Computability, Noncomputability, and Hyperbolic Systems

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Abstract

In this paper we study the computability of the stable and unstable manifolds of a hyperbolic equilibrium point. These manifolds are the essential feature which characterizes a hyperbolic system, having many applications in physical sciences and other fields. We show that (i) locally these manifolds can be computed, but (ii) globally they cannot, since their degree of computational unsolvability lies on the second level of the Borel hierarchy. We also show that Smale's horseshoe, the first example of a hyperbolic invariant set which is neither an equilibrium point nor a periodic orbit, is computable.

1 Introduction

Dynamical systems are powerful objects which can be found in numerous applications. However, this versatility does not come without cost: dynamical systems are objects which are inherently very hard to study.

Recently, digital computers have been successfully used to analyze dynamical systems, through the use of numerical simulations. However, with the known existence of phenomena like the "butterfly effect" – a small perturbation on initial conditions can be exponentially amplified along time – and the fact that numerical simulations always involve some truncation errors, the reliability of such simulations for providing information about the long-term evolution of the systems is questionable.

For instance, numerical simulations suggested the existence of a "strange" attractor for the Lorenz system [Lor63], and it was widely believed that such an attractor existed. However, the formal proof of its existence remained elusive, being at the heart of the 14th from the list of 18 problems that the Fields medalist S. Smale presented for the new millennium [Sma98]. Finally, after a 35 year hiatus, a computer-aided formal proof was achieved in [Tuc98], [Tuc99], but this example painstakingly illustrates the difference between numerical evidence and a full formal proof, even if computer-based.

For the above reasons, it is important to understand which properties can be accurately computed with a computer, and those which cannot. We have previously dwelled on this subject on our paper [GZar] which focused on "stable" dynamical systems having hyperbolic attractors. The latter systems were extensively studied in the XXth century and were thought to correspond to the class of "meaningful" dynamical systems. This happened for several reasons: the "stability" – structural stability – which implies robustness of behavior to small perturbations was believed to be of uttermost importance, since it was thought that only such systems could exist in nature; there were also results (Peixoto's Theorem [Pei62]) which showed that, in the plane, these systems are dense and their invariant sets (which include all attractors) can be fully characterized (they can only be points or periodic orbits).There was hope that such results would generalize for spaces of higher dimensions, but it was shattered by S. Smale, who showed that there exist "chaotic" hyperbolic invariant sets which are neither a point nor a periodic orbit (Smale's horseshoe [Sma67]) and that for dimensions ≥ 3 structurally stable systems are not dense [Sma66].

Moreover, the notion of structural stability was shown to be too strong a requirement. In particular, the Lorenz attractor, which is embedded in a system modeling weather evolution, is not structurally stable, although it is stable to perturbations in the parameters defining the system [Via00] (in other words, robustness may not be needed for *all* mathematical properties of the system).

Despite the fact that systems with hyperbolic attractors are no longer regarded as "the" meaningful class of dynamical systems for spaces of dimension $n \geq 3$, they still play a central role in the study of dynamical systems. In essence, what characterizes a hyperbolic set is the existence at each point of an invariant splitting of the tangent space into stable and unstable directions, which generate the local stable and unstable manifolds (see Section 2).

In this paper we will study the computability of the stable and unstable manifolds for hyperbolic equilibrium points. The stable and unstable manifolds are constructs which derive from the stable manifold theorem (see Theorem 2). This theorem states that for each hyperbolic equilibrium point x_0 (see Section 2 for a definition) of an open subset of $\mathbb{R}^n$, there exists a k-dimensional manifold S such that a trajectory on S will converge to x_0 at an exponential rate as $t \to +\infty$. S is called a (local) stable manifold. Similarly one can obtain an $(n-k)$-dimensional manifold, a (local) unstable manifold U, using similar conditions when $t \to -\infty$. (The stable and unstable manifolds of a hyperbolic equilibrium point are depicted in Fig. 1.) It is important to note that classical proofs on existence of S and U are non-constructive. Thus computability of these manifolds does not follow straightforwardly from these proofs.

We address the following basic question: given a dynamical system and some hyperbolic equilibrium point, are stable and unstable manifolds computable? We will show that the answer is twofold: (i) we can compute a stable manifold S and an unstable manifold U given by the stable manifold theorem which are, in some sense, local, since there are (in general) trajectories starting in points which do not lie in S that will converge to x_0 (the set of all these point united with S would yield the global stable manifold – see Section 2), but (ii) the global stable and unstable manifolds are not, in general, computable from the description of the system and the hyperbolic point. Since classical proofs on existence of S and U are non-constructive, a different approach is needed if one wishes to construct an algorithm that computes S and U. Our approach aiming at a constructive proof makes use of function-theoretical treatment of the resolvents (see the first paragraph of Section 4 for more details).

A most likely interpretation for these results is that, since the definition of hyperbolicity is local, computability also applies locally. However global homoclinic tangles can occur [GH83], leading to chaotic behavior for such systems. So it should not be expected that the global behavior of stable and unstable manifolds be globally computable in general, as indeed our results show.

In the end of the paper we also show that the prototypical example of an hyperbolic invariant set which is neither a fixed point nor a periodic orbit – the Smale horseshoe – is computable.

The computability of simple attractors – hyperbolic periodic orbits and equilibrium points – was studied in our previous papers [Zho09], [GZar] for planar dynamics, as well as the computability of their respective domains of attraction.

A number of papers study dynamical systems, while not exactly in the context used here. For example the paper [Moo90] shows that shift maps can simulate Turing machines. Since many systems of physical inspiration contain within their dynamics horseshoe maps which are equivalent to the shift, the long-term behavior of such systems is undecidable. In [BBKT01] it is shown that a problem concerning a long-term property (stability) of a class of dynamical systems whose dynamics is piecewise linear is also undecidable. In [Col05] it is shown that the reachable set (set of all points which can be reached from trajectories starting in a given set) is lower semi-computable and only computable in some special cases. In [BY06] it is shown that Julia sets are not computable from the maps defining them. The articles [Spa07], [Spa08], [Hoy07], [HKL08], [GHRar], and other papers by the same authors, provide interesting results about dynamical systems. They use a symbolic dynamics and/or a statistical approach and are interested in computability questions about entropy, measures, and related notions. Another interesting paper is [Zho09], where Zhong shows that the problem of calculating the domain of attraction of dynamical systems defined by C^∞-functions is not, in general, computable. See [BGZar] for a more detailed review of the literature.

2 Dynamical systems

In short, a dynamical system is a pair consisting of a state space where the action occurs and a function f which defines the evolution of the system along time. See [HS74] for a precise definition. In general one can consider two kinds of dynamical systems: discrete ones, where time is discrete and one obtains the evolution of the system by iterating the map f, and continuous ones, where the evolution of the system along time is governed by a differential equation of the type

$$\dot{x} = f(x) \tag{1}$$

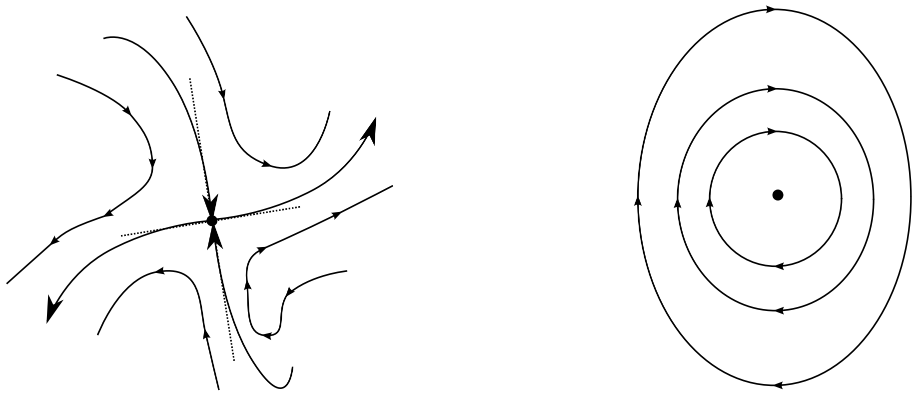

Figure 1: The figure on the left represents dynamics near a hyperbolic equilibrium point. The figure on the right depicts the dynamics near a non-hyperbolic equilibrium point.

where t is the independent variable (the "time") and $\dot{x}$ denotes the derivative $dx(t)/dt$. Continuous-time systems can be translated to discrete-time using time-one maps and, in some cases, the Poincaré map, and vice-versa (using the suspension method). Therefore to study dynamical systems one can focus on continuous-time ones.

Although the essential feature of hyperbolic attractors is the existence of an invariant splitting of the tangent space into stable and unstable directions generating the local stable and unstable manifolds, for a hyperbolic equilibrium point x_0, it can be described equivalently in terms of the linearization of the flow around x_0. We recall that x_0 is an equilibrium point of (1) iff $f(x_0) = 0$.

Definition 1 *An equilibrium point x_0 of (1) is hyperbolic if none of the eigenvalues of $Df(x_0)$ has zero real part.*

According to the value of the real part of these eigenvalues, one can determine the behavior of the linearized flow near x_0: if the eigenvalue has negative real part, then the flow will converge to x_0 when it follows the direction given by the eigenvector associated to this eigenvalue. A similar behavior will happen when the eigenvalue has positive real part, with the difference that convergence happens when $t \to -\infty$. See Fig. 1. The space generated by the eigenvectors associated to eigenvalues of $Df(x_0)$ with negative real part is called the stable subspace $E^s_{Df(x_0)}$, and the space generated by the eigenvectors associated to eigenvalues with positive real part is called the unstable subspace $E^u_{Df(x_0)}$.

We now state the Stable Manifold Theorem (as it appears in [Per01]).

