Improving Classical Authentication with Quantum Communication

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(Dated: 11 February 2010)

Abstract

We propose a quantum-enhanced protocol to authenticate classical messages, with improved security with respect to the classical scheme introduced by Brassard in 1983. In that protocol, the shared key is the seed of a pseudo-random generator (PRG) and a hash function is used to create the authentication tag of a public message. We show that a quantum encoding of secret bits offers more security than the classical XOR function introduced by Brassard. Furthermore, we establish the relationship between the bias of a PRG and the amount of information about the key that the attacker can retrieve from a block of authenticated messages. Finally, we prove that quantum resources can improve both the secrecy of the key generated by the PRG and the secrecy of the tag obtained with a hidden hash function.
I. INTRODUCTION

The authentication of public messages is a fundamental problem nowadays for bipartite and network communications. The scenario is the following: Alice sends a (classical) message to Bob through a public channel, together with an authentication tag through a private or public channel. The tag will allow Bob to verify if the message he received via the public channel has been tampered with or if it is indeed the authentic message, originally sent by Alice. A third character, Eve, wants to sabotage this scheme by intercepting Alice’s message and sending her own message to Bob, together with a false tag which will convince Bob he is receiving the authentic message. For instance, one could imagine that Alice is sending to Bob her bank account number, to which Bob will transfer some money, and Eve wants to interfere in the communication in such a way that Bob will receive her bank account number believing it is Alice’s one, thus giving his money to Eve. The use of authentication tags allows to separate the secrecy problem in message transmission from the authentication problem and it is useful even if a secure communication channel is available [2].

In 1983, G. Brassard proposed a computationally secure scheme of classical authentication tags based on the sharing of short secret keys [1]. Brassard’s scheme is itself an improvement of the Wegman-Carter protocol [2]. Brassard showed that a relatively short seed of a PRG can be used as a secret key shared between Alice and Bob which will allow the exchange of computationally secure authentication tags. This method yields a much more practical protocol, where the requirements on the seed length grow reasonably with the number of messages we want to authenticate, as opposed to the Wegman-Carter proposal.

The security of PRGs is based on the alleged hardness of some problems of number theory, e.g., the factorization of a large number with classical computers. However, several of these problems are provably solvable if quantum computers are available. Consequently, the security of the PRGs might be compromised. Assuming Alice and Bob communicate quantically, can Eve yet menacing the PRG security? This question is our main motivation to write this article.

In this work, we extend Brassard’s protocol to include quantum-encoded authentication tags, which we prove will offer, under certain conditions, information-theoretical security for
the authentication of classical messages. We observe that our scheme can authenticate the quantum channel itself, which is an important part of the quantum cryptography: in fact, it is the crucial first step of quantum key distribution protocols.

II. PRELIMINARIES

In this section we set up basic notation, briefly review the description of the Brassard’s protocol and describe our new proposal. We conclude the section with a negative result on the robustness of an attackable PRG when its output is hidden by a specific quantum coding.

We denote \( M \) the set of messages and \( T \) the set of tags, where \( \log |M| \gg \log |T| \). As hash functions are an important ingredient for all protocols described here we start by presenting their formal definition [6]:

**Definition II.1 (\( \varepsilon \) – almost strongly universal-2 hash functions)** Let \( M \) and \( T \) be finite sets and call functions from \( M \) to \( T \) hash functions. Let \( \varepsilon \) be a positive real number. A set \( H \) of hash functions is \( \varepsilon \)–almost strongly universal-2 if the following two conditions are satisfied

1) The number of hash functions in \( H \) that takes an arbitrary \( m \in M \) to an arbitrary \( t \in T \) is exactly \( |H|/|T| \).

2) The fraction of those functions that also takes \( Y' \neq Y \) in \( M \) to an arbitrary \( T' \in T \) (possibly equal to \( T \)) is no more than \( \varepsilon \).

The number \( \varepsilon \) is related to the probability of guessing the correct tag with respect to an arbitrary message \( Y \). Notice that the smaller \( \varepsilon \) is, the larger is \( |H| \). For additional details

![FIG. 1: Brassard’s classical authentication protocol [1]](image-url)
on universal-2 functions we point the reader to [2]. Brassard’s protocol (see Figure 1) makes use of two secret keys. The first one, \(U(l)\), specifies a fixed universal-2 hash function \(h \in \mathcal{H}\), where \(l = \lceil \log |\mathcal{H}| \rceil\). The second specifies the seed \(X(n) \in \mathbb{Z}_2^n\), for a PRG, a sequence of \(n\) bits.

