Monoidal Indeterminates and Categories of Possible Worlds

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Given any symmetric monoidal (closed) category $\mathbf{C}$ and any suitable collections $\mathcal{W}$ of objects of $\mathbf{C}$, it is shown how to construct $\mathbf{C}[\mathcal{W}]$, a *polynomial* such category, the result of freely adjoining to $\mathbf{C}$ a system of monoidal indeterminates for every element of $\mathcal{W}$. It is then shown that all of the categories of "possible worlds" used to treat languages that allow for dynamic creation of "new" variables, locations, or names are instances of this construction and hence have appropriate universality properties.

1. Introduction

The concept of a *polynomial algebra* $R[x]$, constructed from an algebra R by freely adjoining an *indeterminate* element x, is familiar from algebra. Similarly, Lambek and Scott (Lambek and Scott, 1986, Part I, Section 5) show how to construct a cartesian (or cartesian closed) polynomial category $\mathbf{C}[x]$ from a base cartesian (closed) category $\mathbf{C}$ by freely adjoining an indeterminate arrow $x\colon 1 \longrightarrow a$. Just as the polynomial algebra $R[x]$ is the "most general" such extension of R, the polynomial category $\mathbf{C}[x]$ is the most general cartesian (closed) extension of $\mathbf{C}$ containing indeterminate x.

Such properties are proved as *universality* results. For example, let $R_x\colon \mathbf{C} \longrightarrow \mathbf{C}[x]$ be the embedding of $\mathbf{C}$ into $\mathbf{C}[x]$ and consider any cartesian (closed) functor $F\colon \mathbf{C} \longrightarrow \mathbf{D}$ and any $d\colon 1 \longrightarrow F(a)$ in $\mathbf{D}$; then there exists a *unique* cartesian (closed) functor $F|^d_x$ from $\mathbf{C}[x]$ to $\mathbf{D}$ such that $(F|^d_x)(x) = d$ and $F|^d_x \cdot R_x = F$:

$$\begin{tikzcd} & \mathbf{C}[x] \arrow[dr, dotted, "F|^d_x"] & \\ \mathbf{C} \arrow[ur, "R_x"] \arrow[rr, "F"'] & & \mathbf{D} \end{tikzcd}$$

In this work, we develop comparable methodology for *symmetric monoidal* categories (Mac Lane, 1971). Given a symmetric monoidal category $\mathbf{C}$ and a suitable collection $\mathcal{W}$ of $\mathbf{C}$-objects, we show how to construct the symmetric monoidal polynomial category that results from freely adjoining indeterminates $x_w: I \longrightarrow w$ for every $w \in \mathcal{W}$; if $\mathbf{C}$ is closed, so is the polynomial category.

We believe this construction has many applications. As our leading example, we consider the categories of "possible worlds" that have been used in the semantics of imperative programming languages. John Reynolds and Frank Oles (Reynolds, 1981a; Oles, 1982; Oles, 1985; Oles, 1997; O'Hearn and Tennent, 1992) show how block-structured storage management in Algol-like languages (O'Hearn and Tennent, 1997) may be explicated using a semantics based on functor categories $\mathbf{W} \Rightarrow \mathbf{S}$, where $\mathbf{W}$ is a suitable category of "worlds" characterizing local aspects of storage structure, and $\mathbf{S}$ is a conventional semantic category of sets or domains. Every programming-language type θ is interpreted as a functor $[\![\theta]\!]: \mathbf{W} \longrightarrow \mathbf{S}$ and every programming-language term-in-context $\pi \vdash X: \theta$ is interpreted as a natural transformation $[\![\pi \vdash X: \theta]\!]: [\![\pi]\!] \stackrel{.}{\longrightarrow} [\![\theta]\!]$.

Oles gives two presentations of his category of worlds and shows that they are equivalent. Reynolds presents what *seems* to be a different category of worlds; however, it has recently been shown (Hermida and Tennent, 2007) that, under reasonable assumptions, it is in fact equivalent to Oles's category.

