in the algebraic counterpart of $N$.

We start by presenting the language of $N$. It is obtained from a single-sorted signature $\Sigma_N = \langle S, F \rangle$ such that:

- $S = \{\phi\}$;
- $F_{\epsilon\phi} = \emptyset$;
- $F_{\phi\phi} = \{\neg, \sim\}$;
- $F_{\phi^2\phi} = \{\to, \vee, \wedge\}$;
- $F_{\phi^n\phi} = \emptyset$, for all $n > 2$.

As usual, we can define $\bot = (\varphi \wedge (\neg\varphi))$ and $\top = (\varphi \to \varphi)$, where $\varphi \in L_{\Sigma_M}(X)$ is some fixed but arbitrary formula. The connective $\sim$ is intended to represent strong negation and the remainder connectives are intented to represent the usual intuitionistic connectives. We can define the intuitionistic equivalence as usual as $\xi_1 \leftrightarrow \xi_2 = (\xi_1 \to \xi_2) \wedge (\xi_2 \to \xi_1)$ and we can also define a *strong implication* $(\xi_1 \Rightarrow \xi_2) = (\xi_1 \to \xi_2) \wedge (\sim \xi_2 \to \sim \xi_1)$.

The structural single-sorted deductive system of $N$ consists of the following axioms:

*i)* $\xi_1 \to (\xi_2 \to \xi_1)$;

*ii)* $(\xi_1 \to (\xi_2 \to \xi_3)) \to ((\xi_1 \to \xi_2) \to (\xi_1 \to \xi_3))$;

*iii)* $(\xi_1 \wedge \xi_2) \to \xi_1$;

*iv)* $(\xi_1 \wedge \xi_2) \to \xi_2$;

*v)* $\xi_1 \to (\xi_2 \to (\xi_1 \wedge \xi_2))$;

*vi)* $\xi_1 \to (\xi_1 \vee \xi_2)$;

*vii)* $\xi_2 \to (\xi_1 \vee \xi_2)$;

*viii)* $(\xi_1 \to \xi_3) \to ((\xi_2 \to \xi_3) \to ((\xi_1 \vee \xi_2) \to \xi_3))$;

*ix)* $(\xi_1 \to \xi_2) \to ((\xi_1 \to \neg\xi_2) \to \neg\xi_1)$;

*x)* $\neg\xi_1 \to (\xi_1 \to \xi_2)$;

*xi*) $\sim(\xi_1 \rightarrow \xi_2) \leftrightarrow (\xi_1 \wedge \sim \xi_2)$;

*xii*) $\sim(\xi_1 \wedge \xi_2) \leftrightarrow (\sim \xi_1 \vee \sim \xi_2)$;

*xiii*) $\sim(\xi_1 \vee \xi_2) \leftrightarrow (\sim \xi_1 \wedge \sim \xi_2)$;

*xiv*) $(\sim \neg \xi_1) \leftrightarrow \xi_1$;

*xv*) $(\sim\sim \xi_1) \leftrightarrow \xi_1$;

*xvi*) $(\sim \xi_1 \vee \neg \xi_1) \leftrightarrow \neg \xi_1$;

and the rule:

$$(MP) \qquad \frac{\xi_1 \quad \xi_1 \rightarrow \xi_2}{\xi_2}.$$

Note that the axioms *i*) - *x*) are the usual axioms for IPL. Axioms *xi*) - *xvi*) express the relation between strong negation and the other connectives.

It is well-known that $N$ is algebraizable [Ras81]. It is interesting that it is not the intuitionistic equivalence $\leftrightarrow$ that is used as the set of equivalence formulas in the algebraization of $N$. This is mainly due to the fact that $\leftrightarrow$ does not have the congruence property with respect to strong negation. The equivalence used to algebraize $N$ is the strong equivalence $(\xi_1 \Leftrightarrow \xi_2) = (\xi_1 \Rightarrow \xi_2) \wedge (\xi_2 \Rightarrow \xi_1)$.

