A note on comprehensive factorization

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1 Introduction

The purpose of this note is to give a revised account of the comprehensive factorization of a functor. Such factorization was explicitly introduced in [SW73]. It states that every functor $F : \mathbb{A} \to \mathbb{B}$ can be (uniquely) factored as $F = m \circ e$ where $e : \mathbb{A} \to \mathbb{D}$ is *initial* and $m : \mathbb{D} \to \mathbb{B}$ is a *discrete cofibration*. The non-trivial part of the argument is to show e initial. Unfortunately, *ibid.* fails to give a proof of this. But in fact, the result is correct, and it had been proved earlier by Paré in [Par73], without acknowledging it as a comprehensive factorization though.

The term 'comprehensive' comes from the fact that we use a *comprehension schema* to obtain the factorization. We review the general setting for this (a fibration admitting comprehension and sums) in §2. We then go on to review in some detail how the comprehensive factorization works for the simple case of sets and functions (§3), whereby we meet simple 1-dimensional analogues of the constructions involved in the case of functors in §4. There are no original results; we present the background analysis (which is missing in the quoted references) and reorganise the relevant results from [Par73] to obtain the solution.

2 General setting: comprehension schema on a fibration

Consider a fibration $p : \mathbb{E} \to \mathbb{B}$. We think of it in logical terms: $\mathbb{B}$ is a category of 'sets' and 'functions' and the objects of $\mathbb{E}$ are 'predicates' or 'attributes', while the vertical arrows correspond to 'entailments'. We also assume a 'truth' predicate, which amounts to a fibred terminal object. The concise description is as a right-adjoint-right-inverse $p \dashv \top : \mathbb{B} \to \mathbb{E}$.

In set theory, given a predicate ϕ on a set X, the comprehension axiom says we can form a set $\{x \in X \mid \phi(x)\}$, the 'extent' of X. In the abstract setting, comprehension is characterised as a right adjoint $\top \dashv \{_\} : \mathbb{E} \to \mathbb{B}$.

In order to carry out the familiar 'image factorization' of a function, we need to assume 'direct images' in the fibration. These amount to sums, $\Sigma_f \dashv f^* : \mathbb{E}_J \to \mathbb{E}_I$ for every $f : I \to J$, satisfying Beck-Chevalley. In the presence of such direct images, we can reformulate the comprehension scheme as follows:

$$\mathbb{B}/J \underset{\{_\}}{\overset{Im}{\underset{\top}{\rightleftarrows}}} \mathbb{E}_J$$

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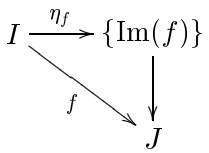
where $Im : \mathbb{B}/J \to \mathbb{E}_J$ takes $f : I \to J$ to $\Sigma_f(\top_I)$ (its image). This is in fact the original formulation in [Law70]. The above adjunction gives the desired factorization:



where η_f is the unit of the adjunction. We shall see in the two specific situations below that the construction of the $\{Im(f)\}$ will guarantee automatically its 'monicness' (for the chosen notion of mono). The main work will be to characterise the 'epis' and show η_f is one such. As it happens in the cases below, the 'comprehension' adjunction can usually be decomposed into several ones (essentially two) and the task reduces then to verifying that the respective units are 'epis', and as we shall see there are quite different reasons why they are so.

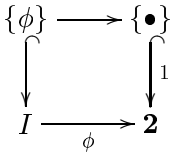
3 The $\mathcal{S}et$ case: epi-mono factorization of a function

Consider the fibration $p : Fam(\mathbf{2}) \to \mathcal{S}et$, where $\mathbf{2}$ is the usual 2-element $\{0, 1\}$ Boolean algebra (the subobject classifier in $\mathcal{S}et$). The 'truth' predicate is the functor $\top : \mathcal{S}et \to Fam(\mathbf{2})$ with action $I \mapsto (i{:}\, I \mapsto 1)$.

Comprehension has the usual form: given $\phi : I \to \mathbf{2}$,

$$\{\phi\} = \{i \in I \mid \phi(i) = 1\}$$

therefore a subset of I. It is more conveniently expressed by the following pullback



where we see that the left arrow is a mono by construction.