The main ingredient of our first quantum-enhanced protocol proposed here (see Figure 2) is replacing the classical gate XOR of Brassard protocol by a quantum encoder similar to that used in the BB84 protocol [11]. After some developments, we shall verify that the key \(U(l)\) is no longer necessary. Assume that Alice and Bob agree on two orthonormal bases \(B_0\) and \(B_1\) for the 2-dimensional Hilbert space,

\[
B_0 = \{|0\rangle, |1\rangle\} \quad \text{and} \quad B_1 = \left\{ |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), |−\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \right\}
\]

These bases will be used to prepare four quantum states. We shall refer to this preparation process as quantum coding. For each bit of the \(k = \lceil \log |T| \rceil\) bits long tag \(T_Y = h(Y)\), Alice prepares a quantum state \(|\psi\rangle = |\psi\rangle (X_i, (T_Y)_i)\) determined by the bit \(X_i\) from the PRG and the corresponding bit \((T_Y)_i\) of 2-radix representation of the tag \(T_Y\). Then, if the bit \(X_i = 0\), Alice prepares \(|\psi\rangle\) using basis \(B_0\), such that

\[
|\psi\rangle = \begin{cases} 
|0\rangle & \text{if } (T_Y)_i = 0 \\
|1\rangle & \text{if } (T_Y)_i = 1.
\end{cases} \quad (1)
\]

Similarly, if the bit \(X_i = 1\), Alice prepares \(|\psi\rangle\) using basis \(B_1\), such that

\[
|\psi\rangle = \begin{cases} 
|+\rangle & \text{if } (T_Y)_i = 0 \\
|−\rangle & \text{if } (T_Y)_i = 1
\end{cases} \quad (2)
\]

After the qubits generation, Alice sends the separable state \(|\psi_Y\rangle^\otimes k\) to Bob through a noiseless quantum channel and the message \(Y\) through an unauthenticated classical channel. At the reception, Bob performs measurements to obtain a sequence of \(k\) bits from the quantum encoded version of \(h(Y)\). For the \(i\)-th received qubit, Bob measures it using the basis \(B_0\) or \(B_1\) depending on the \(i\)-th bit of \(X\) is 0 or 1, respectively, recovering a \(k\)-bit long string \(T' = h' (|\psi\rangle^\otimes k)\).

Because the quantum channel is assumed to be perfect, Bob recognizes that the message is authentic if \(h' = h(Y_B)\), where \(Y_B\) is the message received from the classical channel. Otherwise, Bob assumes that Eve tried to send him an unauthentic message. This concludes
the authentication protocol for one message. Throughout this article it is always assumed that the above coding rule is public.

Even though we assume a noise-free quantum channel, we observe that if the quantum channel is noisy, the only piece of information requiring error-protecting coding is the block of bits \((T_Y)_i\) of the tag \(T_Y\). The sequence of bases to be prepared by Alice and Bob is known a priori, determined locally by the sequence of bits from the PRG. A future task is evaluating the effects of the utilization of error-correcting codes to the bits of \(T_Y\).

FIG. 2: First proposal of quantum-enhanced authentication scheme

In a warning against alleged collective attacks, we notice that our analysis allows Eve to make general procedures (suggested in Figure 2 by the block labeled POVM) without being detected. Our results are robust to such powerful and unrealistic assumption for the attacker. Note that our quantum scheme aims at minimizing the key length for one-way transmission. Another example of such an approach is given in [15]. Next we focus on crucial aspects of the PRGs.

**Weak pseudo-random generators**

Clearly, it is important to understand how secure the authentication code described above is. As we shall see, the security of the authentication code is deeply related with to quality of the pseudo-random generator. The quality of a pseudo-random generator is evaluated by the hardness to discriminate its pseudo-random sequence output from a truly random sequence or by the hardness to find its seed. The first quality evaluation relates to the PRG’s robustness against *distinguishing attacks*, the second relates to the so-called *state recovery attacks*. In [8] it is shown that a state recovery attack is a subclass of the distinguishing attacks.
As a matter of fact, if the pseudo-random generator can be attacked by a quantum computer so does the authentication code. To set this result we refer to Figure 3, that describes a simple scheme to assist us the proof. In this scheme, we simply allow Eve to compare a sequence \( \{Y_i\} \) of classical bits with the corresponding sequence \( \{Z_i\} \) obtained from the measurement apparatus POVM.

Recall that a pseudo-random generator is a polynomial-time family of functions \( G_n : \mathbb{Z}_2^n \times \mathbb{N} \rightarrow \mathbb{Z}_2 \) where \( \mathbb{Z}_2 \) is the set \{0, 1\} and \( G_n \) is the pseudo-generator for seeds with size \( n \), that is, \( G_n(X^{(n)}, i) \) returns the \( i \)-th bit generated from \( n \) bits long seed \( X^{(n)} \).

Pseudo-random generators are expected to fulfill an indistinguishability property that we will not detail here for the sake of simplicity (more details on [7]). In the following definition we write \( X^{(n)}_{p(n)} = (G(X^{(n)}, i_1), G(X^{(n)}, i_2), \ldots, G(X^{(n)}, i_{p(n)})) \) to denote a subsequence of \( p(n) \) (not necessarily contiguous) bits generated by \( G \).

