Representable multicategories

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Abstract

We introduce the notion of *representable multicategory*, which stands in the same relation to that of monoidal category as fibration does to contravariant pseudofunctor (into $\mathcal{C}at$). We give an abstract reformulation of multicategories as monads in a suitable Kleisli bicategory of spans. We describe representability in elementary terms via *universal arrows*. We also give a doctrinal characterisation of representability based on a fundamental monadic adjunction between the 2-category of multicategories and that of strict monoidal categories. The first main result is the coherence theorem for representable multicategories, asserting their equivalence to strict ones, which we establish via a new technique based on the above doctrinal characterisation. The other main result is a 2-equivalence between the 2-category of representable multicategories and that of monoidal categories and strong monoidal functors. This correspondence extends smoothly to one between bicategories and a several object version of representable multicategories.

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1 Introduction

Among his vast seminal body of work, Grothendieck established a paradigm we might call *coherence via universality*. He introduced in [Gro71] the notion of *fibration*, a functor $p : \mathbb{E} \rightarrow \mathbb{B}$ characterised by the existence of cartesian liftings for any pair $(X \in \mathbb{E}, u : I \rightarrow pX)$ in $\mathbb{B}$. Here we have a strict structure, namely the category $\mathbb{E}$ with *strictly associative* composition and identities, within which there are *universally characterised* elements, namely the cartesian morphisms, which subsume a coherent structure. This latter structure is a pseudofunctor $F : \mathbb{B}^{op} \rightarrow \mathcal{C}at$, with its coherent isomorphisms $\delta_{f,g} : Fg \circ Ff \xrightarrow{\sim} F(f \circ g)$ (for composable f, g) and $\gamma_A : id_{FA} \xrightarrow{\sim} F(id_A)$ $(A \in \mathbb{B})$. The coherence relations these isomorphisms must satisfy are best understood associating to F its so called *Grothendieck construction* $(p_F : \mathcal{G}(F) \rightarrow \mathbb{B})$: the axioms on δ's and γ's correspond to the associativity and identity ones for composition in the category $\mathcal{G}(F)$. This construction (a lax colimit) sets up a biequivalence between the 2-category of fibrations over $\mathbb{B}$ and that of pseudofunctors from $\mathbb{B}^{op}$ to $\mathcal{C}at$.

Here we pursue the above paradigm introducing the notion of **representable multicategory**. Just as a category is a (directed) graph with a monoid structure (see §3 below), a multicategory is a multigraph with a monoid structure (see §4 below), thus endowed with a strictly associative and unitary composition. A multigraph in turn consists of a set of arrows, with associated source and target in a set of objects. While the target of an arrow is a single object (just like in a graph), its source is a *sequence* of objects.

The coherent structure we seek to subsume is that of a *monoidal category*, as described for instance in [Mac71, Kel82]. In its traditional finitary presentation, it consists of a category $\mathbb{C}$ endowed with a unit object I and a functor $\otimes : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$, together with natural isomorphisms

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \xrightarrow{\sim} A \otimes (B \otimes C) \quad \lambda_A : A \otimes I \xrightarrow{\sim} A \quad \rho_A : I \otimes A \xrightarrow{\sim} A$$

for A, B, C objects of $\mathbb{C}$, subject to coherence axioms (the celebrated pentagon for the α's and the triangle for the left and right unit isomorphisms λ

and ρ). Such coherence conditions are best expressed by the fact (the basic coherence result for monoidal categories) that between any two possible n-ary 'multiplications' from $\mathbb{C}^n$ to $\mathbb{C}$ defined by I,$\otimes$ (and a well bracketing of n elements) there is a *unique* coherent isomorphism made up by tensoring of α's, λ's and ρ's. We are thus led naturally to consider the structure defined by such n-ary operations and their compositions, namely a multicategory (see §9.2 below).

In the multicategory resulting from such a construction, a morphism $f : \langle A_1, \ldots, A_n \rangle \rightarrow A$ amounts to a morphism $\hat{f} : A_1 \otimes \ldots \otimes A_n \rightarrow A$ in $\mathbb{C}$ for some multiplication of $A_1, \ldots, A_n$. Notice in particular the distinguished class of morphisms $h : \langle A_1, \ldots, A_n \rangle \rightarrow A_1 \otimes \ldots \otimes A_n$ formed by tensoring coherent isomorphisms. Such morphisms witness the monoidal structure inherent in this multicategory. We are then faced with the task of giving a universal characterisation of such morphisms, which will effectively give us our first definition of representable multicategory (see §8.1). This is a good time to explain our choice of terminology: *representability* in a multicategory means that the 'multilinear' structure of its morphisms (and their composition) is universally represented by a ('multilinear classifying') tensor. To pin down this intuition, consider the classical example of a representable multicategory, namely that whose objects are bimodules over a commutative ring and whose arrows are multilinear morphisms between such. Here, the tensor product of bimodules gives a representation of such morphisms as linear ones (see Example 2.2.(1)).

Having just outlined the analogy between 'fibrations vs. pseudofunctors' on the one hand and 'representable multicategories vs. monoidal categories' on the other, it is appropriate to point out that just as fibrations arise naturally in practice while pseudofunctors do require a *choice* (not always available, for instance if we are working internally in a topos), the same is true of representable multicategories. Any monoidal category occurring in practice is either strict (typically with tensor product given by some composition) or has a universally specified tensor structure, which only after performing a *choice* can be cast into the finite presentation above. The 2-equivalence between the two structures enables us to claim that any monoidal category arises by such a choice process. Of course, a pseudofunctor can be a convenient presentation of a fibration and the same consideration applies to monoidal categories with respect to representable multicatgories.

Let us put the subject of this paper in context, indicating briefly our motivation for its development. Over recent years the quest for a suitable definition of *weak n-category* (that is, an n-dimensional categorical structure with coherently associative and unitary compositions) has intensified. Among the various approaches which had emerged to tackle this delicate notion we could mention the topological ones of [Joy97, Tam96] which deal with such n-dimensional gadgets via their *nerves*, imposing suitable filling conditions on certain cellular structures. Another approach is that of Batanin [Bat98b], based on homotopical algebra, more specifically on the theory of *operads*. He considers *contractible* operads to organise the 'spaces' of coherent operations, so that weak n-categories amount to algebras for such operads.

The approach most relevant to our present concerns is that of [BD98]. Its most prominent feature is that the operations of composition are not given explicitly as structure, but rather *specified by a universal property*. This is, in principle, the most attractive approach from a category theoretic perspective. Of course, to specify something universally there must be an ambient structure, which in the case of *ibid.* are the so-called *n-opetopes*, cellular structures whose cells have a specified finite boundary with a single cell of (immediate) lower dimension designated as codomain (n-opetopes are defined in terms of operads, although the opetopes carry no explicit operations themselves).

The approach in *ibid.* just outlined motivated [HMP98b], which in turn initiated the work presented in this paper. There and in [HMP98a], the authors develop an alternative (substantially simpler) method to set-up what ought to be the above mentioned n-opetopes, that is, the ambient cellular structures required to specify n-category compositions via the existence of *universal cells*. The main aim of our theory of representable multicategories (or rather, their higher-dimensional extension which we will take up in a subsequent paper) is to have a solid basis for such a universal approach to weak n-categories. Notice that a major difference between our proposal (here examined thoroughly in its lower dimensional instance) and that of [BD98] is our consideration of a multicategory structure on the cellular structures we consider, in order to have a proper framework for the notion of universality, *viz.* our representability condition. The present paper may be considered then as the first instalment of our approach to *higher-dimensional category theory*, laying down a substantial part of its conceptual and technical foundations.

Overview of the paper: In §2 we recall the elementary definition of multicategory and present a few basic examples. In §3 we recall Bénabou's view of categories as graphs (endospans) with monoid structure, *i.e.* a category in $\mathbb{B}$ is a monad in the bicategory of spans $\mathbf{Spn}(\mathbb{B})$. This is the abstract point of view we apply in §4 to reformulate multicategories (internally). The main abstract construction is a Kleisli bicategory of spans $\mathbf{Spn}_{\mathbf{T}}(\mathbb{B})$ relative to a *cartesian monad* $\mathbf{T}$ on a category with pullbacks $\mathbb{B}$ (Def. 4.2). An internal multicategory in $\mathbb{B}$ is identified with a monad in $\mathbf{Spn}_{\mathbf{M}}(\mathbb{B})$ where $\mathbf{M}$ is the free-monoid monad on $\mathbb{B}$. The point of this abstract reformulation is that composition in $\mathbf{Spn}_{\mathbf{M}}(\mathbb{B})$ embodies the 'tree grafting' involved in multicategory composition. In order to get a better grasp of the latter, we deal briefly with free multicategories in §5, where we characterise them in terms of finite trees and their grafting composition (Prop. 5.2).

Having achieved the basic identification of Def. 4.3, we proceed to set up the 2-category of multicategories. In order to do so, we continue the reformulation of internal category theory (along the lines of [Bén67] mentioned above), giving an algebraic interpretation of natural transformations in terms of morphisms of bimodules (Prop. 6.1), as well as an analysis of their horizontal and vertical compositions. We then instantiate this algebraic formulation of transformations in the context of multicategories, arriving to the notion of transformations satisfying a multinaturality condition (Def. 6.6) as the relevant 2-cells for multicategories. We conclude the 2-categorical setup giving a (multinatural) hom-set isomorphism characterisation of adjunctions between multicategories (Prop. 6.9).

In §7 we arrive at the first major correspondence in this paper, namely the monadic 2-adjunction between the 2-category of strict monoidal categories and that of multicategories (Thm. 7.2).

In §8 we introduce *representable multicategories*. We give two equivalent definitions: one in terms of *universal arrows* closed under composition (Def. 8.1) and the other in terms of *strong universal arrows* (Def. 8.3). This parallels the corresponding situation for fibrations. We also examine the basic examples of multicategories in order to present concrete instances of representability. Along these lines, and mostly to show some intrinsic internal manipulation of the notion of representability, we construct the formal dual of a representable multicategory in §8.2. In §8.3 we arrive to one of our main contributions, namely the adjoint characterisation of representability (Thm. 8.12): a multicategory is representable iff it is an adjoint pseudo-algebra for the 2-monad on $\mathcal{M}ulticat$ induced by the adjunction of §7.

In §9.1 we briefly review both the usual finite presentation of monoidal categories and outline the corresponding infinite presentation (*i.e.* with coherently associative and unitary n-fold tensor products). In §9.2 we establish one of our main results (Thm. 9.8), which asserts that the 2-categories of representable multicategories and (non-strict) monoidal categories are 2-equivalent. The basic tool is a 'Grothendieck contruction' which turns a (non-strict) monoidal category into a representable multicategory (Def. 9.2 and Prop. 9.4). In the opposite direction, we readily turn a representable multicategory into a monoidal category by making a *choice* of universal arrows (Def. 9.6).

In §10 we tackle the important issue of *coherence*. §10.1 shows how to normalise a representable multicategory, that is, to make a choice of universal arrows so that the linear ones are identities. In §10.2 we obtain our second main result, *viz.* the coherence theorem (10.8) which asserts that strict representable multicategories (and their strict morphisms) form a (bicategorically) reflective sub-2-category of that of representable multicategories, in such a way that the unit of this reflection exhibits any representable multicategory as equivalent to a strict one. The key technical lemma 10.4 provides the ingredients for such a coherence result. This technique relies fundamentally on the doctrinal (adjoint) characterisation of representability and sheds new light upon coherence issues for universally characterised structures.

We conclude in §11 showing how the correspondence between representable multicategories and monoidal categories extends to deal with bicategories. The essential insight is Prop. 11.1 which recasts (strict) monoidal categories as *categories in the category of monoids*. We can thus deal with 2-categories as strict monoidal categories (Prop. 11.2) by considering monoids on spans and thus obtain the desired correspondence (Prop. 11.4) between bicategories and representable multicategories (relative to the ambient monoidal category of endospans on the object-of-objects of the bicategory).

The technical appendix A examines the functoriality of the 'bicategory of spans' construction, based on its universal characterisation (Thm. A.2) and shows that our main abstract construction (Def. 4.2) is indeed a Kleisli bicategory associated to a pseudo-comonad (Prop. A.5).

The table below shows the correspondence between concepts of fibred category theory and representable multicategories. This correspondence is far reaching, but we only indicate it here up to the extent of the concepts we introduce (for instance, we do not address in this paper the issue of the Yoneda structure on multicategories, related to that of monoidal categories). It would certainly provide guidance into this subject to the reader familiar with the basic concepts of fibrations [1].

Fibred category theory	Representable multicategory theory
category over $\mathbb{B}$ vertical morphism fibre (strong) cartesian morphism fibration fibrewise dual	multicategory linear morphism $\overline{\mathbb{M}}$ category of linear morphisms (strong) universal arrow representable multicategory linear dual
pseudofunctor pseudonatural transformation modification	(non-strict) monoidal category strong monoidal functor monoidal transformation
$\mathbb{B}\downarrow \_ : \mathcal{C}at/\mathbb{B} \to \mathcal{C}at/\mathbb{B}$: free *split* fibration 2-monad, fibration = (adjoint) $\mathbb{B}\downarrow\_$-pseudo-algebra	$\mathbf{T} : \mathcal{M}ulticat \to \mathcal{M}ulticat$: free *strict* rep. multicategory 2-monad, representable multicategory = (adjoint) $\mathbf{T}$-pseudo-algebra
coherence theorem: fibration $\equiv$ split fibration	coherence theorem: representable multicategory $\equiv$ strict representable multicategory

2 Multicategories

The notion of **multicategory** was introduced in [Lam69] (see also [Lam89, Lin71]) as a framework for a syntactic calculus. A multicategory can be seen as an algebraic structure which models a (restricted kind of) many-sorted algebraic theory: briefly put, the terms are construed as arrows $t : \langle X_1, \ldots, X_n\rangle \to Y$, where $X_1,\ldots,X_n$ are the sorts of the free (non repeating) variables in t (more precisely, the arrows of a multicategory correspond to equivalence classes of terms, modulo provable equality in the theory).

[1] Notice that in a representable multicategory we do not have a *base* category. Hence we are in the presence of (a variant of) Bénabou's proposed notion of *foliation*. This latter amounts to considering the total category of a fibration without a specified base, whereby we retain the cartesian/vertical factorization of morphisms instead.

The salient feature of a multicategory is that its arrows have a *sequence* of objects as domain. This leads to a correspondingly more involved notion of composition than the binary one in ordinary categories. The presentation here is oriented to its abstract algebraic reformulation in §4 below.

2.1. Definition. A **multicategory** M consists of the following data:

1. a set of *objects* M_0, whose elements we write $X, Y, Z, \ldots$
2. a set of *arrows* M_1 whose elements we write $f, g, h, \ldots$, with specified *codomain* (an object) and *domain* (a sequence of objects). We write $f : \langle X_1, \ldots, X_n \rangle \rightarrow Y$ to specify the domain and codomain of f.
3. an *identity* arrow $id_X : \langle X \rangle \rightarrow X$, for each object X.
4. a *composition* operation, taking an arrow $f : \langle X_1, \ldots, X_n \rangle \rightarrow Y$ and a sequence of arrows

$$f_1 : \langle X_{11}, \ldots, X_{1m_1} \rangle \rightarrow X_1, \ldots, f_n : \langle X_{n1}, \ldots, X_{nm_n} \rangle \rightarrow X_n$$

and assigning their composite

$$f\langle f_1, \ldots, f_n \rangle : \langle X_{11}, \ldots, X_{1m_1}, \ldots, X_{n1}, \ldots, X_{nm_n} \rangle \rightarrow Y$$

subject to the following axioms

unit For $f : \langle X_1, \ldots, X_n \rangle \rightarrow Y$,

$$\begin{aligned} f\langle id_{X_1}, \ldots, id_{X_n} \rangle &= f \\ id_Y \langle f \rangle &= f \end{aligned}$$

associativity

$$\frac{\begin{array}{c} f : \langle X_1, \ldots, X_n \rangle \rightarrow Y, \\ f_1 : \langle X_{11}, \ldots, X_{1m_1} \rangle \rightarrow X_1, \ldots, f_n : \langle X_{n1}, \ldots, X_{nm_n} \rangle \rightarrow X_n, \\ f_{11} : \langle X_{111}, \ldots, X_{11p_{11}} \rangle \rightarrow X_{11}, ., f_{nm_n} : \langle X_{nm_n1}, \ldots, X_{nm_np_{nm_n}} \rangle \rightarrow X_{nm_n} \end{array}}{(f\langle f_1, \ldots, f_n \rangle)\langle f_{11}, \ldots, f_{nm_n} \rangle = f\langle f_1\langle f_{11}, \ldots, f_{1m_1} \rangle, ., f_n\langle f_{n1}, \ldots, f_{nm_n} \rangle \rangle}$$

Notice that the domain of a composite is the concatenation of the (sequence of) sequences of the domains of the arrows $f_1, \ldots, f_n$. The composition and axioms for a multicategory are best visualised by representing the arrows as trees. We will come back to this point in §5 below.

2.2. Examples.

1. We recall a typical example of a multicategory. Consider a ring R with unit (a monoid in the category Ab of abelian groups with respect to its monoidal closed structure). The multicategory $R-\mathsf{mod}$ has as objects R-bimodules (with left and right actions), written $M, N, \ldots$. An arrow $f : \langle M_1, \ldots, M_n\rangle \rightarrow N$ is a multilinear morphism, that is a morphism $f : M_1 \times \ldots \times M_n \rightarrow N$ in Ab (using $\times$ to denote the tensor product in Ab) such that the following diagram commutes

$$M_1 \times R \ldots \times R \times M_n \overset{(\ldots)\times l_n}{\underset{r_1\times(\ldots)}{\overset{\vdots}{\rightrightarrows}}} M_1 \times \ldots \times M_n \xrightarrow{f} N$$

where the n parallel arrows[2] are products of the left l_i and right actions r_i of the modules M_i (except for M_1 which only contributes its right action r_1 and similarly for M_n and its left action l_n), *e.g.* for $n = 3$ we have

$$\begin{tikzcd} M_1 \times R \times M_2 \times R \times M_3 \arrow[r, shift left=2, "M_1\times l_2\times l_3"] \arrow[r, "r_1\times M_2\times l_3" description] \arrow[r, shift right=2, "r_1\times r_2\times M_3"'] & M_1 \times M_2 \times M_3 \arrow[r, "f"] & N \end{tikzcd}$$

and the morphism f is also a bimodule morphism ($M_1 \times \ldots \times M_n$ being a bimodule by the left action of M_1 and the right action of M_n)

Composition is given by

$$f\langle f_1, \ldots, f_n\rangle(x_{11}, \ldots, x_{nm_n}) = f(f_1(x_{11}), \ldots, f_n(x_{nm_n}))$$

There is different (but equivalent) presentation of the multilinearity condition in the definition of $\mathcal{V}$-form in [DS97, §7].

2. Given a category $\mathbb{C}$, we define a multicategory $\mathbb{C}_\blacktriangleright$ whose morphisms are discrete cocones in $\mathbb{C}$. Explicitly, $\mathbb{C}_\blacktriangleright$ has

   objects those of $\mathbb{C}$

   arrows $f : \langle x_1, \ldots, x_n\rangle \rightarrow y$ is a tuple of morphisms $\langle f_i : x_i \rightarrow y\rangle$, for $1 \leq i \leq n$ (a cocone on the discrete diagram of x_i's)

   identities those of $\mathbb{C}$

   composition $$\frac{f_1 : \vec{x}_1 \rightarrow y_1, \ldots, f_n : \vec{x}_n \rightarrow y_n \qquad f : \vec{y} \rightarrow z}{f \circ \langle f_1, \ldots, f_n\rangle = \langle f_i \circ f_{ij}\rangle_{1\leq i\leq n, 1\leq j\leq m_i} : \vec{x}_1 \cdot \ldots \cdot \vec{x}_n \rightarrow z}$$

3. Any category $\mathbb{C}$ can be trivially regarded as a multicategory $J\mathbb{C}$ with the same objects and with morphisms $J\mathbb{C}(\langle x\rangle, y) = \mathbb{C}(x, y)$ and $J\mathbb{C}(\vec{x}, y) = \emptyset$ if $\vec{x}$ is not a singleton. Composition and identities are inherited from $\mathbb{C}$.

[2] There are $n-1$ copies of R. Each copy can act on its rigth (r) or its left (l). Hence the relevant morphisms are in 1-1 correspondence with monotone functions $[n-1] \rightarrow [r < l]$. There are n of these, each determined (as a sequence) by the possible first occurrence of l.

