Analytic calculi for monadic PNmatrices

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Abstract. Analytic calculi are a valuable tool for a logic, as they allow for effective proof-search and decidability results. We study the axiomatization of generalized consequence relations determined by monadic partial non-deterministic matrices (PNmatrices). We show that simple axiomatizations can always be obtained, using inference rules which can have more than one conclusion. Further, we prove that these axiomatizations are always analytic, which seems to raise a contrast with recent non-analyticity results for sequent-calculi with PNmatrix semantics.

1 Introduction

PNmatrices were introduced in [5], as a generalization of non-deterministic matrices (Nmatrices) [1,2]. Adding non-determinism and also partiality to the traditional notion of logical matrix (see [17]) has proven quite relevant in a myriad of recent compositional results in logic [3,5,11,13,6], namely as semantical counterparts of certain families of sequent-calculi. However, while Nmatrices still inherit from logical matrices a local semantical form of analyticity (a well formed valuation on a set of formulas closed for subformulas can always be extended to a full valuation), the partiality allowed by PNmatrices spoils this property. It turns out that for the sequent-calculi using these semantical tools, partiality seems to devoid them of a usable (even if generalized) subformula property capable of guaranteeing analyticity (and elimination of non-analytic cuts) [5,11].

Concerning other types of calculi, traditional Hilbert-style calculi are clearly not an option if any form of analyticity is expected. However, a very simple and powerful (and too often neglected) generalization of Hilbert-calculi has been proposed in [16]. Along with their proposal, Shoesmith and Smiley already showed that every logical matrix can be given a *multiple conclusion* axiomatization with rules of the form $\frac{\Gamma}{\Delta}$ where both Γ (read conjunctively, as usual) and Δ (read disjunctively) are sets of formulas. This axiomatization is finite for a given finite

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matrix, in contrast with the existence of finite matrices whose logic cannot be finitely axiomatized in Hilbert-style [18].

In a recent paper [14], we have extended the result of [16] and shown that multiple conclusion axiomatizations can be easily obtained for every Nmatrix with the additional expressiveness requirement of being *monadic*. Further, we showed that the calculi obtained enjoy a suitably generalized subformula property which allowed us to prove the analyticity of the axiomatizations obtained. Herein, we go a step further and analyze the case of PNmatrices. It turns out that the same exact methods can be used to obtain sound and complete axiomatizations of the (generalized) consequence relations of any monadic PNmatrix. However, perhaps expectedly, the calculi obtained are not analytic, again due to partiality. What is really interesting, though, is that in the framework of multiple conclusion calculi it is extremely easy to remedy this situation: with the addition of a few sound rules, the calculus not only remains complete, of course, but it also becomes analytic.

The paper is organized as follows. Section 2 introduces and illustrates multiple conclusion logics and calculi (after [16] and [14]), as well as analyticity; and then recalls the fundamental aspects of (P)Nmatrices (after [1] and [5]), and the key property of being monadic (see [16,14]). Section 3 defines the calculi to be associated with each monadic PNmatrix and proves our main results, i.e., their completeness and analyticity. Along these sections we will illustrate our methods using the implication-free fragment of Kleene's strong three-valued logic (see [12]). In Section 4 we present a detailed example, one of the problematic paraconsistent logics of [11]. We close, in Section 5, discussing the results obtained, their import and limitations, and possible extensions of this work.

2 Preliminaries

In any context, given a function $h : X \to Y$ and $Z \subseteq X$ we use $h(Z)$ to denote the set $\{h(z) : z \in Z\}$.

A propositional *signature* Σ is an $\mathbb{N}$-indexed set $\Sigma = \{\Sigma^{(k)} : k \in \mathbb{N}\}$, where each $\Sigma^{(k)}$ contains the *k-ary connectives* of Σ. As usual, we may write $\copyright \in \Sigma$ when $\copyright \in \Sigma^{(k)}$ for some $k \in \mathbb{N}$. The language $L_\Sigma(P)$ is the carrier of the absolutely free Σ-algebra generated over a given set of propositional variables P. Elements of $L_\Sigma(P)$ are called *formulas*. Notationwise, we use $A, B, C, \dots$ to denote formulas, and $\Gamma, \Delta, \Omega, \dots$ to denote sets of formulas. For convenience, we often use commas and write Γ, Δ instead of $\Gamma \cup \Delta$, or Γ, A instead of $\Gamma \cup \{A\}$, or A, B instead of $\{A, B\}$.

Given a formula $A \in L_\Sigma(P)$, we denote by $\mathsf{var}(A)$ (resp. $\mathsf{sub}(A)$) the set of propositional variables (resp. subformulas) of A. A *substitution* is a mapping $\sigma : P \to L_\Sigma(P)$, uniquely extendable into an endomorphism $\cdot^\sigma : L_\Sigma(P) \to L_\Sigma(P)$. We also use $\overline{\Gamma}$ to denote $L_\Sigma(P) \setminus \Gamma$. If $A, B_1, \dots, B_n \in L_\Sigma(P)$ is such that $\mathsf{var}(A) \subseteq \{p_1, \dots, p_n\}$ then we use $A(B_1, \dots, B_n)$ to denote the formula A^σ where σ is any substitution such that $\sigma(p_1) = B_1, \dots, \sigma(p_n) = B_n$.

For added self-containment, as well as to fix notation, we recall the main notions regarding multiple conclusion logics and calculi, as well Nmatrices, PN-matrices and monadicity.

2.1 Multiple conclusions

Fixed a signature Σ, a *(schematic) (multiple conclusion) inference rule* is a pair $\langle \Gamma, \Delta \rangle \in \wp(L_\Sigma(P)) \times \wp(L_\Sigma(P))$, usually simply written as $\frac{\Gamma}{\Delta}$, where Γ is the set of *premises* and Δ the set of *conclusions*. A *(multiple conclusion) calculus* is a set of inferences rules. A calculus is *finitary* if each of its rules has finitely many premises and conclusions.

Example 1. Consider a signature Σ containing a unary connective $\neg$ and a binary connective $\wedge$. The following four rules define a calculus R_1.

$$\frac{q\,,\ \neg q\,,\ \neg(p\wedge q)}{p\,,\ \neg p\,,\ p\wedge q}\ r_{\wedge ab0} \qquad \frac{q\,,\ \neg q\,,\ p\wedge q}{p\,,\ \neg p\,,\ \neg(p\wedge q)}\ r_{\wedge ab1}$$

$$\frac{q\,,\ \neg q}{p\,,\ \neg p\,,\ p\wedge q\,,\ \neg(p\wedge q)}\ r_{\wedge aba} \qquad \frac{q\,,\ \neg q\,,\ p\wedge q\,,\ \neg(p\wedge q)}{p\,,\ \neg p}\ r_{\wedge abb}$$

The next three rules define another calculus, R_2.

$$\frac{p\,,\ q\,,\ \neg q}{\neg p\,,\ p\wedge q}\ r_{\wedge 120} \qquad \frac{p\,,\ q\,,\ \neg q\,,\ p\wedge q}{\neg p\,,\ \neg(p\wedge q)}\ r_{\wedge 121} \qquad \frac{p\,,\ q\,,\ \neg q\,,\ p\wedge q\,,\ \neg(p\wedge q)}{\neg p}\ r_{\wedge 122}$$

△

Inference rules can be used in derivations of conclusions from premises. However, contrarily to the case of Hilbert-style rules where derivations correspond to sequences of formulas resulting from premises by application of instances of rules, in this generalized setting derivations must now have a tree structure [16]. In order to show that Δ follows from Γ using the rules in R one must be able to build a tree starting from formulas in Γ and branching out whenever applying an instance of a rule, in such a way that all branches of the tree finally include some formula of Δ.