Theorem 2 *[Stable Manifold Theorem]Let E be an open subset of $\mathbb{R}^n$ containing the origin, let $f \in C^1(E)$, and let ϕ_t be the flow of the system (1). Suppose that $f(0) = 0$ and that $Df(0)$ has k eigenvalues with negative real part and $n-k$ eigenvalues with positive real part (*i.e. *0 is a hyperbolic equilibrium point). Then there exists a k-dimensional differentiable manifold S tangent to the stable subspace $E^s_{Df(0)}$ such that for all $t \geq 0$, $\phi_t(S) \subseteq S$ and for all $x_0 \in S$*

$$\lim_{t \to +\infty} \phi_t(x_0) = 0;$$

and there exists an $n-k$ dimensional differentiable manifold U tangent to the unstable subspace $E^u_{Df(0)}$ such that for all $t \leq 0$, $\phi_t(S) \subseteq S$ and for all $x_0 \in U$

$$\lim_{t\to-\infty} \phi_t(x_0) = 0.$$

From the local stable and unstable manifolds given by the Stable Manifold Theorem, one can define the *global stable and unstable manifolds* of (1) at a hyperbolic equilibrium point x_0 by

$$W^s_f(x_0) = \bigcup_{j=0}^{\infty} \phi_{-j}(S) \tag{2}$$
$$W^u_f(x_0) = \bigcup_{j=0}^{\infty} \phi_j(U),$$

respectively, where

$$\phi_t(A) = \{x(t)|x \text{ is a solution of (1) with } x(0) \in A\}.$$

We note that $W^s(x_0)$ and $W^u(x_0)$ are F_σ-sets of $\mathbb{R}^n$ (recall that a subset $F \subseteq \mathbb{R}^n$ is called an F_σ-set if $F = \bigcup_{j=0}^{\infty} A_j$, where A_j, $j \in \mathbb{N}$, is a closed subset of $\mathbb{R}^n$). This is due to the fact that S and U are compact (if they are not closed, intersect them with a sufficiently small closed ball with center x_0 and pick these intersections as S and U in the definition of $W^s(x_0)$ and $W^u(x_0)$, respectively) and ϕ_{-j} and ϕ_j are continuous functions.

We end this section by introducing Smale's horseshoe [Sma67]. We refer the reader to [GH83] for a more thorough discussion of this set. In essence, Smale's horseshoe appears when we consider a map f defined over $S = [0,1]^2$. This map is bijective and performs a linear vertical expansion of S, and a linear horizontal contraction of S, by factors $\mu > 1$ and $0 < \lambda < 1$, respectively, followed by a folding.

Definition 3 *In the conditions defined above (see [GH83] or [HSD04] for precise definitions), the Smale horseshoe is the set Λ given by*

$$\Lambda = \bigcap_{j=-\infty}^{+\infty} f^j(S).$$

This set is invariant for the function f (i.e. $f(\Lambda) = \Lambda$).

3 Computable analysis

We now introduce concepts related to computability. The theory of computation is rooted in the seminal work of Turing, Church, and others, which provided a framework in which to achieve computation over discrete identities or, equivalently, over the integers.

However, this definition was not enough to cover computability over continuous structures, and was then developed by other authors such as Turing himself [Tur36], Grzegorczyk [Grz57], or Lacombe [Lac55] to originate *computable analysis*.

The idea underlying computable analysis to compute over a set A is to encode each element of A by a countable sequence of symbols from a finite alphabet Σ, using representations (see [Wei00] for a complete development). A represented space is a pair $(X;\delta)$ where X is a set, $\mathrm{dom}(\delta) \subseteq \Sigma^{\mathbb{N}}$, and $\delta :\subseteq \Sigma^{\mathbb{N}} \to X$ is an onto map ("$\subseteq \Sigma^{\mathbb{N}}$" is used to indicate that the domain of δ may be a subset of $\Sigma^{\mathbb{N}}$). Every $q \in \mathrm{dom}(\delta)$ such that $\delta(q) = x$ is called a δ-name of x (or a name of x when δ is clear from context). Thus to perform a computation over a represented space, it suffices to work with names only. To compute with names, we use Type-2 machines [Wei00], which are similar to Turing machines, but (i) have a read-only tape, where the input (i.e. the sequence encoding it) is written; (ii) have a write-only output tape, where the head cannot move back and the sequence encoding the output is written. At any finite amount of time we can halt the computation, and we will have a partial result on the output tape. The more time we wait, the more accurate this result will be. Other equivalent approaches to computable analysis can be found in [PER89], [Ko91].

The notion of computable maps between represented spaces now arises naturally.

Definition 4 *A map $\Phi : (X;\delta_X) \to (Y;\delta_Y)$ between two represented spaces is computable if there is a computable map $\phi :\subseteq \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}}$ such that $\Phi \circ \delta_X = \delta_Y \circ \phi$. Informally speaking, this means that there is a Type-2 machine such that on any name of $x \in X$, the machine computes as output a name of $f(x) \in Y$.*

In this paper, we use the following particular representations for real numbers; for points in $\mathbb{R}^n$; for open and closed subsets of $\mathbb{R}^n$; and for functions in $C^1(E;\mathbb{R}^n)$, the set of all continuously differentiable functions defined on an open subset $E \subseteq \mathbb{R}^n$ with ranges in $\mathbb{R}^n$ (these representations can be found in [Wei00] with various prefixes such as ρ-names for real numbers; since we only use certain particular representations, we omit the prefixes for readability):

Definition 5 *(1) A sequence $\{r_k\}$ of rational numbers is called a name of a real number x if there are three functions a, b and c from $\mathbb{N}$ to $\mathbb{N}$ such that for all $k \in \mathbb{N}$, $r_k = (-1)^{a(k)} \frac{b(k)}{c(k)+1}$ and*

$$|r_k - x| \leq \frac{1}{2^k}. \tag{3}$$

(2) A double sequence $\{r_{l,k}\}_{l,k\in\mathbb{N}}$ of rational numbers is called a name of a sequence $\{x_l\}_{l\in\mathbb{N}}$ of real numbers if there are three functions a, b, c from $\mathbb{N}^2$ to $\mathbb{N}$ such that, for all $k, l \in \mathbb{N}$, $r_{l,k} = (-1)^{a(l,k)} \frac{b(l,k)}{c(l,k)+1}$ satisfies

$$|r_{l,k} - x_l| \leq \frac{1}{2^k}.$$

(3) A sequence $\{(r_{1k}, r_{2k}, \ldots, r_{nk})\}_{k\in\mathbb{N}}$ of rational vectors is called a name of a point $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ if $\{r_{jk}\}_{k\in\mathbb{N}}$ is a name of x_j, $1 \leq j \leq n$.

(4) A real number x (a sequence $\{x_l\}_{l\in\mathbb{N}}$ of real numbers) is called computable if it has a computable name, i.e. the functions a, b, and c are computable or, equivalently, there is a Type-2 machine that computes a name of x ($\{x_l\}_{l\in\mathbb{N}}$, respectively) without any input.

There are different representations for real numbers; several of them give rise to the same set of computable real numbers (e.g. a name of a real number x_0 could be a sequence of intervals containing x_0 with diameter $1/k$ for $k \geq 1$, etc.). A notable exception is the decimal expansion of x_0, which cannot be used since it can lead to undesirable behavior [Tur37] because this representation does not respect the topology of the space $\mathbb{R}$.

Definition 6 *1. Let $E \subseteq \mathbb{R}^n$ be an open subset. A name of E is a sequence of rational open balls $B(a_k, r_k) = \{x \in \mathbb{R}^n : |x - a_k| < r_k\}$, where $a_k \in \mathbb{Q}^n$ and $r_k \in \mathbb{Q}$, such that*

$$E = \cup_{k=0}^{\infty} B(a_k, r_k).$$

Without loss of generality one can also assume that for any $k \in \mathbb{N}$, the closure of $B(a_k, r_k)$, denoted as $\overline{B(a_k, r_k)}$, is contained in E. The set E is called recursively enumerable (r.e. for short) open if $\{a_k\}$ and $\{r_k\}$ are computable sequences.

2. *Let $A \subseteq \mathbb{R}^n$ be a closed subset. A sequence $\{x_k\}$, $x_k \in \mathbb{R}^n$, is called a name of A if it is dense in A. A is called r.e. closed if $\{x_k\}$ is a computable sequence.*

3. *An open set $E \subseteq \mathbb{R}^n$ is called computable (or recursive) if E is r.e. open and its complement E^c is r.e. closed. Similarly, a closed set $A \subseteq \mathbb{R}^n$ is called computable if A is r.e. closed and its complement A^c is r.e. open.*

4. *Let $K \subseteq \mathbb{R}^n$ be a compact subset. A pair $(r, \{x_k\})$, $r \in \mathbb{Q}$ and $x_k \in \mathbb{R}^n$, is called a name of K if $K \subset B(0, r)$ and $\{x_k\}$ is a name of K as a closed set. K is called computable if $K \subset B(0, r)$ and it is computable as a closed set.*

In the rest of the paper, we will work exclusively with C^1 functions $f : E \to \mathbb{R}^n$, where E is an open subset of $\mathbb{R}^n$. Thus it is desirable to present an explicit representation for such functions.

Definition 7 *Let $E = \bigcup_{k=0}^{\infty} B(a_k, r_k)$, $a_k \in \mathbb{Q}^n$ and $r_k \in \mathbb{Q}$, be an open subset of $\mathbb{R}^n$ (assuming that the closure of $B(a_k, r_k)$ is contained in E) and let $f : E \to \mathbb{R}^n$ be a continuously differentiable function. Then a (C^1) name of f is a double sequence $\{P_{l,k}\}$ of polynomials ($P_{l,k} : \mathbb{R}^n \to \mathbb{R}^n$) with rational coefficients satisfying $d_k(P_{l,k}, f) < 2^{-l}$, where*

$$d_k(P_{l,k}, f) = \sup_{x \in \overline{B(a_k, r_k)}} |P_{l,k}(x) - f(x)| + \sup_{x \in \overline{B(a_k, r_k)}} |DP_{l,k}(x) - Df(x)|.$$

Thus, f is computable if there is a (type-2) Turing machine that outputs $P_{l,k}$ (more precisely coefficients of $P_{l,k}$), on input l, k, satisfying $d_k(P_{l,k}, f) < 2^{-l}$.

We observe that the above representation of f is well-defined and contains information on both f and Df in the sense that $\{P_{l,k}\}$ is a name of f while $\{DP_{l,k}\}$ is a name of Df. See [ZW03] for further details.

Throughout the rest of this paper, unless otherwise mentioned, we will assume that, in (1), f is continuously differentiable on an open subset of $\mathbb{R}^n$ and we will use the above name for f.

4 Computable stable manifold theorem

The stable manifold theorem states that near a hyperbolic equilibrium point x_0, the nonlinear system

$$\dot{x} = f(x(t)) \tag{4}$$

has stable and unstable manifolds S and U tangent to the stable and unstable subspaces $\mathbb{E}^s_A$ and $\mathbb{E}^u_A$ of the linear system

$$\dot{x} = Ax \tag{5}$$

where $A = Df(x_0)$ is the gradient matrix of f at x_0. The classical proof of the theorem relies on the Jordan canonical form of A. To reduce A to its Jordan form, one needs to find a basis of generalized eigenvectors. Since the process of finding eigenvectors from corresponding eigenvalues is not continuous in general, it is a non-computable process. Thus if one wishes to construct an algorithm that computes some S and U of (4) at x_0, a different method is needed. We will make use of an analytic, rather than algebraic, approach to the eigenvalue problem that allows us to compute S and U without calling for eigenvectors. The analytic approach is based on function-theoretical treatment of the resolvents (see, *e.g.*, [SN42], [Kat49], [Kat50], and [Rob95]).