There is an equivalent presentation of multicategories, upon the same data specified in 2.1.(1) and (2) (called a **multigraph**), with a *placed binary composition*

$$\frac{f : \langle X_1, \ldots, X_n\rangle \to Y_j \quad g : \langle Y_1, \ldots, Y_j, \ldots, Y_m\rangle \to Z}{g \circ_j f : \langle Y_1, \ldots, X_1, \ldots, X_n, \ldots, Y_m\rangle \to Z}$$

subject to the evident identity and associativity axioms plus the following *interchange law*:

$$\frac{f : \langle X_1, \ldots, X_m\rangle \to Y_i, g : \langle X'_1, \ldots, X'_p\rangle \to Y_j, h : \langle Y_1, ., Y_i, \ldots, Y_j, ., Y_n\rangle \to Z \ (i < j)}{(h \circ_i f) \circ_{j+m-1} g = (h \circ_j g) \circ_i f : \langle Y_1, \ldots, X_1, \ldots, X_m, \ldots, X'_1, \ldots, X'_p, \ldots, Y_n\rangle \to Z}$$

Clearly, the interchange law allows us to define a composition operation as in definition 2.1 as follows:

$$f\langle f_1, \ldots, f_n\rangle = (f \circ_n f_n) \ldots \circ_1 f_1$$

and conversely, given such kind of composition we obtain a binary one by whiskering with identities

$$g \circ_i f = g\langle id_{X_1}, \ldots, f, \ldots, id_{X_n}\rangle$$

This alternative description of multicategories, with binary operations of placed composition is taken up in [HMP98b], where a generalisation of the notion of multicategory, allowing a so-called non-standard amalgamation, is introduced.

As we will see in §8.1, this binary composition is important to characterise strong universal arrows.

3 Categories as graphs with monoid structure

A *graph* in a category $\mathbb{B}$ amounts to the data given in the following diagram:

$$\begin{tikzcd} C_1 \arrow[r, shift left, "d_1"] \arrow[r, shift right, "c_1"'] & C_0 \end{tikzcd}$$

where C_1 is the **object of arrows** or 1-*cells*, C_0 is the **object of objects** or 0-*cells*, and the morphisms d_1 and c_1 specify the **domain** and **codomain** of an arrow, respectively.

We are interested in graphs as the underlying data for categories. More precisely, we are interested in regarding a category as a 'graph with units and multiplication'. To this end, it is more convenient to regard a graph as a span

$$\begin{tikzcd} & C_1 \arrow[dl, "d_1"'] \arrow[dr, "c_1"] & \\ C_0 & & C_0 \end{tikzcd}$$

in a category $\mathbb{B}$. If $\mathbb{B}$ admits pullbacks, we regard the above span as an endomorphism in the bicategory $\mathbf{Spn}(\mathbb{B})$ (see §A, Definition A.1), which we write (d_1, C_1, c_1). Therefore the hom-category $\mathbf{Spn}(\mathbb{B})(C_0, C_0)$ is a monoidal category under composition of spans. We can then identify a category in $\mathbb{B}$ with object of objects C_0 as a *monoid* in $\mathbf{Spn}(\mathbb{B})(C_0, C_0)$. More concisely, a category in $\mathbb{B}$ amounts to a *monad* in $\mathbf{Spn}(\mathbb{B})$:

$$\boxed{\text{category in } \mathbb{B} \equiv \text{monad in } \mathbf{Spn}(\mathbb{B})}$$

$$\begin{tikzcd}
& & C_1 \circ C_1 \arrow[dl] \arrow[dr] \arrow[dd, "comp"] & & \\
& C_1 \arrow[dl, "d_1"'] & & C_1 \arrow[dr, "c_1"] & \\
C_0 & & C_1 \arrow[ll, "d_1"'] \arrow[rr, "c_1"] & & C_0 \\
& & C_0 \arrow[u, "\iota"'] \arrow[ull, "id"] \arrow[urr, "id"'] & &
\end{tikzcd}$$

This point of view of internal categories as monoid structures on graphs was pioneered by Bénabou in [Bén67]; it is an essential insight which lies at the heart of our present work.

Having identified categories with monads, we could expect the rest of the structure (namely functors and natural transformations) to follow from this identification. This is not quite straightforward: Street's original formulation of the 2-category of monads in a 2-category [Str72] was designed to account for the category-of-algebras construction, and his definition does not apply to obtain the 2-category of internal categories in a direct way. Street has recently elaborated an alternative version to sort out this difficulty [Str99] (see also Remark 6.4). We will address these issues in §6.1.

4 Multicategories revisited

We now proceed to give a monadic definition of multicategories in the style of §3 above. The crucial point is the asymmetry in the notion of multigraph. A multigraph is a span

$$\begin{tikzcd}
& C_1 \arrow[dl, "d_1"'] \arrow[dr, "c_1"] & \\
\mathbf{M}C_0 & & C_0
\end{tikzcd}$$

where $\mathbf{M}C_0$ is the free monoid on C_0, *i.e.* the set of sequences of elements of C_0. So the question arises as to how to compose such a span with itself to endow it with a monoid structure. The answer is to set up a Kleisli bicategory of spans. The abstract analysis of how such a bicategory arises is given in §A. Here we give an explicit description of it.

4.1. Definition. Let $\mathbb{B}$ be a category with pullbacks, and $\mathbf{T} = (T, \eta, \mu)$ a monad on it. We say $\mathbf{T}$ is **cartesian** if

- The functor $T : \mathbb{B} \to \mathbb{B}$ preserves pullbacks
- The transformations $\eta : id \Rightarrow T$ and $\mu : T^2 \Rightarrow T$ are cartesian, *i.e.* the naturality squares are pullbacks.

4.2. Definition. Given a category $\mathbb{B}$ with pullbacks and a cartesian monad $\mathbf{T} = (T, \eta, \mu)$ on it, the bicategory $\mathbf{Spn_T}(\mathbb{B})$ consists of

objects those of $\mathbb{B}$

morphisms a morphism from X to Y is a span

$$\begin{tikzcd}
 & R \arrow[dl, "d_R"'] \arrow[dr, "c_R"] & \\
TX & & Y
\end{tikzcd}$$

2-cells a 2-cell between morphisms is a morphism between the corresponding spans, so that $\mathbf{Spn_T}(\mathbb{B})(X,Y) = \mathbf{Spn}(\mathbb{B})(TX, Y)$

The identity span on X is

$$\begin{tikzcd}
 & X \arrow[dl, "\eta_X"'] \arrow[dr, "id"] & \\
TX & & X
\end{tikzcd}$$

and composition is given by

$$\begin{tikzcd}
 & R \arrow[dl, "d_R"'] \arrow[dr, "c_R"] & & & S \arrow[dl, "d_S"'] \arrow[dr, "c_S"] & \\
TX & & Y & TY & & Z
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & R\bullet S \arrow[dl, "\overline{d_S}"'] \arrow[dr, "\overline{Tc_R}"] & & \\
 & & TR \arrow[dl, "Td_R"'] \arrow[dr, "Tc_R"] & & S \arrow[dl, "d_S"'] \arrow[dr, "c_S"] & \\
 & T^2X \arrow[dl, "\mu_X"'] & & TY & & Z \\
TX & & & & &
\end{tikzcd}$$

where the square is a pullback. Horizontal composition of 2-cells is clearly induced by that of morphisms, while the vertical composition is inherited from $\mathbf{Spn}(\mathbb{B})$.

We hasten to remark that the exactness conditions imposed in the definition of cartesian monad are required for the above composition to be coherently associative and unitary.

We can now give an internal algebraic definition of multicategory.

4.3. Definition. Let $\mathbb{B}$ be a category with finite limits, which admits free monoids such that the corresponding free monoid monad $\mathbf{M} = (M, \eta, \mu)$ is cartesian.

- a **multigraph** in $\mathbb{B}$ is an *endomorphism* in $\mathbf{Spn}_{\mathbf{M}}(\mathbb{B})$.
- a **multicategory** in $\mathbb{B}$ is a *monad* in $\mathbf{Spn}_{\mathbf{M}}(\mathbb{B})$

Let us state formally the correspondence between the above abstract definition and our previous set-theoretic one (2.1).

4.4. Proposition. *A monad in* $\mathbf{Spn}_{\mathbf{M}}(\mathcal{S}et)$ *is a multicategory in the sense of 2.1.*

Proof. First, let us note that the free monoid monad in any elementary topos (with a natural numbers object) is cartesian [Bén90]. The rest of the proof is routine but it is enlightening to see precisely how composition in $\mathbf{Spn}_{\mathbf{M}}(\mathcal{S}et)$ gives the appropriate domain for the composition operation of a multicategory:

- the composite span has

  $$C_1 \bullet C_1 = \{(\langle f_1, \ldots, f_n \rangle, f) \mid \langle c_1(f_1), \ldots, c_1(f_n) \rangle = d_1(f)\}$$

  with the domain of $(\langle f_1, \ldots, f_n \rangle, f)$ being $d_1(f_1) \cdot \ldots \cdot d_1(f_n)$ (sequence concatenation) and the codomain being that of f.
- the unit for the monoid structure on C_1 assigns to each object $X \in C_0$ an arrow with domian $\eta_{C_0}(X) = \langle X \rangle$ and codomain X, as required for the identity arrow in a multicategory.

$\square$

The above internal characterisation of multicategories was also found independently from the author by Burroni [Bur71] and Leinster [Lei97]. We came up with it as it was essential to our higher-dimensional generalisation, which we will elaborate in a follow-up article (there have been a number of presentations of this higher-dimensional version *cf.*[Her97]).

5 Free multicategories and trees

In order to aid the visualisation of multicategories and computations in them, we give an explicit account of free multicategories (in $\mathcal{S}et$) in terms of finite trees. The language of trees has proved convenient in this context, specially in the description of *operads* (a seemingly more complicated notion of multicategory, but on the same level of abstraction) [Bur71, GK94, Bat98b].

The notion of morphism (or functor) between multicategories is the evident one, given by a morphism of the underlying multigraphs preserving composition and identitites (see §6).

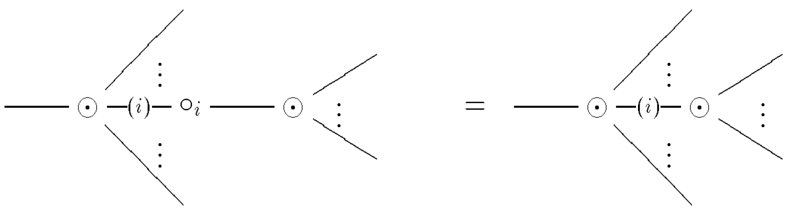
We start with a simple analysis of free categories, to illustrate the argument. The free category on a graph has the same objects while its arrows are sequences of composable arrows from the graph. Given a graph G, the canonical morphism into the terminal graph $\mathbf{1}$ (one object, one arrow) induces a functor $\ell : \mathsf{C}G \to \mathsf{C}\mathbf{1}$ between the respective free categories. The free category $\mathsf{C}\mathbf{1}$ is the one-object category corresponding to the free monoid on one generator, *i.e.* the natural numbers $\mathbf{N}$. Hence the functor ℓ above assigns to each sequence of arrows from the graph its *length*. It is convenient then to regard the arrows of $\mathsf{C}G$ as labellings of those of $\mathsf{C}\mathbf{1}$; the latter give the 'shapes' of the former. Evidently, the shape of a sequence is simply its length.

As observed in [Str96], we can characterise a free category $\mathbb{C}$ on a graph by the existence of a functor $l : \mathbb{C} \to \mathsf{C}\mathbf{1}$ with a 'unique lifting of decompositions' property. We will reproduce this observation in the context of multicategories below.

The terminal multigraph has one object. It should also have a unique arrow for every (domain, codomain) pair. We thus can identify it with the following span

$$\begin{tikzcd}
 & \mathbf{N} \arrow[dl, "id"'] \arrow[dr, "!"] & \\
\mathbf{N} & & \mathbf{1}
\end{tikzcd}$$

For a given multigraph G, the unique morphism into the terminal one $\mathbf{1}$ induces a 'shape' functor $s : \mathcal{M}G \to \mathcal{M}\mathbf{1}$ between the corresponding free multicategories. So we must describe the arrows of $\mathcal{M}\mathbf{1}$ to know what the shapes are. But these are nothing but finite trees, with composition given by grafting as illustrated below.



The domain of such a finite tree is simply its *frontier*, *i.e.* its sequence of input edges. Hence the arrows of the free multicategory $\mathcal{M}G$ are finite trees labelled with the data of the multigraph G, *cf.* [Bur71]. See *e.g.* [Bat98b] for a formal definition of this kind of tree. Here we will give a *characterisation of free multicategories* which makes patent the role of finite trees and their grafting composition.

The category $\mathcal{M}ulticat$ has a small family of *generators*, namely

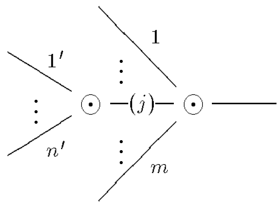
$$G = \{i \succ \bullet \mid i \in \mathbf{N}\} \tag{1}$$

where $i \succ \bullet$ is the *generic arrow of arity* i, *i.e.* the multicategory $i \succ \bullet$ has $i+1$ objects $\{1,\ldots,i,\bullet\}$ and only one non-identity arrow $\imath : \langle 1,\ldots,i\rangle \to \bullet$. This family is *dense*, but we shall not need this fact here.

We now set up the *generic placed composition* by the following pushout in $\mathcal{M}ulticat$ (assuming $m \geq 1$)

$$\begin{tikzcd}
\{\circ\} \arrow[r, "\circ\mapsto j"] \arrow[d, "\circ\mapsto\bullet"'] & m \succ \bullet \arrow[d] \\
n \succ \bullet \arrow[r] & n \succ^j (m \succ \bullet)
\end{tikzcd}$$

where $\{\circ\}$ is the multicategory with only one object and no non-identity arrows. Hence $n \succ^j (m \succ \bullet)$ is the following tree:



and the composite arrow has arity $n+m-1$, which corresponds to a morphism $dec_{m \prec_j n} : ((n+m-1) \succ \bullet) \to (n \succ^j (m \succ \bullet))$.

Notice that this morphism *decomposes* an arrow of arity $n+m-1$ into a j-placed composite of two arrows of arities m and n.

5.1. Definition. A morphism of multicategories $f : \mathbb{M} \to \mathbb{N}$ has **unique lifting of decompositions** (uld for short) if it is orthogonal to the set

$$\{dec_{m \prec_j n} \mid m, n \in \mathbf{N}, 1 \leq j \leq m\}.$$

In more detail, the above definition means that given morphisms b and a as in the diagram

$$\begin{tikzcd}
(n+m-1) \succ \bullet \arrow[r, "dec_{m \prec_j n}"] \arrow[d, "b"'] & n \succ^j (m \succ \bullet) \arrow[d, "a"] \arrow[dl, dashed, "d"] \\
\mathbb{M} \arrow[r, "f"'] & \mathbb{N}
\end{tikzcd}$$

there is a unique morphism $d : n \succ^j (m \succ \bullet) \to \mathbb{M}$. We can now characterise free multicategories in terms of uld functors as follows.

5.2. Proposition (characterisation of free multicategories). *A multicategory* $\mathbb{M}$ *is free over a multigraph iff it admits a uld morphism* $\mathsf{shape} : \mathbb{M} \to \mathcal{M}\mathbf{1}$

This states that in a free multicategory, whose morphisms are labelled trees, given an arrow and a decomposition of its shape, there is a unique way to lift such decomposition to the given arrow (labelled tree). Furthermore, the existence of such a unique lifting characterises freeness. Indeed, to determine the multigraph which generates $\mathbb{M}$, since we are given its objects (those of $\mathbb{M}$), we have to determine its arrows. These are simply the 'indecomposable' arrows of $\mathbb{M}$, *i.e.* those morphisms of multicategories $\imath_{\mathbb{M}} : [i \succ \bullet] \to \mathbb{M}$ which do not factor through any morphism *dec*. The uld property of shape ensures that all the arrows of $\mathbb{M}$ are obtained as placed composites of these generating arrows.

6 The 2-category of multicategories

We proceed to set up our universe of discourse for multicategories. Throughout this section we work within a category $\mathbb{B}$ satisfying the conditions of Definition 4.3.

It is fairly evident what a morphism of multicategories should be. Given multicategories $\mathbb{M}$ and $\mathbb{N}$ a morphism between them is a morphism between their underlying multigraphs

$$\begin{tikzcd}
& M_1 \arrow[rr, "f_1"] \arrow[dr, "c"] \arrow[ddl, "d"'] & & N_1 \arrow[dr, "c"] \arrow[ddl, "d"'] & \\
& & M_0 \arrow[rr, "f_0"'] & & N_0 \\
\mathbf{M}M_0 \arrow[rr, "\mathbf{M}(f_0)"'] & & \mathbf{M}N_0 & &
\end{tikzcd}$$

which preserves composition and identities.

The forgetful functor $U : \mathcal{M}ulticat \to \mathbb{B}$ taking a multicategory $\mathbb{M}$ to its object of objects M_0 is a fibration. The cartesian morphisms are the fully faithful ones, and hence we have the familiar bijective-on-objects/fully faithful factorization of morphisms between multicategories as for ordinary categories. Notice that change of base along $f : X \to Y$ (obtained as a limit) induces a strong monoidal functor $f^* : \mathbf{Spn_M}(\mathbb{B})(Y, Y) \to \mathbf{Spn_M}(\mathbb{B})(X, X)$,

since $\mathbf{M}$ preserves pullbacks, *cf.* Prop. 6.5.

What is a good notion of 2-cell between morphisms of multicategories? It turns out that a 2-cell is given by data identical to that of natural transformations in ordinary category theory, but the naturality condition is a 'multilinear' one. In order to see how this comes about, we give an algebraic reformulation of natural transformations in internal category theory in the following subsection, which can later on be reinstantiated in the context of multicategories.

6.1 Natural transformations revisited

Having recalled in §3 the view of internal categories as monads in $\mathbf{Spn}(\mathbb{B})$, we are left with the task to account in this context for functors and natural transformations. Bearing in mind that monads are monoids, it is not surprising that bimodules and change of base come into the picture.

Let us reexamine the usual notion of natural transformation. Consider small categories $\mathbb{C}$ and $\mathbb{D}$ and functors $F, G : \mathbb{C} \rightarrow \mathbb{D}$. A natural transformation $\alpha : F \Rightarrow G$ between them is given by a function $\alpha : B_0 \rightarrow C_1$ with $\alpha_X : FX \rightarrow GX$ subject to the 'naturality condition'

$$\begin{tikzcd} FX \arrow[r, "\alpha_X"] \arrow[d, "Fh"'] & GX \arrow[d, "Gh"] \\ FY \arrow[r, "\alpha_Y"'] & GY \end{tikzcd}$$

for every morphism $h : X \rightarrow Y$ in C_1. Let us write $\alpha_h : FX \rightarrow GY$ for the diagonal of the above square:

$$\alpha_h = Gh \circ \alpha_X = \alpha_Y \circ Fh$$

We have thus the following commuting diagram:

$$\begin{tikzcd} & C_1 \arrow[rr, "\alpha_{(\_)}"] \arrow[dr, "c"] \arrow[ddl, "d"'] & & D_1 \arrow[dr, "c"] \arrow[ddl, "d"'] & \\ & & C_0 \arrow[rr, "G_0"'] & & D_0 \\ C_0 \arrow[rr, "F_0"'] & & D_0 & & \end{tikzcd}$$

We now seek conditions to ensure that such $\alpha_{(\_)}$ is completely determined by its action on identity arrows $\alpha_{id_X} = \alpha_X$. To this end notice

$$\begin{aligned} \alpha_h &= \alpha_{h \circ id_X} = Gh \circ \alpha_{id_X} \\ &= \alpha_{id_Y \circ h} = \alpha_{id_Y} \circ Fh \end{aligned}$$

Since $\mathbb{C}$ is a monoid (in $\underline{\mathbf{Spn}(\mathcal{S}et)(C_0, C_0)}$), it acts on itself both on the left and on the right by composition, so that it is a $(\mathbb{C}, \mathbb{C})$-bimodule. Similarly, $\mathbb{D}$ is a $(\mathbb{D}, \mathbb{D})$-bimodule, which by change of base along the functors F, G becomes a $(\mathbb{C}, \mathbb{C})$-bimodule, which we write $\langle F, G\rangle^* \mathbb{D}$. The above equations show that $\alpha_{(\_)}$ commutes with the actions of these bimodules, so that it is a morphism in $(\mathbb{C}, \mathbb{C}) - \mathsf{bimod}$, the category of bimodules and morphisms between them which commute with both actions. We have thus

6.1. Proposition. *Given categories $\mathbb{C}$ and $\mathbb{D}$ and functors $F, G : \mathbb{C} \rightarrow \mathbb{D}$ we have*

$$(\mathbb{C}, \mathbb{C}) - \mathsf{bimod}(\mathbb{C}, \langle \mathsf{F}, \mathsf{G}\rangle^* \mathbb{D}) \cong Nat - Transf\,(F, G)$$

where $Nat - Transf\,(F, G)$ denotes the set of natural transformations between the functors F and G.