**Definition II.2** We say that a pseudo-random generator \( G \) is *attackable in (quantum/probabilistic) polynomial time* if there exists a (quantum/probabilistic) polynomial time algorithm \( P \) and polynomial \( p \) such that if \( P \) is fed with a subsequence of \( p(n) \) (not necessarily contiguous) generated bits \( X^{(n)}_{p(n)} \) of \( G \) we have that:

\[
H(X^{(n)}|P(X^{(n)}_{p(n)})) \in O(2^{-n}).
\]

For a pseudo-random generator to be attackable, there must exist an algorithm (quantum or probabilistic) that receives a subsequence of \( p(n) \) generated bits (not necessarily contiguous) and is able to compute the seed up to a negligible uncertainty. We observe that the security/randomness of the pseudo-random generator can not be grounded in the fact that the attack can only be performed to a contiguous subsequence of generated bits. This is due to the fact that the generator could always hide some bits if the attack required this type of sequences. A simple example of a pseudo-random generator that can be attackable in polynomial time are the pseudo-number generators based on linear congruence [8].

**Theorem II.3** If a pseudo-random generator \( G \) is *attackable in (quantum/probabilistic) polynomial time* then the scheme presented in Figure 3 is not secure in polynomial-time for a quantum adversary that has access to \( Y = \{Y_i\} \).

**Proof.** Since \( G \) is attackable there exists a quantum polynomial time algorithm \( P \) and a polynomial \( p \) such that if \( P \) is fed with \( p(n) \) bits of the string \( X \) generated by \( G \) then \( P \)
computes (up to negligible uncertainty) the seed $X^{(n)}$ of $G$. So it is enough to show that Eve, upon capturing the qubits generated by $QC$, is able to recover (with non-negligible probability) $p(n)$ bits of $X$.

Indeed, assume that Eve has captured $8p(n)$ qubits $|\psi\rangle_i, i : 1 \ldots 8p(n)$ and has measured them in a random basis (that is, either the computational or the diagonal basis). Eve can now verify if $Z_i = Y_i$. If this occurs Eve does not now if the basis chose to encode the $Y_i$ bit was the basis she measured or if she got with $\frac{1}{2}$ probability the correct bit due to encoding in the other basis. However, if the outcome is different (that is, $Y_i \neq Z_i$), then she knows that the basis at the $i - th$ bit is the basis she did not choose the measure, because no mismatch would be possible if the encoding was performed with the same basis. In the latter case, she knows that $X_i$ is either 0 or 1 depending if she measured in the diagonal or the computational basis, respectively. Moreover, this happens with $1/4$ probability. So the probability of Eve not obtaining $p(n)$ elements of $X$ by measuring $8p(n)$ qubits is given by the cumulative function of a binomial distribution with $1/4$ Bernoulli trial, $8p(n)$ trials and success of at most $p(n)$. By Hoeffding’s inequality this probability is upper-bounded by $\exp\left(-2\frac{(8p(n)/4-p(n))^2}{p(n)}\right) = \exp(-2p(n))$ which decreases exponentially with $n$, and so in other words, Eve has an exponentially increasing probability of obtaining $p(n)$ bits of $X$ with $8p(n)$ qubits measurements. Since $G$ is attackable by knowing $p(n)$ bits of $X$, Eve is able to perform this attack up to negligible probability. □

**Corollary 1** If a pseudo-random generator $G$ is attackable then the scheme presented in Figure 2 is not secure in polynomial-time for a quantum adversary that has access to hash function $h$.

**Proof.** Eve is able to calculate $h(Y)$ from $Y$ that is public. Therefore she can apply Theorem II.3 by observing a number $N$ of tags such that $N \log |\mathcal{T}| \geq 8p(n)$. □
Although Theorem II.3 points that the quantum coding of Figure 3 is not better asymptotically than the classical coding (where we simply replace the quantum coder QC by a XOR gate), it seems harder to attack the quantum scheme. We will now show that this is true for the simple case where the encoder is fed by an independent and identically distributed (i.i.d.) Bernoulli sequence. The following example illustrates that this is true even for a very simple generator.

**Example II.4 (State Recovery Attack for Linear Congruential Generator (LCG))**

Let \( A \) be a positive integer and \( \mathbb{Z}_A \) the set of integers modulo \( A \). The seed of the LCG is the vector \( X^{(n)} = (A, s_0, a, b) \), where \( s_0, a, b \in \mathbb{Z}_A \). The length of the seed is \( n = 4\left\lceil \log A \right\rceil \).

A binary pseudorandom sequence with length \( N \times \left\lceil \log A \right\rceil \) bits is obtained from the 2-radix expansion of the sequence \( s = \{s_1, s_2, \ldots, s_N\} \) created by the following recursion:

\[
\begin{align*}
    s_i &= as_{i-1} + b \mod A, \quad i = 2, 3, \ldots, N \tag{3}
\end{align*}
\]

It is well known (see [8]) that for all \( i, i = 1, 2, \ldots, N - 3 \), the numbers

\[
\delta_i = \det \begin{bmatrix}
    s_i & s_{i+1} & 1 \\
    s_{i+1} & s_{i+2} & 1 \\
    s_{i+2} & s_{i+3} & 1
\end{bmatrix}
\]

are multiple of \( A \). As a consequence, the greatest common divisor GCD of some \( \delta_i \)'s gives the value of \( A \). The rest of the seed, that is \( a, b \) and \( s_0 \), follow then from a system of linear equations. In practice five values of \( \delta_i \) are enough.

