Preprint.
Notre Dame Journal of Formal Logic, 1999.

Limits for paraconsistent calculi

Walter A. Carnielli♑ and João Marcos♋

CLE and IFCH
State University of Campinas, UNICAMP
C.P. 6133, 13.081-970 — Campinas, SP, Brazil

ABSTRACT. This paper discusses how to define logics as deductive limits of sequences of other logics. The case of da Costa's hierarchy of increasingly weaker paraconsistent calculi, known as C_n, $1\leq n\leq\omega$, is carefully studied. The calculus C_ω, in particular, constitutes no more than a lower deductive bound to this hierarchy, and differs considerably from its companions. A long standing problem in the literature (open for more than 35 years) is to define the deductive limit to this hierarchy, that is its greatest lower deductive bound. The calculus C_{min}, stronger than C_ω, is first presented as a step towards this limit. As an alternative to the bivaluation semantics of C_{min} presented thereupon, possible-translations semantics are then introduced and suggested as the standard technique both to give this calculus a more reasonable semantics and to derive some interesting properties about it. Possible-translations semantics are then used to provide both a semantics and a decision procedure for C_{Lim}, the real deductive limit of da Costa's hierarchy. Possible-translations semantics also make it possible to characterize a precise sense of duality: as an example, $\mathcal{D}_{min}$ is proposed as the dual to C_{min}.

KEY WORDS: Deductive limits, possible-translations semantics, combination of logics, translations between logical systems, non-classical logics.

1. The problem

While formulating the first important hierarchy of paraconsistent calculi, known as C_n, $1\leq n<\omega$, da Costa [12] also introduced another calculus, C_ω, axiomatized by exactly those schemas common to all C_n. One may regard C_ω as a kind of *syntactic limit* of the calculi in the hierarchy.

Axiomatization. The kernel of each of the calculi C_n includes the Intuitionistic Positive Calculus **(Int⁺)**, which may be axiomatized by the following sch emas:

(1) $A\to(B\to A)$
(2) $(A\to B)\to((A\to(B\to C))\to(A\to C))$
(3) $A\to(B\to(A\wedge B))$
(4) $(A\wedge B)\to A$
(5) $(A\wedge B)\to B$
(6) $A\to(A\vee B)$
(7) $B\to(A\vee B)$
(8) $(A\to C)\to((B\to C)\to((A\vee B)\to C))$

having as its only rule *modus ponens* **(MP)**: A, $A\to B$ / B. Adding to **(Int⁺)** the *excluded middle*, and the *reduction of negations*, respectively, in the following form:

♑ supported by a CNPq Research grant. *e-mail:* <carniell@cle.unicamp.br>
♋ supported by a CNPq graduate fellowship. *e-mail:* <vegetal@cle.unicamp.br>
The authors wish to thank Richard Epstein and an anonymous referee for many helpful comments on a previous version of this paper.

(9) $A \vee \neg A$
(10) $\neg\neg A \to A$

one shall obtain C_ω. Each C_n may now be constructed from C_ω by the addition of two schemas more:

(11*n*) $B^{(n)} \to ((A \to B) \to ((A \to \neg B) \to \neg A))$
(12*n*) $(A^{(n)} \wedge B^{(n)}) \to ((A \wedge B)^{(n)} \wedge (A \vee B)^{(n)} \wedge (A \to B)^{(n)})$

We remember that G° abbreviates the formula $\neg(G \wedge \neg G)$, that G^n, $0 \leq n < \omega$, is recursively defined by $G^0 \overset{\text{def}}{=} G$ and $G^{n+1} \overset{\text{def}}{=} (G^n)^\circ$, and that $G^{(n)}$, $1 \leq n < \omega$, by $G^{(1)} \overset{\text{def}}{=} G^1$ and $G^{(n+1)} \overset{\text{def}}{=} G^{(n)} \wedge G^{n+1}$. One may understand the formula $G^{(n)}$ as saying that the proposition G is *well-behaved*, and so **(11)** may be regarded as a form of *paraconsistent reductio ad absurdum* and **(12)** as regulating the *propagation of well-behavior*.

What about the semantics to the calculi C_n, $1 \leq n \leq \omega$? Arruda [3] has shown that none of these calculi is characterizable by finite matrices. Nevertheless, they may be characterized by non-truth-functional bivaluations. For a given C_n, $n<\omega$, let v_n be a function from the well-formed formulas of C_n into $\{0, 1\}$, such that:

val[i] $v_n(A \wedge B) = 1 \;\Leftrightarrow\; v_n(A) = 1$ and $v_n(B) = 1$;
val[ii] $v_n(A \vee B) = 1 \;\Leftrightarrow\; v_n(A) = 1$ or $v_n(B) = 1$;
val[iii] $v_n(A \to B) = 1 \;\Leftrightarrow\; v_n(A) = 0$ or $v_n(B) = 1$;
val[iv] $v_n(A) = 0 \;\Rightarrow\; v_n(\neg A) = 1$;
val[v] $v_n(\neg\neg A) = 1 \;\Rightarrow\; v_n(A) = 1$;
val[vi] $v_n(A^{n-1}) = v_n(\neg A^{n-1}) \;\Leftrightarrow\; v_n(A^n) = 0$;
val[vii] $v_n(A) = v_n(\neg A) \;\Leftrightarrow\; v_n(\neg A^\circ) = 1$;
val[viii] $v_n(A) \neq v_n(\neg A)$ and $v_n(B) \neq v_n(\neg B) \;\Rightarrow\; v_n(A\#B) \neq v_n(\neg(A\#B))$, where $\# \in \{\wedge, \vee, \to\}$.

For each C_n, $1 \leq n < \omega$, we call the function v_n so defined an *n-valuation*. In [14] and [17] the strong soundness and completeness of the semantics given by the set of all such n-valuations is proven. These valuations also help us to show that each C_n is strictly weaker than any of its predecessors, i.e. denoting by $Th(S)$ the set of theorems of a calculus S, we have:

$$Th(C_n) \subset Th(C_m), \text{ if } 1 \leq m < n < \omega.$$

Indeed, the formula $(G^{m-1} \wedge \neg G^{m-1})^{(m)}$, or the axioms **(11*m*)** and **(12*m*)**, for instance, hold in C_m but do not hold in any C_n, $n > m \geq 1$.