The functor-category framework has also been exploited to analyze non-interference in Reynolds's specification logic (Reynolds, 1981b; Tennent, 1990; O'Hearn, 1990; O'Hearn and Tennent, 1993), block expressions in Algol-like languages (Tennent, 1985), and passivity in a variant of Reynolds's Syntactic Control of Interference (Reynolds, 1978; O'Hearn et al., 1999). These applications used a related but significantly different category of worlds, due to Tennent.

Several authors (Moggi, 1990; Pitts and Stark, 1993; Sieber, 1994; Stark, 1996; Fiore et al., 2002) have used *finite sets* (of locally available "locations" or "names") as worlds, with injections as the morphisms.

What is noteworthy about all of this work is that the categories of worlds involved have been developed in *ad hoc* fashion and their properties have not been well understood. We show here that all of these categories of worlds are instances of our monoidal polynomial construction and have *universality* properties.

2. Monoidal Polynomial Categories

Given a symmetric monoidal category $\mathbf{C}$ with unit I and structural isomorphisms

$$\begin{aligned}
&\lambda_x: I \otimes x \;\cong\; x\\
&\rho_x: x \otimes I \;\cong\; x\\
&\alpha_{x,y,z}: (x \otimes y) \otimes z \;\cong\; x \otimes (y \otimes z)\\
&\sigma_{x,y}: x \otimes y \cong y \otimes x
\end{aligned}$$

subject to the usual coherence axioms (Mac Lane, 1971; Kelly, 1982), and a collection $\mathcal{W} \subseteq |\mathbf{C}|$ that includes I and is closed under tensoring and isomorphisms, we want to construct a category with the same objects as $\mathbf{C}$ and morphisms $(f, w): x \longrightarrow y$ for every $w \in \mathcal{W}$ and $f: x \otimes w \longrightarrow y$ in $\mathbf{C}$.

If $(\rho_x, I): x \longrightarrow x$ is considered to be the *identity* on x and the *composition* of $(f, w): x \longrightarrow y$ and $(g, w'): y \longrightarrow z$ is the morphism $(h, w \otimes w'): x \otimes (w \otimes w') \longrightarrow z$ defined as follows:

$$x \otimes (w \otimes w') \xrightarrow{\alpha^{-1}} (x \otimes w) \otimes w' \xrightarrow{f \otimes w'} y \otimes w' \xrightarrow{g} z$$

the associativity and identity requirements for a category are satisfied only up to isomorphism; that is, we have only a *bicategory* (or pseudocategory) (Borceux, 1994), with the 2-cells from $(f, w): x \longrightarrow y$ to $(f', w'): x \longrightarrow y$ being morphisms $h: w \longrightarrow w'$ such that

$$\begin{tikzcd}
 & y & \\
x \otimes w \arrow[ur, "f"] \arrow[rr, "x \otimes h"'] & & x \otimes w' \arrow[ul, "f'"']
\end{tikzcd}$$

A standard technique (Bénabou, 1967) to obtain a category from a bicategory is to consider equivalence classes of morphisms as follows: $(f, w) \simeq (f', w')$ iff there is an isomorphism $\theta: w \cong w'$ such that $f' \circ (x \otimes \theta) = f$. Then taking equivalence classes $[f, w]_{\simeq}$ as morphisms yields a conventional category, which we denote by $\mathbf{C}[\mathcal{W}]$. Thus, the hom-sets of $\mathbf{C}[\mathcal{W}]$ are given as equivalence classes:

$$\mathbf{C}[\mathcal{W}](x, y) = \coprod_{w \in \mathcal{W}} [\mathbf{C}(x \otimes, w, y)]_{\simeq} \tag{1}$$

Proposition 2.1. If $\mathbf{C}$ is symmetric monoidal, $\mathbf{C}[\mathcal{W}]$ has a symmetric monoidal structure.

Proof. The tensor product of objects x and y is $x \otimes y$, as in $\mathbf{C}$, and the same is true for the unit I. The tensor product of morphisms $[f, w]: x \longrightarrow y$ and $[f', w']: x' \longrightarrow y'$ is the morphism $[g, w \otimes w']: x \otimes x' \longrightarrow y \otimes y'$ where g is defined as follows:

$$x \otimes x' \otimes w \otimes w' \xrightarrow{x \otimes \sigma_{x', w} \otimes w'} x \otimes w \otimes x' \otimes w' \xrightarrow{f \otimes f'} y \otimes y'$$

(omitting associativity isos). Verification that this action is functorial involves only functoriality of $\otimes$, naturality of σ, and the coherence conditions on σ. □

Note that a symmetry (or, more generally, a braiding) is needed to tensor morphisms as above.