Let us first show that the stable and unstable subspaces are computable from A for the linear hyperbolic systems $\dot{x} = Ax$. We begin with several definitions. Let $\mathfrak{A}_H$ denote the set of all $n \times n$ matrices such that the linear differential equation $\dot{x} = Ax$, $x \in \mathbb{R}^n$, is hyperbolic, where a linear differential equation $\dot{x} = Ax$ is defined to be hyperbolic if all the eigenvalues of A have nonzero real part. The Hilbert-Schmidt norm is used for $A \in \mathfrak{A}_H$: $\|A\| = \sqrt{\sum_{i=1}^n \sum_{j=1}^n |a_{ij}|^2}$, where a_{ij} is the ij^{th} entry of A. For each $A \in \mathfrak{A}_H$, define the stable subspace $\mathbb{E}^s_A$ and unstable subspace $\mathbb{E}^u_A$ to be

$$\begin{aligned}\mathbb{E}^s_A = \text{span}\,\{v \in \mathbb{R}^n : & \ v \text{ is a generalized eigenvector for an eigenvalue } \lambda \text{ of } A \\ & \text{with } Re(\lambda) < 0\}\end{aligned}$$

$$\begin{aligned}\mathbb{E}^u_A = \text{span}\,\{v \in \mathbb{R}^n : & \ v \text{ is a generalized eigenvector for an eigenvalue } \lambda \text{ of } A \\ & \text{with } Re(\lambda) > 0\}\end{aligned}$$

(We recall that v is a generalized eigenvector for an eigenvalue λ of multiplicity m if v is a nonzero solution of the equation $(A - \lambda I)^k = 0$, for some integer k satisfying $1 \le k \le m$.) Then $\mathbb{R}^n = \mathbb{E}^s_A \bigoplus \mathbb{E}^u_A$. The stable subspace $\mathbb{E}^s_A$ is the set of all vectors which contract exponentially forward in time while the unstable subspace $\mathbb{E}^u_A$ is the set of all vectors which contract exponentially backward in time.

As mentioned above, the process of finding eigenvectors from corresponding eigenvalues is not computable; thus the algebraic approach to $\mathbb{E}^s_A$ and $\mathbb{E}^u_A$ is a non-computable process. Of course, this doesn't necessarily imply that it is impossible to compute $\mathbb{E}^s_A$ and $\mathbb{E}^u_A$ from A, but rather this particular classical approach fails to be computable. So even for the linear hyperbolic system $\dot{x} = Ax$, one needs a different approach to treat the stable/unstable subspace when computability is concerned. The approach used below to treat the stable/unstable subspace is analytic, rather than algebraic. Let $p_A(\lambda)$ be the characteristic polynomial for A, γ_1 be any closed curve in the left half of the complex plane that surrounds (in its interior) all eigenvalues of A with negative real part and is oriented counterclockwise, and γ_2 any closed curve in the right half of the complex plane that surrounds all eigenvalues of A with positive real part, again with counterclockwise orientation. Then

$$P_A^1\mathbb{R}^n = \mathbb{E}_A^s, \qquad P_A^2\mathbb{R}^n = \mathbb{E}_A^u$$

where

$$P_A^1 v = \frac{1}{2\pi i}\int_{\gamma_1}(\xi I - A)^{-1} v d\xi, \quad P_A^2 v = \frac{1}{2\pi i}\int_{\gamma_2}(\xi I - A)^{-1} v d\xi$$

(See Section 4.6 of [Rob95].)

Theorem 8 *The map $H^s : \mathfrak{A}_H \to \mathcal{A}(\mathbb{R}^n) = \{X | X \subseteq \mathbb{R}^n$ is a closed subset of $\mathbb{R}^n\}$, $A \mapsto \mathbb{E}_A^s$, is computable, where $\mathfrak{A}_H$ is represented "entrywise": $\rho = (\rho_{ij})_{i,j=1}^n$ is a name of A if ρ_{ij} is a name of a_{ij}, the ijth entry of A.*

The above theorem can be seen as a strengthening of a seemingly unrelated result [ZB04, Theorem 19]. The latter result can be used to show that the map H^s is computable, provided that one has access to an additional parameter as input: the number of eigenvalues (including multiplicities) associated to the space $\mathbb{E}_A^s$. Our results show that this extra parameter is not needed. Its usage in [ZB04] is due to the fact that eigenvectors are not computable from A. Here we avoid this problem altogether by not using eigenvectors on the construction of the mapping H^s.

Proof. First we observe that the map $A \mapsto$ the eigenvalues of A, $A \in \mathfrak{A}_H$, is computable. Assume that $\lambda_1, \ldots, \lambda_k, \mu_{k+1}, \ldots, \mu_l$, $k \le l \le n$ (some eigenvalues may have multiplicity greater than one), are eigenvalues of A with $Re(\lambda_j) < 0$ for $1 \le j \le k$ and $Re(\mu_j) > 0$ for $k+1 \le j \le l$. Then from the names of the eigenvalues, one can compute two rectangular closed curves γ_A^1 and γ_A^2, where γ_A^1 is in the left half of the complex plane that surrounds all eigenvalues λ_j with $1 \le j \le k$ and γ_A^2 is in the right half of the complex plane that surrounds all eigenvalues μ_j with $k+1 \le j \le l$, and both γ_A^1 and γ_A^2 are oriented counterclockwise. From γ_A^1 and γ_2^A one can further compute the maps $P_A^1, P_A^2 : \mathbb{R}^n \to \mathbb{R}^n$, where $P_A^1 v = \frac{1}{2\pi i}\int_{\gamma_A^1}(\xi I - A)^{-1} v d\xi$ and $P_A^2 v = \frac{1}{2\pi i}\int_{\gamma_A^2}(\xi I - A)^{-1} v d\xi$, since all the operations used to define P_A^1 and P_A^2 are computable. We note that, on the one hand, $\mathbb{E}_A^s = (P_A^2)^{-1}(\{0\})$, and on the other hand, $\mathbb{E}_A^s = \overline{\mathbb{E}_A^s} = \overline{P_A^1\mathbb{R}^n}$, where $\overline{K}$ denotes the closure of the set K. Then by Theorem 6.2.4 of [Wei00], $\mathbb{E}_A^s$ is both r.e. and co-r.e. closed, thus computable. The same argument shows that $\mathbb{E}_A^u$ is computable from A. ∎

Next we present an effective version of the stable manifold theorem. Consider the nonlinear system

$$\dot{x} = f(x(t)) \tag{6}$$

Assume that (6) defines a dynamical system, that is, the solution $x(t, x_0)$ to (6) with the initial condition $x(0) = x_0$ is defined for all $t \in \mathbb{R}$. Since the system (6) is autonomous, if p is a hyperbolic equilibrium point, without loss of generality, we may assume that p is the origin 0.

Theorem 9 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-computable function (meaning that both f and Df are computable). Assume that the origin 0 is a hyperbolic equilibrium point of (6) such that $Df(0)$ has k eigenvalues with negative real part and $l - k$ eigenvalues with positive real part, $0 < k \leq l \leq n$ (some eigenvalues may have multiplicity greater than one). Let $x(t, x_0)$ denote the solution of (6) with the initial value x_0 at $t = 0$. Then there is a (Turing) algorithm that computes from f a compact subset $S \subset \mathbb{R}^n$ containing 0 (S is a manifold except possibly at its boundary of the same dimension as of $\mathbb{E}^s_{Df(0)}$) and two positive rational numbers γ and ϵ such that*

(i) For any $x_0 \in S$, $x(t, x_0) \in S$ for all $t \geq 0$;

(ii) For any $x_0 \in S$ and $t \geq 0$, $|x(t, x_0)| \leq \gamma e^{-\epsilon t}$; consequently, $\lim_{t\to+\infty} x(t, x_0) = 0$.

The proof of Theorem 9 is presented at the end of this section. In that proof, we make use of a number of lemmas and auxiliary results. We begin by recalling that, under the assumption that f is C^1-computable, the solution map $x : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $(t, a) \mapsto x(t, a)$, is computable ([GZB09]).

Let $f(x) = Ax + F(x)$, where $A = Df(0)$ and $F(x) = f(x) - Ax$. Then the equation (6) can be written in the form of

$$\dot{x} = Ax + F(x) \tag{7}$$

The first step in constructing the desired algorithm is to break the flow e^{At} governed by the linear equation $\dot{x} = Ax$ into the stable and unstable components, denoted as $I_{\Gamma_1}(t)$ and $I_{\Gamma_2}(t)$ respectively. By making use of an integral formula, we are able to show that the breaking process is computable from A. This result is presented in the following lemma. But before presenting this result, we need a few definitions. Assume that $\lambda_1, \ldots, \lambda_k, \mu_{k+1}, \ldots, \mu_l$, $k \leq l \leq n$, are eigenvalues of A with $Re(\lambda_j) < 0$ for $1 \leq j \leq k$ and $Re(\mu_j) > 0$ for $k+1 \leq j \leq l$, where $Re(z)$ denotes the real part of the complex number z. Then two rational numbers $\sigma > 0$ and $\alpha > 0$ can be computed from the eigenvalues of A such that $Re(\lambda_j) < -(\alpha+\sigma)$ for $1 \leq j \leq k$ and $Re(\mu_j) > \sigma$ for $k+1 \leq j \leq l$. We break α into two parts for later use: Let α_1 and α_2 be two rational numbers such that

$$0 < \alpha_1 < \alpha \quad \text{and} \quad \alpha_1 + \alpha_2 = \alpha. \tag{8}$$

Let M be a natural number such that $M > \max\{\alpha+\sigma, 1\}$ and $\max\{|\lambda_j|, |\mu_i| : 1 \leq j \leq k, k+1 \leq i \leq l\} \leq M - 1$. We now construct two simple piecewise-smooth close curves Γ_1 and Γ_2 in $\mathbb{R}^2$: Γ_1 is the boundary of the rectangle with the vertices $(-\alpha - \sigma, M)$, $(-M, M)$, $(-M, -M)$, and $(-\alpha - \sigma, -M)$, while Γ_2 is the boundary of the rectangle with the vertices (σ, M), (M, M), $(M, -M)$, and $(\sigma, -M)$. Then Γ_1 with positive direction (counterclockwise) encloses in its interior all the λ_j for $1 \leq j \leq k$ and Γ_2 with positive direction encloses all the μ_j for $k+1 \leq j \leq l$ in its interior. Now define

$$I_{\Gamma_1}(t) = -\frac{1}{2\pi i}\int_{\Gamma_1} e^{\xi t}(A - \xi I_n)^{-1} d\xi \quad \text{and} \quad I_{\Gamma_2}(t) = -\frac{1}{2\pi i}\int_{\Gamma_2} e^{\xi t}(A - \xi I_n)^{-1} d\xi \tag{9}$$