In what follows we describe the equivalent algebraic semantics of $N$, the class of $N$-*lattices*. Let $\mathcal{N}$ be the class of all $\Sigma_N$-algebras $\mathbf{A}$ such that:

- $\langle \mathbf{A}, \wedge_{\mathbf{A}}, \vee_{\mathbf{A}}, \top_{\mathbf{A}}, \bot_{\mathbf{A}} \rangle$ is a bounded distributive lattice;

  and it also satisfies the following equations:

- $\xi_1 \rightarrow (\xi_2 \rightarrow \xi_1) \approx \top$;
- $(\xi_1 \rightarrow (\xi_2 \rightarrow \xi_3)) \rightarrow ((\xi_1 \rightarrow \xi_2) \rightarrow (\xi_1 \rightarrow \xi_3)) \approx \top$;
- $(\xi_1 \wedge \xi_2) \rightarrow \xi_1 \approx \top$;

- $(\xi_1 \wedge \xi_2) \to \xi_2 \approx \top$;
- $\xi_1 \to (\xi_2 \to (\xi_1 \wedge \xi_2)) \approx \top$;
- $\xi_1 \to (\xi_1 \vee \xi_2) \approx \top$;
- $\xi_2 \to (\xi_1 \vee \xi_2) \approx \top$;
- $(\xi_1 \to \xi_3) \to ((\xi_2 \to \xi_3) \to ((\xi_1 \vee \xi_2) \to \xi_3)) \approx \top$;
- $(\xi_1 \to \xi_2) \to ((\xi_1 \to \neg\xi_2) \to \neg\xi_1) \approx \top$;
- $\neg\xi_1 \to (\xi_1 \to \xi_2) \approx \top$;
- $\sim(\xi_1 \to \xi_2) \leftrightarrow (\xi_1 \wedge \sim\xi_2) \approx \top$;
- $\sim(\xi_1 \wedge \xi_2) \leftrightarrow (\sim\xi_1 \vee \sim\xi_2) \approx \top$;
- $\sim(\xi_1 \vee \xi_2) \leftrightarrow (\sim\xi_1 \wedge \sim\xi_2) \approx \top$;
- $(\sim\neg\xi_1) \leftrightarrow \xi_1 \approx \top$;
- $(\sim\sim\xi_1) \leftrightarrow \xi_1 \approx \top$;
- $(\sim\xi_1 \vee \neg\xi_1) \leftrightarrow \neg\xi_1 \approx \top$.

We briefly recall some important properties of $N$-lattices, namely with respect to their connection with Heyting algebras. We just present the results that are useful for our study. For the reader interested in a more detailed study on $N$-lattices we point to [Ras81, Vak77, Sen84].

In [Vak77] Vakarelov introduces a construction of $N$-lattices from Heyting algebras. The algebras obtained by this construction are called *twist algebras*. We now introduce the precise notion of twist algebra and present some interesting results connecting $N$-lattices and twist algebras.

Let $\Gamma = \langle S, F' \rangle$ be the subsignature of $\Sigma_N$ such that $F'_{\phi\phi} = \{\neg\}$ and $F'_{ws} = F_{ws}$ for every $ws \in S^*$ such that $ws \neq \phi\phi$. Note that the subsignature $\Gamma$ is nothing but the intuitionistic reduct of the signature $\Sigma_N$.

Given a $\Gamma$-algebra $\mathbf{A}$, consider the set $A^{\bowtie} = \{\langle a, b \rangle : a, b \in A \text{ and } a \wedge_{\mathbf{A}} b = \bot_{\mathbf{A}}\}$. We can define a $\Sigma$-algebra $\mathbf{A}^{\bowtie} = \langle A^{\bowtie}, \wedge_{\mathbf{A}^{\bowtie}}, \vee_{\mathbf{A}^{\bowtie}}, \to_{\mathbf{A}^{\bowtie}}, \neg_{\mathbf{A}^{\bowtie}}, \sim_{\mathbf{A}^{\bowtie}}, \bot_{\mathbf{A}^{\bowtie}}, \top_{\mathbf{A}^{\bowtie}} \rangle$ over the set $A^{\bowtie}$ by defining the operations as follows:

- $\langle a_1, b_1\rangle \wedge_{\mathbf{A}^{\bowtie}} \langle a_2, b_2\rangle = \langle a_1 \wedge_{\mathbf{A}} a_2,\ b_1 \vee_{\mathbf{A}} b_2\rangle$;
- $\langle a_1, b_1\rangle \vee_{\mathbf{A}^{\bowtie}} \langle a_2, b_2\rangle = \langle a_1 \vee_{\mathbf{A}} a_2,\ b_1 \wedge_{\mathbf{A}} b_2\rangle$;
- $\langle a_1, b_1\rangle \rightarrow_{\mathbf{A}^{\bowtie}} \langle a_2, b_2\rangle = \langle a_1 \rightarrow_{\mathbf{A}} a_2,\ a_1 \wedge_{\mathbf{A}} b_2\rangle$;
- $\neg_{\mathbf{A}^{\bowtie}}\langle a, b\rangle = \langle \neg_{\mathbf{A}} a,\ a\rangle$;
- $\sim_{\mathbf{A}^{\bowtie}} \langle a, b\rangle = \langle b,\ a\rangle$;
- $\top_{\mathbf{A}^{\bowtie}} = \langle \top_{\mathbf{A}}, \bot_{\mathbf{A}}\rangle$;
- $\bot_{\mathbf{A}^{\bowtie}} = \langle \bot_{\mathbf{A}}, \top_{\mathbf{A}}\rangle$.