Direct images are described as follows: given $f : I \to J$ and $\phi : I \to \mathbf{2}$,

$$\Sigma_f(\phi) = (j{:}\, J \mapsto \bigvee_{[i \in f^{-1}(j)]} \phi(i))$$

Now we have the setup to factorize a function $f : I \to J$ as $f = m \circ e$ where $m : \{Im(f)\} \to J$ is the mono determined by its comprehensive image. What about e? We expect e to be an epi in the usual sense. In fact, epis in $\mathcal{S}et$ are simply surjections, which can be expressed logically as

$$e : I \to Z \text{ epi iff } \Sigma_e(\top_I) = \top_Z$$

as this effectively means (by the above formula for Σ_e) that the fibers $e^{-1}(z)$ are non-empty.

Let us examine the comprehension adjunction in more detail. First we notice that **2** is the arrow category $0 \to 1$. We have the (obviously full and faithful) inclusion $\iota : \mathbf{2} \to \mathcal{S}et$ sending 0 to $\emptyset$ and 1 to $\{\bullet\}$. This inclusion has a right adjoint $\pi_0 : \mathcal{S}et \to \mathbf{2}$ with action

$$I \mapsto \bigvee_{[i \in I]} 1$$

Of course, this boils down to 1 iff $I \neq \emptyset$, but the above expression is more telling for the pursposes of its generalization in §4. The unit of this reflection $\eta_I : I \to \pi_0(I)$ is an epi, since it is the coequalizer

$$I \times I \underset{\pi'}{\overset{\pi}{\rightrightarrows}} I \xrightarrow{\eta_I} \pi_0(I)$$

Notice also that $e : I \to Z$ is an epi iff $\forall z\colon Z . \pi_0(e^{-1}(z)) = 1$.

Back to the comprehension adjunction, we can now express as the composite of adjoints

$$\mathcal{S}et/J \underset{\mathrm{Gr}}{\overset{(_)^{-1}}{\underset{\simeq}{\rightleftarrows}}} \mathcal{S}et^J \underset{\iota^J}{\overset{\pi_0^J}{\underset{\perp}{\rightleftarrows}}} \mathbf{2}^J$$

hence the unit $\eta_f : f \to \{Im(f)\}$ is the composite

$$I \xrightarrow{\sim} \mathrm{Gr}(f^{-1}) \xrightarrow{\mathrm{Gr}(\eta^J_{f^{-1}})} \{\mathrm{Im}(f)\}$$

where the first factor is an isomorphism and the second is epic (since it is epic fibrewise). Hence η_f is epic, as we wanted to prove.

3.1. Remark. The argument for the epi-mono factorization we have given goes through for any elementary topos $\mathcal{E}$. The identification of $\mathcal{E}/J$ with $\mathcal{E}^J$ requires considering J as an internal discrete category in $\mathcal{E}$. Of course, **2** must be replaced by the subobject classifier Ω.

4 The $\mathcal{C}at$ case: the initial-discrete cofibration factorization of a functor

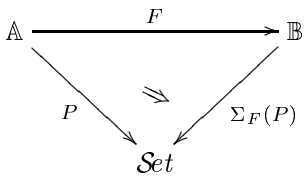
Consider now the fibration $p : Fam(\mathcal{S}et) \to \mathcal{C}at$, whose fibre at a small category $\mathbb{C}$ is the category of $\mathcal{S}et$-valued functors $[\mathbb{C}, \mathcal{S}et]$, with reindexing given by composition. The 'truth' predicate is the functor $\top : \mathcal{C}at \to Fam(\mathcal{S}et)$ whose value at $\mathbb{C}$ is the constant functor

$$\top_{\mathbb{C}} = \mathbb{C} \longrightarrow \mathbf{1} \xrightarrow{\{\bullet\}} \mathcal{S}et$$