There is a natural embedding of $\mathbf{C}$ into $\mathbf{C}[\mathcal{W}]$: $f: x \longrightarrow y \ \mapsto \ [f \circ \rho_x, I]_{\simeq}: x \longrightarrow y$. Since $\rho_I = \lambda_I: I \times I \cong I$, this mapping is functorial (by the coherence axioms for the structural isomorphisms of the monoidal structure on $\mathbf{C}$), so we get a functor $R_{\mathcal{W}}: \mathbf{C} \longrightarrow \mathbf{C}[\mathcal{W}]$. A morphism $[f, w]: x \longrightarrow y$ with $w \cong I$ is termed *raw*. Raw morphisms yield a broad subcategory (i.e., with the same objects as the ambient category) of $\mathbf{C}[\mathcal{W}]$, the essential image of $R_{\mathcal{W}}$.

Proposition 2.2. $R_{\mathcal{W}}: \mathbf{C} \longrightarrow \mathbf{C}[\mathcal{W}]$ is (strongly) symmetric monoidal; i.e., it preserves the structure up to coherent isomorphism.

Proof. The coherent structural isomorphisms are the "raw" images of those in $\mathbf{C}$ under $R_{\mathcal{W}}$ and functoriality ensures that the coherence axioms hold as well; this makes $R_{\mathcal{W}}$ strongly symmetric monoidal. □

3. Properties of $\mathbf{C}[\mathcal{W}]$

3.1. *Universality*

The most significant feature of $\mathbf{C}[\mathcal{W}]$ is that it has, for every $w \in \mathcal{W}$, a "global element" $x_w = [\lambda_w, w]: I \longrightarrow w$. These morphisms will be termed *monoidal indeterminates.*

Definition 3.1. Given a monoidal category $\mathbf{D}$ and a monoidal subcategory $\mathbf{S}$, a *system of monoidal indeterminates for* $\mathbf{D}$ *(with respect to* $\mathbf{S}$*)* is a monoidal transformation $d: \kappa_I \Rightarrow J_{\mathbf{S}}$, where $\kappa_I: \mathbf{S} \longrightarrow \mathbf{D}$ is the strong monoidal functor constantly I, and $J_{\mathbf{S}}: \mathbf{S} \longrightarrow \mathbf{D}$ is the inclusion functor.

See (Kelly, 1982) for explanations of monoidal-categorical concepts, such as *monoidal transformation* and *strong monoidal functor*.

The x_w defined above are then a system of monoidal indeterminates for $\mathbf{C}[\mathcal{W}]$ with respect to $R_{\mathcal{W}}(\mathbf{W})$. We will show below (Theorem 3.2) that $\mathbf{C}[\mathcal{W}]$ is freely generated by this system of monoidal indeterminates.

Definition 3.2. For any object $x \in \mathbf{C}$, $e^w_x: [\mathrm{id}_{x\otimes w}, w]: x \longrightarrow x \otimes w$ is termed the *expansion morphism at* x *(with respect to* w*).*

The terminology will be justified in Section 4.1.

Lemma 3.1 (Expansion–Raw Morphism Factorization).

(1) The expansion morphisms e^w_x are natural in x with respect to raw morphisms.
(2) Every $\mathbf{C}[\mathcal{W}]$ morphism $[f, w]: x \longrightarrow y$ factors uniquely (up to isomorphism) as an expansion $e^w_x: x \longrightarrow x \otimes w$ followed by a raw morphism $[f \circ \rho_{x\otimes w}, I]: x \otimes w \longrightarrow y$.

