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Fibring Labelled First-order Based Logics

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Fibrilação de Lógicas Etiquetadas de base Primeira Ordem

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Provas concluídas em:

Resumo: A dedução e a semântica etiquetada para lógicas de base primeira ordem são desenvolvidas tendo em conta a fibrilação. Um teorema de correcção para sistemas lógicos etiquetados é provado e são definidos os critérios para um sistema etiquetado ser apresentação de uma lógica. De seguida identificam-se os conjuntos canónicos e os conjuntos apropriados de fórmulas etiquetadas, e prova-se que estruturas canónicas se baseadas em conjuntos canónicos são bem definidas e se baseadas em conjuntos apropriados satisfazem as regras do sistema. Mostra-se também que um sistema etiquetado rico é completo assim como sistemas cheios e apropriados. Define-se de seguida a classe dos sistemas ligados e estudam-se duas suas subclasses: os sistemas com negação clássica, local ou não local, e os sistemas com localidade. Mostra-se que os sistemas com negação clássica são apropriados bem como os sistemas com localidade que satisfazem uma restrição adicional. Os resultados são analisados, e ilustrados com exemplos variados. De seguida, define-se a fibrilação e estuda-se a preservação da correcção, consequência, consequência semântica e completude. A terminar estuda-se a fibrilação de sistemas lógicos etiquetados para, a lógica modal de primeira ordem e um fragmento positivo da lógica básica da relevância, e prova-se a sua correcção e completude com base nos resultados de preservação.

Palavras-chave: Fibrilação; Sistemas de Dedução Etiquetados; Completude; Lógicas de base Primeira Ordem; Lógicas Relevantes; Lógicas Modais.

Fibring Labelled First-order Based Logics

Abstract: Labelled deduction and semantics for first-order based logics are developed, taking into account fibring. A soundness theorem is established and the criteria for a labelled logic system to be a labelled presentation of a logic are defined. After that, canonical sets and appropriate sets are defined and it is proved that canonical structures are well-defined whenever based on canonical sets, and satisfy all the rules of the underlying deduction system whenever based on appropriate sets. General completeness theorems for rich labelled logic systems, and full and appropriate labelled logic systems, are established. Connected logic systems are introduced and two subclasses identified: systems with a classical negation, local or non-local, and systems with locality. It is shown that systems with a classical negation are appropriate as well as systems with locality satisfying an additional restriction. These results are discussed and illustrated with sound and complete labelled logic systems for first-order modal logic, relevance logic and some positive fragments of them. Then, fibring is defined, and preservation of soundness, entailment, consequence, and completeness, are studied. Finally, labelled logic systems for first-order modal logic and a fragment of basic relevance logic are fibred, and relying on the preservation results, their soundness and completeness are shown.

Keywords: Fibring; Labelled Deduction Systems; Completeness; First-Order based Logics; Relevant Logics; Modal Logics.

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Contents

Chapter 1

Introduction

Combination of logics is a very important research topic both from a theoretical point of view and from a practical point of view, [13, 14, 30, 47, 24]. From a theoretical point of view because (i) there are logics that can be seen as a combination of other simpler logics, (ii) it allows to build new logics from some known ones, [75, 37, 31, 26, 24, 6, 74, 54, 71, 35]. The point (i) is important since, if the technique of combination used has transference results associated, we can establish properties about the logic seen as the combination of the others just by relying on properties of the component logics. The point (ii) is important since we can invent new logics, either with interesting properties or with good proof theoretic attributes or simply driven by the curiosity about the combination. From a practical point of view combination of logics has very important applications on software engineering, artificial intelligence and computational linguistics, [36, 16, 15, 57, 40, 23].

In this dissertation we study the fibring technique of combining logics when applied to labelled first-order based logics.

1.1 Context

The core of the work presented in this thesis was developed in the scope of the project *FibLog*, a project of the *Center for Logic and Computation*, led by Professor Amílcar Sernadas, on fibring and other constructions for combining logics, with applications to the development of mixed logics, such as logics of hybrid systems and logics of authentication.

The techniques used throughout the thesis rely and are greatly influenced by the work of the *Center for Logic and Computation*, formerly *Logic and Computation Group*.

1.2 Roots

The pillars of this thesis are fibring, first-order based logics and labelled deduction.

Fibring Fibring is a special kind of combination of logics, introduced by Gabbay [44, 42, 45, 29], and further investigated by the *FibLog* project team led by Sernadas [61, 80, 75, 72, 74, 73, 25, 27]. Among the results obtained in our group we can refer the categorial characterization of fibring of propositional based logics endowed with Hilbert style deduction systems [72], the non-truth functional semantics of fibring of propositional based logics endowed with Hilbert style deduction systems [25], the study of preservation of completeness and soundness by fibring of logics endowed with Hilbert style deduction systems, both first-order based [74] and propositional based [80], and in the fibring of propositional based logics endowed with labelled deduction [61].