Lemma 10 *$I_{\Gamma_1}(t)$ and $I_{\Gamma_2}(t)$ are the stable and unstable components of e^A, respectively. Moreover $I_{\Gamma_1}(t)$ and $I_{\Gamma_2}(t)$ are computable from f.*

Proof. Since $F(x) = f(x) - Ax$, it follows that $F(0) = 0$, $DF(0) = 0$, F and DF are both computable because f and Df are computable functions by assumption. Thus there is a computable modulus of continuity $d : \mathbb{N} \to \mathbb{N}$ such that

$$|F(x) - F(y)| \leq 2^{-m}|x - y| \quad \text{whenever} \quad |x| \leq 2^{-d(m)} \quad \text{and} \quad |y| \leq 2^{-d(m)}; \tag{10}$$

in particular, $|F(x)| \leq 2^{-m}|x|$ whenever $|x| \leq 2^{-d(m)}$. Since Df is computable, all entries in the matrix A are computable; consequently, the coefficients of the characteristic polynomial $\det(A - \lambda I_n)$ of A are computable, where $\det(A - \lambda I_n)$ denotes the determinant of $A - \lambda I_n$ and I_n is the $n \times n$ unit matrix. Thus all eigenvalues of A are computable, for they are zeros of the computable polynomial $\det(A - \lambda I_n)$ (actually the eigenvalues are computable uniformly in the characteristic polynomial's coefficients by a nontrivial result due to Ernst Specker [Spe69]). We observe that for any $\xi \in \Gamma_1 \bigcup \Gamma_2$, the matrix $A - \xi I_n$ is invertible. Since the function $g : \Gamma_1 \bigcup \Gamma_2 \to \mathbb{R}$, $g(\xi) = ||(A - \xi I_n)^{-1}||$, is computable (see for example [Zho09]), where $(A - \xi I_n)^{-1}$ is the inverse of the matrix $A - \xi I_n$, the maximum of g on $\Gamma_1 \bigcup \Gamma_2$ is computable. Let $K_1 \in \mathbb{N}$ be an upper bound of this computable maximum. Now for any $t \in \mathbb{R}$, from (5.47) of [Kat95],

$$\begin{aligned} e^{At} &= -\frac{1}{2\pi i}\int_{\Gamma_1} e^{\xi t}(A - \xi I_n)^{-1} d\xi - \frac{1}{2\pi i}\int_{\Gamma_2} e^{\xi t}(A - \xi I_n)^{-1} d\xi \\ &= I_{\Gamma_1}(t) + I_{\Gamma_2}(t) \end{aligned} \tag{11}$$

We recall that e^{At} is the solution to the linear equation $\dot{x} = Ax$. Since A is computable and integration is a computable operator, it follows that I_{Γ_1} and I_{Γ_2} are computable. A simple calculation shows that $||-\frac{1}{2\pi i}\int_{\Gamma_1} e^{t\xi}(A - \xi I_n)^{-1} d\xi|| \leq 4K_1Me^{-(\alpha+\sigma)t}/\pi$ for $t \geq 0$ and $||-\frac{1}{2\pi i}\int_{\Gamma_2} e^{t\xi}(A - \xi I_n)^{-1} d\xi|| \leq 4K_1Me^{\sigma t}/\pi$ for $t \leq 0$. Let

$$K = 4MK_1 \tag{12}$$

Then

$$||I_{\Gamma_1}(t)|| \leq Ke^{-(\alpha+\sigma)t} \text{ for } t \geq 0 \quad \text{and} \quad ||I_{\Gamma_2}(t)|| \leq Ke^{\sigma t} \text{ for } t \leq 0 \tag{13}$$

The two estimates in (13) show that $I_{\Gamma_1}(t)$ and $I_{\Gamma_2}(t)$ are respectively the stable and unstable components of e^{At}. ■

The second step in the construction of the algorithm is to compute a set $\tilde{S} \subset \mathbb{R}^n$ such that the condition (ii) of Theorem 9 is satisfied for any initial value $x_0 \in \tilde{S}$, that is, the solution $x(t, x_0)$ of (6) with the initial value $x_0 \in \tilde{S}$ converges to 0 exponentially as $t \to \infty$. In order to find, effectively, an explicit exponential convergence rate, we make use of a fixed-point argument.

Lemma 11 *Consider the integral equation*

$$u(t, a) = I_{\Gamma_1}(t)a + \int_0^t I_{\Gamma_1}(t - s)F(u(s, a))ds - \int_t^\infty I_{\Gamma_2}(t - s)F(u(s, a))ds \tag{14}$$

where the constant vector $a \in \mathbb{R}^n$ is a parameter. If $u(t,a)$ is a continuous solution to the integral equation, then it satisfies the differential equation (7) ([Per01]).

Proof. See Appendix 1. ■

We observe that the solution to the integral equation is the fixed point of the operator defined on the right hand side of the equation (14). Now let us compute an integer m_0 such that $2^{-m_0} \leq \frac{\sigma}{4K^2}$ (σ, K, and the function d used in the next equation are defined in the proof of Lemma 10) then set

$$r = 2^{-d(m_0)}/2K, \qquad B = B(0,r) = \{a \in \mathbb{R}^n : |a| < r\} \tag{15}$$

where the computable function d is defined in (10).

Lemma 12 *For any $a \in B$ and $t \geq 0$, define the successive approximations as follows:*

$$\begin{aligned} u^{(0)}(t,a) &= 0 \qquad (16)\\ u^{(j)}(t,a) &= I_{\Gamma_1}(t)a + \int_0^t I_{\Gamma_1}(t-s)F(u^{(j-1)}(s,a))ds \\ &\quad - \int_t^\infty I_{\Gamma_2}(t-s)F(u^{(j-1)}(s,a))ds, \quad j \geq 1 \end{aligned}$$

then the following three inequalities hold for all $j \in \mathbb{N}$:

$$|u^{(j)}(t,a) - u^{(j-1)}(t,a)| \leq K|a|e^{-\alpha_1 t}/2^{j-1} \tag{17}$$

$$|u^{(j)}(t,a)| \leq 2^{-d(m_0)}e^{-\alpha_1 t} \tag{18}$$

and for any $\tilde{a} \in B$,

$$|u^{(j)}(t,a) - u^{(j)}(t,\tilde{a})| \leq 3K|a - \tilde{a}| \tag{19}$$

Proof. See Appendix 2. ■

Lemma 13 *The sequence $\{u^{(j)}(t,a)\}_{j=1}^\infty$ defined in Lemma 12 is a computable Cauchy sequence effectively convergent to the solution $u(t,a)$ of the integral equation (14), uniformly in $t \geq 0$ and $a \in B$.*

Proof. The conclusion follows from (16) and (17). ■

It follows from this lemma that the solution operator $\Phi : \mathbb{R}_+ \times B(0,r) \to \mathbb{R}^n$, $(t,a) \mapsto u(t,a)$, is computable, where $\mathbb{R}_+$ denote the set of all non-negative real numbers. Furthermore, (18) and (19) imply that for all $t \geq 0$ and $a, \tilde{a} \in B$,

$$|u(t,a)| \leq 2^{-d(m_0)}e^{-\alpha_1 t}, \qquad |u(t,a) - u(t,\tilde{a})| < 3K|a - \tilde{a}| \tag{20}$$

We note that the first inequality in (20) shows that $u(t,a)$ satisfies condition (ii) of Theorem 9 with $\gamma = 2^{-d(m_0)}$ and $\epsilon = \alpha_1$ for all $t \geq 0$ and $a \in B(0,r)$.

As we already mentioned that if $u(t,a)$ is a continuous solution to the integral equation (14), then it satisfies the equation (7). Thus if $u(0,a) = a$, then $u(t,a)$ is the solution to the initial value problem $\dot{x} = f(x)$, $x(0,a) = a$;

in other words, $u(t,a) = x(t,a)$, which implies that $x(t,a)$ would then satisfy the inequality (20) and, subsequently, the desired condition (ii) of Theorem 9. However, we note that $u(0,a) = I_{\Gamma_1}(0)a - \int_0^\infty I_{\Gamma_2}(-s)F(u(s,a))ds$, which may or may not be equal to a. And so our next attempt is to find a set $\tilde{S}$ such that for any $a \in \tilde{S}$, $u(0,a) = a$. We start by recalling the projection maps to $\mathbb{E}_A^s$ and $\mathbb{E}_A^u$ (the stable and unstable subspaces) and some properties concerning resolvents.

Let $P_1 = I_{\Gamma_1}(0) = -\frac{1}{2\pi i}\int_{\Gamma_1}(A - \xi I_n)^{-1}d\xi$ and $P_2 = I_{\Gamma_2}(0) = -\frac{1}{2\pi i}\int_{\Gamma_2}(A - \xi I_n)^{-1}d\xi$. Then $P_1\mathbb{R}^n = \mathbb{E}_A^s$, $P_2\mathbb{R}^n = \mathbb{E}_A^u$, $\mathbb{R}^n = \mathbb{E}_A^s \bigoplus \mathbb{E}_A^u$, $P_jP_k = \delta_{jk}P_j$ ($\delta_{jk} = 1$ if $j = k$ and $\delta_{jk} = 0$ if $j \neq k$), and $P_1 + P_2 = I$ is the identity map on $\mathbb{R}^n$ (*c.f.* §1.5.3 and §1.5.4, [Kat95]; §4.6 of [Rob95]). Since A is computable, so are P_1 and P_2. Moreover, $I_{\Gamma_1}(t)P_2 = 0$ for any $t \in \mathbb{R}$ as the following calculation shows: Let $R(\xi)$ denote $(A - \xi I_n)^{-1}$. Then we have

$$\begin{aligned}
R(\xi_1) - R(\xi_2) &= R(\xi_1)(A - \xi_2 I_n)R(\xi_2) - R(\xi_1)(A - \xi_1 I_n)R(\xi_2) \\
&= R(\xi_1)[(A - \xi_2 I_n) - (A - \xi_1 I_n)]R(\xi_2) \\
&= R(\xi_1)(\xi_1 - \xi_2)I_n R(\xi_2) \\
&= (\xi_1 - \xi_2)R(\xi_1)R(\xi_2).
\end{aligned}$$