Theorem 3.2 (Universality). Given a symmetric monoidal category $\mathbf{D}$, a strong monoidal functor $F: \mathbf{C} \longrightarrow \mathbf{D}$, and a system $d: \kappa_I \Rightarrow J_{F(\mathbf{W})}$ of monoidal indeterminates for $\mathbf{D}$, there exists an essentially unique strong monoidal functor $F|^d_x: \mathbf{C}[\mathcal{W}] \longrightarrow \mathbf{D}$ and a monoidal iso 2-cell $\theta: (F|^d_x \cdot R_{\mathcal{W}}) \Rightarrow F$ as follows:

$$\begin{tikzcd}
& \mathbf{C}[\mathcal{W}] \arrow[dr, dotted, "F|^d_x"] \arrow[d, phantom, "\theta \Downarrow"] & \\
\mathbf{C} \arrow[ur, "R_{\mathcal{W}}"] \arrow[rr, "F"'] & {} & \mathbf{D}
\end{tikzcd}$$

such that $F|^d_x \cdot x \cong d$.

The isomorphism $F|^d_x \cdot x \cong d$ here is a convenient abbreviation for the following commutativity of monoidal transformations:

$$\begin{tikzcd}
\kappa_I \arrow[r, "\gamma^{-1}"] \arrow[d, "d"'] & F|_x^d \cdot \kappa_I \arrow[d, "F|_x^d \cdot x"] \\
F \cdot J_{\mathbf{W}} & F|_x^d \cdot R_{\mathcal{W}} \cdot J_{\mathbf{W}} \arrow[l, "\theta \cdot J_{\mathbf{W}}"]
\end{tikzcd}$$

where $\gamma\colon F|_x^d \cdot \kappa_I \Rightarrow \kappa_I$ is a structural isomorphism associated with $F|_x^d$.

Proof. It is clear that the action of $F|_x^d$ on objects should be $(F|_x^d)(y) = y$. For a morphism $[f, w]\colon y \longrightarrow z$ factored as $[f \circ \rho_{y\otimes w}, I] \circ e_y^w$, we get $(F|_x^d)[f, w] =$

$$Fy \xrightarrow{\rho_{Fy}^{-1}} Fy \otimes I \xrightarrow{Fy \otimes d_w} Fy \otimes Fw \cong F(y \otimes w) \xrightarrow{Ff} Fz$$

Functoriality follows from the coherence axioms for the structural isomorphisms associated with F and the monoidality of the transformation d. It is easy to see that $F|_x^d$ is strong monoidal, with the same structural isomorphisms as F. We can take $\theta = \mathrm{id}$, but the general statement requires a general θ as we want $F|_x^d$ characterised only up to strong monoidal isomorphism.

The coherence conditions on F imply that $\gamma^{-1}(F|_x^d)(x^w) = \gamma^{-1}(F|_x^d)[\lambda_w, w] = d_w$ and $\gamma^{-1}(F|_x^d)[f, \rho_y] = F(f)$, for any morphism $f\colon y \longrightarrow z$ in $\mathbf{C}$. $\square$

The following special case will prove useful in Section 4.2 in characterizing the "states" functor in the semantics of imperative languages:

Corollary 3.3 (Universality on Coproduct-Monoidal Categories). If the monoidal structure on $\mathbf{D}$ is given by finite co-products $(0, +)$, there is an essentially unique strong monoidal functor $\hat{F}\colon \mathbf{C}[\mathcal{W}] \longrightarrow D$ extending a strong monoidal functor $F\colon \mathbf{C} \longrightarrow \mathbf{D}$:

$$\begin{tikzcd}
& \mathbf{C}[\mathcal{W}] \arrow[dr, dotted, "\hat{F}"] \arrow[d, phantom, "\Downarrow"] & \\
\mathbf{C} \arrow[ur, "R_{\mathcal{W}}"] \arrow[rr, "F"'] & {} & \mathbf{D}
\end{tikzcd}$$

Proof. Since the unit 0 is initial, there is a *unique* choice of global elements $!_w\colon 0 \longrightarrow F(w)$ for any $w \in \mathcal{W}$, and $\hat{F} = F|_x^{!_w}$. $\square$

3.2. *Indeterminate to a Single Object*

A special case of interest is the construction of the symmetric monoidal category generated by a category $\mathbf{C}$ and an indeterminate $x_w\colon I \longrightarrow w$ for a *single* object w. By tensoring such an indeterminate with itself and using the isomorphism $\lambda_I = \rho_I\colon I \otimes I \cong I$, one obtains indeterminates for all tensor powers w^i of w. Therefore, for any symmetric monoidal category $\mathbf{C}$ and object w, $\mathbf{C}[\mathcal{W}]$ for $\mathcal{W} = \{w^i\}_{i\geq 0}$ is a symmetric monoidal category with an indeterminate element $x\colon I \longrightarrow w$, and is universal among such.