What about the horizontal and vertical composition of natural transformations? Interestingly enough, horizontal composition comes about most naturally in the present algebraic reformulation. Namely, given horizontally composable natural transformations

$$
\begin{tikzcd}
\mathbb{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathbb{D} \arrow[r, bend left, "H"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathbb{E}
\end{tikzcd}
$$

given by morphisms of bimodules $\alpha_{(\_)} : \mathbb{C} \to \langle F, G\rangle^* \mathbb{D}$ and $\beta_{(\_)} : \mathbb{D} \to \langle H, K\rangle^* \mathbb{E}$, their horizontal composite corresponds to

$$\mathbb{C} \xrightarrow{\alpha_{(\_)}} \langle F, G\rangle^* \mathbb{D} \xrightarrow{\langle F,G\rangle^* \beta_{(\_)}} \langle HF, KG\rangle^* \mathbb{E}$$

as a morphism of $(\mathbb{C},\mathbb{C})-\mathsf{bimod}$. This really amounts to the ordinary composition of the morphisms $\alpha_{(\_)}$ and $\beta_{(\_)}$ in the ambient category $\mathbb{B}$.

Vertical composition is more delicate. The immediate general structure which enables it is the following.

6.2. Definition. Let $\mathbb{B}$ be a category with pullbacks and (pullback stable) coequalizers. Consider internal categories $\mathbb{C}$ and $\mathbb{D}$ in $\mathbb{B}$ and a monoid R in $(\mathbb{D},\mathbb{D})-\mathsf{bimod}$, *i.e.* equipped with unit $\eta : D_1 \to R$ and multiplication $\mu : R\hat{\circ}R \to R$, where $R\hat{\circ}R$ denotes the composition in $(\mathbb{D},\mathbb{D})-\mathsf{bimod}$ given by the following coequalizer

$$R \circ D_1 \circ R \underset{R \circ r}{\overset{l \circ R}{\rightrightarrows}} R \circ R \twoheadrightarrow R\hat{\circ}R$$

with l and r being the left and right actions respectively of R as a bimodule.

We define a category as follows

$[\mathbb{C}, R]$	**objects**	functors $f : \mathbb{C} \to \mathbb{D}$
	morphisms	a morphism $\alpha : f \Rightarrow g$ is a bimodule morphism $\alpha : C_1 \to \langle f, g\rangle^* R$

The identity on a functor f is

$$C_1 \xrightarrow{f_1} D_1 \xrightarrow{\eta} R$$

and composition of $\alpha : f \Rightarrow g$ and $\beta : g \Rightarrow h$ is given by the composite

$$C_1 \simeq C_1 \hat{\circ} C_1 \xrightarrow{\alpha\hat{\circ}\beta} \langle f, g\rangle^*(R) \hat{\circ} \langle g, h\rangle^*(R) \xrightarrow{can} \langle f, h\rangle^*(R\hat{\circ}R) \xrightarrow{\langle f,h\rangle^*(\mu)} \langle f, h\rangle^*(R)$$

where the first isomorphism comes from the fact that C_1 is the identity $(\mathbb{C},\mathbb{C})$-bimodule, *i.e.* its action is

$$(f : X \to Y) \longmapsto [(id_X, f) \sim (f, id_Y)]$$

and the morphism can is canonically induced as follows

$$\begin{tikzcd}
\langle f,g\rangle^*(R)\circ C_1\circ\langle g,h\rangle^*(R) \arrow[r, shift left, "\bar{l}\circ\langle g,h\rangle^*(R)"] \arrow[r, shift right, "\langle f,g\rangle^*(R)\circ\bar{r}"'] \arrow[d, "-\circ g_1\circ-"'] & \langle f,g\rangle^*(R)\circ\langle g,h\rangle^*(R) \arrow[r, two heads] \arrow[d, "-\circ-"] & \langle f,g\rangle^*(R)\hat{\circ}\langle g,h\rangle^*(R) \arrow[d, dashed, "can"] \\
R\circ D_1\circ R \arrow[r, shift left, "l\circ R"] \arrow[r, shift right, "R\circ r"'] & R\circ R \arrow[r, two heads] & R\hat{\circ}R
\end{tikzcd}$$

Applying the above definition for $R = D_1$ as a $(\mathbb{D},\mathbb{D})$-bimodule, we get $[\mathbb{C}, D_1]$ as the category of functors between $\mathbb{C}$ and $\mathbb{D}$ with natural transformations between them under vertical composition. It is worth illustrating why bimodule composition is relevant to achieve this. First let us notice that the composition for $\mathbb{D}$, written $\bullet : D_1 \circ D_1 \to D_1$, which is its multiplication as a monoid in $\mathbf{Spn}(\mathbb{B})(D_0, D_0)$, endows D_1 with a multiplication in $(\mathbb{D},\mathbb{D})-\mathsf{bimod}$ as shown in the following diagram

$$\begin{tikzcd}
D_1\circ D_1\circ D_1 \arrow[r, shift left, "\bullet\circ D_1"] \arrow[r, shift right, "D_1\circ\bullet"'] & D_1\circ D_1 \arrow[r, two heads] \arrow[dr, "\bullet"'] & D_1\hat{\circ}D_1 \arrow[d, dashed, "\mu"] \\
& & D_1
\end{tikzcd}$$

Of course, μ above is an isomorphism. For $\mathbb{B} = \mathcal{S}et$ we can explicitly describe $D_1\hat{\circ}D_1$ as follows

$$D_1\hat{\circ}D_1(X,Y) = \{(g\circ f, h) \sim (f, h\circ g) \mid f : X\to Z, g : Z\to Z', h : Z'\to Y\}$$

and therefore

$$\mu((g\circ f, h) \sim (f, h\circ g)) = h\circ g\circ f : X\to Y$$

Hence the composite of $\alpha : C_1\to\langle f,g\rangle^* D_1$ and $\beta : C_1\to\langle g,h\rangle^* D_1$ is illustrated in the following diagram, where $k : X\to Y$ is in C_1

$$\begin{tikzcd}
FX \arrow[r, dotted, "\alpha_X"] \arrow[d, dotted, "Fk"'] \arrow[dr, "\alpha_k"] & GX \arrow[r, dotted, "\beta_X"] \arrow[d, "Gk"] \arrow[dr, "\beta_k"] & HX \arrow[d, dotted, "Hk"] \\
FY \arrow[r, dotted, "\alpha_Y"'] & GY \arrow[r, dotted, "\beta_Y"'] & HY
\end{tikzcd}
=
\begin{tikzcd}
FX \arrow[r, dotted, "\beta_X\circ\alpha_X"] \arrow[d, dotted, "Fk"'] \arrow[dr, "(\beta\circ\alpha)_k"] & HX \arrow[d, dotted, "Hk"] \\
FY \arrow[r, dotted, "\beta_Y\circ\alpha_Y"'] & HY
\end{tikzcd}$$

Thus to compose α and β at $k : X\to Y$, we regard it as $(id_X, k) \sim (k, id_Y)$ in $C_1\hat{\circ}C_1$, which upon applying $\alpha\hat{\circ}\beta$ becomes $(\alpha_X, \beta_k) \sim (\alpha_k, \beta_Y)$ in

$$\langle f,h\rangle^*(D_1\hat{\circ}D_1)(X,Y) = (D_1\hat{\circ}D_1)(FX, HY)$$

and finally use the above multiplication μ.

6.3. Remark. Following the above elementary description of how the vertical composite of natural transformations operates, it is clear that we can bypass the composites of bimodules involved and work directly with span composites. We can thus dispense with the requirement that $\mathbb{B}$ had coequalizers. The above abstract description is conceptually more insightful though, and shows how 'pasting of squares' involves bimodules in an essential way, becoming crucial at higher dimensions.

6.4. Remark. Upon reading a previous version of this paper, Ross Street pointed out that he had developed a similar 'arrow to arrow' view of modifications, with a view to extending his formal theory of monads framework to account for the 2-category of internal categories [Str99]. We could have carried out our reformulation at this more abstract level, although the conceptual framework is the same.

6.2 Transformations between morphisms of multicategories

We are now ready to introduce 2-cells for multicategories, reinstantiating the above abstract treatment of natural transformations. To apply it, we need to consider change-of-base for bimodules.

6.5. Proposition. *Given multicategories* $\mathbb{M}$ *and* $\mathbb{N}$, *and morphisms* $f, g : \mathbb{M} \to \mathbb{N}$, *there is a change-of-base functor* $\langle f, g\rangle^* : (\mathbb{N}, \mathbb{N}) - \mathsf{bimod} \to (\mathbb{M}, \mathbb{M}) - \mathsf{bimod}$,

whose action at a bimodule R *is specified by the following limit diagram*

$$\begin{tikzcd}
 & \langle f,g\rangle^*(R) \arrow[rr, dashed, "-"] \arrow[dr, dashed, "c"] \arrow[dl, dashed, "d"'] & & R \arrow[dr, "c"] \arrow[dd, "d"'] & \\
 & & M_0 \arrow[rr, "g_0"] & & N_0 \\
\mathbf{M}M_0 \arrow[rrr, "\mathbf{M}f_0"'] & & & \mathbf{M}N_0 &
\end{tikzcd}$$

The result follows easily from the fact that $\mathbf{M}$, being cartesian, preserves such finite limits. Hence the following definition is valid.

6.6. Definition. Given multicategories $\mathbb{M}$ and $\mathbb{N}$, and morphisms $f, g : \mathbb{M} \to \mathbb{N}$, a transformation between them $\alpha : f \Rightarrow g$ is a morphism $\alpha : M_1 \to \langle f, g\rangle^*(N_1)$ in $(\mathbb{M}, \mathbb{M}) - \mathsf{bimod}$

We now spell out in elementary terms what such a transformation amounts to. We work in $\mathbb{B} = \mathcal{S}et$. A transformation $\alpha : f \Rightarrow g$ assigns to an arrow $h : \langle X_1, \ldots, X_n\rangle \to Y$ in $\mathbb{M}$ an arrow $\alpha_h : \langle fX_1, \ldots, fX_n\rangle \to gY$. Given the condition of preservation of bimodule structure, such assignment is completely determined by its action on identity arrows $\alpha_Y = \alpha_{id_Y} : \langle fY\rangle \to gY$. The arrows α_Y are then subject to the following multinaturality condition

$$\begin{tikzcd}
\langle fX_1, \ldots, fX_n\rangle \arrow[r, "\langle \alpha_{X_1}, \ldots, \alpha_{X_n}\rangle"] \arrow[d, "f(h)"'] & \langle gX_1, \ldots, gX_n\rangle \arrow[d, "g(h)"] \\
fY \arrow[r, "\alpha_Y"'] & gY
\end{tikzcd}$$

α_h being the diagonal of the above commuting square ('thick on top'). Vertical and horizontal composites are given just as for ordinary natural transformations: given $\beta : g \Rightarrow k$, $(\beta \circ \alpha)_Y = \beta_Y \circ \alpha_Y$ and given $\gamma : p \Rightarrow q$ horizontally composable with α,

$$(\gamma * \alpha)_Y = \gamma_{gY} \circ p(\alpha_Y) = q(\alpha_Y) \circ \gamma_{fY}$$

The interchange law follows from the same routine calculation as in $\mathcal{Cat}$. This concludes our definition of the 2-category $\mathcal{Multicat}$.

We will see several 2-functors involving $\mathcal{Multicat}$. Let us point out a simple example: the construction in Ex.2.2.(2) extends evidently to functors and natural transformations to yield a 2-functor $(\_)_{\blacktriangleright} : \mathcal{Cat} {\rightarrow} \mathcal{Multicat}$.

6.7. Definition. Given a multicategory $\mathbb{M}$, we obtain an ordinary category with the same objects, whose arrows are those of $\mathbb{M}$ whose source is a singleton. Diagramatically

$$\begin{tikzcd} & \overline{M_1} \arrow[dl] \arrow[dr] & & \\ M_0 \arrow[dr, "\eta_{M_0}"'] & & M_1 \arrow[dl, "d"] \arrow[dr, "c"] & \\ & \mathbf{M}(M_0) & & M_0 \end{tikzcd}$$

where the square is a pullback, giving the object of morphisms $\overline{M_1}$ of the resulting category, which we write $\overline{\mathbb{M}}$ and call the **underlying category of linear morphisms**.

6.8. Remark. The above definition extends in the evident way to morphisms and transformations, yielding a 2-functor $\overline{(\_)} : \mathcal{Multicat} {\rightarrow} \mathcal{Cat}$. The construction of Example 2.2.(3), which regards any category as a multicategory with linear morphisms only, yields a 2-functor $J : \mathcal{Cat} {\rightarrow} \mathcal{Multicat}$, which is left adjoint to the functor $\overline{(\_)}$

6.3 Adjunctions in $\mathcal{Multicat}$

Since $\mathcal{Multicat}$ is a 2-category, we have a meaningful notion of adjunction in it. We now proceed to give an elementary 'hom-set isomorphism' characterisation of such adjunctions, as we need such description for our doctrinal version of representability for multicategories in §8.1. We work in $\mathbb{B} = \mathcal{Set}$.

6.9. Proposition. *Given multicategories $\mathbb{M}$ and $\mathbb{N}$, and morphisms $f : \mathbb{M} {\rightarrow} \mathbb{N}$ and $g : \mathbb{N} {\rightarrow} \mathbb{M}$, f is left adjoint g iff there is an isomorphism*

$$\mathbb{N}(\langle fx_1, \ldots, fx_n\rangle, y) \cong \mathbb{M}(\langle x_1, \ldots, x_n\rangle, gy)$$

multinatural in both arguments.

Proof. Given a unit $\eta : 1 {\Rightarrow} gf$ and a counit $\epsilon : fg {\Rightarrow} 1$, we get the above isomorphism

$$(\langle x_1, \ldots, x_n\rangle \xrightarrow{h} gy) \longmapsto (\langle fx_1, \ldots, fx_n\rangle \xrightarrow{f(h)} fgy \xrightarrow{\epsilon_y} y)$$

while its converse is given by

$$(\langle x_1, \ldots, x_n\rangle \xrightarrow{\langle \eta_{x_1}, \ldots, \eta_{x_n}\rangle} \langle gfx_1, \ldots, gfx_n\rangle \xrightarrow{g(\overline{h})} gy) \longleftarrow\!\!\mid (\langle fx_1, \ldots, fx_n\rangle \xrightarrow{\overline{h}} y)$$

Conversely, given the above multinatural isomorphism ϕ, we get $\eta_x = \phi^{-1}(id_{fx})$ and $\epsilon_y = \phi(id_{gy})$. Multinaturality of η and ϵ so defined follows from that of ϕ.

$\square$

7 The fundamental adjunction between $\mathcal{M}ulticat$ and $\mathcal{M}on\mathcal{C}at$

In Example 2.2.(1), we could have relied on the presence of a tensor product and described the multilinear arrows $f : \langle M_1, \ldots, M_n\rangle \to N$ in terms of the induced linear maps $\hat{f} : M_1 \otimes \ldots \otimes M_n \to N$. This suggests we can always associate a multicategory to a monoidal category. In this section we will study this process for strict monoidal categories. We write $\mathcal{M}on\mathcal{C}at$ for the 2-category of strict monoidal categories (that is, monoids in $\mathcal{C}at$), strict monoidal functors and monoidal transformations. Recall we are working in an ambient category $\mathbb{B}$ with finite limits and a cartesian free-monoid monad $\mathbf{M}$ (as in Definition 4.3). In particular, a monoid in $\mathbb{B}$ amounts to an $\mathbf{M}$-algebra, $m : \mathbf{M}A \to A$. A monoid is a discrete monoidal category. Let us see its associated multicategory.

7.1. Proposition. *Let $\mathbb{B}$ be a category as in Definition 4.3.*

1. *$m : \mathbf{M}A \to A$ is an $\mathbf{M}$-algebra iff* $\begin{array}{ccc} & \mathbf{M}A & \\ {}^{id}\swarrow & & \searrow^{m} \\ \mathbf{M}A & & A \end{array}$ *is a multicategory (i.e. a monoid in $\mathbf{Spn}_{\mathbf{M}}(\mathbb{B})(A, A)$).*

2. *The above correspondence yields a full and faithful functor $\mathcal{M}on(\mathbb{B}) \hookrightarrow \mathcal{M}ulticat(\mathbb{B})$ from the category of monoids in $\mathbb{B}$ to that of multicategories.*

Both statements in the above proposition are verified by a simple inspection of the diagrams involved.

<u>From monoidal categories to multicategories:</u> A monoidal category $\mathbb{C}$, as a monoid in $\mathcal{C}at(\mathbb{B})$, consists of a category $\begin{array}{ccc} & C_1 & \\ {}^{d}\swarrow & & \searrow^{c} \\ C_0 & & C_0 \end{array}$ where C_0 and C_1 are monoids and both d and c are monoid morphisms. So we can apply the construction of Proposition 7.1 to C_0 and obtain what should be the underlying discrete multicategory of the multicategory associated to $\mathbb{C}$. In fact, the latter is obtained as follows

$$
\begin{tikzcd}[column sep=small]
 & C_1 \arrow[dl, "d"'] \arrow[dr, "c"] & & {} \arrow[r, mapsto] & {} & & & \mathbf{M}(C_0)\circ C_1 \arrow[dl, "p"'] \arrow[dr, "q"] & & \\
C_0 & & C_0 & & & & \mathbf{M}(C_0) \arrow[dl, "id"'] \arrow[dr, "m_0"] & & C_1 \arrow[dl, "d"'] \arrow[dr, "c"] & \\
 & & & & & \mathbf{M}(C_0) & & C_0 & & C_0
\end{tikzcd}
$$

where the square is a pullback and m_0 is the structure map of C_0 as an $\mathbf{M}$-algebra. We have thus defined an assignment from monoidal categories to multicategories, which extends readily to a 2-functor $U : \mathcal{M}on\mathcal{C}at \to \mathcal{M}ulticat$.

<u>From multicategories to monoidal categories:</u> Given a multicategory, we assign to it a (strict) monoidal category as follows:

$$\begin{tikzcd} & D_1 \arrow[dl, "d"'] \arrow[dr, "c"] & \\ \mathbf{M}(D_0) & & D_0 \end{tikzcd} \longmapsto \begin{tikzcd} & & \mathbf{M}(D_1) \arrow[dl, "\mathbf{M}d"'] \arrow[dr, "\mathbf{M}c"] & \\ & \mathbf{M}^2(D_0) \arrow[dl, "\mu_{D_0}"'] & & \mathbf{M}(D_0) \\ \mathbf{M}(D_0) & & & \end{tikzcd}$$

The resulting span is a category as we know from the Kleisli biadjunction in §A (*cf.* Prop. A.5 and its subsequent elaboration). The objects in the span are free monoids. The fact that the resulting domain morphism $\mu \circ \mathbf{M}d$ is a morphism of monoids follows from an easy diagram chase involving the naturality of μ and the associativity law for the monad $\mathbf{M}$.

The above assignment defines a 2-functor $F : \mathcal{M}ulticat \to \mathcal{M}on\mathcal{C}at$.

7.2. Theorem. *The 2-functor $F : \mathcal{M}ulticat \to \mathcal{M}on\mathcal{C}at$ is left 2-adjoint to the 2-functor $U : \mathcal{M}on\mathcal{C}at \to \mathcal{M}ulticat$, and this adjunction is 2-monadic (over $\mathcal{M}ulticat$).*

Proof. The unit $\eta : \mathbb{D} \Rightarrow UF\mathbb{D}$ is given by the pair $(\eta_{D_0}, \langle \mathbf{M}\eta_{D_0} \circ d, \eta_{D_1} \rangle)$ as shown

$$\begin{tikzcd}[column sep=small] & D_1 \arrow[rrrrr, "\langle \mathbf{M}\eta_{D_0} \circ d{,} \eta_{D_1} \rangle)"] \arrow[rrrrrrd, dotted, "\eta_{D_1}"'] \arrow[ddl, "d"'] \arrow[ddr, "c"] & & & & & \mathbf{M}^2(D_0)\circ\mathbf{M}(D_1) \arrow[dll, "p"'] \arrow[dr, "q"] & & \\ & & & & \mathbf{M^2(D_0)} \arrow[dl, "id"'] \arrow[dr, "\mu_{D_0}"] & & & \mathbf{M}(D_1) \arrow[dll, "\mu_{D_0}\mathbf{M}d"'] \arrow[dr, "\mathbf{M}c"] & \\ \mathbf{M}(D_0) \arrow[rrr, bend right, "\mathbf{M}\eta_{D_0}"'] & & D_0 \arrow[rrrrrr, bend right, "\eta_{D_0}"'] & \mathbf{M}^2(D_0) & & \mathbf{M}(D_0) & & & \mathbf{M}(D_0) \end{tikzcd}$$

The counit $\epsilon : FU\mathbb{C} \Rightarrow \mathbb{C}$ is given by the pair

$$(m_0 : \mathbf{M}C_0 \to C_0, m_1 \circ \mathbf{M}q : \mathbf{M}(\mathbf{M}C_0 \circ C_1) \to C_1)$$

as shown

$$
\begin{tikzcd}
 & & \mathbf{M}(\mathbf{M}C_0 \circ C_1) \arrow[dl, "\mathbf{M}p"'] \arrow[dr, "\mathbf{M}q"] \arrow[rrrr, bend left, "m_1 \circ \mathbf{M}q"] & & & & C_1 \arrow[ddl, "d"'] \arrow[ddr, "c"] & \\
 & \mathbf{M}^2(C_0) \arrow[dl, "\mu_{C_0}"'] \arrow[dr, "\mathbf{M}m_0"] & & \mathbf{M}(C_1) \arrow[dl, "\mathbf{M}d"'] \arrow[dr, "\mathbf{M}c"] \arrow[urrr, dotted, "m_1"'] & & & & \\
\mathbf{M}C_0 \arrow[rrrrr, bend right, "m_0"'] & & \mathbf{M}C_0 & & \mathbf{M}C_0 \arrow[rrr, bend right, "m_0"'] & C_0 & & C_0
\end{tikzcd}
$$

The above data sets up a 2-natural isomorphism

$$
\mathcal{M}ulticat(\mathbb{D}, U\mathbb{C}) \cong \mathcal{M}on\mathcal{C}at(F\mathbb{D}, \mathbb{C})
$$

As for monadicity, we verify directly that UF-algebras correspond to monoidal categories. We construct explicitly a pseudo-inverse for the canonical comparison functor $U : \mathcal{M}on\mathcal{C}at \to UF-\mathsf{alg}$.