The algebra $\mathbf{A}^{\bowtie}$ is called a *full twist algebra over* $\mathbf{A}$. A *twist algebra* is a subalgebra of a full twist algebra. The following theorem is due to Vakarelov [Vak77].

**Theorem 5.8.1.** *If* $\mathbf{A}$ *is a Heyting algebra then* $\mathbf{A}^{\bowtie}$ *is a* $N$*-lattice.*

Given a $N$-lattice $\mathbf{A}$ we can consider an equivalence relation $\theta_{\mathbf{A}}$ over $\mathbf{A}$ defined as $\langle a, b\rangle \in \theta_{\mathbf{A}}$ iff $(a \leftrightarrow_{\mathbf{A}} b) = \top_{\mathbf{A}}$. It is well-known that this equivalence relation, that corresponds to intuitionistic equivalence in $\mathbf{A}$, is not a congruence relation, in general. This is due to the fact that the congruence condition might fail for strong negation. Despite this fact, $\theta_{\mathbf{A}}$ is compatible with all the intuitionistic operations and is therefore a $\Gamma$-congruence.

We can then consider the $\Gamma$-algebra $\mathbf{A}_{\bowtie} = (\mathbf{A}_{|\Gamma})/_{\theta}$. Sendlewski [Sen90] proves that $\mathbf{A}_{\bowtie}$ is a Heyting algebra and that it is the least Heyting algebra that can be obtained by factorization. It is usually called the *Heyting algebra associated with* $\mathbf{A}$ or the *untwist algebra of* $\mathbf{A}$. For more results concerning the constructions $(.)^{\bowtie}$ and $(.)_{\bowtie}$ we point to [Vak77, Sen90, Kra98].

We proceed by studying the $\Gamma$-behavioral algebraizability of $N$. Recall that $\Gamma$ is the subsignature of $\Sigma_N$ representing the intuitionistic reduct. Intuitively, we are taking the strong negation out of the original signature, thus keeping just the intuitionistic connectives. Therefore, the intuitionistic equivalence will play a key role in the $\Gamma$-behavioral algebraization of $N$.

**Theorem 5.8.2.** $N$ *is* $\Gamma$*-behaviorally algebraizable.*

*Proof.* Recall that Theorem 3.3.2 gives a sufficient condition for a logic to be Γ-behaviorally algebraizable.

In this proof we use $\Delta = \{(\xi_1 \leftrightarrow \xi_2)\}$. The following conditions are all well-known to hold in IPL, and therefore in every axiomatic extension of IPL, which is the case of $N$:

*i)* $\vdash_N \delta_1 \Delta \delta_1$;

*ii)* $\delta_1 \Delta \delta_2 \vdash_N \delta_2 \Delta \delta_1$;

*iii)* $\delta_1 \Delta \delta_2, \delta_2 \Delta \delta_3 \vdash_N \delta_1 \Delta \delta_3$;

*iv)* $\delta_1 \Delta \delta_2 \vdash_N (\neg \delta_1) \Delta (\neg \delta_2)$;

*v)* $\delta_1 \Delta \delta_2, \delta_3 \Delta \delta_4 \vdash_N (\delta_1 \to \delta_3) \Delta (\delta_2 \to \delta_4)$;

*vi)* $\delta_1 \Delta \delta_2, \delta_3 \Delta \delta_4 \vdash_N (\delta_1 \wedge \delta_3) \Delta (\delta_2 \wedge \delta_4)$;

*vii)* $\delta_1 \Delta \delta_2, \delta_3 \Delta \delta_4 \vdash_N (\delta_1 \vee \delta_3) \Delta (\delta_2 \vee \delta_4)$;

*viii)* $\delta_1, \delta_1 \Delta \delta_2 \vdash_N \delta_2$;

*ix)* $\delta_1, \delta_2 \vdash_N \delta_1 \Delta \delta_2$.

Recall that, in this case, the set of defining equations can be defined as

$$\Theta(\xi) = \{\xi \approx (\xi \leftrightarrow \xi)\}.$$

□

We now describe the Γ-behaviorally equivalent algebraic semantics of $N$, the class $K_N^\Gamma$. Recall that $K_N^\Gamma$ is a class of algebras over the extended two-sorted signature $\Sigma_N^o = \langle \{\phi, v\}, F^o \rangle$ obtained from $\Sigma_N$. This class $K_N^\Gamma$ can be described using Theorem 4.1.3 together with the construction presented at the end of Section 4.1. Although $\Sigma_N^o$ does not have operations on the sort $v$, we can define operations that correspond to the operations in Γ, in every algebra of $K_N^\Gamma$.