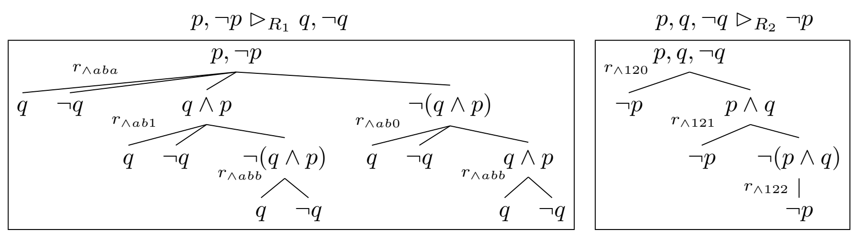
Given a rooted tree t let $<^{\mathsf{t}}$ be the (partial) order induced by t on its nodes by the relation of descendency. We denote: the set of nodes of t as $\mathsf{nodes}^{\mathsf{t}}$; the root of t as $\mathsf{root}^{\mathsf{t}}$ (the minimal element in $<^{\mathsf{t}}$); the set of leaf nodes of t as $\mathsf{leaves}^{\mathsf{t}}$ (the maximal elements in $<^{\mathsf{t}}$); the set of immediate children of node s as $\mathsf{childn}^{\mathsf{t}}(s)$ (the minimal descendants of node s); and the ancestors of node s as $\mathsf{ancest}^{\mathsf{t}}(s) = \{s' : s' <^{\mathsf{t}} s\}$. A tree t is said to be *bounded* if $\mathsf{nodes}^{\mathsf{t}} \setminus \mathsf{leaves}^{\mathsf{t}} = \bigcup_{s\in\mathsf{leaves}^{\mathsf{t}}} \mathsf{ancest}^{\mathsf{t}}(s)$, which means that every branch of the tree has a maximal element (leaf).

We say that a bounded rooted tree t labelled by $\ell : \mathsf{nodes}^{\mathsf{t}} \to \wp(L_\Sigma(P)) \cup \{*\}$ is an *R-derivation* provided that for each node $s \notin \mathsf{leaves}^{\mathsf{t}}$ we have that $\ell(s) \subseteq L_\Sigma(P)$ and there is a rule $\frac{\Gamma}{\Delta} \in R$ and a substitution $\sigma : P \to L_\Sigma(P)$ such that $\Gamma^\sigma \subseteq \ell(s)$ and:

– if $\Delta = \emptyset$ then $\mathsf{childn}^{\mathsf{t}}(s) = \{s^*\}$ and $\ell(s^*) = *$,
– if $\Delta \neq \emptyset$ then $\mathsf{childn}^{\mathsf{t}}(s) = \{s^A : A \in \Delta^\sigma\}$ and each $\ell(s^A) = \ell(s) \cup \{A\}$.

Given $\Gamma, \Delta \subseteq L_\Sigma(P)$, we say that an R-derivation t is a *R-proof* of Δ from Γ whenever $\ell(\mathsf{root}^{\mathsf{t}}) \subseteq \Gamma$ and $\ell(s) \cap \Delta \neq \emptyset$ for every $s \in \mathsf{leaves}^{\mathsf{t}}$ with $\ell(s) \neq *$. Note that leaves labelled by $*$ signal *discontinued* branches of a derivation. It should be noted that whenever R is finitary it is sufficient to consider finite proof trees. We write $\Gamma \rhd_R \Delta$ whenever there exists an R-proof of Δ from Γ.

Example 2. Below, we depict examples of derivations, namely of $p, \neg p \rhd_{R_1} q, \neg q$, and of $p, q, \neg q \rhd_{R_2} \neg p$, using the calculi defined in Ex. 1. Note that we label each child node with only the new formula, instead of the whole set, which can be collected from the labels of its ancestors.



$\triangle$

A *generalized consequence relation* on $L_\Sigma(P)$, or *Scottian*, or *multiple conclusion consequence relation*, after [15,16] is a relation $\rhd \subseteq \wp(L_\Sigma(P)) \times \wp(L_\Sigma(P))$ satisfying the properties below for every $\Gamma, \Delta, \Gamma', \Delta' \subseteq L_\Sigma(P)$:

(O) if $\Gamma \cap \Delta \neq \emptyset$ then $\Gamma \rhd \Delta$,
(D) if $\Gamma \rhd \Delta$ then $\Gamma, \Gamma' \rhd \Delta, \Delta'$,
(C) if $\Gamma, \Omega \rhd \overline{\Omega}, \Delta$ for each $\Omega \subseteq L_\Sigma(P)$, then $\Gamma \rhd \Delta$,
(S) if $\Gamma \rhd \Delta$ then $\Gamma^\sigma \rhd \Delta^\sigma$ for each substitution $\sigma : P \to L_\Sigma(P)$.

Furthermore, if R is finitary then $\rhd_R$ further satisfies the following property for every $\Gamma \subseteq L_\Sigma(P)$:

(F) if $\Gamma \rhd \Delta$ then there exist finite sets $\Gamma_0 \subseteq \Gamma$ and $\Delta_0 \subseteq \Delta$ such that $\Gamma_0 \rhd \Delta_0$.

Property (C) is best known as *cut for sets* or *transitivity*, though we prefer to call it *case exhaustion*. The other properties are usually known as *overlap* (O) or *reflexivity*, *dilution* (D) or *monotonicity*, *substitution invariance* (S) or *structurality*, and *finitariness* (F) (see [15,16,17]).

Proposition 1. *For a calculus R over signature Σ, $\rhd_R$ is the smallest generalized consequence relation on $L_\Sigma(P)$ which contains R.*

Proof. The proof is a straightforward generalization of [16, Theorem 3.5], where the finitary case is dealt with using a property known as *cut for formulas* ($\mathrm{C^F}$), which is equivalent to (C) for finitary consequence relations[1]. $\square$

[1] Cut for formulas demands, for every $\Gamma, \Delta, \{A\} \subseteq L_\Sigma(P)$:
($\mathrm{C^F}$) if $\Gamma, A \rhd \Delta$ and $\Gamma \rhd A, \Delta$ then $\Gamma \rhd \Delta$.

In the context of a given generalized consequence relation $\rhd$, we denote by $\rhd^{\mathrm{T}} = \rhd \cap (\wp(L_\Sigma(P)) \times L_\Sigma(P))$ the Tarskian *companion* of $\rhd$. Recall that, in general, there may be many different generalized consequence relations with exactly the same companion [16].

We will say that a calculus R defines an *axiomatization* of a generalized consequence relation $\rhd$ when $\rhd_R = \rhd$. Of course, in such a case, R can also be used as a calculus for $\rhd^{\mathrm{T}}$.

Fix a signature Σ and a calculus R. Given $\Lambda \subseteq L_\Sigma(P)$ we write[2] $\Gamma \rhd_R^\Lambda \Delta$ when there exists an R-proof of Δ from Γ where all occurring formulas are in Λ. Controlling the possible formulas appearing in a derivation is key to defining a suitable notion of *analyticity* for multiple conclusion axiomatizations.