In the remainder of this section, we describe additional properties of $\mathbf{C}[\mathcal{W}]$, with a view to the role this structure plays in categorical logic and semantics. Some readers might prefer to skip ahead to the applications in Section 4.

3.3. *Closed Structure and Duals*

Proposition 3.4. If $\mathbf{C}$ is a *closed* symmetric monoidal category, so is $\mathbf{C}[\mathcal{W}]$; furthermore, $R_{\mathcal{W}}: \mathbf{C} \to C[\mathcal{W}]$ preserves the closed structure.

Proof. Given the formulation of the hom-sets of $\mathbf{C}[\mathcal{W}]$ in equation (1), $\mathbf{C}[\mathcal{W}]$ inherits closed structure from $\mathbf{C}$ via the isomorphism

$$\textstyle\coprod_{w\in\mathcal{W}} \mathbf{C}\big((x\otimes y)\otimes w, z\big) \cong \coprod_{w\in\mathcal{W}} \mathbf{C}(x\otimes w, y \Rightarrow z)$$

which is compatible with the equivalence relation $\simeq$. It is then clear that $R_{\mathcal{W}}$ preserves the closed structure. □

Corollary 3.5. If $\mathbf{C}$ is *compact* closed (i.e., every object c admits a dual c^* such that $\mathbf{C}(x\otimes c, y) \cong \mathbf{C}(x, c^*\otimes y)$), so is $\mathbf{C}[\mathcal{W}]$; furthermore, $R_{\mathcal{W}}$ preserves duals.

3.4. *Traces*

The notion of *trace* (Joyal et al., 1996; Hasegawa, 2004) in a monoidal category is also compatible with the addition of monoidal indeterminates.

Proposition 3.6. If $\mathbf{C}$ admits a trace, so does $\mathbf{C}[\mathcal{W}]$; furthermore, $R_{\mathcal{W}}$ preserves traces.

Proof. A trace function

$$Tr^u_{x,y}: \mathbf{C}(x\otimes u, y\otimes u) \to \mathbf{C}(x,y)$$

for $\mathbf{C}$ is compatible with the equivalence $\simeq$ by dinaturality

$$\Big[\textstyle\coprod_{w\in W} Tr^u_{x\otimes w,y}\Big]_{\simeq} : \Big[\textstyle\coprod_{w\in W} \mathbf{C}\big((x\otimes w)\otimes u, y\otimes u\big)\Big]_{\simeq} \to \Big[\textstyle\coprod_{w\in W} \mathbf{C}(x\otimes w, y)\Big]_{\simeq}$$

and therefore induces a trace function on $\mathbf{C}[\mathcal{W}]$, evidently preserved by $R_{\mathcal{W}}$. □

4. Applications

4.1. *Oles's Category of Possible Worlds*

For $\mathbf{C}$ a symmetric monoidal category, let $\mathbf{C}_{\text{iso}}$ be the underlying groupoid of isomorphisms, which retains the symmetric monoidal structure of $\mathbf{C}$. When the monoidal structure is cartesian, $\mathbf{C}_{\text{iso}}[\mathcal{W}]$ is precisely $\mathbf{O}_{\mathcal{W}}$, Oles's category of possible worlds for storable data types $\mathcal{W}$ (Oles, 1982; Oles, 1985; Oles, 1997). We actually consider the equivalent presentation based on equivalence relations, as used in, for example, (Tennent, 1991). The expansion morphisms of Definition 3.2 and the factorization–raw morphism factorization of Proposition 3.1 are based on and generalize Oles's work.