In this particular case, we can define the operations $\wedge^o, \vee^o, \to^o, \neg^o, \top^o, \bot^o$ on the sort $v$ that correspond to the intuitionistic connectives. For the sake of notation we denote them by $\sqcap, \sqcup, \sqsupset, -, 1, 0$ respectively.

The class $K_N^\Gamma$ is constituted by all $\Sigma_N^o$-algebras $\mathbf{B}$ such that:

$\langle B_v, \sqcap_\mathbf{B}, \sqcup_\mathbf{B}, \sqsupset_\mathbf{B}, -_\mathbf{B}, 1_\mathbf{B}, 0_\mathbf{B} \rangle$ is a Heyting algebra

and $\mathbf{B}$ $\Gamma$-behaviorally satisfies the following axioms:

*i)* $\sim (\xi_1 \rightarrow \xi_2) \ \approx \ (\xi_1 \wedge \sim \xi_2)$;

*ii)* $\sim (\xi_1 \wedge \xi_2) \ \approx \ (\sim \xi_1 \vee \sim \xi_2)$;

*iii)* $\sim (\xi_1 \vee \xi_2) \ \approx \ (\sim \xi_1 \wedge \sim \xi_2)$;

*iv)* $(\sim \neg \xi_1) \ \approx \ \xi_1$;

*v)* $(\sim\sim \xi_1) \ \approx \ \xi_1$;

*vi)* $(\sim \xi_1 \vee \neg \xi_1) \ \approx \ \neg \xi_1$.

We have observed that, in some sense, the class of algebra $K_N^\Gamma$ explicitly describes the well-known relation between $N$-lattices and Heyting algebras. We now clarify this statement.

To be more specific, we prove that we can canonically define a $N$-lattice $\mathbf{B}^{\bowtie}$, given $\mathbf{B} \in K_N^\Gamma$. We can also define, for every $N$-lattice $\mathbf{A}$, a $\Sigma_N^o$-algebra $\mathbf{A}_{\bowtie}$ such that $\mathbf{A}_{\bowtie} \in K_N^\Gamma$. We have an abuse of notation when we write $\mathbf{B}^{\bowtie}$ and $\mathbf{A}_{\bowtie}$. But, as we will see, this can be well-justified with the key role of the constructions $(.)^{\bowtie}$ and $(.)_{\bowtie}$ in the definitions of $\mathbf{B}^{\bowtie}$ and $\mathbf{A}_{\bowtie}$ respectively.

Let $\mathbf{B} \in K_N^\Gamma$ and recall that $\equiv_\Gamma$ denotes the $\Gamma$-behavioral equivalence over $\mathbf{B}$. We can then define a $\Sigma_N$-algebra

$$\mathbf{B}^{\bowtie} = \langle B^{\bowtie}, \wedge_{\mathbf{B}^{\bowtie}}, \vee_{\mathbf{B}^{\bowtie}}, \rightarrow_{\mathbf{B}^{\bowtie}}, \neg_{\mathbf{B}^{\bowtie}}, \sim_{\mathbf{B}^{\bowtie}}, \top_{\mathbf{B}^{\bowtie}}, \bot_{\mathbf{B}^{\bowtie}} \rangle$$

where

$$B^{\bowtie} = \{\langle [a]_{\equiv_\Gamma}, [\sim_\mathbf{B} a]_{\equiv_\Gamma} \rangle : a \in B_\phi\}$$

and such that the operations are defined as in the construction $(.)^{\bowtie}$.

**Theorem 5.8.3.** *Given* $\mathbf{B} \in K_N^\Gamma$ *we have that* $\mathbf{B}^{\bowtie}$ *is a* $N$*-lattice.*

*Proof.* Let $h^* : X \to B^{\bowtie}$ be an assignment. Take $h : X \to B_\phi$ such that $h(x) = a$ where $h^*(x) = \langle [a]_{\equiv_\Gamma}, [\sim_{\mathbf{B}} a]_{\equiv_\Gamma} \rangle$. Using induction on the structure of a formula, it is easy to prove that $h^*(\varphi) = \langle [h(\varphi)]_{\equiv_\Gamma}, [\sim_{\mathbf{B}} h(\varphi)]_{\equiv_\Gamma} \rangle$, for every formula $\varphi$.