Let $\Phi \subseteq L_\Sigma(P)$. We say that R is *Φ-analytic* if when $\Gamma \rhd_R \Delta$ then $\Gamma \rhd_R^{\Upsilon_\Phi} \Delta$ with $\Upsilon = \mathsf{sub}(\Gamma \cup \Delta)$ and $\Upsilon_\Phi = \Upsilon \cup \{A^\sigma : A \in \Phi, \sigma : P \to \Upsilon\}$. Intuitively, this means that an R-proof of Δ from Γ needs only to use formulas which are subformulas of $\Gamma \cup \Delta$, or instances of Φ with such subformulas. Hence, formulas in Υ_Φ can be seen as a certain notion of *generalized subformula*. Clearly, a Φ-analytic calculus R is *consistent* (i.e., $\emptyset \not\rhd_R \emptyset$) if and only if the rule $\frac{\emptyset}{\emptyset} \notin R$. Analyticity is even more interesting for finite sets Φ, as in these cases we know that deciding the logic is in coNP, and there is an algorithm for proof-search in $\mathsf{EXPTIME}$ (see the discussion in the conclusion of [14]).

2.2 Logical matrices, non-determinism, partiality, and monadicity

A *partial non-deterministic matrix* $\mathbb{M}$ over a signature Σ, or *Σ-PNmatrix*, is a tuple $\langle V, D, \cdot_{\mathbb{M}} \rangle$ where V is a set (of *truth-values*), $D \subseteq V$ is the set of *designated* values and, for each $k \in \mathbb{N}$ and $© \in \Sigma^{(k)}$, $\cdot_{\mathbb{M}}$ gives the *interpretation function* $©_{\mathbb{M}} : V^k \to \wp(V)$ of $©$ in $\mathbb{M}$. Given $X \subseteq V$ we will use $\overline{X}$ to denote $V \setminus X$. In particular, the values in $\overline{D}$ shall be referred to as *undesignated*. Whenever the interpretation function is always different from the empty set we say the PNmatrix is *total*, or *proper*, or simply say it is an *Nmatrix*. The common deterministic notion of a *logical matrix* is recovered by considering (P)Nmatrices for which the interpretation function always yields a singleton.

Given a PNmatrix $\mathbb{M} = \langle V, D, \cdot_{\mathbb{M}} \rangle$, each set $X \subseteq V$ defines a *sub-PNmatrix (or simple refinement)* of $\mathbb{M}$ defined by $\mathbb{M}_X = \langle X, D_X, \cdot_X \rangle$ with $D_X = D \cap X$ and $©_X(x_1, \dots, x_k) = ©_{\mathbb{M}}(x_1, \dots, x_k) \cap X$ for each $© \in \Sigma^{(k)}$ and $x_1, \dots, x_k \in X$. We denote by $\mathcal{T}_{\mathbb{M}}$ the set of all subsets of the values of each non-empty total sub-PNmatrix of $\mathbb{M}$, that is,

$$\mathcal{T}_{\mathbb{M}} = \bigcup_{\substack{\emptyset \neq X \subseteq V \\ \mathbb{M}_X \text{ total}}} \wp(X).$$

[2] Note that in general $\rhd_R^\Lambda$ is not a generalized consequence relation. It still satisfies properties (D) and (C), but only weaker versions of (O) and (S).

Example 3. The Tarskian consequence relation of the implication-free fragment of Kleene's strong three-valued logic can be defined over a signature with one unary connective $\neg$ and two binary connectives $\wedge, \vee$ by means of two three-valued matrices: those arising from the three-valued chain with only the top element designated, or both non-bottom elements designated [12]. Equivalently, the logic is given by the P(N)matrix $\mathbb{K} = \langle\{0, a, b, 1\}, \{b, 1\}, \cdot_{\mathbb{K}}\rangle$ defined by the following truth-tables, where we omit brackets for non-empty (in this case, singleton) sets.

$\wedge_{\mathbb{K}}$	0	a	b	1
0	0	0	0	0
a	0	a	$\emptyset$	a
b	0	$\emptyset$	b	b
1	0	a	b	1

$\vee_{\mathbb{K}}$	0	a	b	1
0	0	a	b	1
a	a	a	$\emptyset$	1
b	b	$\emptyset$	b	1
1	1	1	1	1

	$\neg_{\mathbb{K}}$
0	1
a	a
b	b
1	0

Note that $\mathcal{T}_{\mathbb{K}} = \{X \subseteq \{0, a, b, 1\} : \{a, b\} \nsubseteq X\}$. The three-valued matrices mentioned above clearly correspond to $\mathbb{K}_{\{0,a,1\}}$ and $\mathbb{K}_{\{0,b,1\}}$, respectively. △

A $\mathbb{M}$-*valuation* is a function $v : L_\Sigma(P) \to V$ such that for each $© \in \Sigma^{(k)}$ and $A_1, \ldots, A_k \in L_\Sigma(P)$ we have $v(©(A_1, \ldots, A_k)) \in ©_{\mathbb{M}}(v(A_1), \ldots, v(A_k))$. Note that this implies that $v(L_\Sigma(P)) \in \mathcal{T}_{\mathbb{M}}$. We extend the interpretation in a PNmatrix $\mathbb{M}$ to any formula $A \in L_\Sigma(P)$ with $\mathsf{var}(A) \subseteq \{p_1 \ldots, p_n\}$ by letting $A_{\mathbb{M}}(x_1, \ldots, x_n) = \{v(A) : v \text{ is an } \mathbb{M}\text{-valuation}, v(p_1) = x_1, \ldots, v(p_n) = x_n\}$.

As is well known, if $\mathbb{M} = \langle V, D, \cdot_{\mathbb{M}}\rangle$ is a matrix then every function $f : Q \to V$ with $Q \subseteq P$ can be extended to a $\mathbb{M}$-valuation (in an essentially unique way for all formulas A with $\mathsf{var}(A) \subseteq Q$). When $\mathbb{M}$ is a Nmatrix, however, we know from [2] that a function $f : \Gamma \to V$ with $\Gamma \subseteq L_\Sigma(P)$ can be extended to a $\mathbb{M}$-valuation provided that $\mathsf{sub}(\Gamma) \subseteq \Gamma$ and that $f(©(A_1, \ldots, A_k)) \in ©_{\mathbb{M}}(f(A_1), \ldots, f(A_n))$ whenever $©(A_1, \ldots, A_k) \in \Gamma$. In case $\mathbb{M}$ is a PNmatrix, in general, one does not even have such a guarantee, unless $f(\Gamma) \in \mathcal{T}_{\mathbb{M}}$ [5] (take, for instance, $\Gamma = \{p, q\}$ and $f(p) = a, f(q) = b$ in the PNmatrix $\mathbb{K}$ of Ex. 3).

Every $\mathbb{M}$-valuation v defines a set $\Omega_v \subseteq L_\Sigma(P)$ with $\Omega_v = \{A : v(A) \in D\}$. Of course, it follows that $\overline{\Omega_v} = \{A : v(A) \notin D\}$. Let $\Gamma, \Delta \subseteq L_\Sigma(P)$ be arbitrary sets of formulas. We write $\Gamma \rhd_{\mathbb{M}} \Delta$ if every $\mathbb{M}$-valuation v is such that $\Gamma \cap \overline{\Omega_v} \neq \emptyset$ or $\Delta \cap \Omega_v \neq \emptyset$. It is well known that $\rhd_{\mathbb{M}}$ is a generalized consequence relation, and $\rhd^{\mathrm{T}}_{\mathbb{M}}$ the usual Tarskian consequence relation defined from a (partial) (non-deterministic) matrix. If $\mathbb{M}$ is finite (i.e., its underlying set of truth-values is finite) then $\rhd_{\mathbb{M}}$ and $\rhd^{\mathrm{T}}_{\mathbb{M}}$ are known to be finitary. Every Tarskian, or Scottian consequence relation is known to be characterized by a set of logical matrices [17,16] (as usual, as the intersection of the consequence relations characterized by each of the matrices). Still, only logics satisfying *cancellation* can be given by a single logical matrix [17]. Easily, every logic can be given by a single PNmatrix, as one can use partiality to merge a set of matrices (or Nmatrices) into a single PNmatrix, as in Ex. 3 above. This ability of PNmatrices adds to the power of non-determinism already present in Nmatrices. In [7], we have completely characterized those Tarskian logics definable by finitely many finite matrices. However, there are logics which cannot be defined by finitely many finite matrices but can still be defined by one finite Nmatrix [1,13].