Using the last equation we obtain that for any $v \in \mathbb{R}^n$,

$$\begin{aligned}
I_{\Gamma_1}(t)P_2 v &= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1} e^{\xi t}R(\xi)d\xi \int_{\Gamma_2} R(\xi')v d\xi' \qquad (21) \\
&= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1}\int_{\Gamma_2} e^{\xi t}R(\xi)R(\xi')v d\xi d\xi' \\
&= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1}\int_{\Gamma_2} e^{\xi t}\frac{R(\xi) - R(\xi')}{\xi - \xi'} v d\xi d\xi' \\
&= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1} e^{\xi t}\left(\int_{\Gamma_2}\frac{R(\xi)}{\xi - \xi'}d\xi' - \int_{\Gamma_2}\frac{R(\xi')}{\xi - \xi'}d\xi'\right) v d\xi \\
&= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1} e^{\xi t}\left(-\int_{\Gamma_2}\frac{R(\xi')}{\xi - \xi'}d\xi'\right) v d\xi \\
&= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_2} R(\xi')\left(\int_{\Gamma_1}\frac{e^{\xi t}}{\xi' - \xi}d\xi\right) v d\xi' = 0
\end{aligned}$$

A similar computation shows that for any $t \in \mathbb{R}$,

$$I_{\Gamma_2}(t)P_1 v = 0 \quad \text{and} \quad P_2 I_{\Gamma_2}(t)v = I_{\Gamma_2}(t)v, \quad v \in \mathbb{R}^n \qquad (22)$$

Now let us use these results to compute the projections of $u(0,a)$ in $\mathbb{E}_A^s$ and $\mathbb{E}_A^u$: for any $a \in \mathbb{R}^n$,

$$\begin{aligned}
P_1 u(0,a) &= P_1\left(I_{\Gamma_1}(0)a - \int_0^\infty I_{\Gamma_2}(-s)F(u(s,a))ds\right) \\
&= P_1P_1 a - \int_0^\infty P_1 I_{\Gamma_2}(-s)F(u(s,a))ds \\
&= P_1 a \qquad (23)
\end{aligned}$$

and

$$\begin{aligned}P_2u(0,a) &= P_2\left(I_{\Gamma_1}(0)a - \int_0^\infty I_{\Gamma_2}(-s)F(u(s,a))ds\right)\\ &= P_2P_1a - P_2\left(\int_0^\infty I_{\Gamma_2}(-s)F(u(s,a))ds\right)\\ &= -\int_0^\infty I_{\Gamma_2}(-s)F(u(s,a))ds\end{aligned}$$

We note that for any $a \in \mathbb{R}^n$,

$$\begin{aligned}I_{\Gamma_1}(t)a &= I_{\Gamma_1}(t)(P_1a + P_2a)\\ &= I_{\Gamma_1}(t)P_1a + I_{\Gamma_1}(t)P_2a\\ &= I_{\Gamma_1}(t)P_1a\end{aligned}$$

which implies that if the solution $u(t,a)$ of the integral equation (14) is constructed by successive approximations (16), then P_2a does not enter the computation for $u(t,a)$ and thus may be taken as zero. Therefore the projection of $u(0,a)$ in $\mathbb{E}^u_A$ satisfies the equation

$$P_2u(0,a) = -\int_0^\infty I_{\Gamma_2}(-s)F(u(s,P_1a))ds. \tag{24}$$

Next we define a map $\phi_A : \mathbb{E}^s_A(r) \to \mathbb{E}^u_A$, $b \mapsto -\int_0^\infty I_{\Gamma_2}(-s)F(u(s,b))ds$ for $b \in \mathbb{E}^s_A(r)$, where r is the rational number defined in (15) and $\mathbb{E}^s_A(r) = \{b \in \mathbb{E}^s_A : |b| \le r/2\}$. We observe that the compact set $\mathbb{E}^s_A(r)$ is computable since the closed set $\mathbb{E}^s_A$ is computable (proved in Theorem 8). Obviously the map ϕ_A is computable. By Theorem 6.2.4 [Wei00] the compact set $\phi_A[\mathbb{E}^s_A(r)]$ is computable.

Now we define the set $\tilde{S}$:

$$\tilde{S} = \{b + \phi_A(b) : b \in \mathbb{E}^s_A(r)\} \subset \mathbb{R}^n$$

Then $\tilde{S}$ is a k-dimensional manifold. Since both $\mathbb{E}^s_A(r)$ and $\phi_A[\mathbb{E}^s_A(r)] \subseteq \mathbb{E}^u_A$ are computable compact sets and $\mathbb{R}^n = \mathbb{E}^s_A \oplus \mathbb{E}^u_A$, it follows that the set $\tilde{S}$ is a computable compact set as well. Moreover, for any $a \in \tilde{S}$, $a = b + \phi_A(b)$ for some $b \in \mathbb{E}^s_A(r)$. Since $\mathbb{R}^n = \mathbb{E}^s_A \bigoplus \mathbb{E}^u_A$, $b \in \mathbb{E}^s_A$ and $\phi_A(b) \in \mathbb{E}^u_A$, it follows that

$$b = P_1a \quad \text{and} \quad \phi_A(b) = \phi_A(P_1a) = P_2a \tag{25}$$

From (23), (24), and (25) it follows that for any $a \in \tilde{S}$, $u(0,a) = P_1u(0,a) + P_2u(0,a) = P_1a + \phi_A(P_1a) = P_1a + P_2a = a$. Thus for any $a \in \tilde{S}$, $x(t,a) = u(t,a)$ is the solution to the initial value problem $\dot{x} = f(x)$, $x(0,a) = a$.

Lemma 14 *$\tilde{S} \subset B(0,r)$.*

Proof. For any $a \in \tilde{S}$, it follows from the definition of $\tilde{S}$ that there exists $b \in \mathbb{E}^s_A(r)$ such that $a = b + \phi_A(b)$; in particular, $|b| \le r/2 = 2^{-d(m_0)}/4K$. Since $\phi_A(b) = -\int_0^\infty I_{\Gamma_2}(-s)F(u(s,b))ds$ and, from (10) and the proof presented in

Appendix 2, $|F(x)| \leq 2^{-m_0}|x|$ if $|x| \leq 2^{-d(m_0)}$ and $|u(s,b)| \leq 2K|b|e^{-\alpha_1 s} \leq 2K|b| < 2^{-d(m_0)}$ for any $s \geq 0$, which in turn implies the following estimate:

$$\begin{aligned}
|\phi_A(b)| &= \left|-\int_0^\infty I_{\Gamma_2}(-s)F(u(s,b))ds\right| \\
&\leq 2^{-m_0} \cdot 2K|b| \cdot \left|\int_0^\infty I_{\Gamma_2}(-s)ds\right| \\
&\leq 2^{-m_0} \cdot 2K^2|b| \cdot \left|\int_0^\infty e^{-\sigma s}ds\right| \quad \text{(see (13))} \\
&= 2^{-m_0} \cdot 2K^2|b| \cdot \frac{1}{\sigma} \\
&\leq \frac{\sigma}{4K^2} \cdot 2K^2|b|\frac{1}{\sigma} \\
&= \frac{|b|}{2} \leq \frac{r}{4}
\end{aligned}$$

(recall that m_0 is chosen such that $2^{-m_0} \leq \frac{\sigma}{4K^2}$). Thus $|a| = |b + \phi_A(b)| \leq r/2 + r/4 < r$, which implies that $a \in B(0,r)$. This proves that $\tilde{S} \subset B(0,r)$. ■

We need one more auxiliary result before we can present the construction of the desired set S.

Lemma 15 *If $x(t,x_0)$ is a solution to the differential equation (7) whose initial value $x(0) = x_0$ satisfies the conditions that $0 < |x_0| < 2^{-d(m_0)}/4K^2$ but $x_0 \notin \tilde{S}$, then there exists $t' > 0$ such that $|x(t',x_0)| > 2^{-d(m_0)}$.*

Proof. Assume otherwise that $|x(t,x_0)| \leq 2^{-d(m_0)}$ for all $t \geq 0$. We show in the following that this condition implies $x_0 \in \tilde{S}$, which is a contradiction.

Since $x(t,x_0)$ is the solution to $\dot{x} = Ax + F(x)$ with the initial value x_0, it follows that (*c.f.* Theorem 4.8.2 [Rob95])

$$x(t,x_0) = e^{At}x_0 + \int_0^t e^{(t-s)A}F(x(s,x_0))ds$$

which, using (11), can be rewritten as

$$\begin{aligned}
&x(t,x_0) \qquad (26)\\
&= I_{\Gamma_1}(t)x_0 + I_{\Gamma_2}(t)x_0 + \int_0^t I_{\Gamma_1}(t-s)F(x(s,x_0))ds + \int_0^t I_{\Gamma_2}(t-s)F(x(s,x_0))ds \\
&= I_{\Gamma_1}(t)x_0 + \int_0^t I_{\Gamma_1}(t-s)F(x(s,x_0))ds - \int_t^\infty I_{\Gamma_2}(t-s)F(x(s,x_0))ds + I_{\Gamma_2}(t)b
\end{aligned}$$

where $b = x_0 + \int_0^\infty I_{\Gamma_2}(-s)F(x(s,x_0))ds$ (*c.f.* §1.5.3 [Kat95]). Note that b is well defined since we assume that $|x(t,x_0)| \leq 2^{-d(m_0)}$ for all $t \geq 0$, then, from (10), $|F(x(t,x_0))|$ is bounded for all $t \geq 0$; consequently, by (13), the integral $\int_0^\infty I_{\Gamma_2}(-s)F(x(s,x_0))ds$ converges. We also note that the first three terms in the above representation for $x(t,x_0)$ are bounded. Moreover, $I_{\Gamma_2}(t)b = I_{\Gamma_2}(t)P_1b + I_{\Gamma_2}(t)P_2b = I_{\Gamma_2}(t)P_2b$ since $I_{\Gamma_2}(t)P_1b = 0$ by (22). We claim that if $P_2b \not\equiv 0$, then $I_{\Gamma_2}(t)b$ is unbounded as $t \to \infty$. We make use of the residue formula to prove the claim. Recall that Γ_2 is a closed curve in the right-hand side of the complex plane with counterclockwise orientation that contains in its interior μ_j, $k+1 \leq j \leq l$, where μ_j are the eigenvalues of A with $Re(\mu_j) > 0$, which are the exact singularities of $R(\xi) = (A - \xi I_n)^{-1}$ in the right complex plane. Then by the residue formula,

$$\frac{1}{2\pi i} \int_{\Gamma_2} e^{t\xi} R(\xi) d\xi = \sum_{j=k+1}^{l} e^{\mu_j t} \text{res}_{\mu_j} R \tag{27}$$

where $\text{res}_{\mu_j} R$ is the residue of R at μ_j. Since $Re(\mu_j) > 0$, if $P_2 b \not\equiv 0$, then

$$\begin{aligned} I_{\Gamma_2}(t)b &= I_{\Gamma_2}(t) P_2 b \\ &= -\frac{1}{2\pi i} \int_{\Gamma_2} e^{t\xi} (A - \xi I_n)^{-1} P_2 b d\xi \\ &= -\sum_{j=k+1}^{l} e^{\mu_j t} res_{\mu_j} R P_2 b \end{aligned}$$

is unbounded as $t \to \infty$. This is however impossible because the first three terms in (26) are bounded and we have assumed that $|x(t, x_0)| \leq 2^{-d(m_0)}$ for all $t \geq 0$. Therefore, $P_2 b \equiv 0$; consequently, $I_{\Gamma_2}(t)b = I_{\Gamma_2}(t) P_2 b = 0$. This last equation together with (26) shows that $x(t, x_0)$ satisfies the integral equation (14). Now let $x'(t, x_0)$ be the solution to the integral equation (14) with parameter x_0 and constructed by the successive approximations (16). Then

$$x'(t, x_0) = I_{\Gamma_1}(t) x_0 + \int_0^t I_{\Gamma_1}(t-s) F(x'(s, x_0)) ds - \int_t^\infty I_{\Gamma_2}(t-s) F(x'(s, P_1 x_0)) ds$$