As the axioms of C_ω come from the axioms of a given C_n if we simply erase the schemas **(11*n*)** and **(12*n*)**, exactly the ones dealing with well-behavior, it may seem that a non-truth-functional bivaluation for C_ω would be obtained if we erased clauses **val[vi]** to **val[viii]** of v_n. That is far from true. A complicated, but adequate bivaluation semantics for C_ω, or *ω-valuation*, is provided in [16]. Let's call a *semi-valuation* for C_ω a function s from the wffs of C_ω into $\{0, 1\}$, such that:

sval[i] $s(A \wedge B) = 1 \;\Leftrightarrow\; s(A) = 1$ and $s(B) = 1$;
sval[ii] $s(A \vee B) = 1 \;\Leftrightarrow\; s(A) = 1$ or $s(B) = 1$;
sval[iii] $s(A) = 0 \;\Rightarrow\; s(\neg A) = 1$;
sval[iv] $s(\neg\neg A) = 1 \;\Rightarrow\; s(A) = 1$;
sval[v] $s(A \to B) = 1 \;\Rightarrow\; s(A) = 0$ or $s(B) = 1$;
sval[vi] $s(B) = 1 \;\Rightarrow\; s(A \to B) = 1$.

An ω-valuation v_ω is defined to be a semi-valuation such that the following clause also holds:

sval[vii] For all $A_1, \ldots, A_n$, and all B not of the form $C \rightarrow D$,
$v_\omega(A_1 \rightarrow (A_2 \rightarrow \ldots \rightarrow (A_n \rightarrow B) \ldots)) = 0 \Rightarrow$
there is a semi-valuation s such that $s(A_i)=1$ and $s(B)=0$, $1 \leq i \leq n$.

With the awkward definitions given above, while one might well regard C_ω as a syntactic limit of the hierarchy C_n, one should not also regard the former calculus as a *semantic limit* of the latter.

Clauses **val[i]** to **val[iii]** of an n-valuation inform us that all purely positive classical schemas are valid in each C_n, $n<\omega$. Such is no longer true in C_ω. It is not hard to see, for instance, that the formula $A \vee (A \rightarrow B)$, which we shall call *Dummett's Law* **(DL)**, is not valid in C_ω, though it obviously holds in each C_n, $n<\omega$.

So why should we call C_ω *the* limit of the hierarchy C_n, after all? Under a very reasonable account, we would require that the *limit-calculus* of that hierarchy, which we shall call C_{Lim} hereafter, has as theorems all and only those theorems which are common to all calculi C_n, $1 \leq n < \omega$, that is:

$$\textbf{(Req 1)} \qquad Th(C_{Lim}) = \bigcap_{1 \leq n < \omega} Th(C_n).$$

Clearly, C_ω is not C_{Lim}.

But we do not wish to regard the notion of theoremhood as the cornerstone of our definition of a limit-calculus, as we understand that the notion of derivability, reflected on the consequence operators of our logics, is much more fundamental. Here, in a very general perspective, a *logic* $\mathbf{L}_\#$ will be seen simply as a set (of formulas) $\boldsymbol{L}_\#$ endowed with a consequence operator, $\mathbf{Con}_\#$: $\wp(\boldsymbol{L}_\#) \rightarrow \wp(\boldsymbol{L}_\#)$. Now, the set $\boldsymbol{L}$ of formulas of all C_n coincide. We will require that C_{Lim} should be such that, given any subset Γ of $\boldsymbol{L}$ we have that:

$$\textbf{(Req 2)} \qquad Con_{C_{Lim}}(\Gamma) = \bigcap_{1 \leq n < \omega} Con_{C_n}(\Gamma).$$

It is immediate to see that **(Req 1)** is but a particular case of **(Req 2)**, for $Th(S) = \mathbf{Con}_S(\varnothing)$.

2. First step toward the solution

What if we precisely added **(DL)** to C_ω as a new axiom schema? With this very simple change we obtain a new calculus that we shall call C_{min}. Now we may finally show that C_{min} is, by its turn, closer to the semantic limit of the hierarchy C_n, $1 \leq n < \omega$, once it is characterized exactly by the clauses **val[i]** to **val[v]** of an n-valuation —and so it is a kind of a *minimal* paraconsistent calculus containing all purely positive classical schemas. Let's call min-*valuations* the functions v_{min} subjected to these clauses, and let's define the consequence relation, $\vDash_{min}$, as usual.

THEOREM 2.1 *Let $\Gamma \cup \{A\}$ be a set of formulas of C_{min}. Then:*

$$\Gamma \vdash_{min} A \Rightarrow \Gamma \vDash_{min} A.$$

One just has to check that all axioms **(1)** to **(10)** plus **(DL)** assume only the value 1 in any *min*-valuation, and that **(MP)** preserves validity. This proves soundness.

For completeness we need an auxiliary lemma. Let $\Delta\cup\{G\}$ be a set of formulas of C_{min}. Call Δ a G-*saturated* set if $\Delta \nvdash_{min} G$ and for any formula A of C_{min} such that $A\notin\Delta$ we have $\Delta\cup\{A\} \vdash_{min} G$. First note that any consistent set Γ of formulas of C_{min} such that $\Gamma \nvdash_{min} G$ may be extended to a G-saturated set by the usual Lindenbaum-Asser construction. Now we can prove:

LEMMA 2.2 *Let $\Delta\cup\{G\}$ be a set of formulas of C_{min} with Δ a G-saturated set. Then:*

- ✶ *for any formula A in C_{min},* $\boldsymbol{\Delta \vdash_{min} A \Leftrightarrow A\in\Delta}$**.**
  Consequence of axioms **(1)** and **(2)**, with **(MP)**.
- **(i)** $\boldsymbol{A\wedge B\in\Delta \Leftrightarrow A\in\Delta}$ *and* $\boldsymbol{B\in\Delta}$**.** From ✶, axioms **(3)**, **(4)**, **(5)** and **(MP)**.
- **(ii)** $\boldsymbol{A\vee B\in\Delta \Leftrightarrow A\in\Delta}$ *or* $\boldsymbol{B\in\Delta}$**.** From ✶, axioms **(6)**, **(7)**, **(8)** and **(MP)**.
- **(iii)** $\boldsymbol{A\to B\in\Delta \Leftrightarrow A\notin\Delta}$ *or* $\boldsymbol{B\in\Delta}$**.** From ✶, **(ii)**, axioms **(1)**, **(DL)** and **(MP)**.
- **(iv)** $\boldsymbol{A\notin\Delta \Rightarrow \neg A\in\Delta}$**.** From ✶, axiom **(9)** and **(MP)**.
- **(v)** $\boldsymbol{\neg\neg A\in\Delta \Rightarrow A\in\Delta}$**.** From ✶, axiom **(10)** and **(MP)**.