When axiomatizing the consequence relation determined by a PNmatrix $\mathbb{M}$, we say that a set of rules R is *sound (with respect to* $\mathbb{M}$*)* if $\rhd_R \subseteq \rhd_{\mathbb{M}}$. This means that every $\mathbb{M}$-valuation v *respects* the rules of R, in the sense that for every rule $\frac{\Gamma}{\Delta} \in R$ we have that $\Gamma \cap \overline{\Omega_v} \neq \emptyset$ or $\Delta \cap \Omega_v \neq \emptyset$. Conversely, we say that R is *complete (with respect to* $\mathbb{M}$*)* if $\rhd_{\mathbb{M}} \subseteq \rhd_R$. This means that if $\Gamma \not\rhd_R \Delta$ then there exists a $\mathbb{M}$-valuation v such that $\Gamma \subseteq \Omega_v$ and $\Delta \subseteq \overline{\Omega_v}$. Soundess and completeness jointly imply $\rhd_R = \rhd_{\mathbb{M}}$.

Fix a signature Σ and a Σ-Nmatrix $\mathbb{M} = \langle V, D, \cdot_{\mathbb{M}} \rangle$. We want to use the resources of the logic to distinguish between the different truth-values. Namely, we require that the syntax of the logic, granted the shadow of bivalence present in the contrast between designated and undesignated values, is enough to distinguish among the truth-values [16,8,14]. A pair of non-empty sets of elements $\emptyset \neq X, Y \subseteq V$ are *separated*, $X \# Y$, if $X \subseteq D$ and $Y \subseteq \overline{D}$, or vice versa. A formula S with $\mathsf{var}(S) \subseteq \{p\}$ such that $S_{\mathbb{M}}(x) \# S_{\mathbb{M}}(y)$ is said to *separate* x and y, and called a *monadic separator* for $\mathbb{M}$. The PNmatrix $\mathbb{M}$ is said to be *monadic* if there is a monadic separator for every pair of distinct elements of V.

3 Axiomatizing monadic PNmatrices

We extend to PNmatrices the results obtained in [14] about the construction of analytic calculi for monadic Nmatrices. Granted a monadic PNmatrix $\mathbb{M} = \langle V, D, \cdot_{\mathbb{M}} \rangle$ and some set $\Theta = \{S^{xy} : x, y \in V, x \neq y\}$ of monadic separators for $\mathbb{M}$ such that each S^{xy} separates x and y, a *discriminator* for $\mathbb{M}$ is the V-indexed family $\widetilde{\Theta} = \{\widetilde{\Theta}_x\}_{x \in V}$, with each $\widetilde{\Theta}_x = \{S^{xy} : y \in V \setminus \{x\}\}$. Each $\widetilde{\Theta}_x$ is naturally partitioned into $\Omega_x = \{S \in \widetilde{\Theta}_x : S_{\mathbb{M}}(x) \subseteq D\}$ and $\mho_x = \{S \in \widetilde{\Theta}_x : S_{\mathbb{M}}(x) \subseteq \overline{D}\}$. This partition is easily seen to characterize precisely the truth-values of $\mathbb{M}$.

Lemma 1. *Let $\mathbb{M} = \langle V, D, \cdot_{\mathbb{M}} \rangle$ be a monadic Σ-PNmatrix with discriminator $\widetilde{\Theta}$. For every $\mathbb{M}$-valuation v, $x \in V$ and $A \in L_\Sigma(P)$, we have*

$$v(A) = x \text{ if and only if } v(\Omega_x(A)) \subseteq D \text{ and } v(\mho_x(A)) \subseteq \overline{D}.$$

Proof. Let $v(A) = x$. For each $S \in \widetilde{\Theta}_x$, clearly, we have that $v(S(A)) \in S_{\mathbb{M}}(v(A)) = S_{\mathbb{M}}(x) \subseteq D$ if and only if $S \in \Omega_x$.

Now, let $v(A) = y \neq x$ and consider $S^{xy} \in \widetilde{\Theta}_x$. Since $S^{xy}_{\mathbb{M}}(x) \# S^{xy}_{\mathbb{M}}(y)$, it follows that $v(S^{xy}(A)) \in S^{xy}_{\mathbb{M}}(v(A)) = S^{xy}_{\mathbb{M}}(y) \subseteq D$ if and only if $S^{xy}_{\mathbb{M}}(x) \subseteq \overline{D}$ if and only if $S^{xy} \in \mho_x$. □

Given $X \subseteq V$, let Ω^*_X denote any set built by choosing one element from each Ω_x for $x \in X$, and $\mho^*_X$ denote any set built by choosing one element from each $\mho_x$ for $x \in X$. In particular, if $X = \emptyset$ then $\Omega^*_X = \mho^*_X = \emptyset$ are the only possibilities. On the other hand, if for some $x \in X$ one has $\Omega_x = \emptyset$ then there is no possible choice for Ω^*_X, and similarly there is no possible choice for $\mho^*_X$ whenever $\mho_x = \emptyset$ for some $x \in X$.

Example 4. The PNmatrix $\mathbb{K}$ introduced in Example 3 is monadic. Indeed we have that $\Theta = \{p, \neg p\}$ is a set separators for $\mathbb{K}$, and setting $\widetilde{\Theta}_0 = \widetilde{\Theta}_a = \widetilde{\Theta}_b = \widetilde{\Theta}_1 = \Theta$ defines a discriminator for $\mathbb{K}$. In this case we have that

x	Ω_x	$\mho_x$
0	$\{\neg p\}$	$\{p\}$
a	$\emptyset$	$\{p, \neg p\}$
b	$\{p, \neg p\}$	$\emptyset$
1	$\{p\}$	$\{\neg p\}$

We also have that $\Omega^*_{\{0\}} = \mho^*_{\{1\}} = \{\neg p\}$ and $\Omega^*_{\{1\}} = \mho^*_{\{0\}} = \{p\}$. Furthermore, $\Omega^*_{\{b\}}$ has two possible values, either $\Omega^*_{\{b\}} = \{p\}$ or $\Omega^*_{\{b\}} = \{\neg p\}$. Similarly, $\mho^*_{\{a\}}$ also has the same two possible values. On the contrary, there is no possible choice for $\Omega^*_{\{a\}}$ (nor for Ω^*_X if $a \in X$) or $\mho^*_{\{b\}}$ (nor for $\mho^*_X$ if $b \in X$). △

We now define a set of inference rules respected by any monadic PNmatrix.