Since $\mathbf{B} \in K_N^\Gamma$ we have that $\mathbf{B} \Vdash_\Gamma \varphi \approx \top$ for every theorem $\varphi$ of $N$. So, for every assignment $h'$ over $B_\phi$, we can conclude that $[h'(\varphi)]_{\equiv_\Gamma} = \top_{\mathbf{B}}$. Therefore, given an axiom $\varphi$ of $N$ we have that $h^*(\varphi) = \langle [h(\varphi)]_{\equiv_\Gamma}, [\sim_{\mathbf{B}} h(\varphi)]_{\equiv_\Gamma} \rangle = \langle \top_{\mathbf{B}}, \bot_{\mathbf{B}} \rangle$, which is the unit.

All that remains to prove is that $\mathbf{B}^{\bowtie} = \langle B^{\bowtie}, \wedge_{\mathbf{B}^{\bowtie}}, \vee_{\mathbf{B}^{\bowtie}}, \top_{\mathbf{B}^{\bowtie}}, \bot_{\mathbf{B}^{\bowtie}} \rangle$ is a bounded distributive lattice. This is matter of a direct verification.

□

Consider now given a $N$-lattice $\mathbf{A}$. Recall that we can consider a $\Gamma$-congruence $\theta_{\mathbf{A}}$ over $\mathbf{A}$ defined as $\langle a, b \rangle \in \theta_{\mathbf{A}}$ iff $(a \leftrightarrow_{\mathbf{A}} b) = \top_{\mathbf{A}}$.

We can then consider a $\Sigma_N^o$-algebra $\mathbf{A}^{\bowtie}$ such that:

- $(\mathbf{A}^{\bowtie})_v = (\mathbf{A}_{|\Gamma})/_{\theta_{\mathbf{A}}}$;
- $(\mathbf{A}^{\bowtie})_\phi = \mathbf{A}$;
- $o_{\mathbf{A}^{\bowtie}}(a) = [a]_{\theta_{\mathbf{A}}}$ for every $a \in A$.

**Theorem 5.8.4.** *If* $\mathbf{A}$ *is a* $N$*-lattice then* $\mathbf{A}^{\bowtie} \in K_N^\Gamma$.

*Proof.* First of all, note that it is well-known that $(\mathbf{A}_{|\Gamma})/_{\theta_{\mathbf{A}}}$ is a Heyting algebra [Vak77, Kra98].

From the definition of $N$-lattice we can conclude that $[\sim(\xi_1 \to \xi_2)]_{\theta_{\mathbf{A}}} = [(\xi_1 \wedge \sim \xi_2)]_{\theta_{\mathbf{A}}}$, $[\sim(\xi_1 \wedge \xi_2)]_{\theta_{\mathbf{A}}} = [(\sim \xi_1 \vee \sim \xi_2)]_{\theta_{\mathbf{A}}}$, $[\sim(\xi_1 \vee \xi_2)]_{\theta_{\mathbf{A}}} = [(\sim \xi_1 \wedge \sim \xi_2)]_{\theta_{\mathbf{A}}}$, $[(\sim \neg \xi_1)]_{\theta_{\mathbf{A}}} = [\xi_1]_{\theta_{\mathbf{A}}}$, $[(\sim\sim \xi_1)]_{\theta_{\mathbf{A}}} = [\xi_1]_{\theta_{\mathbf{A}}}$ and $[(\sim \xi_1 \vee \neg \xi_1)]_{\theta_{\mathbf{A}}} = [\neg \xi_1]_{\theta_{\mathbf{A}}}$. Therefore, the result follows from the observation that, by construction, $\theta_{\mathbf{A}}$ is indeed $\equiv_\Gamma$, the $\Gamma$-behavioral equivalence on $\mathbf{A}^{\bowtie}$.

□

We end this example with some conclusions. The first one is that with our approach we are able to make explicit the key role that Heyting algebras play in the algebraic counterpart of $N$. The algebras obtained by behavioral algebraization can be seen as $N$-lattices in a different perspective. Furthermore, our goal is not to provide an alternative to $N$-lattices, but only to provide one more tool for the study of the system $N$ and, in particular, to the study of $N$-lattices.

Note that this example is just a first example of the application of our behavioral theory to the study of algebraizable logics. Of course, due to the large amount of research on $N$-lattices, we did not arrived to a novel major result or conclusion. Nevertheless, in logics with less studied semantics, our approach can help to unveil some interesting algebraic results and, moreover, to shed some light on the relation between different equivalences in a given logic, as it was the case of intuitionistic equivalence and strong equivalence.