COROLLARY 2.3 *The characteristic function of a G-saturated set of formulas of C_{min} gives a* min–*valuation.*

Indeed, let Δ be a G-saturated set and define a function v such that, for any formula A of C_{min}, $v(A)=1$ if $A\in\Delta$, and $v(A)=0$ otherwise. Then it's easy to see that **(i)** to **(v)** satisfy, respectively, **val[i]** to **val[v]**.

THEOREM 2.4 $\Gamma \vDash_{min} A \Rightarrow \Gamma \vdash_{min} A$.

Given a formula A in C_{min} such that $\Gamma \nvdash_{min} A$, one may, by Lindenbaum-Asser's construction, extend Γ to an A-saturated set Δ. As $\Delta \nvdash_{min} A$, then, by **LEMMA 2.2** ✶, $A\notin\Delta$. By **COROLLARY 2.3**, the characteristic function of Δ is such that for any $B\in\Delta$, $v(B)=1$, while $v(A)\neq 1$. So, $\Delta \nvDash_{min} A$, and in particular $\Gamma \nvDash_{min} A$. This proves completeness.

Comparison of C_ω and C_{min}. So far, we have the following situation:

$$Th(C_\omega)\subset Th(C_{min})\subseteq Th(C_{Lim}).$$

If C_{min} is not the limit-calculus of C_n, it is at least closer to it than C_ω. Surely C_{min} and C_ω share some properties, such as the uncharacterizability by finite matrices.

Given any C_n, $n<\omega$, we may define the *strong negation* of a formula G, denoted by $\sim^{(n)}G$, as $\neg G\wedge G^{(n)}$. It is easy to prove that this negation has all the properties of classical negation (cf. [13]) and so, for example, the formula $G\wedge\sim^{(n)}G$ trivializes C_n. However, in C_ω or C_{min} no such negation is definable. Actually, following a suggestion of Alves [1], we may prove:

PROPOSITION 2.5 *Neither C_ω nor C_{min} are finitely trivializable, i.e. no finite set of formulas may be added to any of these calculi so as to trivialize it.*

This is an immediate consequence of the following facts:

Fact 2.5.1: *In all matrices with which C_{min} is provably sound, the ordering relation $\leq$ between its values defined as "$a\leq b$* iff *$a\to b$* takes a distinguished value" *is a pre-order.*
Just verify it's reflexive and transitive.

Fact 2.5.2: *If C_{min} were finitely trivializable, the ordering defined in* **Fact 2.5.1** *would admit a least element.*

Indeed, supposing Fin to be a formula such that, for any formula G, $C_{min}\cup\{Fin\}\vdash G$, then by the Deduction Theorem one has that $C_{min}\vdash Fin\to G$. There is a min-valuation v and a value a such that $v(Fin)=a$. Let p be an atomic variable not occurring in Fin, and v' a min-valuation such that $v'(p)=b$ for some value b and $v'(q)=v(q)$ for all q atomic and different from p. Then $v'(Fin)=a$. In particular, one has that $C_{min}\vdash Fin\to p$, so $v'(Fin\to p)=a\to b$. But $a\to b$ takes a distinguished value, so $a\leq b$ for all b.

Fact 2.5.3: *There are sound matrices for C_{min} not having the property in* **Fact 2.5.2**.

Define the truth-values to be all the cofinite subsets of the natural numbers, $\mathbb{N}$, and $\mathbb{N}$ itself to be the only distinguished value. The connectives are defined as:

$v(A\to B)=v(A)^C\cup v(B)$; $v(A\vee B)=v(A)\cup v(B)$; $v(A\wedge B)=v(A)\cap v(B)$;

$$v(\neg A)=\begin{cases} v(A)^C\cup\{n\in\mathbb{N}: n\geq max(v(A)^C)+2\}, \text{if } v(A)\subset\mathbb{N}; \\ \mathbb{N}\setminus\{0\}, \text{if } v(A)=\mathbb{N}. \end{cases}$$

Now one just has to check that all axioms of C_{min} assume but the distinguished value $\mathbb{N}$, for any given valuation, and that **(MP)** preserves validity. The only difficult case is that of the axiom $\neg\neg A\to A$, especially if $v(A)\neq\mathbb{N}$. In this case, $v(\neg A)=v(A)^C\cup\{n\in\mathbb{N}: n\geq max(v(A)^C)+2\}$, and $v(\neg\neg A)=v(\neg A)^C\cup\{n\in\mathbb{N}: n\geq max(v(\neg A)^C)+2\}$. But then, $v(\neg A)^C=v(A)\cap\{n\in\mathbb{N}: n\leq max(v(A)^C)+1\}$, and so $max(v(\neg A)^C)=max(v(A^C))+1$, hence $v(\neg\neg A)=[v(A)\cap\{n\in\mathbb{N}: n\leq max(v(A^C))+1\}]\cup\{n\in\mathbb{N}: n\geq max(v(A^C))+3\}$. Notice also that $v(A)=v(A)\cup\{n\in\mathbb{N}: n\geq max(v(A^C))+1\}$. By some simple set-theoretical manipulations one finally obtains $v(\neg\neg A)=v(A)\setminus\{max(v(A^C))+2\}$. It is now easy to verify that in this situation $\neg\neg A\to A$ is satisfied (and, by the way, $A\to\neg\neg A$ is *not* satisfied —perhaps these infinitary matrices will validate *only* the theorems of C_{min}?).

The ordering relation in the case of the matrices above turns to be the subset relation, $\subseteq$, that clearly has not a minimal element in the set of values considered.

In [14] and [17], decision procedures using *quasi-matrices* were provided to each C_n, $n<\omega$. As one might expect from the intricated semantic characterization of C_ω given above, quasi-matrices for C_ω usually are very complicated (cf. [16]). Once more, this is not the case for C_{min}. A decision procedure for a formula G in C_{min} is easily obtained from the method of quasi-matrices for some C_n, $n<\omega$, if one simply erases all steps dealing with well-behavior, considering instead the following algorithm:

Let A be some subformula of G or the negation of some proper subformula of G. Then, evaluating A in a line k of a quasi-matrix for G:

[.#.] If A has form $B\#C$, where # is any binary connective, evaluate it classically.

[¬] If A has the form $\neg B$, and the value of B in k is 0, write 1 under A in this line; if the value of B in k is 1, bifurcate this line and write 0 in the first part and, in the second, write 1.