We proceed to state the universal property of $\mathbf{O}_{\mathcal{W}}$. The embedding from $\mathbf{C}$ into $\mathbf{C}[\mathcal{W}]$ restricts to $R_{\mathcal{W}}: \mathbf{C}_{\text{iso}} \to \mathbf{C}_{\text{iso}}[\mathcal{W}] = \mathbf{O}_{\mathcal{W}}$ and the monoidal transformation $x: \kappa_I \Rightarrow J_{R_{\mathcal{W}}(\mathbf{W})}$

still makes sense when restricted to $\mathbf{W}_{\mathrm{iso}}$. The following corollary of Theorem 3.2 shows that $R_{\mathcal{W}}$ and x are universal:

Corollary 4.1 (Universality of Oles's Category of Worlds). Given a symmetric monoidal category $\mathbf{D}$, a strong monoidal functor $F\colon \mathbf{C}_{\mathrm{iso}} \longrightarrow \mathbf{D}$, and a system $d\colon \kappa_I \Rightarrow J_{F(\mathbf{W}_{\mathrm{iso}})}$ of monoidal indeterminates for $\mathbf{D}$, there exists an essentially unique strong monoidal functor $F|^d_x\colon \mathbf{O}_{\mathcal{W}} \longrightarrow \mathbf{D}$ and a monoidal iso 2-cell $\theta\colon (F|^d_x \cdot R_{\mathcal{W}}) \Rightarrow F$ as follows:

$$\begin{tikzcd} & \mathbf{O}_{\mathcal{W}} = \mathbf{C}_{\mathrm{iso}}[\mathcal{W}] \arrow[dr, dotted, "F|^d_x"] \arrow[d, phantom, "\theta \Downarrow"] & \\ \mathbf{C}_{\mathrm{iso}} \arrow[ur, "R_{\mathcal{W}}"] \arrow[rr, "F"'] & {} & \mathbf{D} \end{tikzcd}$$

such that $F|^d_x \cdot x \cong d$.

Thus, Oles's construction freely adds a system of monoidal indeterminates to the groupoid of isomorphisms of the given category.

In giving a semantics of an Algol-like language, one starts with a collection of "storable" types. The global elements are needed to initialize new local variables. The tensor product allows types to be formed into contexts, where the unit corresponds to the empty context. Our generalization of this construction to the symmetric-monoidal setting is compatible with the subsequent developments of O'Hearn and Reynolds (O'Hearn and Reynolds, 2000).

4.2. *The States Functor*

In (O'Hearn and Tennent, 1992), a functor $S\colon \mathbf{W}^{\mathrm{op}} \longrightarrow \mathbf{S}$ mapping worlds to the sets of states available in that world is discussed. This functor can be seen to be a direct consequence of the universality of Oles's category of worlds. It is induced as follows: to give a strong monoidal functor $S^{\mathrm{op}}\colon \mathbf{O}_{\mathcal{W}} \longrightarrow \mathbf{Set}^{\mathrm{op}}$ with respect to the cartesian monoidal structure in $\mathbf{Set}$, we have to pick objects $S(c)$ for every $c \in \mathbf{C}$ together with "global elements" $\mathbf{Set}^{\mathrm{op}}\big(1, S(c)\big)$. But, as noted in the proof of Corollary 3.3, there is only one such global element, namely the unique map ! from $S(c)$ into the terminal 1.

Therefore, the resulting contravariant functor $S\colon \mathbf{O}_{\mathcal{W}}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ sends "expansions" (tensors of identities and indeterminates) to projections (cartesian tensor of identities and !s), which is the action of S via Oles's description of $\mathbf{O}_{\mathcal{W}}$ (Oles, 1985).

It is worth pointing out that the remaining basic semantic functors for Algol, namely those corresponding to expressions, commands, and variables, are definable from S and constant functors via the *contra-exponentiation* of (O'Hearn and Tennent, 1992). The only other noteworthy ingredient in the semantics of Algol (besides the cartesian closed structure of the functor category) is the use of "initial values" for local variables in the definition of the binder **new**, which come from the presence of monoidal indeterminates in $\mathbf{O}_{\mathcal{W}}$ as indicated above.

4.3. *The Category of Finite Sets and Injections*

Several authors (Moggi, 1990; Pitts and Stark, 1993; Sieber, 1994; Stark, 1996; Fiore et al., 2002) have used the category $\mathbf{F}_{\mathrm{inj}}$ of *finite sets* (of locally available "locations" or "names") with *injections* as the morphisms. We will exhibit this category as an instance of (our version of) the Oles construction described in Section 4.1.