## 5.9 Remarks

In Chapter 5 we present some examples to further illustrate the relevance of our new approach to the algebraization of logics. In the first example, we show that our behavioral approach is indeed an extension of the existing tools of both AAL [FJP03]. Next we prove that our behavioral approach is also an extension of the many-sorted work done in AAL [CG07]. In the many-sorted example we also present some non-behavioral many-sorted definitions and results that can be useful when applying the theory to particular examples of logics. We proceed with the example of paraconsistent logic $\mathcal{C}_1$ of da Costa, whose non-algebraizability in the standard sense is again well-known. We show that it is behaviorally algebraizable and, moreover, we give a algebraic counterpart for each of them. Recall that, although the standard non-algebraizability of $\mathcal{C}_1$ is well-known, there have been some proposals of algebraic counterparts of $\mathcal{C}_1$. Of course, since $\mathcal{C}_1$ is not algebraizable, their precise connection with $\mathcal{C}_1$ could never be established at the light of the standard tools of AAL. We prove that the class of algebras that our approach canonically associates with $\mathcal{C}_1$ coincides with one of these existing proposals, thus explaining its precise connection with $\mathcal{C}_1$. We also study the example of the Carnap-style presentation of modal logic $S5$, whose non-algebraizability in the standard sense is well-known [BP89]. We prove that $S5$ is behaviorally algebraizable and we propose an algebraic counterpart for it. We continue by briefly analyzing the example of first order logic $FOL$, whose standard algebraization is well-studied [BP89, ANS01]. Our approach can be useful to shed light on the essential distinction between terms and formulas. In the example of global logic we follow the exogenous semantic approach for enriching a logic [MSS05] and present a sound a complete deductive system for global logic $GL(\mathcal{L})$ over a given local logic $\mathcal{L}$. We also prove that $GL(\mathcal{L})$ is behaviorally algebraizable independently of $\mathcal{L}$. Moreover, we prove that in the cases where $\mathcal{L}$ is algebraizable we are able to recover the algebraic counterpart of $\mathcal{L}$ from the algebraic counterpart of $GL(\mathcal{L})$. Still following the exogenous semantic approach for enriching a logic we present the example of exogenous propositional probability logic EPPL. We prove that EPPL is behaviorally algebraizable and provide an algebraic counterpart for it. We proceed by exemplifying the power of our approach showing that it can be directly applied to study the algebraization of $k$-deductive systems [Mar04]. Finally, we study the example of Nelson logic $N$, which is algebraizable according to the standard definition [Ras81], but its behavioral algebraization can help to give an extra insight to the role of Heyting algebras in the algebraic counterpart of $N$.

# Chapter 6

# Conclusion

We conclude this dissertation with a summary of its main contributions and limitations. Although we hope to have contributed for a generalization of the scope of applicability of AAL, this work is just a starting point towards a full-blown theory of behavioral tools in AAL. Moreover, in order to make the extended theory useful and assess its merits in full, a comprehensive treatment of interesting examples is essential. Therefore, we will close this concluding session with an outline of future work.

## 6.1 Summary of contributions

As far as contributions are concerned, we should mention three major aspects.

First of all, in Chapter 2 and in the beginning of Chapter 3, we envisaged and developed a behavioral framework in which the intended generalization of the notion of algebraizable logic could be fulfilled. We should refer the importance of the idea of considering an extended signature, since this particular feature allowed us to reason about formulas in a behavioral manner.

This leap was not only motivated by concrete examples, but is also well-supported by the consistent development of behavioral logic, and by the fact that in a logic we can only observe the behavior of terms or other syntactic entities indirectly, through their influence on the logical value of the formulas where they appear.

The second contribution we should mention is the fact that, along with the definition of behaviorally algebraizable logic, we have proposed a novel behavioral extension of the standard tools and results of AAL, using many-sorted behavioral logic instead of unsorted equational logic. Our aim was to broaden its range of application to richer and less orthodox logics. We have introduced the novel notion of behaviorally algebraizable logic and proved some necessary conditions for a logic to be behaviorally algebraizable, namely involving the notion of behaviorally equivalential logic and the notion of set of equivalence formulas. We have shown how behavioral algebraization indeed generalizes the standard notion, while further encompassing in a natural way logics whose algebraization was not possible before. Still, we have proved that the behavioral approach remains non-trivial, and actually within the range of protoalgebraizability.

We then envisaged a behavioral generalization of the key notion of Leibniz operator. This was obtained by replacing congruences by $\Gamma$-congruences, where $\Gamma$ is a subsignature of the original signature. A $\Gamma$-congruence is an equivalence relation compatible with the operations in the subsignature $\Gamma$. This behavioral version of the Leibniz operator proved to be the right one since we were able to engage on a generalization of the Leibniz hierarchy. We introduced behavioral versions of the notion of protoalgebraic logic and of weakly algebraizable logic, along with generalizations of several of their standard characterization results. Characterization results for the class of behaviorally algebraizable and behaviorally equivalential logics were also obtained. Useful intrinsic and sufficient conditions for a logic to be behaviorally algebraizable were obtained. We have dedicated Chapter 3 to these tasks.