To show the adequacy of this procedure, we prove that, for a given formula G:

PROPOSITION 2.6 *Given a bivaluation for C_{min} there is a line of a quasi-matrix for G that corresponds to it.*

PROPOSITION 2.7 *Given a line of a quasi-matrix for G, there is a bivaluation for C_{min} corresponding to it.*

A possible-worlds semantics for C_ω was proposed by Baaz [4], and it seems that only some minor modifications might be in order to turn this semantics adequate for

C_{min}. We will not investigate this problem here. It should be observed, however, that possible-worlds semantics for each C_n, $n<\omega$, have still not been produced.

How can a formula and its negation both be true? We believe the semantics just given to C_{min} does not help much to explain its paraconsistent behavior. We introduce in the following a new kind of semantics with various interesting properties:

(a) it sheds some light upon the paraconsistent behavior of C_{min};
(b) it provides a truth-functional interpretation for the connectives of C_{min};
(c) it gives a simple decision procedure for C_{min};
(d) it makes it possible to semantically characterize C_{Lim}, the real limit-calculus of C_n.

3. New semantics for C_{min}

We first introduce some terminology from the theory of *translations between logics* (cf. [9]). In the end of section **1.** we have proposed to see a logic $\mathbf{L}_\#$ as a structure of the form $<\boldsymbol{L}_\#$, $\mathbf{Con}_\#>$, where $\boldsymbol{L}_\#$ is a set, and $\mathbf{Con}_\#$ a consequence operator on $\boldsymbol{L}_\#$. Now, a *translation from the logic* $\mathbf{L}_1$ *into the logic* $\mathbf{L}_2$ is defined as a homomorphism between these structures, that is, a map $*: \boldsymbol{L}_1 \rightarrow \boldsymbol{L}_2$, such that, given $\Gamma\cup\{A\}\subseteq \boldsymbol{L}_1$:

$$A \in \mathbf{Con}_1(\Gamma) \;\Rightarrow\; A^* \in \mathbf{Con}_2(\Gamma^*).$$

Such a map is called a *conservative translation* if the converse also holds. Of course, if we have, for a given calculus S, $\boldsymbol{L}_1 = \boldsymbol{L}_2 =$ wffs of S, $\mathbf{Con}_1$ denoting its syntactic consequence relation and $\mathbf{Con}_2$ a proposed semantic consequence relation, where $*$ is the identity function, then showing that $*$ is a translation is showing soundness, and showing that $*$ is conservative is showing completeness.

Now consider the "weak-strong" logic $\mathcal{W}_3^S$, given by the following three-valued matrices:

$\wedge$	T	T⁻	F
T	T⁻	T⁻	F
T⁻	T⁻	T⁻	F
F	F	F	F

$\vee$	T	T⁻	F
T	T⁻	T⁻	T⁻
T⁻	T⁻	T⁻	T⁻
F	T⁻	T⁻	F

$\rightarrow$	T	T⁻	F
T	T⁻	T⁻	F
T⁻	T⁻	T⁻	F
F	T⁻	T⁻	T⁻

	$\neg_S$	$\neg_W$
T	F	F
T⁻	F	T⁻
F	T	T

Here T and T⁻ are the distinguished values. One may interpret the value T⁻ as "true by default," i.e., by lack of evidence to the contrary. Given two propositions connected by a conjunction, a disjunction or an implication then the matrices above mean that in these cases we can never be completely sure —the evaluation of $\wedge$, $\vee$ or $\rightarrow$ will not return the value T. We have two negations, $\neg_S$ and $\neg_W$: we call the first one *strong*, and observe that it has a classical behavior, changing definitely the status of propositions —from distinguished to non-distinguished and vice-versa; the other one we call *weak*, and observe that there is a situation in which we can neither confirm nor disconfirm a proposition —negating a proposition true by default, this negation will return another proposition of the same status.

Now let's define the set $\boldsymbol{Tr}$ of all functions $*$ from the formulas of C_{min} into the formulas of $\mathcal{W}_3^S$ subjected to the following clauses:

Tr 1. for atomic p, $p^* = p$, $(\neg p)^* = \neg_W p$;
Tr 2. $(\neg A)^* = \neg_S A^*$ or $(\neg A)^* = \neg_W A^*$, for non-atomic A;
Tr 3. $(A\#B)^* = A^*\#B^*$, where $\# \in \{\wedge, \vee, \rightarrow\}$.

We say the pair **PT** $=<\mathcal{W}_3^S$, $\boldsymbol{Tr}>$ gives a *possible-translations semantics* to C_{min}. If $\vDash_{\mathbf{3}}$ denotes the consequence relation in $\mathcal{W}_3^S$, and $\Gamma\cup\{A\}$ is a set of formulas of C_{min}, we define the **PT**-*consequence relation*, $\vDash_{\mathbf{PT}}$, as:

$$\Gamma \vDash_{\mathbf{PT}} A \overset{def}{\Leftrightarrow} \text{ for all } *\in \boldsymbol{Tr}, \text{ we have } \Gamma^* \vDash_{\mathbf{3}} A^*.$$

We will call a *possible translation* of a formula A in C_{min} any image of it through some function in $\boldsymbol{Tr}$. We may immediately prove the following:

THEOREM 3.1 (Soundness) $\Gamma \vdash_{min} A \Rightarrow \Gamma \vDash_{\mathbf{PT}} A$.

Given a formula A, it is evident that the total number of its possible translations is finite — in fact, it is 2^n, where n is the number of negation symbols in A. So here one just has to test all possible translations of each axiom, from **(1)** to **(10)** and **(DL)**, and then verify that all possible translations of **(MP)** preserve validity.

This result assures us that each $*$ in ***Tr*** *is* indeed a translation from C_{min} into $\mathcal{W}_3^S$, in the sense precised above. We may present a stronger result relating the possible-translations semantics to the bivaluation semantics presented in section **2.**

THEOREM 3.2 (Convenience) *Given a translation* $*$ *in* ***Tr*** *and a valuation* w *in* $\mathcal{W}_3^S$, *then the function* v *such that, for every formula* A *in* C_{min},

$$v(A)=1 \Leftrightarrow w(A^*)\in \{\mathrm{T}, \mathrm{T}^-\},$$

is a min-*valuation.*

Immediate, just verify that **val[i]** to **val[v]** hold.

Note that **THEOREM 3.1** is also provable as a corollary of **THEOREM 3.2**.