By the uniqueness of the solution, $x(t, x_0) = x'(t, x_0)$; in particular, $x_0 = x(0, x_0) = x'(0, x_0) = P_1 x_0 - \int_0^\infty I_{\Gamma_2}(-s) F(x'(s, P_1 x_0)) ds$. Since $P_1 x_0 \in \mathbb{E}_A^s$ and $\|P_1 x_0\| \leq \|P_1\| \|x_0\| \leq K \cdot 2^{-d(m_0)}/4K^2 = r/2$ (recall that $\|P_1\| \leq K$ by (13)), it follows that $P_1 x_0 \in \mathbb{E}_A^s(r)$, which further implies that $x_0 = P_1 x_0 - \int_0^\infty I_{\Gamma_2}(-s) F(x'(s, P_1 x_0)) ds = P_1 x_0 + \phi_A(P_1 x_0) \in \tilde{S}$. This contradicts the fact that x_0 is not on $\tilde{S}$. The proof is complete. ■

Finally we come to the **proof of Theorem 9**. First we define the set S:

$$S = \{x(t, x_0) : x_0 \in \tilde{S}, \, t \geq 0\} \subset \mathbb{R}^n$$

(recall that $x(t, x_0)$ is the solution to the initial value problem $\dot{x} = f(x)$, $x(0) = x_0$). It is immediately clear that $\tilde{S} \subseteq S$. We also note that from the definition of $\tilde{S}$ and (20), for any $a \in \tilde{S}$, $a = u(0, a)$ and, consequently, $|a| = |u(0, a)| \leq 2^{-d(m_0)}$. Thus for any $a \in \tilde{S}$ it follows from the proof of Lemma 12 (see Appendix 2) that

$$|x(t, a)| = |u(t, a)| \leq 2K|a| e^{-\alpha_1 t} \leq 2K \cdot 2^{-d(m_0)} e^{-\alpha_1 t}$$

Let $T_{\tilde{S}} = \frac{\ln(10K^3)}{\alpha_1}$. Since $K, d(m_0), r$, and α_1 are all computable from f, then so is $T_{\tilde{S}}$. It is readily seen that for any $a \in \tilde{S}$ and $t \geq T_{\tilde{S}}$,

$$|x(t, a)| = |u(t, a)| \leq \frac{2^{-d(m_0)}}{5K^2} < \frac{2^{-d(m_0)}}{4K^2}. \tag{28}$$

Then from Lemma 15 we conclude that

$$x(t,a) = u(t,a) \in \tilde{S} \text{ for all } a \in \tilde{S} \text{ and } t \geq T_{\tilde{S}} \tag{29}$$

For if otherwise, there exist $a \in \tilde{S}$ and $t \geq T_{\tilde{S}}$ such that $x(t,a) \notin \tilde{S}$, then it follows from Lemma 15 that there exists $t' > 0$ such that $|x(t', x(t,a))| = |x(t'+t,a)| > 2^{-d(m_0)}$ which contradicts (28) for $t'+t > t \geq T_{\tilde{S}}$. From (29) it follows that

$$S = \{x(t,x_0) : x_0 \in \tilde{S},\ 0 \leq t \leq T_{\tilde{S}}\} = \Phi([0,T_{\tilde{S}}] \times \tilde{S})$$

Since $[0,T_{\tilde{S}}] \times \tilde{S}$ is a computable compact set, $\tilde{S} \subset B(0,r)$, and the solution operator $\Phi : \mathbb{R}_+ \times B(0,r) \to \mathbb{R}^n$, $(t,a) \mapsto u(t,a)$, is computable, it follows from Theorem 6.2.4 of [Wei00] that S is a computable compact set.

It remains to show that S satisfies conditions (i) and (ii) of Theorem 9. We note that for any $x_0 \in S$, there exist $a \in \tilde{S}$ and $t_0 \geq 0$ such that $x_0 = x(t_0,a)$. Then for any $t \geq 0$, since $x(t,x_0) = x(t,x(t_0,a)) = x(t+t_0,a)$, we conclude that $x(t,x_0) \in S$; thus S meets the requirement (i). Now let us set $\gamma = 2^{-d(m_0)}$ and $\epsilon = \alpha_1$. Then it follows from (20) that

$$\begin{aligned} |x(t,x_0)| &= |x(t+t_0,a)| \\ &= |u(t+t_0,a)| \\ &\leq 2^{-d(m_0)} e^{-\alpha_1(t+t_0)} \leq 2^{-d(m_0)} e^{-\alpha_1(t)} = \gamma e^{-\epsilon t}. \end{aligned}$$

Therefore the second condition (ii) is also satisfied by S. The proof of Theorem 9 is now complete.

Theorem 16 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-computable function. Assume that the origin 0 is a hyperbolic equilibrium point of (6) such that $Df(0)$ has k eigenvalues with negative real part and $l-k$ eigenvalues with positive real part, $0 \leq k < l \leq n$. Let $x(t,x_0)$ denote the solution to (6) with the initial value x_0 at $t=0$. Then there is a (Turing) algorithm that computes from f a compact subset $U \subset \mathbb{R}^n$ containing 0 (U is a manifold except possibly at its boundary of the same dimension as of $\mathbb{E}^u_{DF(0)}$) and two positive rational numbers γ and ϵ such that*

(i) For any $x_0 \in U$, $x(t,x_0) \in U$ for all $t \leq 0$;

(ii) For any $x_0 \in U$ and $t \leq 0$, $|x(t,x_0)| \leq \gamma e^{\epsilon t}$; consequently, $\lim_{t \to -\infty} x(t,x_0) = 0$.

Proof. The unstable manifold U can be computed by the same procedure as the construction of S by considering the equation

$$\dot{x} = -Ax - F(x(t))$$

■

The proof can be easily extended to show that the map: $\mathfrak{F}_H \to \mathfrak{K} \times \mathfrak{K}$, $f \mapsto (S_f, U_f)$, is computable, where $\mathfrak{F}_H$ is the set of all C^1 functions having the origin as a hyperbolic equilibrium point, $\mathfrak{K}$ is the set of all compact subsets of $\mathbb{R}^n$, and S_f and U_f are some local stable and unstable manifolds of f at the origin respectively.

5 Computability and non-computability of global stable/unstable manifolds

Although the stable and unstable manifolds can be computed locally as shown in the previous section, globally they may not be necessarily computable.

In [Zho09] it is shown that there exists a C^∞ and polynomial-time computable function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that the equation $\dot{x} = f(x)$ has a sink at the origin $\mathbf{0}$ and the basin of attraction (also called the domain of attraction), $B_f(\mathbf{0})$, of f at $\mathbf{0}$ is a non-computable open subset of $\mathbb{R}^2$. Since a sink is a hyperbolic equilibrium point (with all eigenvalues of the gradient matrix $Df(\mathbf{0})$ having negative real parts) and the basin of attraction at a sink is exactly the global stable manifold at this equilibrium point, we conclude immediately that the global stable manifold is not necessarily computable.

A natural question arises - what is the degree of computational unsolvability of the problem of finding the global stable manifold? We investigate this problem in the remaining of this section. Our approach is similar to [Bra05] and [BHW08], where the Borel hierarchy is used to classify the degree of unsolvability of computational problems.

We recall that the (finite beginning of the) Borel hierarchy is given by

$$\begin{aligned}\Sigma_1^0(X) &= \text{set of all open subsets of } X \\ \Pi_k^0(X) &= \text{set of all complements of sets in } \Sigma_k^0(X) \\ \Sigma_{k+1}^0(X) &= \text{set of all countable unions of sets in } \Pi_k^0(X)\end{aligned}$$

where $k \geq 1$ and X is a topological space. In particular, $\Pi_1^0(X)$ is the set of all closed subsets of X and $\Sigma_2^0(X)$ is the set of all F_σ-subsets of X.

As we have noted, the global stable and unstable manifolds of a hyperbolic equilibrium point are F_σ-sets of $\mathbb{R}^n$ and thus they are elements of $\Sigma_2^0(\mathbb{R}^n)$. The following representation of $\Sigma_2^0(\mathbb{R}^n)$, $\delta_{\Sigma_2^0(\mathbb{R}^n)}$, is introduced in [Bra05] (see also [BHW08]).

Definition 17 *The representation $\delta_{\Sigma_2^0(\mathbb{R}^n)}$ of $\Sigma_2^0(\mathbb{R}^n)$, $\delta_{\Sigma_2^0(\mathbb{R}^n)} :\subseteq \Sigma^{\mathbb{N}} \to \Sigma_2^0(\mathbb{R}^n)$ (recall that Σ is a finite alphabet), is given by*

$$\delta_{\Sigma_2^0(\mathbb{R}^n)}(p) = \bigcup_i C_{(p,i)}$$

where $p \in \Sigma^{\mathbb{N}}$ and $\{C_{(p,i)}\}_{i\in\mathbb{N}}$ is a sequence of closed subsets of $\mathbb{R}^n$. In other words, p is a name, called a $\delta_{\Sigma_2^0(\mathbb{R}^n)}$-name, of the F_σ-set $\bigcup_i C_{(p,i)}$ (see [BHW08] for more details).

The next theorem shows that, given a description of a system and of a hyperbolic equilibrium point as input, it is possible to compute a $\delta_{\Sigma_2^0(\mathbb{R}^n)}$-name of the global stable manifold. This result shows that the degree of unsolvability of computing global stable manifolds is bounded above by Σ_2^0. Since there are open global stable sets which are not computable (as open sets) as mentioned at the beginning of this section (see [Zho09]), Σ_2^0 is also the lower bound.