The comprehension scheme associates to a functor $P : \mathbb{C} \to \mathcal{S}et$ the corresponding discrete cofibration via the Grothendieck construction. It is useful to exhibit this construction via the following comma square

$$\begin{array}{ccc} \mathrm{Gr}(P) & \longrightarrow & \mathbf{1} \\ \downarrow & \Leftarrow & \downarrow{\scriptstyle \{\bullet\}} \\ \mathbb{C} & \xrightarrow[P]{} & \mathcal{S}et \end{array}$$

Direct images amount to left (Kan) extensions: given $F : \mathbb{A} \to \mathbb{B}$ and $P : \mathbb{A} \to \mathcal{S}et$ we have



Recall that such a left extension is computed pointwise by the following colimit formula:

$$\Sigma_F(P)(b) = \varinjlim\ F{\downarrow}b \longrightarrow \mathbb{A} \xrightarrow{P} \mathcal{S}et$$

We know now that our 'monics' are going to be the discrete cofibrations. What about the 'epis'? They turn out to be the so called *initial functors*.

4.1 Initial functors and connected components

For a thorough treatment of these concepts see [Par73].

4.1. Definition. A functor $F : \mathbb{A} \to \mathbb{B}$ is **initial** if for every $\Gamma : \mathbb{B} \to \mathbb{C}$,

$$\varprojlim\ \Gamma F \simeq \varprojlim\ \Gamma$$

either one existing iff the other one does.

4.2. Example. Let S be a complete lattice. Consider a double-indexed downward chain $\Gamma : \omega^{op} \times \omega^{op} \to S$ (*i.e.* $\Gamma(i,j) \geq \Gamma(i',j')$ whenever $(i,j) \leq (i',j')$). The diagonal $\Delta : \omega^{op} \to \omega^{op} \times \omega^{op}$ is initial, *i.e.*

$$\bigwedge_{i,j} \Gamma(i,j) = \bigwedge_{n} \Gamma(n,n)$$

Before proceeding to an explicit (and useful) characterization of initial functors we need to introduce the 'connected components' functor, which will be part of our decomposition of the comprehension adjunction.

The inclusion $\Delta : \mathcal{S}et \to \mathcal{C}at$ (regarding sets as discrete categories) has a left adjoint $\Pi_0 : \mathcal{C}at \to \mathcal{S}et$, which takes a category (actually its underlying graph) to the set of connected components. It can be calculated as a colimit:

$$\Pi_0(\mathbb{C}) = \varinjlim\ \mathbb{C} \longrightarrow \mathbf{1} \xrightarrow{\{\bullet\}} \mathcal{S}et$$

Notice that $\Pi_0(\mathbb{C}) = \{\bullet\}$ iff $\mathbb{C}$ is non-empty and pathwise connected.

4.3. Proposition. *A functor $F : \mathbb{A} \to \mathbb{B}$ is initial iff for every object b of $\mathbb{B}$, the comma category $F{\downarrow}b$ is non-empty and pathwise connected, i.e.* $\Pi_0(F{\downarrow}b) \simeq \{\bullet\}$

This well-known characterization has an elegant expression in terms of left extensions.

4.4. Proposition. *A functor $F : \mathbb{A} \to \mathbb{B}$ is initial iff* $\Sigma_F(\top_{\mathbb{A}}) \simeq \top_{\mathbb{B}}$

We then see that initial functors satisfy the 'logical' description of epis for the fibration $p : Fam(\mathcal{S}et) \to \mathcal{C}at$. We sum up the properties of initial functors we shall need later. They all follow easily from Proposition 4.4 and pasting and cancellation properties of left extensions.

4.5. Proposition.

1. *Initial functors are closed under composition.*
2. *If FG and G are initial, so is F.*
3. *Left adjoint functors are initial.*
4. *Coinverters, coidentifiers and coequalizers are initial.*