Definition 1. *Let $\mathbb{M} = \langle V, D, \cdot_{\mathbb{M}} \rangle$ be a monadic PNmatrix, $\widetilde{\Theta}$ a discriminator for $\mathbb{M}$. We define the set of rules $R^{\widetilde{\Theta}}_{\mathbb{M}} = R_\exists \cup R_{\mathsf{D}} \cup R_\Sigma \cup R_\mathcal{T}$ as follows:*

- *$R_\exists$ contains, for each $X \subseteq V$ and each possible $\mho^*_X$ and $\Omega^*_{\overline{X}}$, the rule*

$$\frac{\mho^*_X(p)}{\Omega^*_{\overline{X}}(p)}$$

- *R_{D} contains, for each $x \in V$, the rule*

$$\frac{\Omega_x(p)}{p, \mho_x(p)} \text{ if } x \in D \qquad \frac{\Omega_x(p), p}{\mho_x(p)} \text{ if } x \notin D$$

- *$R_\Sigma = \bigcup_{©\in\Sigma} R_©$ where, for $© \in \Sigma^{(k)}$, $R_©$ contains, for each $x_1, \ldots, x_k \in V$ and $y \notin ©_{\mathbb{M}}(x_1, \ldots, x_k)$, the rule*

$$\frac{\bigcup_{1\le i\le k} \Omega_{x_i}(p_i)\,,\ \Omega_y(©(p_1 \ldots, p_k))}{\bigcup_{1\le i\le k} \mho_{x_i}(p_i)\,,\ \mho_y(©(p_1 \ldots, p_k))}$$

- *$R_\mathcal{T}$ contains, for each $X \subseteq V$ with $X \notin \mathcal{T}_{\mathbb{M}}$, the rule*

$$\frac{\bigcup_{x_i\in X} \Omega_{x_i}(p_i)}{\bigcup_{x_i\in X} \mho_{x_i}(p_i)}$$

Note that the rules above form a finite collection of finite rules whenever Θ is finite, which is always possible for finite $\mathbb{M}$ over finite Σ. The number of propositional variables used in the inference rules $R^{\widetilde{\Theta}}_{\mathbb{M}} \setminus R_\mathcal{T}$ is $k+1$ where k is the maximum arity of a connective in Σ, when it exists. Further, the number of variables in $R_\mathcal{T}$ is bounded by the number of values of $\mathbb{M}$.

Note also that, often, many of the rules obtained by this general process are useless (e.g., in the sense that they are instances of overlap), or can be substantially simplified, or are simply derivable from other rules.

Example 5. Recall the PNmatrix $\mathbb{K}$ introduced in Ex. 3 and its descriminator $\widetilde{\Theta}$ from Ex. 4. A simplified version of the axiomatization $R^{\widetilde{\Theta}}_{\mathbb{K}}$ consists of the following rules.

$$\frac{p\,,\,q}{p\wedge q}\,r_1 \quad \frac{p\wedge q}{p}\,r_2 \quad \frac{p\wedge q}{q}\,r_3 \quad \frac{\neg p}{\neg(p\wedge q)}\,r_4 \quad \frac{\neg q}{\neg(p\wedge q)}\,r_5 \quad \frac{\neg(p\wedge q)}{\neg p\,,\,\neg q}\,r_6$$

$$\frac{p}{p\vee q}\,r_7 \quad \frac{q}{p\vee q}\,r_8 \quad \frac{\neg(p\vee q)}{\neg p}\,r_9 \quad \frac{\neg(p\vee q)}{\neg q}\,r_{10} \quad \frac{\neg p\,,\,\neg q}{\neg(p\vee q)}\,r_{11} \quad \frac{p\vee q}{p\,,\,q}\,r_{12}$$

$$\frac{p}{\neg\neg p}\,r_{13} \quad \frac{\neg\neg p}{p}\,r_{14} \quad \frac{p\,,\,\neg p}{q\,,\,\neg q}\,r_{15}$$

Note that every rule resulting from $R_\exists$ and R_D is a case of overlap and was omitted. After simplification, the rules r_1–r_6 correspond to $R_\wedge$, r_7–r_{12} to $R_\vee$, r_{13}–r_{14} to $R_\neg$, and r_{15} results from $R_\mathcal{T}$ (with $X = \{a, b\}$).

Notice that the four rules of the calculus R_1 introduced in Ex. 1 (where each $r_{\wedge aby}$ corresponds to $R_\wedge$ for $y \notin (a \wedge_\mathbb{K} b) = \emptyset$) have been omitted, as they are easily derivable from r_{15}. Several other innocuous simplifications have been applied. △

It is not hard to understand in general that the rules proposed in Definition 1 capture the behaviour of the given PNmatrix $\mathbb{M}$. Namely, $R_\exists$ allows one to exclude combinations of separators that do not correspond to truth-values, R_D distinguishes those combinations of separators that characterize designated values from those that characterize undesignated values, R_Σ completely determines the interpretation of connectives in $\mathbb{M}$. The novelty with respect to [14] consists in the rules $R_\mathcal{T}$, which do not apply to Nmatrices, as they guarantee that values are taken within a total sub-PNmatrix of $\mathbb{M}$. The following results rigorously capture these intuitions.

Proposition 2. *Given a monadic PNmatrix $\mathbb{M} = \langle V, D, \cdot_\mathbb{M}\rangle$ with discriminator $\widetilde{\Theta}$, $R^{\widetilde{\Theta}}_{\mathbb{M}}$ is a calculus sound with respect to $\mathbb{M}$.*

Proof. We show that every $\mathbb{M}$-valuation v respects the rules of $R^{\widetilde{\Theta}}_{\mathbb{M}} = R_\exists \cup R_\mathsf{D} \cup R_\Sigma \cup R_\mathcal{T}$. For rules of each type, we show that if v fails to respect a rule then a contradiction can be obtained. Lemma 1 is instrumental, in all cases.

$R_\exists$: If (i) $v(\mho^*_X(p)) \subseteq D$ and (ii) $v(\Omega^*_X(p)) \subseteq \overline{D}$, then it easily follows that (i) for each $x \in X$ there is $y \neq x$ with $S^{xy} \in \mho_x$ and $v(S^{xy}(p)) \in D$ and (ii) for each $x \in \overline{X}$ there is $y \neq x$ with $S^{xy} \in \Omega_x$ and $v(S^{xy}(p)) \in \overline{D}$, and thus Lemma 1 guarantees that (i) $v(p) \notin X$ and (ii) $v(p) \notin \overline{X}$, which contradicts the fact that $v(p) \in V = X \cup \overline{X}$.

R_D: If we have $v(\Omega_x(p)) \subseteq D$ and $v(\mho_x(p)) \subseteq \overline{D}$ then Lemma 1 guarantees that $v(p) = x$, therefore $v(p) \in \overline{D}$ if and only if $x \in D$ is a contradiction.

R_Σ**:** If $v(\Omega_{x_i}(p_i)) \subseteq D$ and $v(\mho_{x_i}(p_i)) \subseteq \overline{D}$ then Lemma 1 guarantees that $v(p_i) = x_i$ for each $1 \leq i \leq k$, further, if $v(\Omega_y(©(p_1 \dots, p_k))) \subseteq D$ and $v(\mho_y(©(p_1 \dots, p_k))) \subseteq \overline{D}$ then Lemma 1 guarantees that $v(©(p_1 \dots, p_k)) = y$, and thus $y = v(©(p_1 \dots, p_k)) \in ©_{\mathbb{M}}(v(p_1), \dots, v(p_k)) = ©_{\mathbb{M}}(x_1, \dots, x_k)$ which contradicts the fact that $y \notin ©_{\mathbb{M}}(x_1, \dots, x_k)$.

$R_\mathcal{T}$**:** If $v(\Omega_{x_i}(p_i)) \subseteq D$ and $v(\mho_{x_i}(p_i)) \subseteq \overline{D}$ then Lemma 1 guarantees that $v(p_i) = x_i$ for each $x_i \in X$, therefore $X \subseteq v(L_\Sigma(P)) \in \mathcal{T}_{\mathbb{M}}$ which contradicts the fact that $X \notin \mathcal{T}_{\mathbb{M}}$. □

Having established the soundness of the calculi $R_{\mathbb{M}}^{\widetilde{\Theta}}$, we now proceed to prove their completeness and analyticity. We first need another auxiliary result.