Figure 4(right) displays a simplified version of the scheme shown in Figure 3, where \( X \) stands for the pseudo-random sequence from the output of the PRG. The left side of Figure 4 displays the situation when a gate XOR is utilized. We notice that the state recovery attack is applicable without change to the XOR-based scheme. It is enough to compute \( X = Z \oplus Y \) before applying the algorithm. In contrast, for the quantum scheme, Eve is submitted to an irreducible uncertainty on the \( X \) values due to quantum coding. In particular, if she employs the procedure described in the proof of the Theorem II.3 it is expected only one fourth of the \( X \)'s are expected to be correct. The problem from Eve's point of view is how to solve the seed from a degraded version of the algorithm input \( X \).
III. COMPARING XOR WITH QUANTUM CODING

In the last section we have considered the problem of the state recovering attack and defined the weakness of a PRG. In this section we make a rigorous comparison between the XOR and the quantum coding performances using information-theoretical measures. To this end consider Figure 4 where both classical and quantum encodings are displayed. The QC denotes the quantum encoder defined before, in (1) and (2), where \( X \) is the variable that sets the basis. The block POVM stands for the measurement apparatus defined by the positive operator-valued measure

\[ Z = \{ E_m(Y) \}_{m \in O} \]

where \( O \) is the set of outcomes. Observe that the measurement may depend on the message \( Y \), which is public. The goal of Eve is to maximize the knowledge of \( X \), that is, minimize the entropy \( H(X|Y, Z) \).

We consider the classical and quantum scheme presented in Figure 4 in two ways: Firstly, we will assume that \( X \) is a sequence of fair and independent Bernoulli random variables, that is, the PRG describing \( X \) is perfect. Secondly, we consider a biased PRG (unfair) to describe \( X \) and introduce blocks of random variables into the analysis.

**Fair input single-sized block**

We start with the simple case of a single-sized block and where \( X \sim \text{Ber} \left( \frac{1}{2} \right) \). In the classical XOR encoding case we have that \( Z = X \oplus Y \) and thus \( H(X|Y, Z) = 0 \), and so Eve has no doubt about \( X \). In the quantum encoding case, the Holevo bound states that

\[ I(X; Z|Y) \leq S(\rho(Y)) - \sum_{i=0}^{1} \frac{1}{2} S(|\phi_i(Y)\rangle\langle\phi_i(Y)|) \]

(4)
where $\rho(Y)$ is the density operator describing the encoding by QC that is
\[
\rho(Y) = \frac{1}{2}|\phi_0(Y)\rangle\langle\phi_0(Y)| + \frac{1}{2}|\phi_1(Y)\rangle\langle\phi_1(Y)|,
\] (5)
where $|\phi_0(0)\rangle = |0\rangle$, $|\phi_1(0)\rangle = |+\rangle$, $|\phi_0(1)\rangle = |1\rangle$ and $|\phi_1(1)\rangle = |\rangle$. We shall need a well known property of the von Neumann entropy (see [4] for more details).

**Proposition III.1** Let $\rho$ be a quantum state and $S(\rho)$ its entropy, then $S(\rho) \geq 0$, and the equality holds iff $\rho$ is a pure state.

Thus, thanks to Proposition III.1 we can simplify (4) to
\[
I(X;Z|Y) \leq S(\rho(Y)).
\] (6)

Moreover, one can compute easily the von Neumann entropy of $S(\rho(Y)) = S(\rho(0)) = S(\rho(1))$ and is
\[
S^* = S(\rho(Y)) = -2\cos^2\left(\frac{\pi}{8}\right) \log \left(\cos\left(\frac{\pi}{8}\right)\right) - 2\log \left(\sin\left(\frac{\pi}{8}\right)\right) \sin^2\left(\frac{\pi}{8}\right).
\] (7)

And so, since $H(X|Z,Y) = H(X|Y) - I(X;Z|Y)$ and $H(X|Y) = 1$, the minimum uncertainty that Eve may attain about $X$ is given by
\[
H(X|Y,Z) = 1 - S(\rho(Y)).
\] (8)

The Holevo bound can be achieved by a simple von Neumann measurement [12, pp.421] described by the Hermitian
\[
A = 0|\psi_\theta\rangle\langle\psi_\theta| + 1|\psi_\theta^\perp\rangle\langle\psi_\theta^\perp|,
\] (9)
with $\psi_\theta = \cos(\theta)|0\rangle + \sin(\theta)|1\rangle$, $\psi_\theta^\perp = \sin(\theta)|0\rangle - \cos(\theta)|1\rangle$ and $\theta = -\frac{\pi}{8}$.

**Fair input $k$-blocks**

First, consider the classical setup, then $H(X^k|Y^k,Z^k) = 0$, since the block $X^k$ is completely determined from the knowledge of $Y^k$ and $Z^k$.