Theorem 18 *The map $\psi :\subseteq C^1(\mathbb{R}^n) \times \mathbb{R}^n \to \Sigma_2^0(\mathbb{R}^n)$, $(f, x_0) \to W_f^s(x_0)$, is computable (using the representations of Section 3 for inputs and the representation $\delta_{\Sigma_2^0(\mathbb{R}^n)}$ for the output), where $(f, x_0) \in dom(\psi)$ if x_0 is a hyperbolic equilibrium point of f, and $W_f^s(x_0)$ is the global stable manifold of f at x_0.*

Proof. From Theorem 9 one can compute from f and x_0 a compact subset S of $\mathbb{R}^n$, which is a local stable manifold of f at x_0. Note that the global stable manifold of f at x_0 is the union of the backward flows of S, *i.e.*,

$$W_f^s(x_0) = \bigcup_{j=0}^{\infty} \phi_{-j}(S)$$

where $\phi_t(a)$ is the flow induced by the equation $\dot{x} = f(x)$ at time t with the initial data $x(0) = a$. Since the sequence $\{\psi_{-j}(a)\}$ is computable from f and a [GZB09], it follows that the sequence $\{\phi_{-j}(S)\}$ of compact sets are computable from f and x_0 (Theorem 6.2.4 [Wei00]). Since a $\Sigma_2^0(\mathbb{R}^n)$-name of $W_f^s(x_0)$ can be computed from the inputs, the map ψ is computable. ■

We note that the function f in the counterexample mentioned at the beginning of this section is C^∞ but non-analytic (recall that an analytic function is a function that is locally given by a convergent power series. One can also define analytic functions in several variables by means of power series in those variables). It is well known that analycity sometimes helps to reduce degree of computational unsolvability. For example, the sequence $\{f^{(n)}\}$ of derivatives of f is computable if f is analytic but may be non-computable if f is only C^∞. Thus we are led to consider the following problem: whether or not the degree of unsolvability of computing global stable/unstable manifolds can be lowered from Σ_2^0 to Π_1^0 in the context of analytic dynamical systems. Our next theorem shows that the answer to this question is negative.

Let us denote by $\omega(\mathbb{R}^n)$ the set of real analytic functions $f : \mathbb{R}^n \to \mathbb{R}^n$ and $\mathcal{A}(\mathbb{R}^n)$ the set of all closed subsets of $\mathbb{R}^n$. With Wijsman topology $\mathcal{W}$ on $\mathcal{A}(\mathbb{R}^n)$ (see [Bee94]), $(\mathcal{A}(\mathbb{R}^n), \mathcal{W})$ is a topological space that is separable and metrizable with a complete metric. In particular, if the Wijsman topology on $\mathcal{A}(\mathbb{R}^n)$ is induced by the (maximum) distance d on $\mathbb{R}^n$ (i.e., $d(x - y) = \|x - y\|$, where $\|x\| = \|(x_1, x_2, \ldots, x_n)\| = \max\{|x_1|, |x_2|, \ldots, |x_n|\}$), then it can be shown that the Wijsman topology on $\mathcal{A}(\mathbb{R}^n)$ is the same as the topology induced by the representation ψ of $\mathcal{A}(\mathbb{R}^n)$ (Definition 5.1.6 [Wei00]).

Theorem 19 *The map $\psi :\subseteq \omega(\mathbb{R}^n) \times \mathbb{R}^n \to \mathcal{A}(\mathbb{R}^n)$, $(f, x_0) \to \overline{W_f^u(x_0)}$ (the closure of $W_f^u(x_0)$), is not continuous and hence not computable, where $(f, x_0) \in dom(\psi)$ if x_0 is a hyperbolic equilibrium point of $(f, x_0) \in dom(\psi)$ if x_0 is a hyperbolic equilibrium point of f, and $W_f^u(x_0)$ is the global unstable manifold of f at x_0.*

Proof. Consider the following system $\dot{x} = f_\mu(x)$ taken from [HSD04],

$$\begin{aligned} x' &= x^2 - 1 \\ y' &= -xy + \mu(x^2 - 1) \end{aligned} \tag{30}$$

where $\mu \in \mathbb{R}$ is a parameter. The system (30) has two equilibria: $z_1 = (-1, 0)$ and $z_2 = (1, 0)$. $Df_\mu(z_1)$ has eigenvalues -2 and 1, associated to the eigenvectors $(-1, -2\mu/3)$ and $(0, 1)$, respectively, and $Df_\mu(z_2)$ has eigenvalues 2 and -1, associated to the eigenvectors $(-1, -2\mu/3)$ and $(0, 1)$, respectively. Therefore both points z_1 and z_2 are saddles (the behavior of the system is sketched in Fig. 2). From this information and the fact that any point $(-1, y)$ or $(1, y)$ can only move vertically, one concludes that the line $x = -1$ is the unstable manifold of

z_1 and the line $x = 1$ gives the stable manifold of z_2. Note that z_1, z_2 and the above manifolds do not depend on μ.

Let us now study the unstable manifold of z_2. It is split by the stable manifold $x = 1$, and hence there is a "right" as well as a "left" portion of the unstable manifold. Let us focus our attention to the "left" portion. From the system (30) one concludes that any point z near z_2, located to its left (i.e. $z < z_2$) will be pushed to the line $x = -1$ at a rate that is independent of the y-coordinate of z.

When $\mu = 0$, the eigenvector of $Df_\mu(z_2)$ associated to the unstable manifold is $(-1, 0)$. Looking at (30), one concludes that the "left" portion of the unstable manifold of z_2 can only be the segment

$$U_{l,0} = \{(x, 0) \in \mathbb{R}^2 | -1 < x < 1\}.$$

Now let us analyze the case where $\mu < 0$. Since $(-1, -2\mu/3)$ is the eigenvector of $Df_\mu(z_2)$ associated to the unstable manifold, as the unstable manifold of z_2 moves to the left, its y-coordinate starts to grow. The "left" portion of the unstable manifold is always above the line $y = 0$ (if it could be $y = 0$, the dynamics of (30) would push the trajectory upwards), and as soon as its x-coordinate is less than 0 (this will eventually happen), the trajectory is pushed upwards with y-coordinate converging to $+\infty$. Notice that the closer μ is to 0, the lesser the y-component of the trajectory grows, and the closer to z_1 the unstable manifold will be.

If $U_{l,\mu}$ represents the left portion of the unstable manifold of z_2 and

$$A = U_{l,0} \cup \{(-1, y) \in \mathbb{R}^2 | 0 \le y\}$$

then one concludes that

$$\lim_{\mu \to 0^-} d(A, U_{l,\mu}) = 0$$

where d is the Hausdorff distance on $\mathbb{R}^n$.

Suppose that ψ is computable. Then, in particular, the map $\chi : \mathbb{R} \to \mathcal{A}(\mathbb{R}^2)$ defined by $\chi(\mu) = \overline{W^u_{f_\mu}(z_2)}$, is also computable. Since computable maps must be continuous (Corollary 3.2.12 of [Wei00]), this implies that χ should be a continuous map. But the point $\mu = 0$ is a point of discontinuity for χ since (without loss of generality, we restrict ourselves to the semi-plane $x < 1$)

$$\lim_{\mu \to 0^-} \chi(\mu) = \lim_{\mu \to 0^-} \overline{U}_{l,\mu} = \overline{A} \neq \overline{U}_{l,0} = \chi(0)$$

and hence this map cannot be computable, which implies that ψ is not computable either. ∎

6 The Smale horseshoe is computable

In this section we show that Smale's horseshoe is computable. We recall that Smale's horseshoe is the invariant set of the horseshoe map (more details on the horseshoe map and on Smale's horseshoe can be found in e.g. [HSD04]).

Theorem 20 *The Smale Horseshoe Λ is a computable (recursive) closed subset in $I = [0, 1] \times [0, 1]$.*

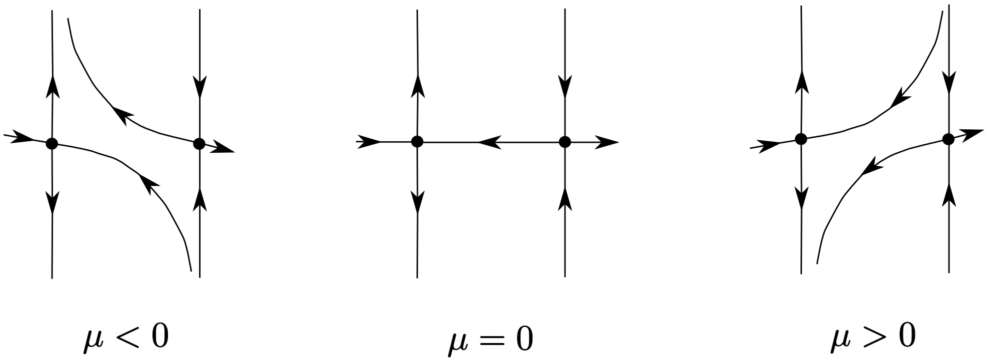

Figure 2: Dynamics of the system (30).

Proof. We show that $\Omega = I \setminus \Lambda$ is a computable open subset in I by making use of the fact [Zho96]: An open subset $U \subseteq I$ is computable if and only if there is a computable sequence of rational open rectangles (having rational corner points) in I, $\{J_k\}_{k=0}^{\infty}$, such that

(a) $J_k \subset U$ for all $k \in \mathbb{N}$,

(b) the closure of J_k, $\bar{J}_k$, is contained in U for all $k \in \mathbb{N}$, and

(c) there is a recursive function $e : \mathbb{N} \to \mathbb{N}$ such that the Hausdorff distance $d(I \setminus \cup_{k=0}^{e(n)} J_k, I \setminus U) \leq 2^{-n}$ for all $n \in \mathbb{N}$.

Let $f : I \to \mathbb{R}^2$ be a map such that $\Lambda = \bigcap_{n=-\infty}^{\infty}(f^n(I) \cap I)$ is the Smale horseshoe. Without loss of generality assume that f performs a linear vertical expansion by a factor of $\mu = 4$ and a linear horizontal contraction by a factor of $\lambda = \frac{1}{4}$. For each $n \in \mathbb{N}$, let $U_n = I \setminus \bigcap_{k=-n}^{n}(f^k(I) \cap I)$. Then $I \setminus \Lambda = \bigcup_{n=0}^{\infty} U_n$. Moreover,

(1) there exists a computable function $\alpha : \mathbb{N} \to \mathbb{N}$ such that U_n is a union of $\alpha(n)$ rational open rectangles in I,

(2) $d(\Lambda, I \setminus \bigcup_{k=0}^{n} U_n) \leq (1/4)^n$.

Define $e : \mathbb{N} \to \mathbb{N}$, $e(n) = \sum_{k=0}^{n} \alpha(k)$, $n \in \mathbb{N}$. Then it follows from the lemma that $I \setminus \Lambda$ is indeed a computable open subset of I. ■

7 Conclusions

We have shown that, locally, one can compute the stable and unstable manifolds of some given hyperbolic equilibrium point, though globally these manifolds are, in general, only semi-computable. It would be interesting to know if these results are only valid for equilibrium points or can be extended e.g. to hyperbolic periodic orbits. In [GZar] we provide an example which shows that the global stable/unstable manifold cannot be computed for hyperbolic periodic orbits.