THEOREM 3.3 (Representability) *Given a* min-*valuation* v_{min}, *there is a translation* $*$ *in* ***Tr*** *and a valuation* w *in* $\mathcal{W}_3^S$ *such that, for every formula* A *in* C_{min},

$$w(A^*)\in \{\mathrm{T}, \mathrm{T}^-\} \Leftrightarrow v_{min}(A)=1.$$

Define p^* as p, and define the valuation w for atomic p as

$w(p^*)=\mathrm{T}$ iff $v(\neg p)=0$;
$w(p^*)=\mathrm{T}^-$ iff $v(p)=1$ and $v(\neg p)=1$;
$w(p^*)=\mathrm{F}$ iff $v(p)=0$.

Define $(\neg p)^*$ as $\neg_{\mathrm{w}} p^*$, and $(A\#B)^*$ as $A^*\#B^*$. For non-atomic A, define $(\neg A)^*$ as $\neg_{\mathrm{w}}A^*$ if $v(A)=v(\neg A)$, and define it as $\neg_{\mathrm{s}}A^*$ otherwise. Now one just has to check that these definitions work.

COROLLARY 3.4 (*Completeness*) $\Gamma \vDash_{\mathbf{PT}} A \Rightarrow \Gamma \vdash_{min} A$.

Thus "weaving" together all the translations in ***Tr***, as we would do with sheaves, we have eventually obtained a conservative translation from C_{min} into the structure **PT**.

The new decision procedure for C_{min} is immediate. Given a formula G in C_{min}, we just have to make all possible translations of it, and test each of them using the matrices of $\mathcal{W}_3^S$. There is an obvious relation between this method and the one of quasi-matrices:

PROPOSITION 3.5 *Given a formula* G *of* C_{min} *and a quasi-matrix for it,* $\mathbf{QM}_G$,

(i) *for given* w *and* $*$ *in* **PT** *there is a line* k *of* $\mathbf{QM}_G$ *that corresponds to them;*
From **THEOREM 3.2** and **PROPOSITION 2.6**.

(ii) *for each line* k *of* $\mathbf{QM}_G$ *there are corresponding* w *and* $*$ *in* **PT**.
From **PROPOSITION 2.7** and **THEOREM 3.3**.

So the apparent superiority of the new testing method over the one with quasi-matrices seems to consist in adding new columns instead of bifurcating the lines. We restore truth-functionality if we only allow each formula of C_{min} to be interpreted as a conjunction of all its possible translations.

A nice application of the possible-translations semantics for C_{min} is to help to easily show the following:

PROPOSITION 3.6 *No negated formula is a theorem of C_{min} (and, consequently, of C_ω).*

Argument 3.6.1: *For any given negated formula $\neg G$ one may find a valuation w and a translation $*$ such that $w((\neg G)^*)=\mathrm{F}$.*
Just pick a w such that $w(p)=\mathrm{T}^-$ for any atomic p, and then translate every negated subformula $\neg A$ of G as $\neg_{\mathrm{w}}A^*$, while translating $\neg G$ itself as $\neg_{\mathrm{s}}G^*$.

Argument 3.6.2: *There are models of C_{min} in which no negated formulas are valid.*
Indeed, one such model is given in **Fact 2.5.3** above.

Either of the arguments above prove **PROPOSITION 3.6**. A modified version of **Argument 3.6.1** was used in [11] to prove that negated formulas are also not theorems of any C_n, unless they have well-behaved subformulas.

4. Not the limit!

It seems the particular axioms **(11*n*)** and **(12*n*)** of C_n can play tricks on us. Using both of them we may prove, for example, some forms of *De Morgan Laws* that we cannot prove without them.

PROPOSITION 4.1 *The following are the only forms of De Morgan Laws provable in each C_n, $1\leq n<\omega$:*

(DM1) $\neg(A\wedge B)\to(\neg A\vee\neg B)$; **(DM3)** $\neg(\neg A\wedge B)\to(A\vee\neg B)$;
(DM2) $\neg(A\wedge\neg B)\to(\neg A\vee B)$; **(DM4)** $\neg(\neg A\wedge\neg B)\to(A\vee B)$.

Note: The syntactic proofs surely require some skill from the reader.

None of them is provable in C_n without the axiom **(11*n*)**.

Just consider the following matrices:

∧	♈	♉	♊
♈	♉	♉	♊
♉	♉	♉	♊
♊	♊	♊	♊

∨	♈	♉	♊
♈	♉	♈	♈
♉	♈	♈	♈
♊	♈	♈	♊

→	♈	♉	♊
♈	♉	♈	♊
♉	♈	♈	♊
♊	♈	♈	♈

	¬
♈	♊
♉	♉
♊	♈

where ♈ and ♉ are distinguished.

None of them is provable in C_n without the axiom **(12*n*)**.

Just consider the same matrices above, changing only the conjunction for:

∧	♈	♉	♊
♈	♉	♈	♊
♉	♈	♈	♊
♊	♊	♊	♊

.

Of course, one does not really need to give independence proofs to show these formulas to be not valid in C_{min}. We have *two* semantics and decision procedures already at our disposal. The formula **(DM1)**, for instance, may be shown to be not valid, *either:*

- if we pick atomic variables p and q as A and B and choose a *min*-valuation v_{min}, such that:

$$v_{min}(p)=v_{min}(q)=1,\ v_{min}(\neg p)=v_{min}(\neg q)=0 \text{ and } v_{min}(\neg(p\wedge q))=1,$$

or

- if we pick atomic variables p and q as A and B and choose a translation $*$ and a valuation w such that:

$$(\neg p)^*=\neg_w p,\ (\neg q)^*=\neg_w q,\ (\neg(p\wedge q))^*=\neg_w(p\wedge q) \text{ and } w(p)=w(q)=\mathrm{T}.$$

Let's give one more full example of those semantics in action, now to prove that:

PROPOSITION 4.2 $(A\wedge\neg A)\to\neg\neg(A\wedge\neg A)$ *is not a theorem of* C_{min}*, though it is indeed a theorem of any* C_n*, and consequently of* C_{Lim}*.*

To see why this formula is provable in any C_n, just take a look at the clause **val[vii]**, in **1.** On the other side, let's turn to the quasi-matrix of the formula $(p\wedge\neg p)\to\neg\neg(p\wedge\neg p)$ in C_{min}:

<table>
<tr><th>p</th><th>¬p</th><th>p∧¬p</th><th>¬(p∧¬p)</th><th>¬¬(p∧¬p)</th><th>(p∧¬p)→¬¬(p∧¬p)</th><th></th></tr>
<tr><td>0</td><td>1</td><td>0</td><td>1</td><td>0</td><td>1</td><td>1</td></tr>
<tr><td rowspan="4">1</td><td>0</td><td>0</td><td>1</td><td>0</td><td>1</td><td>2</td></tr>
<tr><td rowspan="3">1</td><td rowspan="3">1</td><td>0</td><td>1</td><td>1</td><td>3</td></tr>
<tr><td rowspan="2">1</td><td>0</td><td>0</td><td>4</td></tr>
<tr><td>1</td><td>1</td><td>5</td></tr>
</table>

Line ***4*** tells this formula not to be a tautology of C_{min}. Of course this line cannot appear in a quasi-matrix for any C_n. Now let's consider the possible translations of this formula:

① $(p\wedge\neg_w p)\to\neg_s\neg_s(p\wedge\neg_w p)$;
② $(p\wedge\neg_w p)\to\neg_w\neg_s(p\wedge\neg_w p)$;
③ $(p\wedge\neg_w p)\to\neg_s\neg_w(p\wedge\neg_w p)$;
④ $(p\wedge\neg_w p)\to\neg_w\neg_w(p\wedge\neg_w p)$;

p	①	②	③	④	
T	T^-	T^-	T^-	T^-	❶
T^-	T^-	T^-	**F**	T^-	❷
F	T^-	T^-	T^-	T^-	❸

Line ❷ of the ③rd translation shows this formula once more to be invalid in C_{min}. The canonical connection established in **PROPOSITION 3.5** between the two procedures above will tell the reader, for instance, how to transform lines ***4*** and ***5*** of the quasi-matrix above into, respectively, the pairs <③,❷> and <④,❷> of **PT**, and, conversely, how to transform the pairs <①,❶> and <③,❷> of **PT** into the lines ***2*** and ***4*** of the quasi-matrix.

Thence, the situation has turned out to be the following:

$$Th(C_\omega)\subset Th(C_{min})\subset Th(C_{Lim}).$$

We conclude that the calculus C_{min} too, though very interesting by itself, is not the desired limit-calculus of C_n.

An idea. Let's construct from each C_n the calculus $\mathcal{B}_n$, just erasing axiom **(12*n*)**. So even though we still have paraconsistent reductio ad absurdum, we have no propagation of well-behavior. The third part of **PROPOSITION 4.1** guarantees us that no De Morgan Laws are valid in any $\mathcal{B}_n$. Given a specific $\mathcal{B}_n$, it's not hard to prove that an adequate non-truth-functional semantics for it is provided if we just erase clause **val[viii]** of an n-valuation.

Perhaps C_{min} *is* indeed a limit-calculus of the hierarchy $\mathcal{B}_n$, $1\leq n<\omega$? To convince oneself of the negative answer to this question, one should just observe that the clause **val[vii]** is still present for any calculus $\mathcal{B}_n$, and so $(A\wedge\neg A)\rightarrow\neg\neg(A\wedge\neg A)$ is still provable in any $\mathcal{B}_n$. Will C_{min} be characterized as the limit-calculus of some further weakening of the calculi $\mathcal{B}_n$? We cannot answer this question at this time.

5. So where's the limit?

What about some history first? Possible-translations semantics can be situated into the more general setting of *combinations of logics* (for an overview, see [5], and for a categorial approach of possible-translations semantics, see [8]). One of us has initially proposed possible-translations semantics as a way of combining logics with well-known many-valued semantics so as to produce interpretations to some non-classical logics (cf. [6]). A special case of possible-translations semantics is society semantics (cf. [10]). Possible-translations semantics based on three-valued logics and adequate for interpreting slightly stronger versions of the calculi C_n may be found in [7] and [11], and the hierarchy C_n itself is studied in [18].

For each C_m, $1\leq m<\omega$, we may define $\mathbf{PT}_m$, a possible-translations semantics based on three-valued matrices with three conjunctions, three disjunctions, three implications and two negations, together with convenient restrictions over the functions in $\boldsymbol{Tr}_m$. Let's denote the consequence relation defined in $\mathbf{PT}_m$ as $\vDash_m$. So, for a given formula A we would theoretically have a maximum of $2^n.3^{c+d+i}$ possible translations, where n is the number of negations in the formula A, c the number of conjunctions, d of disjunctions, i of implications. We collect these translations into a set $PT(A)$. But remember that for each C_m this set may be restricted and diminished by the conditions over the translations in $\boldsymbol{Tr}_m$. Thus, denoting by $Pt(A,\ m)$ the set of all possible-translations of a formula A in a calculus C_m, we actually have, for any given $1\leq m<n<\omega$:

$$\textbf{(1)}\qquad Pt(A,\ m)\subseteq Pt(A,\ n)\subseteq PT(A).$$

Making use of these possible-translations semantics for C_n, we may now make explicit $\mathbf{PT}_{Lim}$, a possible-translations semantics for C_{Lim}. It is the pair $<\{C_n\}_{1\leq n<\omega}$, $\{*_n\}_{1\leq n<\omega}>$, where each function $*_n$ is an identity map from the formulas of C_{Lim} into the formulas of C_n. The consequence relation in $\mathbf{PT}_{Lim}$ is obviously defined as:

$$\Gamma\vDash_{\boldsymbol{Lim}} A \overset{def}{\Leftrightarrow} \text{ for all } *_n\text{, we have } \Gamma^{*_n}\vDash_n A^{*_n}\text{, i.e. for all } n\text{, we have } \Gamma\vDash_n A.$$

In such a way, one may refer to the calculus C_{Lim} and to the formulas validated in it. One can indeed provide a decision procedure for the formulas of C_{Lim}. Indeed, as a consequence of **(1)**, the set defined as:

$$Pt(A,\ Lim)\overset{\text{def}}{=}\bigcup_{1\leq n<\omega} Pt\,(A,\ n)$$

is finite, and we know its content. So we may effectively test all the formulas in it with the three-valued matrices above mentioned (see [11] or [18]).

The reader should note that while the possible-translations offered for C_{min} in section **3.** was obtained through the suitable combination of an infinite number of fragments of $\mathcal{W}_3^S$ (and similarly in the case of C_n, mentioned above), the possible-translations semantics just proposed for C_{Lim} made use of an infinite number (of possible-translations semantics) of different logics, viz. all the C_n, for $n<\omega$. The whole procedure, nevertheless, is quite the same.