For the quantum setup, the subsystem that Eve owns is described by
\[
|\rho_{Y^k} = \bigotimes_{i=1}^k \left(\frac{1}{2}|\phi_0(Y_i)\rangle\langle\phi_0(Y_i)| + \frac{1}{2}|\phi_1(Y_i)\rangle\langle\phi_1(Y_i)|\right).
\] (10)
By the Holevo bound we get that

\[ H(X^k|Y^k, Z^k) \geq H(X^k) - S(\rho_{Y^k}). \]  

(11)

**Example III.2** Table I illustrates the scenario for \( k = 2 \). Rows are indexed by the four possible values of \( Y^2 \) and columns are indexed by the bases corresponding to the four values of \( X^2 \). Notice that Eve is not able to distinguish which column is being used. Then, her uncertainty is lower bounded by the von Neumann entropy of the quantum system formed by states listed in row indexed by the values of \( Y^2 \) that she can access.

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TABLE I: Encoding for blocks of length 2

Recall the following property concerning the von Neumann entropy.

**Proposition III.3** Let \( \rho \) and \( \sigma \) be quantum states, then \( S(\rho \otimes \sigma) = S(\rho) + S(\sigma) \).

As a consequence of Equation (10) and Proposition III.3, for a sequence of fair Bernoullis we have

\[ S(\rho_{Y^k}) = kS^*, \]  

(12)

where \( S^* \) is given by (8). So we have that

\[ H(X^k|Y^k, Z^k) \geq k - kS^*. \]  

(13)

Again, the equality can be achieved by a simple von Neumann measurement, namely that defined by \( A^{\otimes k} \). This is the best scenario one can imagine to defeat Eve. However, for the protocol to be practical, the \( X \)'s should be generated by a PRG, which is the case we examine next.
Unfair input $k$-blocks

The results above where obtained assuming that $\{X_i\}$ was a sequence of i.i.d. fair Bernoulli random variables. In this section we study the general case, with the purpose of clarifying how the use of a real PRG affects the uncertainty about $X$.

Consider $k-$length blocks $X^k$, $Y^k$ and $Z^k$, where $X^k = X_{i+1}, X_{i+2}, \ldots, X_{i+k}$ is a contiguous subsequence of $\{X_i\}$ and similarly to $Y^k$ and $Z^k$. Note that, to ease notation, we omit the index $i$ in defining $X^k$. However, it is crucial to remark that the probability distribution of $X^k$ is, in general, dependent on $i$. As a matter of fact, $p_{X^k} = (p_0, p_2, \ldots, p_{2^k-1})$ can even degenerate to a distribution with a single component equal to 1, depending on the robustness of the PRG. We shall simplify the notation denoting $p_{X^k}$ by $p$.

Concerning the unfairness of $\{X_i\}$, the best strategy for Eve to get information from $X^k$ is to prepare a measurement (POVM) over the all $k$ qubits sent, given that she knows $Y^k$. Again, the Holevo bound gives us

$$H(X^k|Y^k, Z^k) \geq H(X^k) - S(\rho_{Y^k}) = H(X^k) - H(\lambda)$$

(14)

where $\lambda = (\lambda_1 \ldots \lambda_{2^k})$ is the spectrum of $\rho_{Y^k}$ and

$$\rho_{Y^k} = \sum_{j=0}^{2^k-1} p_j |\phi_j\rangle \langle \phi_j|$$

(15)

where the states $|\phi_j\rangle = \otimes_{i=1}^{k} |\phi_{j,i}(Y_i)\rangle$ and $j_i$ is the $i$-th bit of the binary representation of $j$.

Note that $\rho_{Y^k}$ is a mixture of pure states weighted by the probabilities $p_j$, $j \in \{0, \ldots, 2^k-1\}$. Accordingly, we write $p_j = \Pr[X^k = j]$ where $j$ is seen in its binary representation (e.g., for $k = 2$, $p_0 = \Pr[X^2 = 00]$, $p_1 = \Pr[X^2 = 01]$, $\ldots$). Observe that $S(\rho_{Y_1}) = S(\rho_{Y_Z})$ since there exist a unitary transformation $U$ such that $U \rho_{Y_1} U^{-1} = \rho_{Y_Z}$.

We now establish a relationship between the probability vectors $p_{X^k}$ and the lower bound given in Equation (14).

Denote by $\rho$ the uniform distribution, that is, $q_j = 1/2^k$, $j = 0, \ldots, 2^k-1$. In this section we shall verify that if the probability distribution of a block $X^k$ from the PRG, say $p$, is near enough the distribution $\rho$, for a block of size $k$, then the lower bound of (14) will be kept significantly near of $k - kS^*$, which is the best one can achieve.

Let $\sigma_{Y^k}$ be the density operator corresponding to a $k-$length block $X^k$ generated by a
fair Bernoulli sequence given that the $k$-length block $Y^k$ is known, that is

$$\sigma_{Y^k} = \sum_{j=0}^{2^k-1} q_j |\phi_j\rangle\langle \phi_j|.$$  

(16)

In this section we establish some results relating von Neumann entropy with the trace distance $D(\rho_{Y^k}, \sigma_{Y^k})$ between $\rho_{Y^k}$ and $\sigma_{Y^k}$. Recall that the trace distance between two quantum states $\rho$ and $\sigma$ is defined by

$$D(\rho, \sigma) = \frac{1}{2} \text{tr} |\rho - \sigma|$$

where $|A| = \sqrt{A^\dagger A}$. We shall also need the trace distance between probability vectors, say $a$ and $b$, defined by

$$D(a, b) = \frac{1}{2} \sum_j |a_j - b_j|.$$ 

The trace distance can be used to measure how biased a probability distribution is compared to a fair Bernoulli sampling. Given a probability distribution $p$, we call the bias of $p$ the value $B(p) = D(p, \rho)$ where $\rho$ is the uniform distribution.