Given a multicategory $\mathbb{D}$ with an algebra structure $(m_0, \tilde{m}) : UF(\mathbb{D}) \to \mathbb{D}$, the underlying category of linear morphisms $\overline{\mathbb{D}}$ given by the following diagram (where the square is a pullback, *cf.* Definition 6.7)

$$
\begin{tikzcd}
 & \overline{D_1} \arrow[dl, "\overline{d}"'] \arrow[dr, "j"] & & \\
D_0 \arrow[dr, "\eta_{D_0}"'] & & D_1 \arrow[dl, "d"] \arrow[dr, "c"] & \\
 & \mathbf{M}(D_0) & & D_0
\end{tikzcd}
$$

is a monoidal category. Clearly $m_0 : \mathbf{M}D_0 \to D_0$ makes D_0 a monoid, while the monoid structure on $\overline{D_1}$ is given as follows: consider the diagram

$$
\begin{tikzcd}
\mathbf{M}\overline{D_1} \arrow[r, dashed] \arrow[d, "\mathbf{M}\overline{d}"'] \arrow[rr, bend left, "\mathbf{M}j"] & \mathbf{M}^2 D_0 \circ \mathbf{M}D_1 \arrow[r, "q"'] \arrow[d, "p"] & \mathbf{M}D_1 \arrow[d, "\mu_{D_0}\mathbf{M}d"] \\
\mathbf{M}D_0 \arrow[r, "\eta_{\mathbf{M}D_0}"'] & \mathbf{M}^2 D_0 \arrow[r, "\mu_{D_0}"'] & \mathbf{M}(D_0)
\end{tikzcd}
$$

and we write $\langle \eta_{\mathbf{M}D_0} \mathbf{M}\overline{d}, \mathbf{M}j \rangle : \mathbf{M}\overline{D_1} \to \mathbf{M}^2 D_0 \circ \mathbf{M}D_1$

for the above canonically induced (dashed) arrow into the pullback. The composite morphism $\tilde{m}\langle \eta_{\mathbf{M}D_0} \mathbf{M}\overline{d}, \mathbf{M}j \rangle : \mathbf{M}\overline{D_1} \to \overline{D_1}$

gives the required monoid structure on $\overline{D_1}$ in such a way that both $\overline{d} : \overline{D_1} \to D_0$ and $cj : \overline{D_1} \to D_0$ are monoid morphisms.

This construction yields a 2-functor, which we write $\overline{(\_)} : UF-\mathsf{alg} \to \mathcal{M}on\mathcal{C}at$, pseudo-inverse to U:

- $\overline{(\_)}U \cong 1$ as in the following diagram

$$\begin{tikzcd}
C_1 \arrow[r, dashed] \arrow[d, "d"'] \arrow[rr, bend left, "id"] & \mathbf{M}C_0 \circ C_1 \arrow[r] \arrow[d] & C_1 \arrow[d, "d"'] \arrow[dr, "c"] & \\
C_0 \arrow[r, "\eta_{C_0}"] \arrow[rr, bend right, "id"'] & \mathbf{M}C_0 \arrow[r, "m_0"] & C_0 & C_0
\end{tikzcd}$$

  all three rectangles are pullbacks.

- $U\overline{(\_)} \cong 1$ requires a very delicate argument. Before embarking on the formalities, let us indicate what the isomorphism means: it amounts to saying that a multicategory with a UF-algebra structure can be recovered from its underlying category of linear morphisms. This will be best understood when considering representability (*cf.* §8.1). Consider the following diagram, where the rectangle is a pullback

$$\begin{tikzcd}
D_1 \arrow[r, dashed, "l"] \arrow[dr, "d"'] \arrow[rrr, bend left, "\tilde{m}\langle \mathbf{M}d, id\rangle \eta_{D_1}"] & P \arrow[rr, "i"] \arrow[d, "d'"] & & D_1 \arrow[d, "d"] \\
& \mathbf{M}D_0 \arrow[r, "m_0"'] & D_0 \arrow[r, "\eta_{D_0}"'] & \mathbf{M}D_0
\end{tikzcd}$$

  where the outer rectangle commutes as shown

$$\begin{tikzcd}
D_1 \arrow[r, "\eta_{D_1}"] \arrow[dr, "d"'] & \mathbf{M}D_1 \arrow[r, "\langle \mathbf{M}d, id\rangle"] \arrow[dr, "\mathbf{M}d"] & \mathbf{M}^2 D_0 \circ \mathbf{M}D_1 \arrow[r, "\tilde{m}"] \arrow[d, "p"] & D_1 \arrow[d, "d"] \\
& \mathbf{M}D_0 \arrow[r, "\eta_{\mathbf{M}D_0}"] \arrow[dr, dash, "m_0"'] & \mathbf{M}^2 D_0 \arrow[r, "\mathbf{M}m_0"] & \mathbf{M}D_0 \\
& & {} \arrow[ur, "\eta_{D_0}"'] &
\end{tikzcd}$$

  Let us understand in terms of elements (as if we were in $\mathcal{S}et$) the action of the top morphism in the above diagram:

$$(\vec{x} \xrightarrow{f} y) \overset{\eta_{D_1}}{\longmapsto} (\langle \vec{x} \rangle \xrightarrow{\langle f \rangle} \langle y \rangle) \xmapsto{\tilde{m}\langle \mathbf{M}d, id\rangle} (\langle \otimes\ \vec{x} \rangle \xrightarrow{\tilde{m}\langle f \rangle} y)$$

  so that we transform an arrow f into a linear morphism. The fact that the canonical comparison l above is an isomorphism amounts to

$$\mathbb{D}(\vec{x}, y) \cong \mathbb{D}(\langle \otimes\ \vec{x} \rangle, y)$$

  which corresponds to the fact that $\mathbb{D}$ is representable (*cf.* Corollary 8.6, where the above isomorphism corresponds to the universal/linear factorization in $\mathbb{D}$). The inverse of l is obtained by precomposition (internally in $\mathbb{D}$) with the morphism

$$\mathbf{M}D_0 \xrightarrow{\mathbf{M}\iota} \mathbf{M}D_1 \xrightarrow{\langle \mathbf{M}d, id\rangle} \mathbf{M}^2 D_0 \circ \mathbf{M}D_1 \xrightarrow{\tilde{m}} D_1$$

which provides a canonical choice of universal arrows (*cf.* Definition 8.1). Here $\iota : D_0 \to D_1$ is the 'identity' of the multicategory $\mathbb{D}$. The delicate calculations involve a detailed analysis of composition in the multicategory $UF(\mathbb{D})$ in terms of that of $\mathbb{D}$. We shall indicate the main points:

- To show $l^{-1} \circ l = id : D_1 \to D_1$ we observe that this composite can be written as

  $$D_1 \xrightarrow{h} (\mathbf{M}^2 D_0 \circ \mathbf{M}D_1) \bullet (\mathbf{M}^2 D_0 \circ \mathbf{M}D_1) \xrightarrow{comp} \mathbf{M}^2 D_0 \circ \mathbf{M}D_1 \xrightarrow{\tilde{m}} D_1$$

  using the fact that $\tilde{m}$ is a morphism of multicategories. The first morphism above is

  $$h = \langle \eta_{\mathbf{M}^2 D_0 \circ \mathbf{M}D_1} \langle \mathbf{M}d, id\rangle \mathbf{M}(\iota) d, \langle \mathbf{M}d, id\rangle \eta_{D_1} \rangle : D_1 \to (\mathbf{M}^2 D_0 \circ \mathbf{M}D_1) \bullet (\mathbf{M}^2 D_0 \circ \mathbf{M}D_1)$$

  The argument concludes by showing that $comp \circ h = \eta_{\mathbb{D}}$ and hence its composite with $\tilde{m}$ is the identity by the algebra equation for $\mathbb{D}$.
- To show $l \circ l^{-1} = id : P \to P$ we form l^{-1} as the composite

  $$P \xrightarrow{\langle \eta_{D_1} \tilde{m} \langle \mathbf{M}d, id\rangle \mathbf{M}(\iota) d', i\rangle} D_1 \bullet D_1 \xrightarrow{comp} D_1$$

  and show that

  $$\langle \mathbf{M}d, id\rangle \eta_{D_1} l^{-1} = \eta_{\mathbb{D}} i : P \to \mathbf{M}^2 D_0 \circ \mathbf{M}D_1$$

  by analysing its projections out of the pullback $\mathbf{M}^2 D_0 \circ \mathbf{M}D_1$. Finally,

  $$\begin{aligned} ill^{-1} &= \tilde{m}\langle \mathbf{M}d, id\rangle \eta_{D_1} l^{-1} \\ &= \tilde{m}\eta_{\mathbb{D}} i \\ &= i \end{aligned}$$

□

7.3. Remark. There is a canonical description of the underlying category of linear morphisms $\overline{\mathbb{D}}$, as we have a pullback

$$\begin{tikzcd} D_1 \arrow[r, "\tilde{m}\langle \mathbf{M}d, id\eta_{D_1}\rangle"] \arrow[d, "d"'] & D_1 \arrow[dd, "d"] \\ \mathbf{M}D_0 \arrow[d, "m_0"'] & \\ D_0 \arrow[r, "\eta_{D_0}"'] & \mathbf{M}D_0 \end{tikzcd}$$

given that η_{D_0} is a section of m_0. This convenient description of the identification of a monoidal category as an algebra on a multicategory will be exploited (substantially) in our subsequent treatment of higher-dimensional multicategories.

7.4. Corollary. *The unit of the adjunction* $\eta, \epsilon : F \dashv U : \mathcal{M}on\mathcal{C}at \to \mathcal{M}ulticat$ *is fully faithful.*

Proof. The statement amounts to the fact that the relevant diagram for the unit (shown in the proof of Theorem 7.2) is a limit diagram, which follows from a routine calculation using the fact that the square

$$\begin{tikzcd} D_1 \arrow[r, "\eta_{D_1}"] \arrow[d, "c"'] & \mathbf{M}D_1 \arrow[d, "\mathbf{M}c"] \\ D_0 \arrow[r, "\eta_{D_0}"'] & \mathbf{M}D_0 \end{tikzcd}$$

is a pullback.

□

7.5. Remark. The proof of monadicity of the adjunction in Theorem 7.2 may seem excessively involved. But such an explicit correspondence between strict monoidal categories and UF-algebras illustrates quite precisely the identification of strict representable multicategories and strict monoidal categories, *cf.* Corollary 8.13 and its subsequent elaboration. However, it is important to emphasise that a different (and certainly shorter) proof of monadicity of this 2-adjunction can be obtained directly from results already available in the literature [Str76, Bat98b, Bat98a, Wol74]. The fact that n-categories are monadic over *computads* [Str76, Bat98a] is closely related to the monadicity result above, since a multicategory is a restricted kind of computad.

8 Representable multicategories

8.1 Elementary definition of representability in multicategories

Let us recall that, given a commutative ring R and bimodules M, N and P, a bilinear map $f : \langle M, N\rangle \to P$ is a function $f : M \times N \to P$ linear in each variable, *i.e.* both $f(m, \_)$ and $f(\_, n)$ are morphisms of modules, for all $m \in M$ and $n \in N$, which is furthermore a bimodule morphism (*cf.* Example 2.2.(1)). We then say that a tensor $\otimes$ classifies bilinear maps when there is an isomorphism

$$Bilin(\langle M, N\rangle, P) \cong R\!-\!\mathsf{mod}(M\otimes N, P)$$

natural in P, that is, $M\otimes N$ represents $Bilin(\langle M, N\rangle, \_) : R\!-\!\mathsf{mod} \to \mathcal{S}et$. Of course, we expect such a representation to extend to general multilinear maps $g : \langle M_1, \ldots, M_n\rangle \to P$, *i.e.* we want the binary tensor to induce arbitrary n-ary ones. This amounts to requiring that the universal (bilinear) maps $\pi_{M,N} : \langle M, N\rangle \to M \otimes N$ be closed under multicategory composition, as in the definition below.

Generalising from this example, we want to consider such multilinear classifying tensors in multicategories. So we demand the existence of universal multilinear maps closed under composition. This leads to our first definition of representable multicategory (in $\mathcal{S}et$).

8.1. Definition. A multicategory $\mathbb{M}$ is said to be **representable** if

1. for every tuple of objects $\vec{x} = \langle x_1, \ldots, x_n\rangle$ there exists a **universal arrow** $\pi_{\vec{x}} : \langle x_1, \ldots, x_n\rangle \to \otimes\, \vec{x}$, in the sense that precomposition with this arrow induces an isomorphism

$$\mathbb{M}(\langle x_1, \ldots, x_n\rangle, y) \cong \mathbb{M}(\langle \otimes\, \vec{x}\rangle, y)$$

   natural in y, and

2. universal arrows are closed under composition.

8.2. Remark. The universality condition in (1) above can also be expressed as follows. Recall from Definition 6.7 the underlying category of linear morphisms of a multicategory $\mathbb{M}$, written $\overline{\mathbb{M}}$. For any tuple of objects $\langle x_1, \ldots, x_n\rangle$, we get a functor $\mathbb{M}(\langle x_1, \ldots, x_n\rangle, \_) : \overline{\mathbb{M}} \to \mathcal{S}et$. The existence of a universal arrow for $\vec{x}$ amounts then to the representability of this functor.

As for condition (2), it is a necessary requirement to ensure the associativity of the tensor, *e.g.* $(x \otimes y) \otimes z \cong x \otimes (y \otimes z)$ actually means that both $\pi_{(x\otimes y),z} \circ \langle \pi_{x,y}, id_z\rangle$ and $\pi_{x,y\otimes z} \circ \langle id_x, \pi_{y,z}\rangle$ are universal arrows for $\langle x, y, z\rangle$, *cf.* Definition 9.6.

The traditional construction of tensor products in categories of algebras by means of generators and relations is actually an explicit description of the coequalizer

$$\begin{tikzcd} M \times R \times N \arrow[r, shift left, "l \times N"] \arrow[r, shift right, "M \times r"'] & M \times N \arrow[r, two heads] & M \otimes N \end{tikzcd}$$

referring to the initial example of rings and modules Example 2.2.(1). It is well known that the associativity of the tensor so constructed follows from the fact that coequalizers are preserved under (tensor) product. This is a slightly different way to ensure well-defined n-ary tensors from binary ones. In terms of multicategories, this amounts to requiring that the representation of arrows induced by the tensor be stable in arbitrary contexts, as we make explicit in our second definition of representability below.

8.3. Definition. A multicategory $\mathbb{M}$ is representable if for any tuple of objects $\langle x_1, \ldots, x_n \rangle$, there exists a **strong universal** arrow

$$\pi_{\vec{x}} : \langle x_1, \ldots, x_n \rangle \rightarrow \otimes\, \vec{x},$$

in the sense that it induces an isomorphism

$$\mathbb{M}(\langle z_1, \ldots, z_i, \otimes\, \vec{x}, z_{(i+1)}, \ldots, z_n \rangle, y) \cong \mathbb{M}(\langle z_1, ., z_i, x_1, \ldots, x_n, z_{(i+1)}, ., z_n \rangle, y)$$

natural in y and multinatural in z.

8.4. Remark. The isomorphism in the above definition is induced by placed binary composition of the strong universal arrow as explained in §2. This means that there is a 1-1 correspondence

$$\frac{\langle z_1, \ldots, z_i, \otimes\, \langle \vec{x} \rangle, z_{(i+1)}, \ldots, z_n \rangle \xrightarrow{\quad h \quad} y}{\langle z_1, \ldots, z_i, x_1, \ldots, x_n, z_{(i+1)}, \ldots, z_n \rangle \xrightarrow{h \circ_{(i+1)} \pi_{\vec{x}}} y}$$

This shows the role of placed binary composition in our theory.

Having introduced two definitions of representability for multicategories, we must justify ourselves by showing them equivalent.

8.5. Proposition. *The definitions 8.1 and 8.3 of representable multicategories are equivalent, i.e.*

1. *strong universal arrows are closed under composition*
2. *universal arrows which are closed under composition are strong universal.*

Proof. We refer to the notations of Definition 8.3. Let $\vec{z} = \langle z_1, \ldots, z_i \rangle$ and $\underline{\vec{z}} = \langle z_{(i+1)}, \ldots, z_n \rangle$.

(1) We must verify that a composite of strong universals satisfies the contextual representation given by the isomorphism in 8.3. Let $\vec{x} = \langle x_1, \ldots, x_n \rangle$ whose tensor we write $x = \otimes\, \vec{x}$ and $\vec{y} = \langle y_1, \ldots, y_m \rangle$ whose tensor we write $y = \otimes\, \vec{y}$. Consider the composite $\pi_{x,y} \circ \langle \pi_{\vec{x}}, \pi_{\vec{y}} \rangle$. It gives rise to the following chain of isomorphisms

$$\begin{array}{rcl} \mathbb{M}(\langle \vec{z}, x \otimes y, \underline{\vec{z}} \rangle, \_) & \cong & \mathbb{M}(\langle \vec{z}, x, y, \underline{\vec{z}} \rangle, \_) \\ & \cong & \mathbb{M}(\langle \vec{z}, \vec{x}, y, \underline{\vec{z}} \rangle, \_) \\ & \cong & \mathbb{M}(\langle \vec{z}, \vec{x}, \vec{y}, \underline{\vec{z}} \rangle, \_) \end{array}$$

Hence $\pi_{x,y} \circ \langle \pi_{\vec{x}}, \pi_{\vec{y}} \rangle$ is universal for $\vec{x} \cdot \vec{y}$.

(2) We must show that any arrow $h : \vec{z}, \vec{x}, \underline{\vec{z}} \rightarrow y$ factors as $h = \overline{h} \circ_{(i+1)} \pi_x$ for a unique $\overline{h}$. Consider universal arrows $\pi_{\vec{z}} : \vec{z} \rightarrow z$ and $\pi_{\underline{\vec{z}}} : \underline{\vec{z}} \rightarrow \underline{z}$. The morphism $\pi_{z,x,\underline{z}} \circ \langle \pi_{\vec{z}}, \pi_x, \pi_{\underline{\vec{z}}} \rangle$

is universal for $\vec{z}, \vec{x}, \underline{\vec{z}}$, hence there is unique $\hat{h} : \langle z, x, \underline{z}\rangle \rightarrow y$ such that $h = \hat{h} \circ (\pi_{z,x,\underline{z}} \circ \langle \pi_{\vec{z}}, \pi_x, \pi_{\underline{\vec{z}}}\rangle)$.

Setting $\overline{h} = \hat{h} \circ (\pi_{z,x,\underline{z}} \circ \langle \pi_{\vec{z}}, id_x, \pi_{\underline{\vec{z}}}\rangle)$

yields the required unique factorization.

□

8.6. Corollary (universal/linear factorization). *In a representable multicategory, every morphism $f : \langle x_1, \ldots, x_n\rangle \rightarrow y$ factorizes as $f = \hat{f} \circ \pi_{\vec{x}}$ with $\pi_{\vec{x}} : \langle x_1, \ldots, x_n\rangle \rightarrow \otimes\, \vec{x}$ universal and $\hat{f} : \langle \otimes\, \vec{x}\rangle \rightarrow y$. Hence f is universal iff $\hat{f}$ is an isomorphism.*

We refer to the factorization of f in the above corollary as the **universal/ linear factorization**, with $\hat{f}$ being the **linear factor**.

8.7. Remark. It is an important consequence of Definition 8.1 that given a representable multicategory $\mathbb{M}$, there is a broad sub-multicategory $\mathbb{M}_g$ with the same objects but only universal arrows. Once we establish the correspondence between representable multicategories and monoidal categories (§9.2) we will see that this sub-multicategory corresponds to the underlying groupoid of the monoidal category associated to $\mathbb{M}$. Notice that in a multicategory, only the linear arrows, *i.e.* those with a singleton source, could be invertible.

8.8. Examples. Let us examine the examples in 2.2.

1. The multicategory $R-\mathsf{mod}$ is representable: the universal arrows are the coequalizers of the relevant parallel arrows given by whiskering of the actions. Clearly, binary coequalizers suffice to construct these, and furthermore these are known to be stable under tensoring, which ensures that the universal arrows so constructed are actually strong.