How could we define a non-truth-functional semantics of bivaluations for C_{Lim}? Should we maintain clause **val[vii]** and just erase clauses **val[vi]** and **val[viii]** of an n-valuation? And how could we characterize axiomatically C_{Lim}? Would it be possible to define a strong negation in this calculus, and how? These questions are still open.

Another limit. So far we have been able to define semantically C_{Lim}, the greatest deductive lower bound of the hierarchy C_n, $1\leq n<\omega$. Surely, now we can look for deductive upper bounds for this same hierarchy. C_1 would be such an upper bound, as it is strictly stronger than any of the other calculi which follow it.

But let us note that both da Costa and Jaśkowski, commonly held as the founders of paraconsistent logic, intended their paraconsistent calculi to be so strong as to contain most classical schemas and rules compatible with their paraconsistent character (see [13] and [15]). One such a *maximal* paraconsistent calculus extending each C_n was devised by Sette (see [22]), and is known as $\mathcal{P}^1$. It is interesting to note that $\mathcal{P}^1$ is also a three-valued calculus.

Bearing in mind the objective of approximating the calculus C_1 to the classical, a first obvious strengthening we might propose would be the addition to it as a new axiom of the schema **(AN)**: $A\rightarrow\neg\neg A$. Given a calculus C_n, for $1\leq n<\omega$, we define $C_n^{\neg\neg}$ by the axioms of C_n plus **(AN)**. A possible-translations semantics for a slightly stronger version of the hierarchy $C_n^{\neg\neg}$, $1\leq n<\omega$, was presented in [7], and the model-theoretic properties of a first-order calculus with equality based on $C_1^{\neg\neg}$ was studied by Alves [2]. The greatest deductive lower bound for the hierarchy $C_n^{\neg\neg}$, $1\leq n<\omega$, may be obtained as above.

Nevertheless, the calculus $\mathcal{P}^1$ does not extend any $C_n^{\neg\neg}$, for **(AN)** is not a theorem of $\mathcal{P}^1$. It is possible although to define another three-valued maximal paraconsistent calculus, this time extending the strengthened new hierarchy —and consequently also the previous hierarchy. Such a calculus was called $\mathcal{P}^2$ and was first introduced by Mortensen, in [20], and then rediscovered by one of us, in [18], where one may also learn which axioms may be added to any C_n so as to obtain $\mathcal{P}^1$ and $\mathcal{P}^2$.[©] Mortensen has also raised the question as to whether there could exist other maximal three-valued paraconsistent logics "sufficiently similar" yet distinct from $\mathcal{P}^1$ and $\mathcal{P}^2$. The answer is definitely affirmative: We finish this section noting that in [19] the reader may find the axiomatization and the truth-tables of nothing but 2^{13} such logics.

6. A dual paracomplete calculus

Possible-translations semantics actually opens to us a new possibility of defining logical systems. We may combine logics for specific needs. Do we have a group of interesting logics whose semantical properties we wish to simultaneously preserve? Then look for a way of combining their semantics. Do we want to build a paraconsistent calculus with a

[©] Actually, in [20], Mortensen introduced $\mathcal{P}^2$ under the name $C_{0.2}$, but for some reason he insisted that this logic should have only *one* designated value. Consequently, his completeness proof holds, but the soundness of his system does *not* hold, for **(MP)** will not preserve validity. This problem is nevertheless fixed if we pick *two* designated values, instead of one. More details may be found in [19].

possible-worlds interpretation? Mix possible-worlds interpretations of intuitionistic calculi, as shown in [7]. Do we want a logic that is paraconsistent only at the level of propositions, but not in relation to complex propositions? Carnielli & Lima-Marques [10] have indicated how to combine two copies of classical logic (by means of a particularization of the possible-translations semantics —the so-called *society semantics*) so as to obtain such a logic, and then have shown that the logic they obtained coincided with $\mathcal{P}^1$.

Possible-translations semantics have also been used to investigate the problem of duality between logical systems (for an overview of this topic, see [21]). In [10], the calculi $\mathcal{P}^1$ and $\mathcal{I}^1$ (for the latter, consult [23]) are shown to respect a precise definition of duality. As pointed out by Sylvan [24], one should expect the dual of a paraconsistent calculus to be a paracomplete calculus.® In [11] a hierarchy of paracomplete calculi in some sense dual to a slightly stronger version of the hierarchy C_n is introduced.

And the dual to C_{min}? Intuitively, we would define $\mathcal{D}_{min}$, the dual to C_{min}, as the logic characterized by the possible-translations semantics obtained when we consider the set ***Tr*** of translations subjected to the very same conditions **Tr 1.** to **Tr 3.** as in **3.**, and the following three-valued matrices of $\mathcal{V}_3^S$ (instead of $\mathcal{W}_3^S$):

$\wedge$	T	F^+	F
T	T	F^+	F^+
F^+	F^+	F^+	F^+
F	F^+	F^+	F^+

$\vee$	T	F^+	F
T	T	T	T
F^+	T	F^+	F^+
F	T	F^+	F^+

$\rightarrow$	T	F^+	F
T	T	F^+	F^+
F^+	T	T	T
F	T	T	T

	$\neg_S$	$\neg_W$
T	F	F
F^+	T	F^+
F	T	T

Here T is the only distinguished value. The interpretations to the values and connectives above are “dual” to those given in **3.**

This logic has some very interesting properties:

PROPOSITION 6.1 *$\mathcal{D}_{min}$ is not characterizable by finite matrices.*

PROPOSITION 6.2 *A non-truth-functional bivaluation for $\mathcal{D}_{min}$ is obtainable from a* min-*valuation just substituting clause* **val[iv]**: $v_{min}(A)=0 \Rightarrow v_{min}(\neg A)=1$ *for* **val[iv^d]**: $v_{min}(A)=1 \Rightarrow v_{min}(\neg A)=0$, *and substituting* **val[v]**: $v_{min}(\neg\neg A)=1 \Rightarrow v_{min}(A)=1$ *for* **val[v^d]**: $v_{min}(\neg\neg A)=0 \Rightarrow v_{min}(A)=0$.

PROPOSITION 6.3 *A simple quasi-matrix procedure for $\mathcal{D}_{min}$ is obtained if one only substitutes the rule for negation in C_{min} for:*

> **[¬]** *If A is of the form ¬B, and the value of B in a line k is* 1, *write* 0 *under A in this line; if the value of B in a line k is* 0, *bifurcate this line and write* 0 *in the first part and, in the second, write* 1.