However, the question whether locally these manifolds can be computed remains open.

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Appendix 1. Proof of Lemma 11. Assume that $u(t,a)$ is a continuous solution to the integral equation (14). We show that it satisfies the differential equation (7). To see this, we first establish a relation between I_{Γ_j}, $j=1,2$, and the Jordan canonical form of A:

$$A = C\begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} C^{-1}$$

where C is an $n\times n$ invertible matrix, P is a $k\times k$ matrix with eigenvalues λ_j, $1\le j\le k$, and Q is a $(n-k)\times(n-k)$ matrix with eigenvalues μ_j, $k+1\le j\le n$. By (11), we have

$$e^{At} = I_{\Gamma_1}(t) + I_{\Gamma_2}(t).$$

On the other hand, it is straightforward to show that

$$e^{At} = C\begin{pmatrix} e^{Pt} & 0 \\ 0 & 0 \end{pmatrix} C^{-1} + C\begin{pmatrix} 0 & 0 \\ 0 & e^{Qt} \end{pmatrix} .C^{-1}$$

Since the first k columns of C consists of a basis of the stable subspace $P_1\mathbb{R}^n$ and the last $(n-k)$ columns of C is a basis of the unstable subspace $P_2\mathbb{R}^n$, it then follows from (21) and (22) that

$$I_{\Gamma_2}(t)C\begin{pmatrix} e^{Pt} & 0 \\ 0 & 0 \end{pmatrix} C^{-1} = 0, \quad \text{and}$$

$$I_{\Gamma_1}(t)C\begin{pmatrix} 0 & 0 \\ 0 & e^{Qt} \end{pmatrix} .C^{-1} = 0$$

Thus

$$I_{\Gamma_1}(t) = C\begin{pmatrix} e^{Pt} & 0 \\ 0 & 0 \end{pmatrix} C^{-1}$$

and

$$I_{\Gamma_2}(t) = C\begin{pmatrix} 0 & 0 \\ 0 & e^{Qt} \end{pmatrix} C^{-1}$$

(recall that $\mathbb{R}^n = P_1\mathbb{R}^n \bigoplus P_2\mathbb{R}^n$). Consequently the integral equation (14) can be written in the following form:

$$u(t,a) = C\begin{pmatrix} e^{Pt} & 0 \\ 0 & 0 \end{pmatrix} C^{-1}a + \int_0^t C\begin{pmatrix} e^{P(t-s)} & 0 \\ 0 & 0 \end{pmatrix} C^{-1}F(u(s,a))ds \quad (31)$$
$$-\int_t^\infty C\begin{pmatrix} 0 & 0 \\ 0 & e^{Q(t-s)} \end{pmatrix} C^{-1}F(u(s,a))ds.$$

It is known that if $u(t,a)$ is a continuous solution to (31), then it is the solution to (7) (*c.f.* §2.7 [Per01]). The proof is complete.

Appendix 2. Proof of Lemma 12. For $a, \tilde{a} \in B$ and $t \geq 0$, where $B = B(0,r) = \{x \in \mathbb{R}^n : |x| < r\}$, $r = 2^{-d(m_0)}/2K$ (d is the computable function defined in (10), and m_0 is a positive integer such that $2^{-m_0} \leq \frac{\sigma}{4K}$, define

$$
\begin{aligned}
u^{(0)}(t,a) &= 0 \\
u^{(j)}(t,a) &= I_{\Gamma_1}(t)a + \int_0^t I_{\Gamma_1}(t-s)F(u^{(j-1)}(s,a))ds \\
&\quad - \int_t^\infty I_{\Gamma_2}(t-s)F(u^{(j-1)}(s,a))ds, \quad j \geq 1
\end{aligned}
$$

We show that the following three inequalities hold for all $j \in \mathbb{N}$:

$$
\begin{aligned}
|u^{(j)}(t,a) - u^{(j-1)}(t,a)| &\leq K|a|e^{-\alpha_1 t}/2^{j-1} \\
|u^{(j)}(t,a)| &\leq 2^{-d(m_0)}e^{-\alpha_1 t} \\
|u^{(j)}(t,a) - u^{(j)}(t,\tilde{a})| &\leq 3K|a-\tilde{a}|
\end{aligned}
$$

We argue by induction on j. Since $u^{(0)}(t,a) = 0$ for any a, by (8) and (13) we get $|u^{(1)}(t,a)| = ||I_{\Gamma_1}(t)a|| \leq Ke^{-(\alpha+\sigma)t} \cdot 2^{-d(m_0)}/2K < 2^{-d(m_0)}e^{-\alpha_1 t}$, and $|u^{(1)}(t,a) - u^{(1)}(t,\tilde{a})| = |I_{\Gamma_1}(t)(a-\tilde{a})| \leq Ke^{-(\alpha+\sigma)t}|a-\tilde{a}| \leq K|a-\tilde{a}|$, the three inequalities hold for $j=1$. The estimate (13) is used in calculations here.

Assume that the three inequalities hold for all $k \leq j$. Then for $k = j+1$,

$$
\begin{aligned}
&|u^{(j+1)}(t,a) - u^{(j)}(t,a)| \\
&\leq \int_0^t ||I_{\Gamma_1}(t-s)|| \cdot |F(u^{(j)}(s,a)) - F(u^{(j-1)}(s,a))|ds \\
&+ \int_t^\infty ||I_{\Gamma_2}(t-s)|| \cdot |F(u^{(j)}(s,a)) - F(u^{(j-1)}(s,a))|ds \\
&\leq \int_0^t ||I_{\Gamma_1}(t-s)|| \cdot 2^{-m_0}|u^{(j)}(s,a) - u^{(j-1)}(s,a)|ds \\
&+ \int_t^\infty ||I_{\Gamma_2}(t-s)|| \cdot 2^{-m_0}|u^{(j)}(s,a) - u^{(j-1)}(s,a)|ds \\
&\leq 2^{-m_0}\int_0^t Ke^{-(\alpha_1+\alpha_2+\sigma)(t-s)}\frac{K|a|e^{-\alpha_1 s}}{2^{j-1}}ds + 2^{-m_0}\int_t^\infty Ke^{\sigma(t-s)}\frac{K|a|e^{-\alpha_1 s}}{2^{j-1}}ds \\
&= 2^{-m_0}\frac{K^2|a|}{2^{j-1}}e^{-(\alpha_1+\alpha_2+\sigma)t}\int_0^t e^{(\alpha_2+\sigma)s}ds + 2^{-m_0}\frac{K^2|a|}{2^{j-1}}e^{\sigma t}\int_t^\infty e^{(-(\alpha_1+\sigma)s}ds \\
&= 2^{-m_0}\frac{K^2|a|}{2^{j-1}}e^{-(\alpha_1+\alpha_2+\sigma)t}\frac{e^{(\alpha_2+\sigma)t}-1}{\alpha_2+\sigma} + 2^{-m_0}\frac{K^2|a|}{2^{j-1}}e^{\sigma t}\frac{0-e^{-(\alpha_1+\sigma)t}}{-(\alpha_1+\sigma)} \\
&\leq 2^{-m_0}\frac{K^2|a|}{2^{j-1}}\frac{e^{-\alpha_1 t}}{\alpha_2+\sigma} + 2^{-m_0}\frac{K^2|a|}{2^{j-1}}\frac{e^{-\alpha_1 t}}{\alpha_1+\sigma} \\
&< \frac{K|a|}{2^j}e^{-\alpha_1 t} \quad \text{(recall that } 2^{-m_0} \leq \tfrac{\sigma}{4K})
\end{aligned}
$$

and furthermore,

$$
\begin{aligned}
&|u^{(j+1)}(t,a)| \\
&\le |u^{(j)}(t,a)| + |u^{(j+1)}(t,a) - u^{(j)}(t,a)| \\
&\le |u^{(j)}(t,a)| + \frac{K|a|}{2^j} e^{-\alpha_1 t} \\
&\le \sum_{k=1}^{j} \frac{K|a|}{2^k} e^{-\alpha_1 t} \qquad \text{(induction hypothesis on } u^{(k)}(t,a) \text{ for } k \le j) \\
&\le 2K|a|e^{-\alpha_1 t} \le 2Ke^{-\alpha_1 t} \cdot 2^{-d(m_0)}/2K = 2^{-d(m_0)} e^{-\alpha_1 t}
\end{aligned}
$$

Lastly we show that if $|u^{(k)}(t,a) - u^{(k)}(t,\tilde{a})| \le 3K|a - \tilde{a}|$ holds for all $k \le j$, then it holds for $j+1$.

$$
\begin{aligned}
&|u^{(j+1)}(t,a) - u^{(j+1)}(t,\tilde{a})| \\
&= \left| I_{\Gamma_1}(t)(a-\tilde{a}) + \int_0^t I_{\Gamma_1}(t-s) \left(F(u^{(j)}(s,a)) - F(u^{(j)}(s,\tilde{a})) \right) ds - \right. \\
&\left. - \int_t^\infty I_{\Gamma_2}(t-s) \left(F(u^{(j)}(s,a)) - F(u^{(j)}(s,\tilde{a})) \right) ds \right| \\
&\le |I_{\Gamma_1}(t)(a-\tilde{a})| + \int_0^t ||I_{\Gamma_1}(t-s)|| \cdot \left| F(u^{(j)}(s,a)) - F(u^{(j)}(s,\tilde{a})) \right| ds + \\
&\int_t^\infty ||I_{\Gamma_2}(t-s)|| \cdot \left| F(u^{(j)}(s,a)) - F(u^{(j)}(s,\tilde{a})) \right| ds \\
&\le K|a-\tilde{a}| + 2^{-m_0} \left| u^{(j)}(s,a) - u^{(j)}(s,\tilde{a}) \right| \left(\int_0^t Ke^{-(\alpha+\sigma)(t-s)} ds + \int_t^\infty Ke^{\sigma(t-s)} ds \right) \\
&\le K|a-\tilde{a}| + 2^{-m_0} \cdot 3K|a-\tilde{a}| \cdot \left(\frac{K}{\alpha+\sigma} + \frac{K}{\sigma} \right) \\
&= K|a-\tilde{a}| \left(1 + 2^{-m_0} \frac{3K}{\alpha+\sigma} + 2^{-m_0} \frac{3K}{\sigma} \right) \\
&\le 3K|a-\tilde{a}| \quad (\text{Recall that } 2^{-m_0} \le \tfrac{\sigma}{4K})
\end{aligned}
$$

The proof is complete.

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