2. The multicategory $\mathbb{C}_{\blacktriangleright}$ is representable iff $\mathbb{C}$ admits finite sums. Indeed, universal arrows are (discrete) colimit cocones. These are easily seen to be closed under composition (which yields the coherent associativity of coproducts).

3. The multicategory $J\mathbb{C}$ is *not* representable, as there is no possible choice of universal arrows for non-singleton sequences of objects.

4. An important kind of example, which we have seen in the proof of Theorem 7.2, is the multicategory $U(\mathbb{C})$ for $\mathbb{C}$ a strict monoidal category. Indeed, we have by the mere definition of $U(\mathbb{C})$ an isomorphism

$$U\mathbb{C}(\vec{x}, y) \cong \mathbb{C}(\otimes\, \vec{x}, y)$$

and in particular, for any sequence of objects $\vec{x}$, the identity morphism on its tensor multiple provides a universal arrow:

$$id_{\otimes \vec{x}} \in U\mathbb{C}(\vec{x}, \otimes\, \vec{x})$$

8.2 Duality

Given the asymmetric nature of the arrows in a multicategory, in the sense that their domains are sequences while their codomains are single objects, there is no evident way of formally reversing them. But in the presence of universal arrows, every arrow is fully determined by its linear factor, and linear arrows can be meaningfully regarded in the opposite direction (*i.e.* reversing their orientation). Hence we seek the analogue of the dualisation 2-isomorphism $(\_)^{op} : \mathcal{C}at \to \mathcal{C}at^{co}$ in the context of representable multicategories.

Consider a representable multicategory $\mathbb{M}$ and an arrow $f : \vec{x} \to y$ in it. Choosing a universal arrow of source $\vec{x}$, we have the universal/linear factorization of f as

$$\vec{x} \xrightarrow{\pi_{\vec{x}}} \otimes\, \vec{x} \xrightarrow{\hat{f}} y$$

We write f^o for its formal dual, the cospan

$$\begin{tikzcd} \vec{x} \arrow[r, "\pi_{\vec{x}}"] & \otimes\, \vec{x} & y \arrow[l, dotted, "\hat{f}"'] \end{tikzcd}$$

where we have formally reversed the direction of its linear factor. Since the definition of f^o should not depend on the choice of $\pi_{\vec{x}}$ we are led to consider equivalence classes of cospans, which form then the arrows of the dual of $\mathbb{M}$, which we make explicit in the following definition.

8.9. Definition (Dual of a representable multicategory). Given a representable multicategory $\mathbb{M}$, we define its **linear dual** as the multicategory $\mathbb{M}^{lop}$ with

objects those of $\mathbb{M}$

arrows an arrow from $\vec{x}$ to y is an equivalence class of cospans in $\mathbb{M}$

$$\vec{x} \xrightarrow{f_u} z \xleftarrow{f_l} y$$

which we write $[(f_u, f_l)] : \vec{x} \to y$ where

- $f_u : \vec{x} \to z$ is a universal arrow
- $f_l : \langle y \rangle \to z$ is a linear morphism
- Given another such cospan

$$\vec{x} \xrightarrow{g_u} w \xleftarrow{g_l} y$$

we declare them one-step-equivalent if the unique linear isomorphism $m : z \to w$ mediating between f_u and g_u satisfies $m \circ f_l = g_l$, *i.e.* the following diagram commutes

$$\begin{tikzcd} & z \arrow[dd, dashed, "m"] & \\ \vec{x} \arrow[ur, "f_u"] \arrow[dr, "g_u"'] & & y \arrow[ul, "f_l"'] \arrow[dl, "g_l"] \\ & w & \end{tikzcd}$$

and consider then equivalence classes of one-step-equivalent cospans

identities the identity on x is $[(id_x, id_x)]$

composition

$$\frac{\begin{array}{c}[\vec{x}_1 \xrightarrow{f_u^1} z_1 \xleftarrow{f_l^1} y_1], \ldots, [\vec{x}_n \xrightarrow{f_u^n} z_n \xleftarrow{f_l^n} y_n] \\ [\vec{y} \xrightarrow{f_u} z \xleftarrow{f_l} t]\end{array}}{[\vec{x}_1 \cdot \ldots \cdot \vec{x}_n \xrightarrow{\pi_{\vec{z}} \circ \langle f_u^1, \ldots, f_u^n \rangle} \bigotimes \vec{z} \xleftarrow{(f_l^1 \otimes \ldots \otimes f_l^n) \circ f_l} t]}$$

where

$$\begin{tikzcd}
\langle y_1, \ldots, y_n \rangle \arrow[r, "f_u"] \arrow[d, "\langle f_l^1, \ldots, f_l^n \rangle"'] & z \arrow[d, dashed, "(f_l^1 \otimes \ldots \otimes f_l^n)"] \\
\langle z_1, \ldots, z_n \rangle \arrow[r, "\pi_{\vec{z}}"'] & \bigotimes \vec{z}
\end{tikzcd}$$

8.10. Proposition.

1. *$\mathbb{M}^{lop}$ is representable.*

2. *The above construction extends to morphisms and 2-cells to yield a 2-isomorphism*

   $(\_)^{lop} : \mathcal{R}ep\mathcal{M}ulticat \to \mathcal{R}ep\mathcal{M}ulticat^{co}$

8.11. Remark. Notice that $\overline{\mathbb{M}^{lop}} \cong \overline{\mathbb{M}}^{op}$ since the linear morphisms of both $\mathbb{M}$ and $\mathbb{M}^{lop}$ are the same and composition of linear morphisms in $\mathbb{M}^{lop}$ is essentially that of $\mathbb{M}$.

We will not explore duality any further in this paper. The reader could easily verify that it commutes with the correspondence with monoidal categories in §9.2.

8.3 Doctrinal characterisation of representability

Having given an elementary presentation of representable multicategories, we now exploit the 2-adjunction of Theorem 7.2 to give an intrinsic adjoint characterisation, which among its many technical advantages, makes sense internally in a category $\mathbb{B}$. The value of such characterisation will we further witnessed by our proof of the coherence theorem for representable multicategories in §10.

Given a multicategory $\mathbb{M}$, we can associate to it a strict monoidal category $F\mathbb{M}$. Therefore the multicategory $UF\mathbb{M}$ is representable (see §9.2) and the unit $\eta_{\mathbb{M}} : \mathbb{M} \to UF\mathbb{M}$ gives a full and faithful embedding of $\mathbb{M}$ into it, *cf.* Corollary 7.4. Hence for $\mathbb{M}$ to be representable we must be able to internalise the representability of $UF\mathbb{M}$ in $\mathbb{M}$. We thus arrive to the following essential result, characterising representability for multicategories (in $\mathcal{S}et$) by the existence of an adjoint.

8.12. Theorem (Adjoint characterisation of representability). *A multicategory $\mathbb{M}$ is representable iff the unit $\eta_{\mathbb{M}} : \mathbb{M} \to UF\mathbb{M}$ has a left adjoint.*

Proof. We begin by giving an elementary description of the arrows of $UF\mathbb{M}$.

$$\begin{tikzcd}
 & & & & \mathbf{M}^2(M_0)\circ\mathbf{M}(M_1) \arrow[dll, "p"'] \arrow[dr, "q"] & & \\
 & & P \arrow[dl, "p'"'] \arrow[dr, "q'"] & & & \mathbf{M}(M_1) \arrow[dll, "\mathbf{M}d"'] \arrow[dr, "\mathbf{M}c"] & \\
 & \mathbf{M}^2(M_0) \arrow[dl, "id"'] \arrow[dr, "\mu_{M_0}"] & & \mathbf{M}^2(M_0) \arrow[dl, "\mu_{M_0}"'] & & & \mathbf{M}(M_0) \\
\mathbf{M}^2(M_0) & & \mathbf{M}(M_0) & & & &
\end{tikzcd}$$

The objects of $UF\mathbb{M}$ are tuples or sequences $\vec{x} = \langle x_1, \ldots, x_n \rangle$ of objects of $\mathbb{M}$. Consider a sequence of such sequences $\langle \vec{x}_1, \ldots, \vec{x}_n \rangle$ and write $\vec{x}_1 \cdot \ldots \cdot \vec{x}_n$ for their concatenation (this is the action of μ_{M_0} on such a sequence). Let $\vec{y} = \langle y_1, \ldots, y_m \rangle$.

$$\begin{aligned} UF\mathbb{M}(\langle \vec{x}_1, \ldots, \vec{x}_n \rangle, \vec{y}) \;\cong \{\langle f_1, \ldots, f_m \rangle \mid \;& f_i : \vec{z}_i \to y_i, 1 \leq i \leq m, \\ & \wedge\ \vec{z}_1 \cdot \ldots \cdot \vec{z}_m = \vec{x}_1 \cdot \ldots \cdot \vec{x}_n\} \end{aligned}$$

Consider a left adjoint $\otimes : UF\mathbb{M} \to \mathbb{M}$ to $\eta_{\mathbb{M}}$ and let $\pi : 1 \Rightarrow (\eta_{\mathbb{M}} \otimes)$ be the unit of such adjunction. We claim that $\pi_{\vec{x}} : \vec{x} \to \langle \otimes\, \vec{x} \rangle$ is a (strong) universal arrow in $\mathbb{M}$. First of all notice that such a morphism in $UF\mathbb{M}$ must be a singleton sequence of morphisms from $\mathbb{M}$ since its target its such a sequence, *cf.* the proof of Corollary 7.4. Then using Proposition 6.9, we see that the adjunction of $\eta_{\mathbb{M}}$ and tensor $\otimes$ amounts to the following isomorphism

$$UF\mathbb{M}(\vec{x}_1, \ldots, \vec{x}_n, \langle y \rangle) \cong \mathbb{M}(\langle \otimes\, \vec{x}_1, \ldots, \otimes\, \vec{x}_n \rangle, y)$$

realised (from right to left) by precomposition with $\langle \pi_{\vec{x}_1}, \ldots, \pi_{\vec{x}_n} \rangle$. Since $\eta_{\mathbb{M}}$ is full and faithful, the counit of its adjunction with $\otimes$ must be an isomorphism. In fact, we could harmlessly assume it to be an identity (Prop. 10.3), which effectively amounts to choosing $\pi_{\langle y \rangle} = id_y$. Instantiating the above isomorphism with $\vec{x}_j = \langle x_j \rangle$ for $j \neq i$ and $\vec{x}_i = \vec{x}$ we get

$$UF\mathbb{M}(\langle x_1 \rangle, \ldots, \vec{x}, \ldots, \langle x_n \rangle, \langle y \rangle) \cong \mathbb{M}(\langle x_1, \ldots, \otimes\, \vec{x}, \ldots, x_n \rangle, y)$$

realised by placed binary composition (at i) with π_x. FInally notice that the left hand side is isomorphic to $\mathbb{M}(\langle x_1, \ldots, \vec{x}, \ldots, x_n \rangle, y)$ because we have a singleton sequence as target. This concludes the proof that $\pi_{\vec{x}}$ is strong universal. In fact, tracing the above argument backwards we conclude that the existence of strong universal arrows guarantees the existence of the left adjoint $\otimes$.

□

Let us write $\mathbf{T} = UF : \mathcal{M}ulticat \to \mathcal{M}ulticat$ for the 2-monad induced by the adjunction. Since $\mathbf{T}\mathbb{M}$ is itself representable, the unit $\eta_{\mathbf{T}\mathbb{M}} : \mathbf{T}\mathbb{M} \to \mathbf{T}^2\mathbb{M}$ has a left adjoint. In fact, it is easy to verify that $\mu_{\mathbb{M}} : \mathbf{T}^2\mathbb{M} \to \mathbf{T}\mathbb{M}$ is such a left adjoint: given a sequence of sequences $\langle \vec{x}_1, \ldots, \vec{x}_n \rangle$ of objects of $\mathbb{M}$, $id_{\vec{x}_1 \cdot \ldots \cdot \vec{x}_n} : \langle \vec{x}_1, \ldots, \vec{x}_n \rangle \to \vec{x}_1 \cdot \ldots \cdot \vec{x}_n$ is a universal arrow in $\mathbf{T}\mathbb{M}$. So $\mathbf{T}$ is a 2-monad whose pseudo-algebras are (left) adjoints to the units [Str73, Koc95].

We define $\mathcal{RepMulticat}$ to be the locally full (*i.e.* all 2-cells) sub-2-category of $\mathcal{Multicat}$ consisting of representable multicategories and morphisms between such which preserve universal arrows. We write $\mathsf{Ps}-\mathbf{T}-\mathsf{alg}$ for the 2-category of pseudo-algebras, pseudo-morphisms and all 2-cells between such.

8.13. Corollary. *There is a 2-isomorphism*

$$\mathcal{RepMulticat} \cong \mathsf{Ps}-\mathbf{T}-\mathsf{alg}$$

obtained by choosing, in a representable multicategory, universal arrows for every sequence of objects.

We can see this in the proof of Theorem 8.12, *i.e.* to give an adjunction $\pi, \varepsilon : \otimes \dashv \eta_{\mathbb{M}}$ amounts to giving a *choice* of universal arrows in $\mathbb{M}$ for every sequence of objects in it. We call $\mathbb{M}$ **strict representable** when such a choice can be given which is closed under identities and composition, *i.e.* M_0 is a monoid and the choice of universal arrows $univ : \mathbf{M}M_0 \to M_1$ sets up a morphism of multicategories

$$\begin{tikzcd}
 & \mathbf{M}M_0 \arrow[rr, "univ"] \arrow[dl, "id"'] \arrow[dr, "m_0"] & & M_1 \arrow[dl, "d"'] \arrow[dr, "c"] & \\
 & & M_0 \arrow[rr, "id"'] & & M_0 \\
\mathbf{M}M_0 \arrow[rr, "\mathbf{M}(id)"'] & & \mathbf{M}M_0 & &
\end{tikzcd}$$

cf. Prop. 7.1.(1). Clearly such strict representable multicategories are those of the form $U\mathbb{C}$ for a (strict) monoidal category $\mathbb{C}$. Let $\mathcal{RepMulticat}_s$ denote the (locally full) sub-2-category of $\mathcal{RepMulticat}$ whose objects are the strict representable multicategories and whose morphisms are those which preserve the chosen structure on the nose.

$$\boxed{\mathcal{RepMulticat}_s \cong \mathbf{T}-\mathsf{alg} \simeq \mathcal{MonCat}}$$

8.14. Remark. We have seen that the multicategory $\mathbb{C}_{\blacktriangleright}$ of Example 2.2.(2) is representable precisely when $\mathbb{C}$ admits finite coproducts. Notice that $F\mathbb{C} \simeq \mathsf{Fam_f}(\mathbb{C})$, where $\mathsf{Fam_f}(\mathbb{C})$ is the category of finite families of objects and arrows of $\mathbb{C}$, the well-known finite coproduct completion of $\mathbb{C}$. The category $F\mathbb{C}$ is the corresponding finite *strictly associative* coproduct completion of $\mathbb{C}$.

We will see in §10.2 that every representable multicategory is equivalent to a strict one.

9 The 2-equivalence between representable multicategories and weak monoidal categories

As we mentioned in §1, we intend representable multicategories to stand in the same relationship to (non-strict) monoidal categories as fibrations do to pseudofunctors. Technically, this means that we should set up a correspondence between these two concepts which yields a 2-equivalence between the corresponding 2-categories. In the following subsection §9.1 we review the finite and infinite presentations for monoidal categories, and tackle the correspondence with multicategories in §9.2. We refer to the excellent survey on coherence for monoidal categories and their functors in [JS93, §1] for the results we use in the constructions below.

9.1 Finite vs. infinite presentations of monoidal categories

Recall that given a monoid $(M, \cdot, e)$ (in $\mathcal{S}et$), the associativity and unit axioms for $\cdot$ and e yield unique n-ary multiplication operations $\cdot^n : M^n \to M$ (for all n), with action

$$(x1, \ldots, x_n) \mapsto (x_1 \cdot x_2) \ldots x_n$$

associative and unitary (just like the composition and identities of a multicategory, *cf.* Definition 2.1). Conversely, such a collection of associative and unitary multiplications is completely determined by $\cdot^2$ and $\cdot^0 (= e)$.

A monoid in $\mathcal{C}at$ (with respect to its cartesian monoidal structure) is a *strict monoidal category*. Transporting such structure along an equivalence of categories we obtain a weaker structure, namely that of monoidal category (this is part of the content of the coherence theorem for monoidal categories, which asserts any such is equivalent to a strict one). Taking into account the two presentations for monoids above, viz. finite and infinite, we obtain accordingly both a finite and an infinite presentation of monoidal categories. The equivalence between these presentations is formally expressed by a 2-equivalence between the corresponding 2-categories as we indicate below.

We briefly recall the basic definitions of monoidal categories, their functors and transformations (under their usual finite presentation). For details the reader can consult [Mac71, Kel82, JS93].

- A *monoidal category* is a category $\mathcal{V}$, equipped with
  - functors $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ and $I : 1 \to \mathcal{V}$
  - structural natural isomorphisms

  $$\alpha_{x,y,z} : (x \otimes y) \otimes z \xrightarrow{\sim} x \otimes (y \otimes z) \quad \lambda_x : I \otimes x \xrightarrow{\sim} x \quad \rho_x : x \otimes I \xrightarrow{\sim} x$$

  subject to coherence axioms (the associativity pentagon for α and the triangle relating α, λ and ρ). We write $(\mathcal{V}, \otimes, I, \alpha, \lambda, \rho)$ for the above data.

- A *strong monoidal functor* (F, δ, γ) between monoidal categories $(\mathcal{V}, \otimes, I, \alpha, \lambda, \rho)$ and $(\mathcal{V}', \otimes', I', \alpha', \lambda', \rho')$ consists of a functor $F : \mathcal{V} \to \mathcal{V}'$ together with structural natural isomorphisms

$$\gamma : I' \stackrel{\sim}{\to} FI \qquad \delta_{x,y} : Fx \otimes' Fy \stackrel{\sim}{\to} F(x \otimes y)$$

subject to coherence axioms.

- A *monoidal transformation* between strong monoidal functors $(F, \delta, \gamma))$ and $(F', \delta', \gamma'))$ is a natural transformation $\theta : F \Rightarrow F'$, commuting with δ's and γ's.

We write $w\mathcal{M}on\mathcal{C}at$ for the 2-category of monoidal categories, strong monoidal functors and monoidal natural transformations. Sometimes we may refer to a monoidal category as non-strict to emphasise that we are referring to one of these general structures as opposed to the special strict ones.

As for the infinite presentation of a monoidal category (and their functors), one concise way to go about it goes as follows: consider the free-monoid 2-monad $\mathsf{M} : \mathcal{C}at \to \mathcal{C}at$. A monoidal category in this second sense amounts precisely to a pseudo M-algebra, while strong monoidal functors correspond to strong morphisms of these. The 2-equivalence between the finite and infinite presentations we alluded to above is

$$\boxed{w\mathcal{M}on\mathcal{C}at \cong \mathsf{Ps}-\mathsf{M}-\mathsf{alg}}$$

XSThis equivalence is realised, from left to right, by choosing n-fold 'tensor products' $\otimes^n : \mathcal{V}^n \to \mathcal{V}$ build out of the given $\otimes$ and I. One such choice (consistent with the one made above for monoids) is obtained by *leftmost bracketing*:

$$\otimes^n (x_1, \ldots, x_n) = (x_1 \otimes x_2) \ldots \otimes x_n$$

while the corresponding structural coherent isomorphisms, *e.g.*

$$\otimes^2 (\otimes^3 (x_1, x_2, x_3), \otimes^4 (x_4, x_5, x_6, x_7)) \simeq \otimes^7 (x_1, \ldots, x_7)$$

are obtained from the α's, there being a *unique* such because of Mac Lane's 'all diagrams commute' result [JS93, Cor. 1.6]. Such result is clearly seen in this context as establishing the above equivalence between the finite and infinite presentations.

This infinite presentation of coherent structures is the point of view commonly adopted in homotopical algebra, where they are construed as algebras for operads *cf.* [Bat98b].

We refer the reader to [Kel74] and the references therein for an elaborated explanation of these issues about coherent categorical structures.

9.2 Relating monoidal categories and representable multicategories

Having outlined the correspondence between the finite and infinite presentation of monoidal categories, we are ready to transform these into representable multicategories.

Given a monoidal category, we should perform a 'Grothendieck construction' to obtain a representable multicategory, and conversely, making a choice of universal arrows in one such obtain a monoidal category, both passages yielding equivalences (in fact, isomorphisms) on both sides. Throughout this subsection we will work in $\mathcal{S}et$ so as to have elementary descriptions of the morphisms involved in such correspondence.

Let us start with a strict monoidal category $\mathbb{C}$ and analyse the universal arrows in its associated multicategory $U\mathbb{C}$. A morphism $f : \langle x_1, \ldots, x_n\rangle \to y$ in $U\mathbb{C}$ is a morphism $f : x_1 \otimes \ldots \otimes x_n \to y$ in $\mathbb{C}$. Among such we have a distinguished kind, namely the identities in $\mathbb{C}$, which we construe as $id_{x_1\otimes\ldots\otimes x_n} : \langle x_1, \ldots, x_n\rangle \to x_1 \otimes \ldots \otimes x_n$ in $U\mathbb{C}$. We then have a canonical factorization of f in $U\mathbb{C}$ as $\hat{f} \circ id_{x_1\otimes\ldots\otimes x_n}$ where $\hat{f} : \langle x_1 \otimes \ldots \otimes x_n\rangle \to y$ is f regarded trivially as a linear morphism in $U\mathbb{C}$.