We then engage on the behavioral generalization of the notion of matrix semantics along with several of standard semantical results. This was done in Chapter 4. We have characterized the class of algebras that our behavioral approach canonically associates with a given behaviorally algebraizable logic. We proved a unicity result with respect to the algebraic counterpart of a behaviorally algebraizable logic. We were able to provide a mechanism that allows us to produce the axiomatization of the algebraic counterpart of a behaviorally algebraizable logic $\mathcal{L}$ from the deductive system of $\mathcal{L}$. We have proposed two possible extensions of the notion of matrix semantics to the many-sorted behavioral setting. Along with the first proposal, based on an immediate generalization of the notion of logical matrix, we proved several interesting results. Namely, we were able to canonically associate a class $Alg_\Gamma(\mathcal{L})$ of algebras to every logic $\mathcal{L}$ and moreover, to prove that for a $\Gamma$-behaviorally algebraizable logic this class coincides with its largest $\Gamma$-behaviorally equivalent algebraic semantics. For a logic $\mathcal{L}$ which is non-algebraizable logic and is $\Gamma$-behaviorally algebraizable the advantage of considering $Alg_\Gamma(\mathcal{L})$ instead of $Alg(\mathcal{L})$ is clear: the behavioral consequence associated with $Alg_\Gamma(\mathcal{L})$ has a strong relation with $\mathcal{L}$ and such a feature is not shared by the equational consequence of $Alg(\mathcal{L})$. Of course, when a logic $\mathcal{L}$ is algebraizable and also $\Gamma$-behaviorally algebraizable, the relation between $Alg_\Gamma(\mathcal{L})$ and $Alg(\mathcal{L})$ should be studied more carefully. We proved some interesting relations between these two classes of algebras. We developed a second generalization of the notion of logical matrix around the ideas of valuation semantics. We were able to give an algebraic flavor to this kind of semantics. We proved a completeness theorem with respect to the class $Mod_\Gamma(\mathcal{L})$ of all Γ-valuation models and we proved also a result relating a metalogical property of a logic $\mathcal{L}$ and an algebraic property of $Mod_\Gamma(\mathcal{L})$. From the algebraic counterpart $K$ of a Γ-behaviorally algebraizable logic $\mathcal{L}$ we were able to produce a class $\mathcal{M}_K$ of Γ-valuations, which was shown to be complete with respect to $\mathcal{L}$. Although the main aim of this algebraic version of valuation semantics was its connection with AAL, we developed it as the starting point to the study of a very general and useful class of semantic structures. This algebraic version of valuation semantics has matrix semantics and valuation semantics as particular cases and can be seen as having the best of both approaches.

The third major contribution was the fruitful application of the theory developed in Chapters 3 and 4 to some concrete examples. At this point we should mention that we provided, in Chapters 3 and 4, all the necessary tools to analyze concrete logics. In one hand we gave, in Chapter 3, sufficient conditions to prove that a concrete logic is behaviorally algebraizable, and, in this case, we also provided tools to obtain its behaviorally algebraic counterpart. On the other hand, we gave necessary conditions for behavioral algebraizability which can be used to prove that a given logic is not behaviorally algebraizable. Of course, in practice, it is hard to prove that a given logic fails to fulfill these necessary conditions. Therefore, in Chapter 4, we gave a semantical characterization of behavioral algebraizability that is very useful to prove non-behavioral algebraizability of a given concrete logic.