Lemma 2. *Let $\mathbb{M} = \langle V, D, \cdot_{\mathbb{M}} \rangle$ be a monadic PNmatrix, $\widetilde{\Theta}$ a discriminator for $\mathbb{M}$, and $R_{\mathbb{M}}^{\widetilde{\Theta}} = R_\exists \cup R_\mathsf{D} \cup R_\Sigma \cup R_\mathcal{T}$. For every $\Omega, \Upsilon \subseteq L_\Sigma(P)$ with $\mathsf{sub}(\Upsilon) \subseteq \Upsilon$, we have:*

- **(a)** *if $\Omega \not\rhd_{R_\exists}^{\Upsilon_\Theta} \overline{\Omega}$ then for every $A \in \Upsilon$ there is $x \in V$ such that $\Omega_x(A) \subseteq \Omega$ and $\mho_x(A) \subseteq \overline{\Omega}$,*
- **(b)** *if $\Omega \not\rhd_{R_\mathsf{D}}^{\Upsilon_\Theta} \overline{\Omega}$ then for every $A \in \Upsilon$ and $x \in V$ with $\Omega_x(A) \subseteq \Omega$ and $\mho_x(A) \subseteq \overline{\Omega}$, we have that $x \in D$ iff $A \in \Omega$,*
- **(c)** *if $\Omega \not\rhd_{R_\Sigma}^{\Upsilon_\Theta} \overline{\Omega}$ then for every $© \in \Sigma^{(k)}$, $A = ©(A_1, \dots, A_k) \in \Upsilon$ and $x_1, \dots, x_k \in V$ with $\Omega_{x_i}(A_i) \subseteq \Omega$ and $\mho_{x_i}(A_i) \subseteq \overline{\Omega}$ for each $1 \leq i \leq k$, we have that $\Omega_y(A) \subseteq \Omega$ and $\mho_y(A) \subseteq \overline{\Omega}$ implies $y \in ©_{\mathbb{M}}(x_1, \dots, x_k)$,*
- **(d)** *if $\Omega \not\rhd_{R_\mathcal{T}}^{\Upsilon_\Theta} \overline{\Omega}$ then $\{x \in V : \Omega_x(A) \subseteq \Omega, \mho_x(A) \subseteq \overline{\Omega}$ for $A \in \Upsilon\} \in \mathcal{T}_{\mathbb{M}}$.*

Proof. We prove each of the items.

(a) Assume that for some $A \in \Upsilon$ there is no $x \in V$ with $\Omega_x(A) \subseteq \Omega$ and $\mho_x(A) \subseteq \overline{\Omega}$. Then, we can consider $X = \{x \in V : \mho_x(A) \cap \Omega \neq \emptyset\}$, and $\overline{X} = V \setminus X$. Define $\mho_X^*$ by choosing some $S \in \mho_x$ such that $S(A) \in \mho_x(A) \cap \Omega$ for each $x \in X$, and $\Omega_{\overline{X}}^*$ by choosing some $S \in \Omega_x$ such that $S(A) \in \Omega_x(A) \cap \overline{\Omega}$ for each $x \in \overline{X}$. We have that $\mho_X^*(A) \subseteq \Omega \cap \Upsilon_\Theta$ and $\Omega_{\overline{X}}^*(A) \subseteq \overline{\Omega} \cap \Upsilon_\Theta$. Hence, $\Omega \rhd_{R_\exists}^{\Upsilon_\Theta} \overline{\Omega}$.

(b) Assume that there is $A \in \Upsilon$ such that $\Omega_x(A) \subseteq \Omega$, $\mho_x(A) \subseteq \overline{\Omega}$ and $x \in D$. Then $\Omega_x(A) \subseteq \Omega \cap \Upsilon_\Theta$ and $\mho_x(A) \subseteq \overline{\Omega} \cap \Upsilon_\Theta$. Hence, $\Omega \rhd_{R_\mathsf{D}}^{\Upsilon_\Theta} \overline{\Omega}$. The case where $x \notin D$ is analogous.

(c) Assume that there is $A = ©(A_1, \dots, A_k) \in \Upsilon$, $\Omega_{x_i}(A_i) \subseteq \Omega$ and $\mho_{x_i}(A_i) \subseteq \overline{\Omega}$ for $1 \leq i \leq k$, and for some $y \notin ©_{\mathbb{M}}(x_1 \dots, x_n)$ we have $\Omega_y(A) \subseteq \Omega$ and $\mho_y(A) \subseteq \overline{\Omega}$. Then $\bigcup\limits_{1 \leq i \leq k} \Omega_{x_i}(p_i) \cup \Omega_y(©(p_1 \dots, p_k)) \subseteq \Omega \cap \Upsilon_\Theta$ and $\bigcup\limits_{1 \leq i \leq k} \mho_{x_i}(p_i) \cup \mho_y(©(p_1 \dots, p_k)) \subseteq \overline{\Omega} \cap \Upsilon_\Theta$. Hence, $\Omega \rhd_{R_\Sigma}^{\Upsilon_\Theta} \overline{\Omega}$.

(d) Let $X = \{x \in V : \Omega_x(A) \subseteq \Omega, \mho_x(A) \subseteq \overline{\Omega}$ for $A \in \Upsilon\}$. For each $x_i \in X$ pick $A_i \in \Upsilon$ such that $\Omega_{x_i}(A_i) \subseteq \Omega$ and $\mho_{x_i}(A_i) \subseteq \overline{\Omega}$. Easily, then, $\bigcup\limits_{x_i \in X} \Omega_{x_i}(A_i) \subseteq \Omega \cap \Upsilon_\Theta$ and $\bigcup\limits_{x_i \in X} \mho_{x_i}(A_i) \subseteq \overline{\Omega} \cap \Upsilon_\Theta$, and $\Omega \rhd_{R_\mathcal{T}}^{\Upsilon_\Theta} \overline{\Omega}$ if $X \notin \mathcal{T}_{\mathbb{M}}$.

□

With Proposition 2 and Lemma 2 in hand, it is relatively straightforward to show that $R^{\widetilde{\Theta}}_{\mathbb{M}}$ is a Θ-analytic calculus that provides an axiomatization of the generalized consequence relation determined by $\mathbb{M}$.

Theorem 1. *Given a monadic PNmatrix $\mathbb{M} = \langle V, D, \cdot_{\mathbb{M}}\rangle$ with discriminator $\widetilde{\Theta}$, $R^{\widetilde{\Theta}}_{\mathbb{M}}$ is a Θ-analytic axiomatization of $\rhd_{\mathbb{M}}$.*

Proof. Let $R = R^{\widetilde{\Theta}}_{\mathbb{M}} = R_{\exists} \cup R_{\mathsf{D}} \cup R_{\Sigma} \cup R_{\mathcal{T}}$. Soundness follows from Proposition 2, that is, $\rhd_R \subseteq \rhd_{\mathbb{M}}$. Let us detail the analytic completeness part.