9.1. Proposition. *An arrow $f : \langle x_1, \ldots, x_n\rangle \to y$ is universal in $U\mathbb{C}$ iff the corresponding arrow $f : x_1 \otimes \ldots \otimes x_n \to y$ in $\mathbb{C}$ is an isomorphism.*

This gives us the recipe of how to build a multicategory $\int \mathcal{V}$ out of a monoidal category $(\mathcal{V}, \otimes, I, \lambda, \rho, \alpha)$. In view of Corollary 8.6 and the above proposition, the linear morphisms in $\int \mathcal{V}$ are those of $\mathcal{V}$, while the universal morphisms can be chosen as the coherent isomorphisms (built up from α, λ and ρ using $\otimes$ and I). Let us state this precisely.

9.2. Definition. Given a monoidal category $(\mathcal{V}, \otimes, I, \lambda, \rho, \alpha)$, the multicategory $\int \mathcal{V}$ has

objects those of $\mathcal{V}$

morphisms $\int \mathcal{V}(\langle x_1, \ldots, x_n\rangle, y) = \mathcal{V}((\overbrace{(x_1 \otimes x_2) \ldots \otimes x_n}^{\textit{leftmost bracketing}}), y)$

identities those of $\mathcal{V}$

composition

$$\frac{\vec{x}_1 \xrightarrow{f_1} y_1 \ldots \vec{x}_n \xrightarrow{f_n} y_n \qquad \vec{y} \xrightarrow{f} y}{\begin{array}{c} \overbrace{((x_{11} \otimes x_{1m_1}) \ldots x_{nm_n})} \\ \Big\downarrow coh \\ (\overbrace{(x_{11} \otimes \ldots x_{1m_1})} \ldots \overbrace{(x_{n1} \otimes \ldots x_{nm_n})}) \xrightarrow[\underbrace{(f_1 \otimes \ldots f_n)}]{} (y_1 \otimes \ldots y_n) \xrightarrow{f} y \end{array}}$$

where coh is the unique reassociation isomorphism.

Notice that in the above definition we appeal to Mac Lane's 'all diagrams commute' result [JS93, Cor. 1.6] so that *coh* above is in fact the unique such isomorphism. Of course, we could prescribe a particular choice. This means that we have transformed a finite presentation of a monoidal category into an infinite one, as we indicated in §9.1.

The coherence result quoted above is furthermore essential for the composition to be associative and unitary. Let us understand this more precisely.

We say that a multicategory $\mathbb{M}$ is **posetal** if any two arrows between the same source and target are equal. We observe immediately the following.

9.3. Proposition. *A representable multicategory is posetal iff its underlying category of linear morphisms is.*

The construction in Proposition 7.1.(1) provides a source of posetal representable multicategories. In general, given a multicategory $\mathbb{M}$ we can extract from it a multicategory $\mathbb{M}_d$ with the same objects but whose only linear morphisms are identities. Now the coherence result [JS93, Cor. 1.6] is equivalent to the statement that in $(\int\mathcal{V})_d$ is posetal. Since associativity (and unity) of composition in $\int\mathcal{V}$ reduces to that in $(\int\mathcal{V})_d$, this shows the necessity and sufficiency of such a result to have a well-defined multicategory. We could say then that $\mathbb{M}_d$ embodies all the coherence structure of a multicategory.

We want to extend the above construction of multicategories out of monoidal categories to their functors and transformations.

9.4. Proposition. *The construction $\int\mathcal{V}$ is the assignment on objects of a 2-functor $\int(\_) : w\mathcal{M}on\mathcal{C}at \to \mathcal{R}ep\mathcal{M}ulticat$.*

Proof. Given monoidal categories $(\mathcal{V}, \otimes, I, \lambda, \rho, \alpha)$ and $(\mathcal{V}', \otimes', I', \lambda', \rho', \alpha')$ and a strong monoidal functor $F : \mathcal{V} \to \mathcal{V}'$, with structural isomorphisms $\gamma : I' \xrightarrow{\sim} FI$ and $\delta_{x,y} : Fx \otimes' Fy \xrightarrow{\sim} F(x \otimes y)$ define $\int F : \int\mathcal{V} \to \int\mathcal{V}'$ with action

$$(\overbrace{(x_1 \otimes \ldots x_n)} \xrightarrow{f} y) \quad \mapsto \overbrace{(Fx_1 \otimes \ldots Fx_n)} \xrightarrow[\delta_{(x_1 \ldots x_n)}]{\sim} F(x_1 \otimes \ldots x_n) \xrightarrow{Ff} Fy$$

where $\delta_{(x_1 \ldots x_n)} : (Fx_1 \otimes \ldots Fx_n) \xrightarrow{\sim} F(x_1 \otimes \ldots x_n)$ is the unique such isomorphism built out of $\delta_{x,y}$'s and $\otimes$. The uniqueness of such isomorphism follows from the coherence for strong monoidal functors [JS93, Cor.1.8], which also implies that the following diagram commutes

$$\begin{tikzcd}
\overbrace{((Fx_1 \otimes Fx_2) \ldots \otimes Fx_n)} \arrow[r, "\delta_{((x_1 x_2) \ldots x_n)}"] \arrow[d, "coh"'] & F((x_1 \otimes x_2) \ldots \otimes x_n) \arrow[d, "Fcoh"] \\
(Fx_1 \otimes Fx_2 (\ldots) \otimes Fx_n) \arrow[r, "\delta_{(x_1 x_2 (\ldots) x_n)}"'] & F(x_1 \otimes x_2 (\ldots) x_n)
\end{tikzcd}$$

where in the lower row we consider any bracketing of $x_1, \ldots, x_n$. This implies that $\int F$ is a well-defined morphism of multicategories.

Given a monoidal transformation $\theta : F \Rightarrow G$, with (G, γ', δ') another strong monoidal functor between $\mathcal{V}$ and $\mathcal{V}'$, $\theta_x : \langle Fx \rangle \to Gx$ as a morphism in $\int \mathcal{V}'$ is the component at x of the multinatural transformation $\int \theta : \int F \Rightarrow \int G$. Here we use coherence for θ to ensure multinaturality, *e.g.* given $f : \langle x_1, x_2, x_3 \rangle \to y$ in $\int \mathcal{V}$, that is $f : ((x_1 \otimes x_2) \otimes x_3) \to y$ in $\mathcal{V}$ we have the following commuting diagram

$$\begin{tikzcd}
((Fx_1 \otimes Fx_2) \otimes Fx_3) \arrow[r, "((\theta_{x_1} \otimes \theta_{x_2}) \otimes \theta_{x_3})"] \arrow[d, "\delta_{((x_1 x_2) x_3)}"'] & ((Gx_1 \otimes Gx_2) \otimes Gx_3) \arrow[d, "\delta'_{((x_1 x_2) x_3)}"] \\
F((x_1 \otimes x_2) \otimes x_3) \arrow[r, "\theta_{((x_1 \otimes x_2) \otimes x_3)}"'] \arrow[d, "Ff"'] & G((x_1 \otimes x_2) \otimes x_3) \arrow[d, "Gf"] \\
Fy \arrow[r, "\theta_y"'] & Gy
\end{tikzcd}$$

where the upper rectangle commutes by the coherence conditions for θ.

□

9.5. Remark. From the proof of the above proposition we highlight the following correspondences

- If $\mathcal{V}$ is strict monoidal, then $\int \mathcal{V} \equiv U\mathcal{V}$
- $$\frac{\mathcal{V} \xrightarrow{F} \mathcal{V}' \quad \textit{(lax) monoidal functor}}{\int \mathcal{V} \xrightarrow{\int F} \int \mathcal{V}' \quad \textit{as a morphism of multicategories}}$$
- $$\frac{\begin{tikzcd} \mathcal{V} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{V}' \end{tikzcd} \quad \textit{monoidal transformation}}{\begin{tikzcd} \int \mathcal{V} \arrow[r, bend left, "\int F"] \arrow[r, bend right, "\int G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \int \mathcal{V}' \end{tikzcd} \quad \textit{as a transformation of multicategories}}$$

We now must turn representable multicategories into (non-strict) monoidal categories. This involves a *choice* of universal arrows.

9.6. Definition. Given a representable multicategory $\mathbb{M}$ with a choice of universal arrows (for every sequence of objects), we define a monoidal category $(\mathbb{M})_c$, whose underlying category is $\overline{\mathbb{M}}$ (*cf.* Definition 6.7), that is, whose objects are those $\mathbb{M}$ and whose morphisms are the linear morphisms of $\mathbb{M}$. The monoidal structure $(\otimes, I, \lambda, \rho, \alpha)$ is given as follows

- For objects x and y, their tensor $x \otimes y$ is the codomain of the given universal arrow $\pi_{x,y} : \langle x, y \rangle \to x \otimes y$
- The unit I is the codomain of the universal arrow $\pi_{\langle\ \rangle} : \langle\ \rangle \to I$ for the empty sequence.
- The structural isomorphisms λ, ρ and α are canonically determined as the unique mediating (linear) morphisms between universal arrows in the following diagrams

$$
\begin{tikzcd}
 & \langle\ \rangle \cdot \langle x \rangle = \langle x \rangle \arrow[dl, "{\langle \pi_{\langle\ \rangle}, id_x \rangle}"'] \arrow[dd, "id_x"] \\
\langle I, x \rangle \arrow[d, "\pi_{I,x}"'] & \\
I \otimes x \arrow[r, dashed, "\lambda_x"'] & x
\end{tikzcd}
\qquad
\begin{tikzcd}
\langle x \rangle = \langle x \rangle \cdot \langle\ \rangle \arrow[dd, "id_x"'] \arrow[dr, "{\langle id_x, \pi_{\langle\ \rangle} \rangle}"] & \\
 & \langle x, I \rangle \arrow[d, "\pi_{x,I}"] \\
x & x \otimes I \arrow[l, dashed, "\rho_x"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \langle x, y, z \rangle \arrow[dl, "{\langle \pi_{x,y}, id_z \rangle}"'] \arrow[dr, "{\langle id_x, \pi_{y,z} \rangle}"] & \\
\langle x \otimes y, z \rangle \arrow[d, "\pi_{x \otimes y, z}"'] & & \langle x, y \otimes z \rangle \arrow[d, "\pi_{x, y \otimes z}"] \\
(x \otimes y) \otimes z \arrow[rr, dashed, "\alpha_{x,y,z}"'] & & x \otimes (y \otimes z)
\end{tikzcd}
$$

The coherence axioms for the structural isomorphisms in the above definition, that is the triangle for the units and the associativity pentagon, follow from the uniqueness of mediating morphisms between (codomains of) universal arrows for the same sequence of objects. In the terminology of [JS93], such univeral arrows and the mediating morphisms between them form a *clique* (a category equivalent to the terminal one).

We now extend the above construction to morphisms of representable multicategories and transformations.

9.7. Proposition. *The assignment* $\mathbb{M} \longmapsto (\mathbb{M})_c$ *extends to a 2-functor*

$$(\_)_c : \mathcal{R}ep\mathcal{M}ulticat \rightarrow w\mathcal{M}on\mathcal{C}at$$

Proof. Given a morphism of representable multicategories $F : \mathbb{M} \rightarrow \mathbb{N}$, the strong monoidal functor $F_c : (\mathbb{M})_c \rightarrow (\mathbb{N})_c$ acts like F on objects and morphisms, that is $F_c = \overline{F}$ between the underlying categories of linear morphisms. The structural isomorphisms γ and δ are canonically determined as shown in the following diagrams

$$
\begin{tikzcd}
 & \langle\ \rangle \arrow[dl, "\pi'_{\langle\ \rangle}"'] \arrow[dr, "F\pi_{\langle\ \rangle}"] & \\
I' \arrow[rr, dashed, "\gamma_F"'] & & FI
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \langle Fx, Fy \rangle \arrow[dl, "\pi'_{Fx,Fy}"'] \arrow[dr, "F\pi_{x,y}"] & \\
Fx \otimes Fy \arrow[rr, dashed, "\delta^F_{x,y}"'] & & F(x \otimes y)
\end{tikzcd}
$$

where $F\pi_{\langle\ \rangle}$ and $F\pi_{x,y}$ are universal since F preserves universal arrows. Once again, the coherence axioms follow from uniqueness of mediating morphisms between universal arrows.

Given a transfomation $\theta : F \Rightarrow G : \mathbb{M} \rightarrow \mathbb{N}$, we verify easily that the associated natural transformation $\overline{\theta}_x : \langle Fx \rangle \rightarrow Gx$ is monoidal:

$$
\begin{tikzcd}
 & & FI \arrow[dd, "\theta_I"] \\
\langle\,\rangle \arrow[r, dashed, "\pi'_{\langle\,\rangle}"] \arrow[urr, bend left, "F\pi_{\langle\,\rangle}"] \arrow[drr, bend right, "G\pi_{\langle\,\rangle}"'] & I' \arrow[ur, dashed, "\gamma_F"'] \arrow[dr, dashed, "\gamma_G"] & \\
 & & {}
\end{tikzcd}
\qquad
\begin{tikzcd}
\langle Fx,Fy\rangle \arrow[r, "\pi'_{Fx,Fy}"'] \arrow[rr, bend left, "F\pi_{x,y}"] \arrow[d, "{\langle\theta_x,\theta_y\rangle}"'] & Fx\otimes Fy \arrow[r, dashed, "\delta^F_{x,y}"'] \arrow[d, dashed, "\theta_x\otimes\theta_y"] & F(x\otimes y) \arrow[d, "\theta_{x\otimes y}"] \\
\langle Gx,Gy\rangle \arrow[r, "\pi'_{Gx,Gy}"] \arrow[rr, bend right, "G\pi_{x,y}"'] & Gx\otimes Gy \arrow[r, dashed, "\delta^G_{x,y}"] & G(x\otimes y)
\end{tikzcd}
$$

where all the dashed arrows are canonical mediating morphisms induced by universal arrows.

□

We can finally state the equivalence between representable multicategories and (non-strict) monoidal categories. Let $\mathbf{U} : w\mathcal{M}on\mathcal{C}at \to \mathcal{C}at$ denote the 2-functor which forgets the monoidal structure. Recall that we have also a 2-functor $\overline{(\_)} : \mathcal{R}ep\mathcal{M}ulticat \to \mathcal{C}at$ which takes a multicategory to its 'underlying' category of linear morphisms *cf.* Definition 6.7.

9.8. Theorem. *There is a 2-equivalence*

$$
\begin{tikzcd}
w\mathcal{M}on\mathcal{C}at \arrow[rr, shift left, "\int -"] \arrow[rr, phantom, "\simeq"] \arrow[dr, "\mathbf{U}"'] & & \mathcal{R}ep\mathcal{M}ulticat \arrow[ll, shift left, "(\_)_c"] \arrow[dl, "\overline{(\_)}"] \\
 & \mathcal{C}at &
\end{tikzcd}
$$

Proof. We construct 2-natural isomorphisms
$\varepsilon_{\mathbb{M}} : \mathbb{M} \xrightarrow{\sim} \int(\mathbb{M})_c$
and $\rho_{\mathcal{V}} : \mathcal{V} \xrightarrow{\sim} (\int \mathcal{V})_c$ which are identities on the underlying categories.

- To define

  $\varepsilon_{\mathbb{M}} : \mathbb{M} \to \int(\mathbb{M})_c$ on arrows (being already given as the identity on objects and linear morphisms), consider an arrow $f : \vec{x} \to y$ in $\mathbb{M}$. We take a universal/linear factorisation $f = \hat{f} \circ \pi_{\vec{x}}$. We define

  $$\varepsilon_{\mathbb{M}}(\vec{x} \xrightarrow{\pi_{\vec{x}}} \otimes\, \vec{x}) \;=\; \overbrace{((x_1 \otimes x_2)\ldots \otimes x_n)} \overset{can_{\vec{x}}}{\dashrightarrow} \otimes\, \vec{x}$$

  canonically determined as the unique morphism mediating between the universal arrows defining its domain and codomain. We easily verify that the resulting arrow $\varepsilon_{\mathbb{M}}(f) = \hat{f} \circ can_{\vec{x}}$ is independent of the choice of universal/linear factorisation for f.

- Since $\rho_{\mathcal{V}} : \mathcal{V} \to (\int \mathcal{V})_c$ is already determined as a functor (being the identity), it only remains to define its structural isomorphisms as strong monoidal functor. This effectively means that the two monoidal structures over the (underlying) category $\mathcal{V}$ are isomorphic. This is clear, as $x \otimes y$ and $x \otimes_c y$ are both codomains of universal arrows in $\int \mathcal{V}$ and hence canonically (and therefore coherently) isomorphic.

□

9.9. Remark. It is helpful to realise that the above 2-equivalence shows that $\int(\_) : w\mathcal{M}on\mathcal{C}at \to \mathcal{R}ep\mathcal{M}ulticat$ can be regarded as a kind of 'Grothendieck construction' for monoidal categories, since it effectively transforms a coherent structure (namely a pseudo-algebra) into a universally specified one (*i.e.* an adjoint pseudo-algebra). This is what we regard as the essence of such construction, notwhitstanding the fact that the many-to-one nature of morphisms in a multicategory forces us to consider an infinite presentation of monoidal categories, and we must consequently appeal to Mac Lane's result to obtain one such from the (usual) finite presentation.

Now that we have established the precise equivalence between our notion of representable multicategory and the traditional one of monoidal category, it is worth emphasising this conceptual identification:

Given a category $\mathbb{C}$, a monoidal structure on it *determines* and *is determined by* a representable multicategoy $\mathbb{M}$ with the same objects and linear morphisms as those of $\mathbb{C}$

As we have seen in the definition of $(\_)_c$, this identification allows us to replace arguments which involve coherence axioms (or relations) by the conceptually and technically more convenient arguments by universality, which lie at the heart of category theory. In fact, our proof of the coherence theorem 10.8 for representable multicategories is as good an example as any of this paradigm.

9.10. Corollary. *The equivalence of Theorem 9.8 restricts to one between groupoidal representable multicategories (i.e. all whose arrows are universal) and groupoidal (non strict) monoidal categories (monoidal groupoids), i.e.* $\boxed{(\mathbb{M}_g)_c \equiv (\mathbb{M}_c)_g}$ *(cf. Remark 8.7) where* $(\_)_g : \mathcal{C}at \to \mathcal{G}pd$ *in the right-hand side is the underlying groupoid of a category (right adjoint to the inclusion).*

10 Coherence for representable multicategories and their morphisms

Given the analogy made in §1 between representable multicategories and fibrations, and the correspondence established in Theorem 9.8 between the former and monoidal categories, we should expect to find that every representable multicategory can be turned into an equivalent strict one. We first establish a mild normalisation result for a choice of universal arrows in a representable mulitcategory in §10.1 before tackling the main coherence result in §10.2.

10.1 Normalisation

In any multicategory, a linear morphism $f : \langle X \rangle \to Y$ is universal iff it is an isomorphism. Given a representable multicategory $\mathbb{M}$ with chosen universal arrows $\pi, \varepsilon : \otimes \dashv \eta_{\mathbb{M}} : \mathbb{M} \to \mathbf{T}\mathbb{M}$,

the counit of this adjunction $\varepsilon_X : \langle X \rangle \to \otimes \langle X \rangle$ is an isomorphism. We will show we can always force it to be the identity.

10.1. Definition. A representable multicategory $\mathbb{M}$ with chosen universal arrows

$\pi, \varepsilon : \otimes \dashv \eta_{\mathbb{M}} : \mathbb{M} \to \mathbf{T}\mathbb{M}$ is **normal** if $\varepsilon = id$.

We need the following technical lemma. It states that $\eta_{\mathbb{M}}$ has the opfibration property among representable multicategories. It does in fact prove that $\eta_{\mathbb{M}} : \mathbb{M} \to \mathbf{T}\mathbb{M}$ is the free representable multicategory on $\mathbb{M}$. For simplicity, we work in the ambient category $\mathcal{S}et$.

10.2. Lemma. *Given a representable multicategory $\mathbb{N}$ and the data*

$$\begin{tikzcd} \mathbb{M} \arrow[r, "\eta_{\mathbb{M}}"] \arrow[dr, "f"'] \arrow[dr, phantom, shift left=2, "\theta \Nearrow"] & \mathbf{T}\mathbb{M} \arrow[d, "g"] \\ & \mathbb{N} \end{tikzcd}$$

there exists a morphism of representable multicategories $f' : \mathbf{T}\mathbb{M} \to \mathbb{N}$ (unique up to isomorphism) and a 2-cell $\theta' : f' \Rightarrow g$ (uniquely determined by f' and θ) such that

$$\begin{tikzcd} \mathbb{M} \arrow[r, "\eta_{\mathbb{M}}"] \arrow[dr, "f"'] & \mathbf{T}\mathbb{M} \arrow[d, bend right, "f'"'] \arrow[d, bend left, "g"] \arrow[d, phantom, "\overset{\theta'}{\Rightarrow}"] \\ & \mathbb{N} \end{tikzcd} \quad = \quad \begin{tikzcd} \mathbb{M} \arrow[r, "\eta_{\mathbb{M}}"] \arrow[dr, "f"'] \arrow[dr, phantom, shift left=2, "\theta \Nearrow"] & \mathbf{T}\mathbb{M} \arrow[d, "g"] \\ & \mathbb{N} \end{tikzcd}$$

Furthemore, if g preserves universal arrows and θ is an isomorphism, then θ' is an isomorphism as well.