4.2 The comprehensive factorization of a functor

In order to decompose the comprehension adjunction, let us examine the left adjoint part $\mathrm{Im} = \Sigma_F(\top_{\mathbb{A}}) : \mathcal{C}at/\mathbb{B} \to [B, \mathcal{S}et]$. Bearing in mind the computation of left extensions (into $\mathcal{S}et$) as colimits, we have

$$\Sigma_F(\top_{\mathbb{A}})(b) = \varinjlim\ F{\downarrow}b \longrightarrow \mathbb{B} \longrightarrow \mathbf{1} \xrightarrow{\{\bullet\}} \mathcal{S}et$$

whence

$$\Sigma_F(\top_{\mathbb{A}})(b) = \Pi_0(F{\downarrow}b)$$

We are thus led to consider the functor $M(F) = F{\downarrow}_ : \mathbb{B} \to \mathcal{C}at$. This functor sets up an adjunction

$$\mathcal{C}at/\mathbb{B} \underset{\mathrm{Gr}}{\overset{M}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} [\mathbb{B}, \mathcal{C}at]$$

the right adjoint being (as usual) the Grothendieck construction associating a split cofibration to a covariant $\mathcal{C}at$-valued functor.

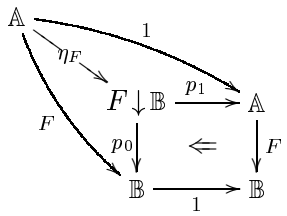
Thus, just like in the $\mathcal{S}et$ case, the comprehension adjunction decomposes as

$$\mathcal{C}at/\mathbb{B} \underset{\mathrm{Gr}}{\overset{M}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} [\mathbb{B}, \mathcal{C}at] \underset{\Delta^{\mathbb{B}}}{\overset{\Pi_0^{\mathbb{B}}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} [\mathbb{B}, \mathcal{S}et]$$

hence the unit $\bar{\eta}_F : F \to \mathrm{Gr}(\Sigma_F(\top_{\mathbb{A}}))$ of the composite adjunction is the composite

$$\mathbb{A} \xrightarrow{\eta_F} \mathrm{Gr}(F\downarrow_) \xrightarrow{\mathrm{Gr}((\eta'_{M(F)})^{\mathbb{A}})} \mathrm{Gr}(\Sigma_F(\top_{\mathbb{A}}))$$

First, let us notice $\mathrm{Gr}(F\downarrow_) \simeq F\downarrow\mathbb{B}$, and η_F is obtained as the unique mediating arrow into the comma object:



It then follows (using universality of the comma object) that $\eta_F \dashv p_1$. By Proposition 4.5.(3), $\eta_F : \mathbb{A} \to F\downarrow\mathbb{B}$ is initial.

As for the second factor $\mathrm{Gr}(\eta'_{M(F)}) : F\downarrow\mathbb{B} \to \mathrm{Gr}(\Sigma_F(\top))$ we have in general that

$$\mathbb{C}^{\to} \underset{d_1}{\overset{d_0}{\Downarrow}} \mathbb{C} \xrightarrow{\eta'_{\mathbb{C}}} \Pi_0(\mathbb{C})$$

is a coidentifier. Hence $\mathrm{Gr}(\eta'_{M(F)})$ is (fibrewise) initial by Proposition 4.5.(4). In turn, Proposition 4.5.(1) allows us to conclude η_F is initial, as required.

We end up noticing we can do a little bit better. Taking discrete objects (which as shown above amounts to a coidentifier) is an unnatural construction in most bicategories of interest, as it involves the identification of objects. We are only interested in the identity of objects (inside a category) up to isomorphism. Hence it is more reasonable to consider the groupoidal reflection of the fibration $d_0 : F\downarrow\mathbb{B} \to \mathbb{B}$. In general, the groupoidal reflection of a category $\mathbb{C}$ is obtained as the following coinverter

$$\mathbb{C}^{\to} \underset{d_1}{\overset{d_0}{\Downarrow}} \mathbb{C} \xrightarrow{\eta''_{\mathbb{C}}} \Pi_1(\mathbb{C})$$

and the above argument guarantees that we still have an initial-groupoidal cofibration factorization for a functor, obtained from the comprehension scheme

$$\mathcal{C}at/\mathbb{B} \underset{\mathrm{Gr}}{\overset{\Pi_1 \circ M}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} [\mathbb{B}, \mathcal{G}pd]$$

We furthermore obtain from this factorization the previous one, noticing that $\Pi_0(\mathbb{C})$ is a skeleton of $\Pi_1(\mathbb{C})$ and hence equivalent.

References

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