**Proposition III.4** Let $\varepsilon > 0$ be an arbitrary real number. If

$$B(p) \leq \varepsilon$$

(17)

then

$$D(\rho_{Y^k}, \sigma_{Y^k}) \leq \varepsilon$$

(18)

where $\rho_{Y^k}$ is the state defined in Equation (15).

**Proof.** Denote $\gamma_j = |\phi_j\rangle\langle \phi_j|$. From the strong convexity of the trace distance we have:

$$D \left( \sum_{j=0}^{2^k-1} p_j \gamma_j, \sum_{j=0}^{2^k-1} \frac{1}{2^k} \gamma_j \right) \leq D(p, \rho) + \sum_{j=0}^{2^k-1} p_j D(\gamma_j, \gamma_j)$$

(19)

$$= D(p, \rho)$$

(20)

which concludes the proof. $\Box$

In the proof of the next proposition we shall apply Fannes’ inequality (see [14] for more details about this equality):

$$|S(\rho) - S(\sigma)| \leq 2D(\rho, \sigma) \ln \left( \frac{N}{2D(\rho, \sigma)} \right)$$

(21)

where it is assumed that $D(\rho, \sigma) \leq 1/(2e)$ and $N$ is the dimension of the Hilbert space dimension where the states live in.
Theorem III.5 If the conditions in Proposition III.4 hold, that is, if $B(p) \leq \varepsilon$, then

$$|H(X^k) - S(\rho_{Y^k}) - (H(X^k) - S(\sigma_k))| = |S(\rho_{Y^k}) - S(\sigma_k)|$$  \hspace{1cm} (22)

$$\leq 2\ln2(k - 1)\varepsilon + 2\varepsilon \ln \frac{1}{\varepsilon}. \hspace{1cm} (23)$$

**Proof.** Observe that the function $-x\ln x$ is monotonous in the interval $(0, 1/e)$. Therefore, assuming $0 \leq \varepsilon \leq 1/e$ and for $N = 2^k$, we have:

$$|S(\rho_{Y^k}) - S(\sigma_k)| \leq (a) 2D(\rho_{Y^k}, \sigma_k)\ln\frac{2^k}{2D(\rho_{Y^k}, \sigma_k)}$$  \hspace{1cm} (24)

$$\leq (b) 2\ln2(k - 1)D(\rho_{Y^k}, \sigma_k) + 2D(\rho_{Y^k}, \sigma_k)\ln\frac{1}{D(\rho_{Y^k}, \sigma_k)}$$  \hspace{1cm} (24)

$$\leq (c) 2\ln2(k - 1)\varepsilon + 2\varepsilon \ln \frac{1}{\varepsilon} \hspace{1cm} (25)$$

where (a) results from Fannes’ inequality, (b) is due to logarithm properties and, (c) is due to Proposition III.4. □

This result states that if a PRG is such that the probability distribution of its output $X^k$, say, $p$ (possibly conditioned on the past), is near enough the fair distribution $\rho$, then Eve’s uncertainty is kept near the maximum $H(X^k|Y^k, Z^k) = k - kS^*$ (see Equation (13)).

Note that the distribution of $p$ is induced by the random secret seed of the PRG, $X^{(n)}$, which is chosen with uniform distribution. Consequently, any practical use of Equation (23) will depend on the Eve’s capability to estimate that distribution and clearly, on the PRG being used. For instance, suppose we want to upper bound the right side of (23) with a given tolerance defined by a positive real number $\delta$. After some simple algebraic manipulation we obtain that

$$k < 1 + \frac{1}{2\ln2} \left(\frac{\delta}{\varepsilon}\right) - \frac{\ln \left(\frac{1}{\varepsilon}\right)}{\ln2}. \hspace{1cm} (26)$$

For the case of $\varepsilon = \delta$ we get the simple bound

$$k < 1 + \frac{1}{2\ln2} - \frac{\ln \left(\frac{1}{\varepsilon}\right)}{\ln2} \leq 1 + \frac{1}{2\ln2} + \frac{(\frac{1}{\varepsilon}) - 1}{\ln2} \leq 1 + \frac{1}{\varepsilon \ln4}. \hspace{1cm} (27)$$

Additionally, when the conditions of Proposition III.4 hold, that is, for bias $B(p) < \varepsilon$, we can rewrite (26) as

$$k < 1 + \frac{1}{2\ln2} \left(\frac{\delta}{B(p)}\right) - \frac{\ln(1/B(p))}{\ln2}. \hspace{1cm} (28)$$
Note that the right-hand side of (28) approximates \( \frac{1}{2\ln 2} \left( \frac{\delta}{B(p)} \right) \) as \( B(p) \) tends to zero. In detail, Equation (28), at the light of Theorem III.5, provides a way to compute the largest block whose uncertainty remains near \( k - kS^* \) (up to \( \varepsilon \)), given an upper bound of the bias of \( p \). However, a word of advice is necessary: from its very definition, \( p \) depends on \( k \) and also on the position \( i \) the block start, because \( X^k = X_{i+1}, X_{i+2}, \ldots, X_{i+k} \). So, the use of Equation (28) to establish a bound of a secure block relies on a bias difficult to compute for standard PRG’s.