Consider the category $\mathbf{F}_{\mathrm{bij}}$ of finite sets and bijections (or permutations). This is known to be the free symmetric monoidal category on one generator (Kelly, 1974), the generator being any one-point set 1, and the monoidal structure being disjoint union (finite co-product). Applying the Oles construction to $\mathbf{F}_{\mathrm{bij}}$ with $\mathcal{W} = \{1^i\}_{i\geq 0}$ freely adds a monoidal indeterminate $x_1 : \emptyset \longrightarrow 1$.

Proposition 4.2. There is an identity-on-objects isomorphism $\mathbf{F}_{\mathrm{bij}}[\mathcal{W}] \cong \mathbf{F}_{\mathrm{inj}}$ and so $(\mathbf{F}_{\mathrm{inj}}, +, \emptyset)$ is the free symmetric monoidal category on one generator 1 with a monoidal indeterminate $x_1 : \emptyset \longrightarrow 1$.

Proof. An injection $f : X \hookrightarrow Y$ corresponds to an isomorphism $X + W \cong Y$ where $W = Y \setminus f(X)$. The universal characterization of $(\mathbf{F}_{\mathrm{inj}}, +, \emptyset)$ now follows from those of $(\mathbf{F}_{\mathrm{bij}}, +, \emptyset)$ and the Oles construction. □

A straightforward consequence of this identification is that the formula

$$B^A(s) = \mathbf{Set}^{\mathbf{F}_{\mathrm{inj}}}\big(A(s + \cdot), B(s + \cdot)\big)$$

for functor exponentiation in (Stark, 1996, Section 5) is an instance of the Exponent Representation Lemma (Lemma 4) of (O'Hearn and Reynolds, 2000), which in fact holds for any $\mathbf{O}_{\mathcal{W}}$ category.

4.4. *Tennent's Category of Possible Worlds*

The following category, $\mathbf{T}$, is described in (Tennent, 1990): the objects are sets, interpreted as the sets of states *allowed* in each possible world, and a morphism from X to Y is a pair f, Q having the following properties:

(i) f is a function from Y to X;
(ii) Q is an equivalence relation on Y; and
(iii) f restricted to any Q-equivalence class is injective; i.e., for all $y, y' \in Y$, if yQy' and $f(y) = f(y')$ then $y = y'$.

Intuitively, f extracts the small stack embedded in a larger one, and Q relates large stacks with identical "extensions." For Oles's category, the function f is required to be *bi*jective on equivalence classes, so that $Y \cong X \times Y/Q$. The motivation for allowing subsets as well as "expansions" was to model the non-interference predicate in Reynolds's specification logic (Tennent, 1990; O'Hearn and Tennent, 1993).

The identity morphism id_X on an object X has as its two components: the identity function on X, and the universally-true binary relation on X. The composition of morphisms $f, Q : X \longrightarrow Y$ and $g, R : Y \longrightarrow Z$ has as its two components: the functional composition of f and g, and the equivalence relation on Z that relates $z_0, z_1 \in Z$ just if they are R-related and Q relates $g(z_0)$ and $g(z_1)$.

Proposition 4.3. Given sets X and Y, there is a one-to-one correspondence between the following sets of data:

(1) pairs (m, W) where W is a set and $m\colon Y \hookrightarrow X \times W$ is a monomorphism;
(2) $\mathbf{T}$-morphisms $f, Q\colon X \longrightarrow Y$.

Proof. From (1) to (2): Let $f\colon Y \longrightarrow X$ be the composite $Y \overset{m}{\hookrightarrow} X \times W \xrightarrow{\pi} X$ and Q be the kernel equivalence of $Y \overset{m}{\hookrightarrow} X \times W \xrightarrow{\pi'} W$; i.e., yQy' iff $\pi'(my) = \pi'(my')$. To show that f is injective on each equivalence class, assume yQy' and $f(y) = f(y')$; then $\pi(my) = \pi(my')$ and $\pi'(my) = \pi'(my')$ and so $my = my'$. But then $y = y'$ because m is monic.