Proof. Clearly, we must set $f'\langle x \rangle = fx$. For $\vec{x} = \langle x_1, \ldots, x_n \rangle$, $1 \neq n$ in $\mathbf{T}\mathbb{M}$ define $f'\vec{x}$ in $\mathbb{N}$ as the codomain of a universal arrow $\pi_{f\vec{x}} : \langle fx_1, \ldots, fx_n \rangle \to f'\vec{x}$. This extends readily to arrows to yield the required morphism $f' : \mathbf{T}\mathbb{M} \to \mathbb{N}$. Next we define $\theta'_{\vec{x}} : f'\vec{x} \to g\vec{x}$ as the unique linear morphism in the following diagram

$$\begin{tikzcd} \langle fx_1, \ldots, fx_n \rangle \arrow[r, "\pi_{f\vec{x}}"] \arrow[d, "\langle \theta_{x_1}, \ldots, \theta_{x_n} \rangle"'] & f'\vec{x} \arrow[d, dashed, "\theta'_{\vec{x}}"] \\ \langle gx_1, \ldots, gx_n \rangle \arrow[r, "g(id)"'] & g\vec{x} \end{tikzcd}$$

where $id : \langle\langle x_1\rangle, \ldots, \langle x_n\rangle\rangle \to \vec{x}$ is the universal arrow in $\mathbf{T}\mathbb{M}$ which yields (via g) the bottom arrow in the diagram. Clearly, if g preserves universal arrows and the θ_{x_i}'s are isomorphisms, so is the induced $\theta_{\vec{x}}$, being the mediating linear morphism between universals.

$\square$

10.3. Proposition. *Every representable multicategory* $\mathbb{M}$ *with a given choice of universals* $\pi, \varepsilon : \otimes \dashv \eta_{\mathbb{M}} : \mathbb{M} \to \mathbf{T}\mathbb{M}$ *can be normalised, which is to say that there is a* $\otimes' : \mathbf{T}\mathbb{M} \to \mathbb{M}$ *with*

$\pi', id : \otimes' \dashv \eta_{\mathbb{M}} : \mathbb{M} \to \mathbf{T}\mathbb{M}$

Proof. Apply Lemma 10.2 to

$$\begin{tikzcd} \mathbb{M} \arrow[r, "\eta_{\mathbb{M}}"] \arrow[dr, "1"'] & \mathbf{T}\mathbb{M} \arrow[d, "\otimes"] \arrow[dl, phantom, "\varepsilon^{-1}"] \\ & \mathbb{M} \end{tikzcd}$$

to obtain $\otimes' : \mathbf{T}\mathbb{M} \to \mathbb{M}$ which is left adjoint to $\eta_{\mathbb{M}}$, *i.e.* $\pi \circ \eta_{\mathbb{M}}(\varepsilon^{-1})', id : \otimes' \dashv \eta_{\mathbb{M}}$. Notice that $\otimes : \mathbf{T}\mathbb{M} \to \mathbb{M}$ preserves universal arrows as it is a left adjoint, *cf.* Corollary 8.13 and [Koc95].

$\square$

10.2 Representable multicategories equivalent to strict ones

We aim to prove now the main coherence result for representable multicategories, namely that every such is equivalent to a strict one. In view of the doctrinal characterisation in §8.3, we already know this from the corresponding coherence result for (non-strict) monoidal categories [JS93, Cor. 1.4]. But the main point of this paper being the replacement of such coherent structures by universal ones, it would be unfortunate if we had to rely on such results which involve only too subtle manipulations of the coherence axioms. That is, we should be able to obtain such kind of results in a more direct fashion, exploiting the fact that our categorical structures are characterised universally, that is, representability for a multicategory is a *property* and not a mere (coherent) structure.

In fact, the proof of the equivalence between representable multicategories and strict ones we are about to give introduces a new technique for this kind of coherence result, which we regard as a substantial contribution to the subject of coherence in category theory.

Where do we start? We know that for any multicategory $\mathbb{M}$, the unit $\eta_{\mathbb{M}} : \mathbb{M} \to \mathbf{T}\mathbb{M}$ gives a full and faithful embedding of $\mathbb{M}$ into a representable one. Moreover if $\mathbb{M}$ is representable, $\eta_{\mathbb{M}}$ has a left adjoint. Of course, we cannot expect this adjunction to be an equivalence, since $\mathbf{T}\mathbb{M}$ knows nothing about the representability of $\mathbb{M}$. So we should exploit this extra information somehow.

Consider a strict representable multicategory $\mathbb{N}$ and a universal-arrow preserving morphism $f : \mathbb{M} \to \mathbb{N}$. There is, by freeness of $\mathbf{T}\mathbb{M}$ and the opfibration lemma 10.2, a unique (up to isomorphism) morphism of representable multicategories $f' : \mathbf{T}\mathbb{M} \to \mathbb{N}$ such that $f' \circ \eta_{\mathbb{M}} = f$. Let us analize what does f' do to the universal arrows coming from $\mathbb{M}$. Given $\pi_{\vec{x}} : \vec{x} \to y$ universal in $\mathbb{M}$, its image under $\eta_{\mathbb{M}}$ is $\langle \pi_{\vec{x}} \rangle : \langle \langle x_1 \rangle \ldots \langle x_n \rangle \rangle \to \langle y \rangle$.

Since $\mathbf{T}\mathbb{M}$ is representable, we may consider the universal/linear factorisation of $\langle \pi \rangle$ and apply f' to it

$$\begin{tikzcd}
\langle\langle x_1\rangle \ldots \langle x_n\rangle\rangle \arrow[r, "\langle \pi_{\vec{x}}\rangle"] \arrow[dr, "id"'] & \langle y\rangle & \arrow[r, mapsto, "f'"] & {} & \langle\langle f x_1\rangle \ldots \langle f x_n\rangle\rangle \arrow[r, "f'\langle \pi_{\vec{x}}\rangle"] \arrow[dr, "f'(id)"'] & f'\langle y\rangle \\
& \vec{x} \arrow[u, dashed, "\pi_{\vec{x}}"'] & & & & f'\vec{x} \arrow[u, "f'\pi_{\vec{x}}"']
\end{tikzcd}$$

We see that both $f'(id)$ and $f'\langle \pi_{\vec{x}} \rangle = f' \eta_{\mathbb{M}}(\pi_{\vec{x}}) = f\pi_{\vec{x}}$ are universal (since both f and f' preserve such arrows). Hence, $f'\pi_{\vec{x}}$ is an *isomorphism*, for it is the mediating arrow between two universals with the same source. In particular, if we take $f = \eta^{\sigma}_{\mathbb{M}} : \mathbb{M} \to \mathbb{M}^{\sigma}$, where $\mathbb{M}^{\sigma}$ denotes the (hypothetic) free strict representable multicategory on $\mathbb{M}$ as a representable multicategory, so that $\eta^{\sigma}_{\mathbb{M}}$ preserves universals, we have that:

$\boxed{(\eta^{\sigma}_{\mathbb{M}})' : \mathbf{T}\mathbb{M} \to \mathbb{M}^{\sigma} \text{ inverts } \pi : 1 \Rightarrow \eta_{\mathbb{M}} \otimes \text{ universally among morphisms into strict representable multicategories}}$

This leads us to consider the coinverter of π:

$$\begin{tikzcd}
\mathbf{T}\mathbb{M} \arrow[rr, bend left, "1"] \arrow[dr, "\otimes"'] & \Downarrow\pi & \mathbf{T}\mathbb{M} \arrow[r, "q"] & \mathbf{T}\mathbb{M}[\pi^{-1}] \\
& {} \arrow[ur, "\eta_{\mathbb{M}}"'] & &
\end{tikzcd}$$

in *Multicat* and then extend it, if necessary, by turning $\mathbf{T}\mathbb{M}[\pi^{-1}]$ into a strict representable multicategory. For the notion of coinverter in a 2-category we refer to [Kel89]. The reader unfamiliar with this notion at the abstract level might think of it as the 2-categorical generalisation of the familiar *category of fractions* construction of [GZ67].

We now want to analyse whether $\eta^{\sigma}_{\mathbb{M}}$ would be an equivalence given the coinverter property of $(\eta^{\sigma}_{\mathbb{M}})'$. We would also like an explicit description of the coinverter without having to indulge in calculations with formal fractions. The following key technical lemma provides answers to both queries.

10.4. Lemma. *Consider an adjunction $\eta, \varepsilon : l \dashv r : C \to D$ in a 2-category, with r full and faithful (which is equivalent to ε being an isomorphism). Consider the coinverter of the unit*

$$\begin{tikzcd}
D \arrow[rr, bend left, "1"] \arrow[dr, "l"'] & \Downarrow\eta & D \arrow[r, "q"] & D[\eta^{-1}] \\
& {} \arrow[ur, "r"'] & &
\end{tikzcd}$$

and the unique morphism $l' : D[\eta^{-1}] \to C$ induced by l.

1. *The morphisms* $l' : D[\eta^{-1}] \to C$ *and* $qr : C \to D[\eta^{-1}]$ *form an adjoint equivalence.*

2. *There is a canonical isomorphism*

$$\begin{tikzcd} D \arrow[r, "q"] \arrow[dr, "J"'] & D[\eta^{-1}] \arrow[d, dashed] \\ & D_{(rl)} \end{tikzcd}$$

where $D_{(rl)}$ *denotes the Kleisli object (= lax colimit) for the (idempotent) monad* $rl : D \to D$ *induced on* D *by the given adjunction.*

Proof. First of all, notice that there is indeed such a $l' : D[\eta^{-1}] \to C$ induced by l, since $\varepsilon l \circ l\eta = 1$ and εl is an isomorphism, and thus so is $l\eta$.

1. Consider the following diagram

$$\begin{tikzcd} D \arrow[rr, bend left, "1"] \arrow[r, "l"'] & {} \arrow[u, phantom, "\Downarrow\eta"] \arrow[r, "r"'] & D \arrow[r, "q"] \arrow[dr, "l"'] & D[\eta^{-1}] \arrow[d, "l'"] \arrow[drr, "1"] \arrow[dr, phantom, "\Uparrow(\hat{q\eta})"] & & \\ & & & C \arrow[r, "r"'] & D \arrow[r, "q"'] & D[\eta^{-1}] \end{tikzcd}$$

where $(\hat{q\eta}) : qrl' \Rightarrow q$ is the isomorphism uniquely induced by $q\eta : qrl \Rightarrow q$. We also have the isomorphism $\varepsilon : lr = l'(qr) \Rightarrow 1$, which establishes the required equivalence.

2. This statement follows from [GZ67, §I.2] in *Cat*, and more generally from [SW78, Prop. 24]

□

The second statement in the above lemma gives us a simple identification of the coinverter in terms of the Kleisli construction for a monad. So we are led to consider such construction in *Multicat* and analyse whether it preserves strict representability.

10.5. Definition (Kleisli multicategory for a monad). Given a multicategory $\mathbb{M}$ and a monad $T = (T, \eta, \mu)$ on it, we define the **Kleisli multicategory** $\mathbb{M}_T$ as follows

objects those of $\mathbb{M}$

arrows a morphism in $\mathbb{M}_T(\vec{x}, y)$ is a morphism $f : \vec{x} \to Ty$ in $\mathbb{M}$

identities the identity on x is $\eta_x : \langle x \rangle \to Tx$

composition

$$\frac{f_1 : \vec{x}_1 \to Ty_1, \ldots, f_n : \vec{x}_n \to Ty_n \quad f : \vec{y} \to Tz}{\mu_z \circ Tf \circ \langle f_1, \ldots, f_n \rangle : \vec{x}_1 \cdot \ldots \cdot \vec{x}_n \to Tz}$$

The verification that the above composition is associative and unitary is a routine calculation, using the multinaturality of μ and the monad equations.

10.6. Proposition. *The construction in Definition 10.5 yields Kleisli objects in $\mathcal{M}ulticat$. Furthermore, if $\mathbb{M}$ is (strict) representable, so is $\mathbb{M}_T$ and $J : \mathbb{M} \to \mathbb{M}_T$ preserves universals.*

Proof. Define $J : \mathbb{M} \to \mathbb{M}_T$ to be the identity on objects and $J(f : \vec{x} \to y) = \eta_y \circ f$. Define $\rho : JT \Rightarrow J$ as $\rho_x = id_{Tx}$. Then

$$\begin{tikzcd} \mathbb{M} \arrow[dr, "J"] \arrow[dd, "T"'] & \\ & \mathbb{M}_T \\ \mathbb{M} \arrow[ur, "J"'] \arrow[u, phantom, "\rho \Uparrow"] & \end{tikzcd}$$

exhibits $\mathbb{M}_T$ as a lax colimit of (T, η, μ) in $\mathcal{M}ulticat$.

If $\pi_{\vec{x}} : \vec{x} \to y$ is universal in $\mathbb{M}$, $J\pi_{\vec{x}} = \eta_y \circ \pi_{\vec{x}} : \vec{x} \to Ty$ is universal in $\mathbb{M}_T$: given $g : \vec{x} \to Tz$ let $g = \hat{g} \circ \pi_{\vec{x}}$ its universal/linear factorisation in $\mathbb{M}$. Then $g = \hat{g} \circ J\pi_{\vec{x}}$ in $\mathbb{M}_T$. Hence if $\mathbb{M}$ is (strict) representable, so is $\mathbb{M}_T$ and J preserves universals, as required.

$\square$

10.7. Remark. A monad (T, η, μ) on $\mathbb{M}$ induces an ordinary monad $(\overline{T}, \overline{\eta}, \overline{\mu})$ on the category of linear morphisms $\overline{\mathbb{M}}$. Then $\overline{\mathbb{M}}_{\overline{T}} \cong \overline{\mathbb{M}_T}$, canonically.

Recall that $\mathcal{R}ep\mathcal{M}ulticat_s$ denotes the (locally full) sub-2-category of $\mathcal{M}ulticat$ consisting of the strict representable multicategories and the morphisms between such which preserve the chosen universals on the nose.

10.8. Theorem (Coherence for representable multicategories). *The inclusion $\mathcal{R}ep\mathcal{M}ulticat_s \hookrightarrow \mathcal{R}ep\mathcal{M}ulticat$ has a left biadjoint whose unit is a (pseudo-natural) equivalence.*

Proof. Given $\mathbb{M}$ a representable multicategory, we apply Lemma 10.4 to the adjunction

$\pi, \varepsilon : \otimes \dashv \eta_{\mathbb{M}}$. By Proposition 10.6, the coinverter $\mathbf{T}\mathbb{M}[\pi^{-1}]$ constructed as the Kleisli multicategory $\mathbf{T}\mathbb{M}_{(\eta_{\mathbb{M}} \otimes)}$ is a strict representable multicategory since $\mathbf{T}\mathbb{M}$ is. The argument above shows that the equivalence

$J\eta_{\mathbb{M}} : \mathbb{M} \to \mathbf{T}\mathbb{M}_{\eta_{\mathbb{M}} \otimes}$

is the unit of the required biadjoint, as we have an equivalence

$$\mathcal{R}ep\mathcal{M}ulticat(\mathbb{M}, \mathbb{N}) \simeq \mathcal{R}ep\mathcal{M}ulticat_s(\mathbf{T}\mathbb{M}_{\eta_{\mathbb{M}} \otimes}, \mathbb{N})$$

induced by precomposition with $J\eta_{\mathbb{M}}$, and hence pseudo-natural in $\mathbb{M}$ and 2-natural in strict representable multicategories $\mathbb{N}$.

□

Let us write $(\_)^\sigma : \mathcal{R}ep\mathcal{M}ulticat \rightarrow \mathcal{R}ep\mathcal{M}ulticat_s$ for the above biadjoint (homomorphism or pseudo-functor) and $\eta^\sigma_{\mathbb{M}} : \mathbb{M} \rightarrow \mathbb{M}^\sigma$ for the unit. Notice that the above bireflection fails to be a biequivalence in that the inclusion is not locally an equivalence. The problem is that given a morphism $f : \mathbb{M} \rightarrow \mathbb{N}$ preserving universals, with both multicategories *strict* we cannot necessarily find an isomorphic morphism f' between them which preserves the chosen universals on the nose. We have seen in Proposition 10.2 that we can do so when $\mathbb{M}$ is free, essentially because we can redefine f on the generators to enforce preservation. In general, the best we can do is to classify morphisms between strict representable multicategories in the sense of the following corollary.

10.9. Corollary (Classification of morphisms). *Given a strict representable multicategory $\mathbb{M}$, the equivalence $\eta^\sigma_{\mathbb{M}} : \mathbb{M} \rightarrow \mathbb{M}^\sigma$ induces an isomorphism*

$$\mathcal{R}ep\mathcal{M}ulticat(\mathbb{M}, \mathbb{N}) \cong \mathcal{R}ep\mathcal{M}ulticat_s(\mathbb{M}^\sigma, \mathbb{N})$$

2-natural in $\mathbb{N}$ (a strict representable multicategory).

Proof. This follows directly from the above biadjunction, noticing that since $\mathbb{M}$ is strict it is therefore normal and hence $\otimes' \circ \eta^\sigma_{\mathbb{M}} = 1$. Therefore precomposition with $\eta^\sigma_{\mathbb{M}}$ is bijective on objects rather than merely essentially surjective, and so the hom-equivalence of the biadjunction restricts to the above isomorphism of categories.

□

10.10. Remark. The above argument for Theorem 10.8 and Corollary 10.9, namely Lemma 10.4 and the existence of Kleisli objects applies equally well to other monads whose algebras are adjoints to units. Among the best known examples we have fibrations, categories with products and categories with coproducts.

11 Bicategories as 'several objects' representable multicategories

It is well-known that a (non-strict) monoidal category can be seen as a one-object bicategory [Bén67]. Conversely it is helpful to think of a bicategory as a 'several objects' monoidal category (we will make this rigorous below) and hence as representable multicategories.

In the strict situation we have monoidal categories as one-object 2-categories. We want to show how we can construe 2-categories as monoidal categories. Let us start by reexamining monoidal categories. A monoidal category $\mathbb{C}$ consists of a span

$$\begin{tikzcd} & C_1 \arrow[dl, "d"'] \arrow[dr, "c"] & \\ C_0 & & C_0 \end{tikzcd}$$

where both C_0 and C_1 are monoids and both d and c are monoid morphisms, because of the functoriality of $\otimes\colon \mathbb{C}^2 \to \mathbb{C}$. Since we are working in a category $\mathbb{B}$ admitting free monoids, the category $\mathcal{M}on(\mathbb{B})$ is monadic over $\mathbb{B}$ and therefore as (finitely) complete as $\mathbb{B}$ is. We can consider then the bicategory $\mathbf{Spn}(\mathcal{M}on(\mathbb{B}))$ and look at monads in it. The above span is then such a monad: the composition of $\mathbb{C}$ preserves the (pairwise) monoid structure of $C_1 \circ C_1$ into C_1 by the functoriality of $\otimes$. We thus arrive at the following alternative view of monoidal categories, as categories in monoids rather than monoids in $\mathcal{C}at$:

11.1. Proposition. $\boxed{\mathcal{M}on\mathcal{C}at(\mathbb{B}) \cong \mathcal{C}at(\mathcal{M}on(\mathbb{B}))}$

Consider a 2-category

$$\begin{tikzcd} & C_2 \arrow[dl, "d_2"'] \arrow[dr, "c_2"] & \\ C_1 \arrow[d, "d_1"'] \arrow[drr, "c_1" near start] & & C_1 \arrow[d, "c_1"] \arrow[dll, "d_1"' near start] \\ C_0 & & C_0 \end{tikzcd}$$

The globularity condition

$$\begin{aligned} c_1 c_2 &= c_1 d_2 \\ d_1 c_2 &= d_1 d_2 \end{aligned}$$

means that we have well defined horizontal domain and codomain maps $d^2, c^2 : C_2 \to C_0$ (as the diagonals of the above commuting rectangles). Hence the top span amounts to a span

$$\begin{tikzcd} & (d^2, C_2, c^2) \arrow[dl, "d_2"'] \arrow[dr, "c_2"] & \\ (d_1, C_1, c_1) & & (d_1, C_1, c_1) \end{tikzcd}$$

in $\mathbf{Spn}(\underline{\mathbf{Spn}(\mathbb{B})(C_0, C_0)})((d_1, C_1, c_1), (d_1, C_1, c_1))$ where we use the notation $(\mathit{domain}, \mathit{top\ object}, \mathit{codomain})$ for in-line spans. Since $\mathbb{C}$ is a 2-category, both (d_1, C_1, c_1) and (d_2, C_2, c_2) are categories under horizontal composition of 1-cells and 2-cells respectively, and d_2 and c_2 are functors. So we have a span of monoids in $\mathbf{Spn}(\mathbb{B})(C_0, C_0)$. Finally, the vertical composition of 2-cells endows the above span with a monoid structure in

$\mathbf{Spn}(\underline{\mathbf{Spn}(\mathbb{B})(C_0, C_0)})((d_1, C_1, c_1), (d_1, C_1, c_1))$ and the interchange law guarantees that such structure preserves the (pairwise, horizontal composite) monoid structure of $(d_1, C_1, c_1) \circ (d_1, C_1, c_1)$. Thus, writing $2\!-\!\mathcal{C}at(\mathbb{B})_{C_0}$ for the category of 2-categories in $\mathbb{B}$ with object-of-objects C_0 and 2-functors which are identities on objects, we get the following identification:

11.2. Proposition. $\boxed{2\!-\!\mathcal{C}at(\mathbb{B})_{C_0} \cong \mathcal{M}on\mathcal{C}at(\underline{\mathbf{Spn}(\mathbb{B})(C_0, C_0)})}$

which is the rigorous formulation of the slogan '2-categories as several-objects monoidal categories' we mentioned at the beginning of this section. As for bicategories, they correspond to (non-strict) monoidal categories, hence

$$\boxed{\mathcal{B}icat(\mathbb{B})_{C_0} \cong w\mathcal{M}on\mathcal{C}at(\underline{\mathbf{Spn}(\mathbb{B})(C_0, C_0)})}$$

11.3. Remark. This point of view of internal 2-categories will prove central in our treatment of laxity for n-categories, which we will develop in a subsequent paper.