The following corollary clarifies the meaning of Theorem III.5 from an asymptotic point of view.

**Corollary 2** Given a PRG, let \( p \) be the probability distribution of a \( k \)-length generated block, and let \( f(k, n) \) and \( g(n) \) positive functions such that:

- \( \lim_{n \to \infty} g(n) = +\infty \)
- \( \lim_{n \to \infty} g(n)f(g(n), n) = 0 \).

Then, if \( B(p_{PRG}) \leq f(k, n) \) and \( k \leq g(n) \),

\[
\lim_{n \to \infty} |S(\rho_{Y_{g(n)}}) - S(\sigma_{g(n)})| = 0.
\]

We now discuss the results above. The idea is that \( n \) is the size of the seed of the PRG and \( k \) is the size of the block. If one chooses \( k \leq g(n) \) for some \( g \) and the bias of the PRG is smaller the \( f(k, n) \) for some \( f \) fulfilling the conditions of Corollary 2, then the information Eve can retrieve from blocks of size \( g(n) \) is as close to the ideal case as desired, just be choosing a larger \( n \). A good PRG is one for which \( n << g(n) \), so that the block size could be larger than the seed and still, little information about the seed is revealed.

In the next section we make a comparison between classical XOR and quantum QC Brassard’s schemes for authentication of classical messages.

### IV. IMPROVING KEY-TAG SECRECY

In the last section we compared Eve’s equivocation on \( X \) for the XOR and QC schemes when she has access both to the message \( Y \) and its quantum encoded version, which she observes from the quantum channel. We concluded that the equivocation is kept above some lower
bound depending on the quality of the PRG. In this section we include a hash function \( h \) in the scheme (see Figure 5) in such a way that Eve only accesses the public message \( Y \) and the quantum encoded version of the tag \( T = h(Y) \). Thanks to that modification we shall demonstrate that is feasible to improve the secrecy of the key and of the tag simultaneously.

By information-theoretic secrecy, as usually, we mean \( I(W;V) = 0 + O(2^{-n}) \) or equivalently, the equivocation \( H(W|V) = H(W) - O(2^{-n}) \), where \( W \) is the secret to be protected and \( V \) is the piece of data available to the eavesdropper. Our derivations will focus in the equivocation \( H(W | V) \) to measure the quality of the scheme. Then, the information to be protected is \( W = (T,X^k) \) and the information available, from Eve’s viewpoint, is \( V = (Y,Z) \). We investigate the uncertainty of the tag \( H(T | Y,Z) \) and the uncertainty of the key \( H(X^k | Y,Z) \). We assume that \( X^k \) is independent of the message \( Y \) and that the hash function is selected from the \( \varepsilon \)-almost universal-2 class of hash functions, which we refer in the following just as \textit{hash functions}.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{Authentication scheme with a single key \( X^{(n)} \)}
\end{figure}

\textbf{Modified classical case}

Consider a simple modified setup where a XOR gate is taken in place of the QC block in the scheme displayed in Figure 5.

If \( \{X_i\} \) is a fair Bernoulli and a \( k \)-block of bits such that \( k = \max\{\lceil \log |T| \rceil, \lceil \log |\mathcal{H}| \rceil \} \) it is utilized per message, then the scheme turns to be equivalent to the Wegman-Carter one-time pad scheme. Indeed, in this situation \( h \) is in fact drawn uniformly from \( \mathcal{H} \), then

\begin{align*}
H(T,X^k|Y,Z^k) & \overset{a}{=} H(T|Y,Z^k) + H(X^k|T,Y,Z^k) \\
& \overset{b}{=} H(T|Y,Z^k) \\
& \overset{c}{=} H(T|Y)
\end{align*}
Where equality (a) is due to chain rule for Shannon entropy, (b) is due to the fact that in the classical setup $X^k = T \oplus Z^k$. The equality (c) is harder to obtain, indeed it follows from the properties of the $\varepsilon$–almost universal-2 class of hash function. Note that $T = h_{X^k}(Y)$ has a uniform distribution. Moreover $T|_{x_k} = h_{x_k}(Y)$ has also uniform distribution, and therefore, $T$ is independent of $X^k$. Since $Z^k = f(X^k, Y)$ we have that $H(T|Y, Z^k) = H(T|Y)$. Equality (d) is also due to the properties of hash functions. On the other hand, if $\{X_i\}$ comes from a PRG, the Eve’s uncertainty on the tag can, eventually, decreases by observing the random variable $Z^k$. Indeed, in general, $H(T|Y, Z^k) < H(T|Y)$. Consequently, unconditional secrecy relative to $T$, $H(T|Y, Z^k) = \log |T|$ cannot be assured.

Uncertainty of the tag in the quantum case

In this subsection we introduce a condition to attain unconditional security of the tag in terms of conditioned mutual information between $T$ and the $k$–block of bits of the key.