Given $\Gamma, \Delta \subseteq L_{\Sigma}(P)$, it is clear that $\rhd_R^{\Upsilon_{\Theta}} \subseteq \rhd_R \subseteq \rhd_{\mathbb{M}}$ for $\Upsilon = \mathsf{sub}(\Gamma \cup \Delta)$. We show that if $\Gamma \not\rhd_R^{\Upsilon_{\Theta}} \Delta$ then $\Gamma \not\rhd_{\mathbb{M}} \Delta$. Knowing that $\Gamma \not\rhd_R^{\Upsilon_{\Theta}} \Delta$, by property (C), we get that there is $\Omega \subseteq L_{\Sigma}(P)$ such that $\Gamma, \Omega \not\rhd_R^{\Upsilon_{\Theta}} \Delta, \overline{\Omega}$. Now, using Lemma 2 **(a)**, **(b)** and **(c)**, one can build a function $f : \Upsilon \to V$ with $f(A) \in D$ iff $A \in \Omega$, and such that $f(©(A_1, \dots, A_k)) \in ©_{\mathbb{M}}(f(A_1), \dots, f(A_k))$ whenever $©(A_1, \dots, A_k) \in \Upsilon$. At last, Lemma 2 **(d)** guarantees that $f(\Upsilon) \in \mathcal{T}_{\mathbb{M}}$, and we conclude that f can be extended to a full $\mathbb{M}$-valuation and thus $\Gamma \not\rhd_{\mathbb{M}} \Delta$. □

We must emphasize here that the $R_{\mathcal{T}}$ rules play no role when we are interested in proving just the completeness of $R^{\widetilde{\Theta}}_{\mathbb{M}}$. Indeed, taking $\Upsilon = L_{\Sigma}(P)$, we can use Lemma 2 **(a)**, **(b)** and **(c)**, which only depend on $R_{\exists} \cup R_{\mathsf{D}} \cup R_{\Sigma}$, and directly obtain a valuation over $\mathbb{M}$. The precise role of the $R_{\mathcal{T}}$ rules can be made clearer. In [14], we showed that the local demands necessary in order to guarantee the extension of valuation functions to full valuations over Nmatrices, were sufficient to show that the axiomatization $R = R^{\widetilde{\Theta}}_{\mathbb{M}} \setminus R_{\mathcal{T}}$ would grant an Θ-*analytic* calculus for $\rhd_{\mathbb{M}}$. Expectedly, this property may not be true, in general, if $\mathbb{M}$ is a PNmatrix. At this point, we could simply have adopted the strategy delineated in [11], decomposing the given PNmatrix into its total sub-Nmatrices, then providing analytic calculi for each of them, and finally using these calculi together in order to deal with $\rhd_{\mathbb{M}}$. But we could do better. The rules in $R_{\mathcal{T}}$ are sound, and therefore they must be derivable in $R^{\widetilde{\Theta}}_{\mathbb{M}} \setminus R_{\mathcal{T}}$. Ex. 2, on the left, examplifies precisely this fact, given the explanation in Ex. 5 above. In general, however, the derivation we manage to obtain is not Θ-analytical. In the example, we see that in order to obtain the derivation we need to use the rules of the connectives that lead to the relevant partial entry of the PNmatrix (just conjunction, in this case), thus loosing the generalized subformula property. However, notably, adding $R_{\mathcal{T}}$ to the axiomatization restored analyticity.

4 A detailed example

In this section we consider as a full fledge example a *logic of formal inconsistency* [9], over a signature containing two single unary connective $\neg$ and $\circ$ and three binary connectives $\wedge$, $\vee$ and $\to$. Namely, we take the logic resulting from adding the axioms $\circ p \to \circ(p \wedge q)$ and $(\neg p \vee \neg q) \to \neg(p \wedge q)$ to the basic logic of formal inconsistency $\mathcal{KB}$ [4,10].

As shown in [11], the logic is characterized by the three-valued PNmatrix $\mathbb{P} = \langle \{0,1,2\}, \{1,2\}, \cdot_{\mathbb{P}} \rangle$ defined by the following truth-tables (again we omit brackets for non-empty sets.).

$\wedge_{\mathbb{P}}$	0	1	2
0	0	0	0
1	0	1	$\emptyset$
2	0	2	2

$\vee_{\mathbb{P}}$	0	1	2
0	0	1, 2	1, 2
1	1, 2	1, 2	1, 2
2	1, 2	1, 2	1, 2

$\to_{\mathbb{P}}$	0	1	2
0	1, 2	1, 2	1, 2
1	0	1, 2	1, 2
2	0	1, 2	1, 2

	$\neg_{\mathbb{P}}$	$\circ_{\mathbb{P}}$
0	1, 2	1, 2
1	0	1, 2
2	1, 2	0

It is clear that $\mathcal{T}_{\mathbb{P}} = \{X \subseteq \{0,1,2\} : \{1,2\} \not\subseteq X\}$. This happens because there is an empty entry in the truth-tables of $\mathbb{P}$, namely $1 \wedge_{\mathbb{P}} 2 = \emptyset$, which implies that no $\mathbb{P}$-valuation v can have $v(A) = 1$ and $v(B) = 2$ for $A, B \in L_{\Sigma}(P)$. Thus, all the truth-table entries corresponding to applications of any of the binary connectives $\wedge, \vee, \to$ to a pair of values formed with 1 and 2 are irrelevant and could be empty as well, namely $2 \wedge_{\mathbb{P}} 1, 1 \vee_{\mathbb{P}} 2, 2 \vee_{\mathbb{P}} 1, 1 \to_{\mathbb{P}} 2, 2 \to_{\mathbb{P}} 1$. This would not change the logic, but would potentially introduce subtle differences in the rules obtained directly by our method.

It is easy to see that $\Theta = \{p, \neg p\}$ is a set of separators for $\mathbb{P}$, and therefore it is monadic. Furthermore, setting $\widetilde{\Theta}_0 = \{p\}$ and $\widetilde{\Theta}_1 = \widetilde{\Theta}_2 = \{p, \neg p\}$ defines a discriminator for $\mathbb{P}$, which yields the following partitions:

x	Ω_x	$\mho_x$
0	$\emptyset$	$\{p\}$
1	$\{p\}$	$\{\neg p\}$
2	$\{p, \neg p\}$	$\emptyset$

Note that there is no possible choice for Ω^*_X if $0 \in X$, and also no possible choice for $\mho^*_X$ if $2 \in X$. Applying Definition 1 we obtain, after simplification of the axiomatization $R^{\widetilde{\Theta}}_{\mathbb{P}}$, the following inference rules.

$$\frac{p\,,\,q}{p \wedge q}\,r_1 \qquad \frac{p \wedge q}{p}\,r_2 \qquad \frac{p \wedge q}{q}\,r_3 \qquad \frac{\neg p}{\neg(p \wedge q)}\,r_4$$

$$\frac{p}{p \vee q}\,r_5 \qquad \frac{q}{p \vee q}\,r_6 \qquad \frac{p \vee q}{p\,,\,q}\,r_7 \qquad \frac{p\,,\,p \to q}{q}\,r_8 \qquad \frac{q}{p \to q}\,r_9 \qquad \frac{}{p\,,\,p \to q}\,r_{10}$$

$$\frac{}{p\,,\,\circ p}\,r_{11} \qquad \frac{p}{\neg p\,,\,\circ p}\,r_{12} \qquad \frac{p\,,\,\neg p\,,\,\circ p}{}\,r_{13} \qquad \frac{}{p, \neg p}\,r_{14} \qquad \frac{p\,,\,q\,,\,\neg q}{\neg p}\,r_{15}$$

Note that every rule resulting from $R_{\exists}$ and R_{D} is a case of overlap and was omitted. After simplification, the rules r_1–r_4 correspond to $R_{\wedge}$, r_5–r_7 to $R_{\vee}$, r_8–r_{10} to $R_{\to}$, r_{11}–r_{13} to $R_{\circ}$, and r_{14} to $R_{\neg}$, whereas r_{15} results from $R_{\mathcal{T}}$ (with $X = \{1,2\}$).