11.4. Corollary. $\boxed{\mathcal{B}icat(\mathbb{B})_{C_0} \cong \mathcal{R}ep\mathcal{M}ulticat(\underline{\mathbf{Spn}(\mathbb{B})(C_0, C_0)})}$

The only point to note here is that we must guarantee that the free monoid construction in $\mathbf{Spn}(\mathbb{B})(C_0, C_0)$ (which amounts to the free category on a graph) yields a cartesian monad. This is easily seen in $\mathcal{S}et$. The general treatment will appear in a sequel paper.

11.5. Example. Given a small category $\mathbb{C}$ we define a multicategory $\mathbf{m}\text{-}\mathbf{Spn}(\mathbb{C})$ in $\mathbf{Spn}(\mathcal{S}et)(C_0, C_0)$ as follows

objects The span $(s, \mathsf{spans}(\mathbb{C}), \mathsf{t})$ on C_0, where $\mathsf{spans}(\mathbb{C})$ is the set of spans in $\mathbb{C}$

$$\begin{tikzcd} & R \arrow[dl, "d_R"'] \arrow[dr, "c_R"] & \\ X & & Y \end{tikzcd}$$

with $s(d_R, R, c_R) = X$ and $t(d_R, R, c_R) = Y$

arrows An object in $\mathbf{M}(s, \mathsf{spans}(\mathbb{C}), \mathsf{t})$ is a finite sequence of composable spans

$$\begin{tikzcd} & R_1 \arrow[dl, "d_1"'] \arrow[dr, "c_1"] & & & & R_n \arrow[dl, "d_n"'] \arrow[dr, "c_n"] & \\ X & & Z_1 & \cdots & Z_{(n-1)} & & Y \end{tikzcd}$$

and an arrow $f : \langle R_1, \ldots, R_n \rangle \to R$ is a sequence of arrows $\langle f_i : R \to R_i \rangle$ in $\mathbb{C}$ such that

$$d_i f_i = c_{i-1} f_{i-1} (1 < i \leq n) \qquad d_1 f_1 = d \qquad c_n f_n = c$$

which is to say that the f_i's form a *cone* over the above diagram, with vertex R.

identities $id_R : R \to R$ in $\mathbb{C}$

composition $f \circ \langle g_i \rangle = \langle f_i \circ (g_i)_j \rangle (1 \leq i \leq n, 1 \leq j \leq m_i)$, with our usual notational conventions.

A routine diagram chasing shows the above multicategory composite is well-defined and associative, just like in Example 2.2.(2).

11.6. Proposition. *The multicategory* **m-Spn**$(\mathbb{C})$ *is representable iff* $\mathbb{C}$ *admits pullbacks*

Proof. Universal arrows in **m-Spn**$(\mathbb{C})$ correspond to limit cones. Hence **m-Spn**$(\mathbb{C})$ iff $\mathbb{C}$ admits limits of such diagrams. But these diagrams are *finite simply connected*, and Paré's characterisation in [Par90] shows that such diagrams are precisely the ones whose limits can be constructed by pullbacks.

□

It is clear that the bicategory associated to the representable multicategory **m-Spn**$(\mathbb{C})$ is non other than **Spn**$(\mathbb{C})$.

A The functoriality of the Spn(_) construction

In this technical supplement we briefly describe the functoriality of **Spn**(_) partly to justify our claim that $\mathbf{Spn_T}(\mathbb{B})$ is a Kleisli bicategory, but also to shed some light as to how such a construction arises.

We start by recalling the definition of the bicategory of spans on a category with pullbacks, introduced in [Bén67].

A.1. Definition. Given a category $\mathbb{B}$ with pullbacks, the **bicategory of spans Spn**$(\mathbb{B})$ consists of

objects those of $\mathbb{B}$

morphisms a morphism from X to Y is a span

$$\begin{tikzcd} & R \arrow[dl, "d_R"'] \arrow[dr, "c_R"] & \\ X & & Y \end{tikzcd}$$

2-cells a 2-cell between morphisms is a morphism between the top objects of the spans, commuting with the domain and codomain morphisms:

$$\begin{tikzcd} & R \arrow[dl, "d_R"'] \arrow[dr, "c_R"] \arrow[dd, "f"] & \\ X & & Y \\ & S \arrow[ul, "d_S"] \arrow[ur, "c_S"'] & \end{tikzcd}$$

The identity span on X is

$$\begin{tikzcd}[column sep=small] & X \arrow[dl, "id"'] \arrow[dr, "id"] & \\ X & & X \end{tikzcd}$$

and composition is given by

$$\dfrac{\begin{tikzcd}[column sep=small] & R \arrow[dl, "d_R"'] \arrow[dr, "c_R"] & & & S \arrow[dl, "d_S"'] \arrow[dr, "c_S"] & \\ X & & Y & Y & & Z \end{tikzcd}}{\begin{tikzcd}[column sep=small] & & R{\circ}S \arrow[dl, "\overline{d_S}"'] \arrow[dr, "\overline{c_R}"] & & \\ & R \arrow[dl, "d_R"'] \arrow[dr, "c_R"] & & S \arrow[dl, "d_S"'] \arrow[dr, "c_S"] & \\ X & & Y & & Z \end{tikzcd}}$$

where the square is a pullback. Horizontal composition of 2-cells is clearly (canonically) induced by that of morphisms, while the vertical composition is inherited from $\mathbb{B}$.

Now we establish the universal property of $\mathbf{Spn}(\mathbb{B})$ (which is folklore although we know no references for it).

A.2. Theorem (Universal characterisation of $\mathbf{Spn}(\mathbb{B})$). *Consider a category $\mathbb{B}$ with pullbacks and the functor $\eta_{\mathbb{B}} : \mathbb{B} \to \mathbf{Spn}(\mathbb{B})$ given by*

$$x \xrightarrow{f} y \quad \longmapsto \quad \begin{tikzcd}[column sep=small] & x \arrow[dl, "id"'] \arrow[dr, "f"] & \\ x & & y \end{tikzcd}$$

1. *The functor $\eta_{\mathbb{B}} : \mathbb{B} \to \mathbf{Spn}(\mathbb{B})$ is universal among functors from $\mathbb{B}$ to bicategories $\mathcal{K}$, $F : \mathbb{B} \to \mathcal{K}$, which send the morphisms of $\mathbb{B}$ to **maps** (1-cells with a right adjoint) satisfying the Beck-Chevalley condition. This means that such a functor factors as $F = \hat{F} \circ \eta_{\mathbb{B}}$, for a unique (up to isomorphism) homomorphism $\hat{F} : \mathbf{Spn}(\mathbb{B}) \to \mathcal{K}$.*

2. *Given two such functors $F, G : \mathbb{B} \to \mathcal{K}$ and a 2-natural transformation $\alpha : F \Rightarrow G$, there is a unique lax transformation $\hat{\alpha} : \hat{F} \Rightarrow \hat{G}$ such that $\hat{\alpha}\eta_{\mathbb{B}} = \alpha$. Furthermore, if for every morphism $f : x \to y$ in $\mathbb{B}$ the pair (α_x, α_y) induces a pseudo-map of adjoints from $Ff \dashv (Ff)^*$ to $Gf \dashv (Gf)^*$, the corresponding 2-cell $\hat{\alpha}$ is pseudo-natural.*

Instead of going through the details of the proof, which relies purely on the fact that every morphism in $\mathbf{Spn}(\mathbb{B})$ factors as

$$
\begin{tikzcd}[column sep=small]
 & R \arrow[dl, "d"'] \arrow[dr, "c"] & \\
x & & y
\end{tikzcd}
\quad = \quad
\begin{tikzcd}[column sep=small]
 & R \arrow[dl, "d"'] \arrow[dr, "id"] & & R \arrow[dl, "id"'] \arrow[dr, "c"] & \\
x & & R & & y
\end{tikzcd}
$$

and the righthand side is $\eta_{\mathbb{B}}(c) \circ \eta_{\mathbb{B}}(d)^*$, we will spell out the resulting functoriality of $\mathbf{Spn}(\_)$.

Since $\mathbf{Spn}(\_)$ operates on categories with pullbacks it seems natural to take as domain of variation the 2-category $\mathbf{Pbk}$ of categories with pullbacks, pullback preserving functors and cartesian transformations. Let $\mathcal{B}icat$ denote the tricategory of bicategories, homomorphisms, pseudo-natural transformations and modifications [GPS95].

A.3. Proposition. *The bicategory of spans construction extends to a trihomomorphism*

$$\mathbf{Spn}(\_) : \mathbf{Pbk} \to \mathcal{B}icat$$

Proof. Given a pullback preserving functor $F : \mathbb{B} \to \mathbb{C}$, the homomorphism $\mathbf{Spn}(F) : \mathbf{Spn}(\mathbb{B}) \to \mathbf{Spn}(\mathbb{C})$ acts as follows

$$
\begin{tikzcd}[column sep=small]
 & R \arrow[dl, "d"'] \arrow[dr, "c"] & \\
x & & y
\end{tikzcd}
\quad \longmapsto \quad
\begin{tikzcd}[column sep=small]
 & FR \arrow[dl, "Fd"'] \arrow[dr, "Fc"] & \\
Fx & & Fy
\end{tikzcd}
$$

and given a cartesian transformation $\alpha : F \Rightarrow G$ with G pullback preserving, $\mathbf{Spn}(\alpha) : \mathbf{Spn}(F) \Rightarrow \mathbf{Spn}(G)$ is defined at an object x in $\mathbf{Spn}(\mathbb{B})$ as the span

$$
\begin{tikzcd}[column sep=small]
 & Fx \arrow[dl, "id"'] \arrow[dr, "\alpha_x"] & \\
Fx & & Gx
\end{tikzcd}
$$

With the above explicit description, it is immediate that $\mathbf{Spn}(\_)$ preserves composition of functors on the nose, while it preserves both vertical and horizontal composition of 2-cells up to canonical isomorphism induced by universality of pullbacks. □

It is important to notice that the operation which reverses spans

$$
\begin{tikzcd}[column sep=small]
 & R \arrow[dl, "d"'] \arrow[dr, "c"] & \\
x & & y
\end{tikzcd}
\quad \longmapsto \quad
\begin{tikzcd}[column sep=small]
 & R \arrow[dl, "c"'] \arrow[dr, "d"] & \\
y & & x
\end{tikzcd}
$$

induces an isomorphism $\sigma : \mathbf{Spn}(\mathbb{B}) \xrightarrow{\sim} \mathbf{Spn}(\mathbb{B})^{op}$ and we thus obtain a trihomorphism $\sigma\mathbf{Spn}(\_) : \mathbf{Pbk} \to \mathcal{B}icat^{co}$ (reversing the 2-cells in $\mathcal{B}icat$). Since trihomomorphisms preserve pseudo-monads (*cf.* [DS97, Prop. 5]), we get the following corollary.

A.4. Corollary. *A cartesian monad* $\mathbf{T} = (T, \eta, \mu)$ *on a category with pullbacks* $\mathbb{B}$ *induces both a (psuedo)monad* $(\mathbf{Spn}(T), \mathbf{Spn}(\eta), \mathbf{Spn}(\mu))$ *and a (pseudo)comonad* $(\mathbf{Spn}(T), \sigma\mathbf{Spn}(\eta), \sigma\mathbf{Spn}(\mu))$ *on* $\mathbf{Spn}(\mathbb{B})$.

Proof. Immediately from Prop. A.3, since a cartesian monad is simply a monad in **Pbk**.

$\square$

We will now focus our attention in the pseudo-comonads so induced. We can readily perform the (bi)Kleisli construction, which resolves a pseudo-comonad by a biadjunction.

A.5. Proposition. *Given a cartesian monad* $\mathbf{T} = (T, \eta, \mu)$ *on a category with pullbacks* $\mathbb{B}$*, the (bi)Kleisli bicategory of the pseudo-comonad* $(\mathbf{Spn}(T), \sigma\mathbf{Spn}(\eta), \sigma\mathbf{Spn}(\mu))$ *on* $\mathbf{Spn}(\mathbb{B})$ *is* $\mathbf{Spn_T}(\mathbb{B})$ *of Def. 4.2.*

Proof. Define a homomorphism $J : \mathbf{Spn}(\mathbb{B}) \to \mathbf{Spn_T}(\mathbb{B})$ as follows

$$
\begin{tikzcd}[column sep=small]
 & R \arrow[dl, "d"'] \arrow[dr, "c"] & & & & x \arrow[dl, "\eta_x"'] \arrow[dr, "id"] & & R \arrow[dl, "d"'] \arrow[dr, "c"] & & & & R \arrow[dl, "\eta_x d"'] \arrow[dr, "c"] & \\
x & & y & \longmapsto & Tx & & x & & y & = & Tx & & y
\end{tikzcd}
$$

and a pseudo-natural transformation $\rho : J \Rightarrow J\mathbf{Spn}(T)$, whose component at X is the span

$$
\begin{tikzcd}
 & TX \arrow[dl, "id"'] \arrow[dr, "id"] & \\
TX & & TX
\end{tikzcd}
$$

and we have thus defined a (bi)colax cocone

$$
\begin{tikzcd}
\mathbf{Spn}(\mathbb{B}) \arrow[dd, "\mathbf{Spn}(T)"'] \arrow[dr, "J"] & \\
\arrow[r, phantom, "\Swarrow\rho"] & \mathbf{Spn_T}(\mathbb{B}) \\
\mathbf{Spn}(\mathbb{B}) \arrow[ur, "J"'] &
\end{tikzcd}
$$

with structural isomorphisms

$$
\begin{tikzcd}
\mathbf{Spn}(\mathbb{B}) \arrow[dd, bend right=50, "1"'] \arrow[dd, bend left=30] \arrow[dr, "J"] & \\
\arrow[r, phantom, "\overset{\Leftarrow}{\sigma\eta}\quad\Swarrow\rho"] & \mathbf{Spn}_T(\mathbb{B}) \\
\mathbf{Spn}(\mathbb{B}) \arrow[ur, "J"'] &
\end{tikzcd}
\overset{\bar{u}}{\cong}
\begin{tikzcd}
\mathbf{Spn}(\mathbb{B}) \arrow[dd, bend right=50, "1"'] \arrow[dr, "J"] & \\
\arrow[r, phantom, "\Swarrow\rho"] & \mathbf{Spn}_T(\mathbb{B}) \\
\mathbf{Spn}(\mathbb{B}) \arrow[ur, "J"'] &
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
 & \mathbf{Spn}(\mathbb{B}) \arrow[dl, "\mathbf{Spn}(T)"'] \arrow[dd, "\mathbf{Spn}(T)"] \arrow[dr, "J"] & \\
\cdot \arrow[dr, "\mathbf{Spn}(T)"'] \arrow[r, phantom, "\overset{\Leftarrow}{\sigma\mu}"] & \arrow[r, phantom, "\Swarrow\rho"] & \mathbf{Spn}_T(\mathbb{B}) \\
 & \mathbf{Spn}(\mathbb{B}) \arrow[ur, "J"'] &
\end{tikzcd}
\overset{\bar{m}}{\cong}
\begin{tikzcd}
 & \mathbf{Spn}(\mathbb{B}) \arrow[dl, "\mathbf{Spn}(T)"'] \arrow[dr, "J"] & \\
\mathbf{Spn}(\mathbb{B}) \arrow[rr, "J"] \arrow[dr, "\mathbf{Spn}(T)"'] & \arrow[u, phantom, "\Swarrow\rho"] \arrow[d, phantom, "\Downarrow\rho"] & \mathbf{Spn}_T(\mathbb{B}) \\
 & \mathbf{Spn}(\mathbb{B}) \arrow[ur, "J"'] &
\end{tikzcd}
$$

and $\bar{u}$ and $\bar{m}$ satisfy coherence axioms as for pseudo-algebras. This (bi)colax cocone is universal, in the sense that there is a biequivalence

$$\mathcal{B}icat(\mathbf{Spn_T}(\mathbb{B}), \mathcal{K}) \simeq Colax\!-\!cocones(\mathbf{Spn}(T), K)$$

essentially surjective on objects up to isomorphism, where $Colax\!-\!cocones(\mathbf{Spn}(T), K)$ denotes the bicategory whose objects are (bi)colax cocones $(L : \mathbf{Spn}(\mathbb{B}) \to \mathcal{K}, \lambda, \bar{u}, \bar{m})$ as above, morphisms are 2-cells $\theta : L \Rightarrow L'$ together with an invertible modification $\overline{\theta\lambda}$

$$\begin{tikzcd}
\mathbf{Spn}(\mathbb{B}) \arrow[d, "\mathbf{Spn}(T)"'] \arrow[dr, bend left, "L"] \arrow[dr, phantom, "\Swarrow\lambda"] & \\
\mathbf{Spn}(\mathbb{B}) \arrow[r, bend left, "L"] \arrow[r, bend right, "L'"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathcal{K}
\end{tikzcd}
\overset{\overline{\theta\lambda}}{\cong}
\begin{tikzcd}
\mathbf{Spn}(\mathbb{B}) \arrow[d, "\mathbf{Spn}(T)"'] \arrow[dr, bend left, "L"] \arrow[dr, "L'"'] \arrow[dr, phantom, bend left, "\Swarrow\theta"] & \\
\mathbf{Spn}(\mathbb{B}) \arrow[r, "L'"'] \arrow[ur, phantom, "\Swarrow\lambda'"] & \mathcal{K}
\end{tikzcd}$$

compatible with the given $\bar{u}$'s and $\bar{m}$ and 2-cells modifications between such θ's compatible with the given modifications $\overline{\theta\lambda}$'s[3].

$\square$

The above Kleisli bicategory construction yields a biadjunction

$$\begin{tikzcd}
\mathbf{Spn}(\mathbb{B}) \arrow[r, bend right, "J"'] \arrow[r, phantom, "\perp"] & \mathbf{Spn_T}(\mathbb{B}) \arrow[l, bend right, "L"']
\end{tikzcd}$$

where L acts as

$$\begin{tikzcd}[column sep=small]
& R \arrow[dl, "d"'] \arrow[dr, "c"] & \\
Tx & & y
\end{tikzcd}
\longmapsto
\begin{tikzcd}[column sep=small]
& T^2x \arrow[dl, "\mu_x"'] \arrow[dr, "id"] & & TR \arrow[dl, "Td"'] \arrow[dr, "Tc"] & \\
Tx & & T^2x & & Ty
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
& TR \arrow[dl, "\mu_x \circ Td"'] \arrow[dr, "Tc"] & \\
Tx & & Ty
\end{tikzcd}$$

Of course such homomorphisms of bicategories map monads in one to the other. In particular $L : \mathbf{Spn_T}(\mathbb{B}) \to \mathbf{Spn}(\mathbb{B})$ takes multicategories in $\mathbb{B}$ to categories in $\mathbb{B}$; the induced 2-functor $L : \mathcal{M}ulticat(\mathbb{B}) \to \mathcal{C}at(\mathbb{B})$ is none other than the 2-functor $F : \mathcal{M}ulticat \to \mathcal{M}on\mathcal{C}at$ of Theorem 7.2. In the other direction the 2-functor induced by J, $J : \mathcal{C}at(\mathbb{B}) \to \mathcal{M}ulticat(\mathbb{B})$ is the inclusion taking a category to the multicategory whose only morphisms are the linear ones, given in Example 2.2.(3).

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[3]Such data corresponds to: colax cocone = pseudo-algebra, morphism = strong morphism of pseudo-algebras.

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