**Proposition IV.1** If $I(T; X^k|Y, Z^k) = H(T)$ then the tag is secure in the information theoretical sense, that is, $H(T|Y, Z^k) = H(T)$.

**Proof.**

From the standard chain rule of Shannon entropy we have:

\[
H(T, X^k | Y, Z^k) = H(X^k | Y, Z^k) + H(T | X^k, Y, Z^k) \tag{33}
\]

\[
= H(T|Y, Z^k) + H(X^k|T, Y, Z^k). \tag{34}
\]

Then, comparing (33) and (34) we obtain

\[
H(T|Y, Z^k) \overset{(a)}{=} H(T|X^k, Y, Z^k) + \]

\[
H(X^k|Y, Z^k) - H(X^k|T, Y, Z^k) \tag{35}
\]

\[
\overset{(b)}{=} H(T|X^k, Y, Z^k) + I(T; X^k|Y, Z^k) \tag{36}
\]

\[
\overset{(c)}{=} I(T; X^k|Y, Z^k) \tag{37}
\]

Where (a) is due to a simple manipulation of (33) and (34), (b) is definition of mutual information and (c) follows because the hash function is determined by $X^k$ and so, then
\[ T = h(Y) \] is immediately calculated. That is, \( H(T|X^k, Y, Z^k) = H(T|X^k, Y, T = h(Y)) = 0 \). The results follows from (37). □

Eq. (37) clearly indicates that in order to increase Eve’s uncertainty about \( T \) we must maximize the mutual information between the block \( X^k \) and the tag \( T \). This is the information-theoretical hint that motivates the scheme presented in Figure 5. Note that in this case we make the tag \( T \) depend of \( X^k \), increasing thus their mutual information. In Brassard scheme (see Figure 1) the hash function is fixed in the beginning, and therefore \( I(T; X^k|V') = 0 \)
where \( V' \) is the observation that Eve can perform in Brassard’s scheme.

It is remarkable to be possible to attain unconditional security of the tag using non-fair Bernoulli for \( X \) with the proposed of Figure 5. This fact is in sharp contrast with the classical setup for which only Bernoulli sequences can assure that requirement.

Thus, a good approximation is to use PRG for the sequence of \( X \), and the mutual information \( I(T; X^k|Y, Z^k) \) is as high as the PRG is unbiased, since that mutual information is mediated by the random variable \( Z^k \).

It is clear that, if we are dealing with real PRGs (that do not generate a sequence of fair Bernoullis), then the conditions of Theorem III.5 should be considered in order to evaluate the number of messages that can be authenticated before leaking too much information. Another possibility to apply the scheme of Figure 5 is to spend just \( k = \log |\mathcal{T}| \) key bits per message to protect the current tag. This approach is similar to Brassard’s scheme, but improves it since the tag is protected by the quantum coding. Observe that as \( \log |\mathcal{T}| < \log |\mathcal{H}| \), this scheme is less costly in terms of key consumption.

**Uncertainty of the key in the quantum case**

In this case, the bounds derived in Section III remain valid, namely the inequality (14) that we recall

\[
H(X^k|Y^k, Z^k) \geq H(X^k) - S(\rho_{Y^k}) = H(X^k) - H(\lambda).
\]

(38)

In this case, since the measurement \( Z^k \) is on the quantum encoding of the tag, and not on the quantum encoding of \( Y \), the uncertainty is greater than that of the case discussed in Section III.

So, with the scheme of Figure 5, not only we obtain a high equivocation about the tag, but we also increase the uncertainty of the sequence \( X^k \) and, therefore, also of the seed \( X^{(n)} \)
of the PRG. Observe that Theorem III.5 and inequality 26 are also valid for this scheme, and can be used to get bounds about the size of $k$ for which a threshold of information is leaked to Eve.

V. SUMMARY

In this work we have investigated how quantum resources can improve the security of Brassard’s classical message authentication protocol. We have started by showing that a quantum coding of secret bits offers more security than the classical XOR function introduced by Brassard. Then, we have used this quantum coding to propose a quantum-enhanced protocol to authenticate classical messages, with improved security with respect to the classical scheme introduced by Brassard in 1983. Our protocol is also more practical in the sense that it requires a shorter key than the classical scheme by using the pseudorandom bits to choose the hash function. We then establish the relationship between the bias of a PRG and the amount of information about the key that the attacker can retrieve from a block of authenticated messages. Finally, we prove that quantum resources can improve both the secrecy of the key generated by the PRG and the secrecy of the tag obtained with a hidden hash function.

Acknowledgments

F. M. Assis acknowledges partial support from Brazilian National Council for Scientific and Technological Development (CNPq) under Grants No. 302499/2003-2 and CAPES-GRICES No. 160. P. Mateus and Y. Omar thank the support from project IT-QuantTel, as well as from Fundação para a Ciência e a Tecnologia (Portugal), namely through programs POCTI/POCI/PTDC and projects PTDC/EIA/67661/2006 QSec and PTDC/EEA-TEL/103402/2008 QuantPrivTel, partially funded by FEDER (EU).