PROPOSITION 6.4 *$\mathcal{D}_{min}$ is axiomatized as C_{min}, just substituting the schema* **(9)**: $A\vee\neg A$ *for* **(9^d)**: $A\rightarrow(\neg A\rightarrow B)$, *and substituting the schema* **(10)**: $\neg\neg A\rightarrow A$ *for* **(10^d)**: $A\rightarrow\neg\neg A$.

The proofs of **PROPOSITIONS 6.1 - 6.4** are entirely analogous to the case of C_{min} above. The semantics of $\mathcal{D}_{min}$ also inform that:

® Justus Diller (personal communication) had already pointed out this possibility to one of the authors.

PROPOSITION 6.5 *The following formulas* are not *theorems of* $\mathcal{D}_{min}$:

(i) $A \vee \neg A$ **(iii)** $\neg(A \wedge \neg A)$;
(ii) $\neg\neg A \to A$ **(iv)** $(A \to B) \to ((A \to \neg B) \to \neg A)$.

The fact that $\mathcal{D}_{min}$ does not prove **(i)** and **(ii)** makes it a proper candidate to answer to Brouwer's well-known requirements for *the* Intuitionistic Logic. Some of the more striking differences of $\mathcal{D}_{min}$ from Heyting's Intuitionistic Calculus **(HIC)** reside in the dismissal of **(iii)** and **(iv)** by $\mathcal{D}_{min}$. So, while **(HIC)** rejects a part of positive logic, while maintaining non-contradiction and reductio ad absurdum, $\mathcal{D}_{min}$ rejects both non-contradiction and reductio ad absurdum, while maintaining the whole of positive logic.

References

[1] Alves, E. H., *Logic and Inconsistency: a study of the calculi* C_n, $1 \le n \le \omega$ (in Portuguese), Thesis, State University of São Paulo, FFLCH, 1976, 137p.

[2] Alves, E. H., "Paraconsistent logic and model theory," *Studia Logica*, vol. 43 (1984), pp. 17–32.

[3] Arruda, A. I., "Remarques sur les systèmes C_n," *Comptes Rendus de l'Academie de Sciences de Paris*, Séries A–B (1975) t.280, pp. 1253–6.

[4] Baaz, M., "Kripke-type semantics for da Costa's paraconsistent logic C_ω," *Notre Dame Journal of Formal Logic*, vol. 27 (1986), no.4, pp. 523–7.

[5] Blackburn, P., and M. de Rijke, "Zooming in, zooming out," *Journal of Logic, Language and Information*, vol. 6 (1997), no.1, pp. 5–31.

[6] Carnielli, W. A., "Many-valued logics and plausible reasoning," pp. 328–35 in *Proceedings of the XX International Congress on Many-Valued Logics*, IEEE Computer Society, University of Charlotte, North Carolina, 1990.

[7] Carnielli, W. A., "Possible-translations semantics for paraconsistent logics," pp. 149–63 in *Frontiers in paraconsistent logic:* Proceedings of the I World Congress on Paraconsistency, Ghent, 1998, edited by D. Batens et al., King's College Publications, 2000.

[8] Carnielli, W. A., and M. E. Coniglio, "A categorial approach to the combination of logics," *Manuscrito*, vol. 22 (1999), no.2, pp. 69–94.

[9] Carnielli, W. A., and I. M. L. D'Ottaviano, "Translations between logical systems: a manifesto," *Logique & Analyse*, vol. 157 (1997), pp. 67–81.

[10] Carnielli, W. A., and M. Lima-Marques, "Society semantics and multiple-valued logics," pp. 33–52 in *Advances in Contemporary Logic and Computer Science:* Proceedings of the XI Brazilian Conference on Mathematical Logic, May 6–10, 1996, Salvador, Bahia, Brazil, edited by W. A. Carnielli and I. M. L. D'Ottaviano, col. Contemporary Mathematics, vol. 235, American Mathematical Society, 1999.

[11] Carnielli, W. A., and J. Marcos, "Possible-translations semantics and dual logics," to appear in *Soft Computing*.

[12] da Costa, N. C. A., *Inconsistent Formal Systems* (in Portuguese), Thesis, Universidade Federal do Paraná, 1963. Curitiba: Editora UFPR, 1993, 68p.

[13] da Costa, N. C. A., "On the theory of inconsistent formal systems," *Notre Dame Journal of Formal Logic*, vol. 15 (1974), no.4, pp. 497–510.

[14] da Costa, N. C. A., and E. H. Alves, "A semantical analysis of the calculi C_n," *Notre Dame Journal of Formal Logic*, vol. 18 (1977), no.4, pp. 621–30.

[15] Jaśkowski, S., "Propositional calculus for contradictory deductive systems," *Studia Logica*, vol. 24 (1969), pp. 143–57.

[16] Loparić, A., "A semantical study of some propositional calculi," *The Journal of Non-Classical Logic*, vol. 3 (1986), no.1, pp. 73–95.

[17] Loparić, A., and E. H. Alves, "The semantics of the systems C_n of da Costa," pp. 161–72 in *Proceedings of the Third Brazilian Conference on Mathematical Logic*, edited by A. I. Arruda, N. C. A. da Costa, and A. M. Sette. São Paulo: Sociedade Brasileira de Lógica, 1980.

[18] Marcos, J., *Possible-Translations Semantics* (in Portuguese), Thesis, State University of Campinas, IFCH, 1999, xxviii + 240p. (ftp://www.cle.unicamp.br/pub/thesis/J.Marcos/)

[19] Marcos, J., "8K solutions and semi-solutions to a problem of da Costa." Submitted for publication.

[20] Mortensen, C., "Paraconsistency and C_n," pp. 289–305 in *Paraconsistent Logic:* essays on the inconsistent, edited by G. Priest, R. Routley, and J. Norman, Philosophia Verlag, 1989.

[21] Queiroz, G. S., *On the Duality between Intuitionism and Paraconsistency* (in Portuguese), Thesis, State University of Campinas, IFCH, 1997, 150p.

[22] Sette, A. M., "On the propositional calculus $\mathcal{P}^1$," *Mathematica Japonicae*, vol. 18 (1973), pp. 173–80.

[23] Sette, A. M., and W. A. Carnielli, "Maximal weakly-intuitionistic logics," *Studia Logica*, vol. 55 (1995), pp. 181–203.

[24] Sylvan, R., "Variations on da Costa C systems and dual-intuitionistic logics. I. Analyses of C_ω and CC_ω," *Studia Logica*, vol. 49 (1990), no.1, pp. 47–65.