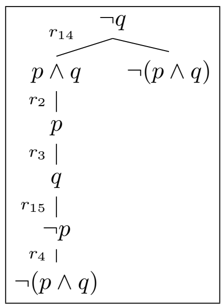
Of course, several innocuous (Θ-analytical) simplifications have been applied. For instance, the three rules of the calculus R_2 introduced in Ex. 1 (where each $r_{\wedge 12y}$ corresponds to $R_{\wedge}$ for $y \notin (1 \wedge_{\mathbb{P}} 2) = \emptyset$) are all subsumed by r_{15}. More interestingly, as already explained in regard to the previous example, rule r_{15} is derivable from the other three rules too, as shown in Ex. 2, on the right. That derivation is, of course, not Θ-analytical.

It is interesting to see that the following two additional, useful, rules

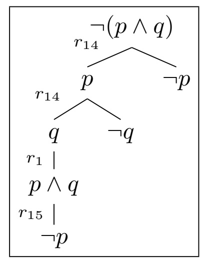
$$\frac{\neg q}{\neg(p \wedge q)}\ r_{16} \qquad\qquad \frac{\neg(p \wedge q)}{\neg p, \neg q}\ r_{17}$$

can be analytically derived from the others.

$\neg q \rhd_{\mathbb{P}} \neg(p \wedge q)$

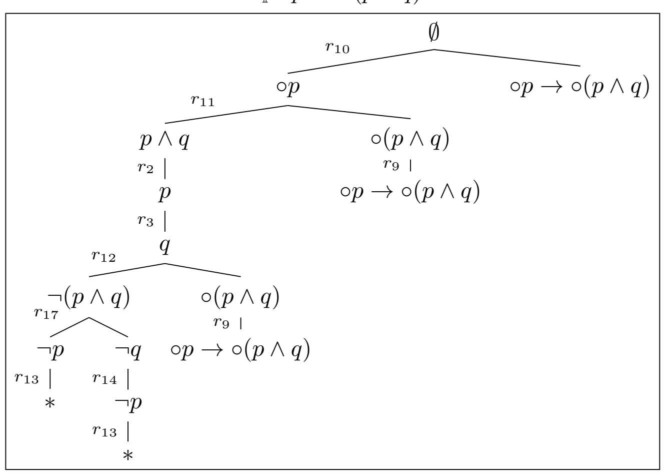


$\neg(p \wedge q) \rhd_{\mathbb{P}} \neg p, \neg q$

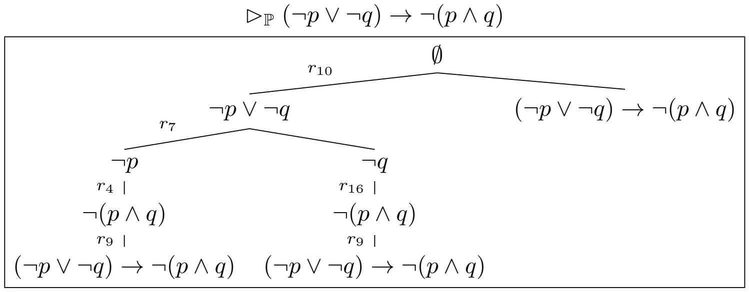


Finally, we present derivations of the two intended axioms.

$\rhd_{\mathbb{P}} \circ p \to \circ(p \wedge q)$



$\rhd_{\mathbb{P}} (\neg p \vee \neg q) \to \neg(p \wedge q)$



5 Conclusion and future work

In this paper we have shown the usefulness of multiple conclusion calculi, namely by proving that one can mechanically obtain analytic calculi for any given monadic PNmatrix. The monadicity requirement is fundamental, here, and corresponds to the *sufficient expressiveness* used in [3,11,8]. However, there is possibly still some room for improvement. Shoesmith and Smiley in [16] also used monadicity for logical matrices, but then showed that a more general notion of separability, using parameters, (readily available for reduced matrices) would suffice. We believe that a deeper understanding of what it means to reduce (P)Nmatrices, as well as how to deal with parameters, may help to generalize the present results.

Continued exploration of further compositional results that may be covered by these techniques is important. Still, we can identify two relatively obvious topics of further work: a detailed implementation of our methods and related proof search and decidability algorithms; and a deeper study of the relationship between multiple conclusion calculi and sequent-calculi, that may render our methods useful in designing analytic sequent-calculi, even when their semantics is not given by proper PNmatrices.

References

1. A. Avron and I. Lev. Non-deterministic multiple-valued structures. *Journal of Logic and Computation*, 15(3):241–261, 2005.
2. A. Avron and A. Zamansky. Non-deterministic semantics for logical systems. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic*, volume 16, pages 227–304. Springer, 2011.
3. Arnon Avron, Jonathan Ben-Naim, and Beata Konikowska. Cut-free ordinary sequent calculi for logics having generalized finite-valued semantics. *Logica Universalis*, 1(1):41–70, 2007.
4. Arnon Avron, Beata Konikowska, and Anna Zamansky. Cut-free sequent calculi for C-systems with generalized finite-valued semantics. *Journal of Logic and Computation*, 23(3):517–540, 2013.
5. M. Baaz, O. Lahav, and A. Zamansky. Finite-valued semantics for canonical labelled calculi. *Journal of Automated Reasoning*, 51(4):401–430, 2013.
6. C. Caleiro, S. Marcelino, and J. Marcos. Combining fragments of classical logic: When are interaction principles needed? *Soft Computing*, 23(7):2213–2231, 2019.
7. C. Caleiro, S. Marcelino, and U. Rivieccio. Characterizing finite-valuedness. *Fuzzy Sets and Systems*, 345:113–125, 2018.
8. C. Caleiro, J. Marcos, and M. Volpe. Bivalent semantics, generalized compositionality and analytic classic-like tableaux for finite-valued logics. *Theoretical Computer Science*, 603:84–110, 2015.
9. W. Carnielli, M. Coniglio, and J. Marcos. Logics of formal inconsistency. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic*, volume 14. Kluwer, 2007.
10. W. Carnielli and J. Marcos. A taxonomy of C-systems. In W. Carnielli, M. Coniglio, and I. D'Ottaviano, editors, *Paraconsistency: The logical way to the inconsistent*,

volume 228 of *Lecture Notes in Pure and Applied Mathematics*, pages 1–94. Marcel Dekker, 2002.
11. Agata Ciabattoni, Ori Lahav, Lara Spendier, and Anna Zamansky. Taming paraconsistent (and other) logics: An algorithmic approach. *ACM Transactions on Computational Logic*, 16(1):5:1–5:23, December 2014.
12. J. Font. Belnap's Four-Valued Logic and De Morgan Lattices. *Logic Journal of the IGPL*, 5(3):1–29, 05 1997.
13. S. Marcelino and C. Caleiro. Disjoint fibring of non-deterministic matrices. In R. de Queiroz J. Kennedy, editor, *Logic, Language, Information and Computation (WoLLIC 2017)*, volume 10388 of *LNCS*, pages 242–255. Springer, 2017.
14. S. Marcelino and C. Caleiro. Axiomatizing non-deterministic many-valued generalized consequence relations. *Synthese, doi 10.1007/s11229-019-02142-8*, to appear.
15. D. Scott. Completeness and axiomatizability in many-valued logic. In L. Henkin, J. Addison, C. Chang, W. Craig, D. Scott, and R. Vaught, editors, *Proceedings of the Tarski Symposium*, volume XXV of *Proceedings of Symposia in Pure Mathematics*, pages 411–435. American Mathematical Society, 1974.
16. D. Shoesmith and T. Smiley. *Multiple-Conclusion Logic*. Cambridge University Press, 1978.
17. R. Wójcicki. *Theory of Logical Calculi*, volume 199 of *Synthese Library*. Kluwer, 1998.
18. A. Wroński. A three element matrix whose consequence operation is not finitely based. *Bulletin of the Section of Logic*, 2(8):68–70, 1979.