In the first example we showed that our behavioral approach is indeed an extension of the existing unsorted and many-sorted tools of AAL. Our results are encouraging in that they allow us to shed new light over logics like $\mathcal{C}_1$, whose non-algebraizability in the standard sense is well-known. In the case of $\mathcal{C}_1$, there have been, nevertheless, some proposals of algebraic counterparts in the literature. Of course, since $\mathcal{C}_1$ is not algebraizable, their precise connection with $\mathcal{C}_1$ could never be established at the light of the standard tools of AAL. Using our behavioral tools we were able to draw the precise connection between $\mathcal{C}_1$ and the class of so-called da Costa algebras. We were also able to obtain the well-known two-valued non-truth-functional semantics for $\mathcal{C}_1$ from the behavioral equivalent algebraic counterpart of $\mathcal{C}_1$. Furthermore, even the behavioral analysis of logics which are algebraizable according to the standard notion proved to be useful. For instance, in the case of Nelson's logic, behavioral algebraization helps to understand better the connection between N-lattices and Heyting algebras. Other interesting examples we have studied are those in the family of exogenous (global and probabilistic) logics. We presented a sound a complete deductive system for global logic $GL(\mathcal{L})$ over a given local logic $\mathcal{L}$. We also proved that $GL(\mathcal{L})$ is behaviorally algebraizable independently of $\mathcal{L}$. Moreover, we were able to recover the algebraic counterpart of $\mathcal{L}$ from the algebraic counterpart of $GL(\mathcal{L})$. We proved that EPPL is behaviorally algebraizable and provided an algebraic counterpart for it. The example of the Carnap-style presentation of modal logic $S5$ was studied. We proved that $S5$ is behaviorally algebraizable and proposed an algebraic counterpart for it. In the example of first order logic FOL, our approach was useful to shed light on the essential distinction between terms and formulas. We exemplified the power of our approach by showing that it can be directly applied to study the algebraization of $k$-deductive systems. This was done in Chapter 5.

## 6.2 Limitations and future work

With respect to limitations, one of the main aspects we should mention is the lack of an exhaustive set of interesting examples. Indeed, in order to make the extended theory useful and assess its merits in full, a comprehensive treatment of interesting examples is essential. Important examples are those separating classes in the behavioral Leibniz hierarchy. Another class of interesting examples are those logics, like $\mathcal{C}_1$ that are not algebraizable according to the standard notion, but to which some algebraic counterpart has been proposed in the literature. Our approach could then be the right framework where the connection between the logic and its algebraic counterpart could be established. Moreover, continuing the work done in the examples of exogenous global and probabilistic logics stemming from [MS06], it would be very interesting to study the behavioral algebraization of exogenous quantum logic (EQPL) [MS06]. This could shed some new light on the connection between EQPL and the traditional algebraic based quantum logic of Birkhoff and von Neumann [BvN36]. However, it seems to us that, in order to be compared with the traditional quantum logic, EQPL needs first to be extended with the notion of evolution.

We have noted several times in the dissertation, that most of the definitions are parametrized by the choice of the subsignature $\Gamma$. However, there is no way of associating a single $\Gamma$ to a given logic. Therefore, we can no longer guarantee uniqueness with respect to equivalences in a logic. We would like to have an exhaustive study of the relationship between existing equivalence sets for a given logic, and their impact on the distinct behavioral algebraizations of the same logic that can be obtained using distinct and non-interderivable equivalences. We should note, nevertheless, that the possibility of behaviorally algebraizing a logic with different subsignatures can help on a better understanding of the logic, as suggested by the example of Nelson's logic.

Another aspect of our work is the intentional mismatch between the signature of a logic and the algebraic signature. The algebras we associate to a given behaviorally algebraizable logic are not over the original signature of the logic, but over an extended signature obtained from it. Moreover, we are not able to associate to a logic a class $Mat^*$ of reduced matrices. This is due to the fact that, since we are dealing with $\Gamma$-congruences, we cannot perform quotients. The best we can do is to use the extended signature and algebras over the extended signature to simulate the quotient. Although the approach using valuation semantics seems to overcome some of these difficulties, it is not clear what should be the natural path to follow in what semantics is concerned. For example, in the case of valuation semantics, we were not able to provide a $\Gamma$-behavioral valuation semantics for every logic, but just for the $\Gamma$-behaviorally algebraizable logics. The existing mismatch in the semantic approach clearly seems to lead to the exploration of more suitable alternatives, including Avron's non-deterministic matrices [Avr05], or perhaps even gaggles [Dun91].

On the methodological side, this paper is just a starting point towards a full-blown theory of behavioral AAL. Such a development will need time and effort to be consolidated. We would like to obtain behavioral versions of other classes of logic in the Leibniz hierarchy, along with their respective characterization results. Still, it is possible to put forth a few directions that should clearly be pursued. For example, the definition of behaviorally equivalential logic presented here is syntactic, but we are pretty sure that model theoretic characterizations (closure properties of the class of reduced models), similar to the standard ones obtained by AAL, could be established. The theory certainly needs many more semantic results, namely involving metalogical properties of a given behaviorally algebraizable logic and algebraic properties of the class of algebras associated with it. The precise role of the parametric variables of the contexts should also deserve further analysis. Another topic that deserves a close look is the application of our approach to the systematic study of the interplay between systems of equivalence and the detachment deduction theorem, as suggested by the example of Nelson's logic. This interplay seems to play an important role in the not very well understood Fregean hierarchy.

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