From (2) to (1): Let W be Y/Q and $m\colon Y \longrightarrow X \times W$ map y to the pair $(fy, [y]_Q)$. To show m is monic, assume $my = my'$; then $fy = fy'$ and yQy', and so $y = y'$. $\square$

This correspondence is applicable to any category in which we can reason about "quotients of equivalence relations"; for instance, the argument can be carried out for any regular category.

Consider now any symmetric monoidal category $\mathbf{C}$ such that $x \otimes \_\colon \mathbf{C} \longrightarrow \mathbf{C}$ preserves monomorphisms; this is automatic when the monoidal structure is cartesian because product projections are cartesian transformations. We may then construct $\mathbf{T}_{\mathbf{C}}$, the Tennent category of worlds relative to $\mathbf{C}$: the objects are those of $\mathbf{C}$, and the morphisms from x to y are equivalence classes $[f, w]_{\simeq}$ with $f\colon y \hookrightarrow x \otimes w$ in $\mathbf{C}$ and the equivalence $\simeq$ is with respect to isomorphisms $w \cong w'$ that make the evident diagram commute. Composition of $f, w\colon x \longrightarrow y$ and $g, w'\colon y \longrightarrow z$ is $(h, w \otimes w')\colon x \longrightarrow z$ where $h\colon z \hookrightarrow x \otimes (w \otimes w')$ is defined as follows:

$$z \xrightarrow{\quad g \quad} y \otimes w' \xrightarrow{\quad f \otimes w' \quad} (x \otimes w) \otimes w' \xrightarrow{\quad \alpha \quad} x \otimes (w \otimes w')$$

This is where we use the assumption that tensoring with an object preserves monomorphisms.

Proposition 4.4. $\mathbf{T}_{\mathbf{C}} \equiv \mathbf{C}^{\mathrm{op}}_{\mathrm{mono}}[\mathcal{W}]$,

where the mono subscript indicates restrictions to the broad monoidal subcategory with only monomorphisms as arrows and the collection $\mathcal{W}$ is that of all objects of $\mathbf{C}$ (assumed to be small).

As a consequence, we obtain a universal characterization of Tennent's category of worlds:

Corollary 4.5 (Universality of Tennent's Category of Worlds). Given a symmetric monoidal category $\mathbf{D}$ such that $d \otimes \_$ preserves monomorphisms,, a strong monoidal functor $F\colon \mathbf{C}^{\mathrm{op}}_{\mathrm{mono}} \longrightarrow \mathbf{D}$, and a system $d\colon \kappa_I \Rightarrow J_{F(\mathbf{W}_{\mathrm{iso}})}$ of monoidal indeterminates for $\mathbf{D}$, there exists an essentially unique strong monoidal functor $F|^d_x\colon \mathbf{T}_{\mathbf{C}} \longrightarrow \mathbf{D}$ and a monoidal iso 2-cell $\theta\colon (F|^d_x \cdot R_{\mathcal{W}}) \Rightarrow F$ as follows:

$$\begin{tikzcd} & \mathbf{T}_{\mathbf{C}} \arrow[dr, dotted, "F|_x^d"] & \\ \mathbf{C}^{\mathrm{op}}_{\mathrm{mono}} \arrow[ur, "R_{|\mathbf{C}|}"] \arrow[rr, "F"'] & \theta \Downarrow & \mathbf{D} \end{tikzcd}$$

such that $F|_x^d \cdot x \cong d$.

5. Discussion

We have described here the construction of a polynomial symmetric monoidal (closed) category, obtained from a symmeteric monoidal (closed) category by freely adjoining a system of monoidal indeterminates. The construction was motivated by our desire to understand the categories of possible worlds that have been used in semantical analyses of languages allowing creation of "new" variables or names. These categories, though originally presented in fairly *ad hoc* fashion, have all been shown here to be polynomial monoidal categories, with corresponding universality properties. Intuitively, the indeterminates represent uninitialized "new" components of the state or name context; the substitution functor $F|_x^d$ then provides the means to produce an "expanded" state or context with *initialized* new variables, for any appropriate choice of initial values d:

$$\begin{tikzcd} \mathbf{C} \arrow[r, shift left, "R_{\mathcal{W}}"] & \mathbf{C}[\mathcal{W}] \arrow[l, shift left, "F|_x^d"] \end{tikzcd}$$

We expect that the methodology introduced here will be useful in other applications.

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