But what is fibring? Before describing fibring, we note that by the fibring of labelled first-order based, **lfob**, logics we mean the collection of all lfob logic systems resulting from the fibring of the suitable pairs of lfob logic systems for each logic. So the important notion is the notion of fibring of lfob logic systems. In the following, without loss of generality, we consider the fibring of lfob logics simply as the fibring of a suitable pair of their lfob logic systems, whenever from that single case, it is straightforward to infer the fibring of all the other suitable pairs of lfob logic systems for that logics. Back to the description of fibring, we can say, in a very informal way, that the fibring of any logic systems $\mathcal{L}_1$ and $\mathcal{L}_2$ is a logic system $\mathcal{L}$ having as its language the complete mixture of the languages of $\mathcal{L}_1$ and $\mathcal{L}_2$. For the deduction part of $\mathcal{L}$, it is necessary that $\mathcal{L}_1$ and $\mathcal{L}_2$ are endowed with the same type of deduction components. If this is the case then the deduction part of $\mathcal{L}$ is obtained by the use of the sets of inference rules of $\mathcal{L}_1$ and $\mathcal{L}_2$. Note that it is important that both deduction systems be schematic, in order for any formula be possibly (if allowed by the provisos) considered in the rules of the other logic system. With respect to the semantics of $\mathcal{L}$, it is essential that $\mathcal{L}_1$ and $\mathcal{L}_2$ have the same type of semantic structures. If this is the case then each model of $\mathcal{L}$ is such that its $\mathcal{L}_1$ part, i.e., the model without the denotations of the connectives from $\mathcal{L}_2$, constitute a model in $\mathcal{L}_1$, and similarly for $\mathcal{L}_2$.

Note that, as is evident from the previous paragraph, fibring has been studied only in the context of what we call, the deductively homogeneous scenario, since we are considering only logic systems endowed with deduction systems of a similar kind. In fact, it has not yet been carried out the study of the fibring of logic systems presented by heterogeneous deduction systems, for example, the fibring of a logic system presented by tableaux with a logic system endowed by a Hilbert style deduction system. Semantically, for fibring to be defined in a class of logic systems, it is necessary that, for any logic system in that class, it is possible to abstract all the properties of a model to a type of interpretation structure common to all logic systems in that class. Note that, for each logic system, the entailment relation based on the class of interpretation structures must be equal to the entailment relation based on the usual models. Fibring is then defined over these structures. Hence, also semantically, we define fibring only in a homogeneous scenario. In fact, the study of the fibring of logics endowed with heterogeneous semantics has not yet been carried out.

Other techniques of combination of logics based on fibring have been and are being developed. We refer here to our work in modulated fibring [75]. In simple terms it differs from fibring on the fact that the relation between the truth values of the components and of the fibring is not the identity but an adjunction. This general technique of combination avoids in a large number of situations the collapse of the combination in one of the systems that are being combined. For instance, in particular, it allows to combine classical logic and intuitionistic logic without a collapse. Note that the fibring of classical logic and intuitionistic logic collapses in classical logic, as was observed by Gabbay in [44]. This problem was open in the literature during some time, until Luis Fariñas del Cerro and Andreas Herzig proposed a non-collapsing combination of that logic systems [31].

First-order based logics The class of first-order based logics is a wide class of logics encompassing first-order logic, first-order modal logic, first-order temporal logic, first-order relevance logic, inter alia, [39, 1, 48, 56, 12, 41, 49, 52, 78]. Moreover, in some of the logics, there are a variety of options concerning its semantics. These options may involve the quantifiers (for instance if their domain of quantification is constant, varying, increasing or decreasing), identity (whether it is present or is not present), terms and predicates (flexible or rigid), among others. For example, for first-order modal logic, multiple systems and variants were proposed due to this variety of options, see [48, 51, 39] for an extensive discussion.

Results on labelled first-order based logics appear in [79, 10, 70, 5]. In [79], Viganò defines proof systems based on labelled natural deduction for a large class of first-order modal logics namely with varying, increasing, decreasing or constant domains. Soundness, completeness and normalization are proved for that logics, and it is established an encoding of that systems in Isabelle. In [70] the author presents labelled natural deduction systems for several predicate modal logics and proves their soundness and completeness.

Labelled deduction The idea behind labelled deduction is to provide more information to the deduction level in the form of labels, in order to obtain deduction systems with better characteristics, [43, 79, 8, 62, 9, 76, 10, 55, 11]. For instance, the usual non-labelled approaches for presenting modal and relevance logics by natural deduction systems do not completely satisfy the criteria for a good encoding in a Logical Framework, see [79, 22, 32, 38], but the labelled natural deduction systems for these logics are well suited for that encoding [79].

In a logic system endowed with labelled deduction, the basic unit of reasoning is the pair label:formula. Moreover, the consequence relation is defined over labelled schema formulae. Note that the label can carry a lot of different information ranging from semantic information like worlds or truth-values, to proof-theoretic information, or to more practical information such as agents in a community, nodes of a net, resources, among others.

Labelled deduction, besides being used to characterize the type of deduction component of a logic system, may also be used to denote a methodology. That methodology provides a general, uniform, modular and natural way, of obtaining labelled deduction systems and semantics for a family of logics satisfying some specific conditions, [43, 28, 79, 20]. The labelled deduction systems obtained in this way are usually Gentzen style deduction systems such as natural deduction systems, sequent systems and tableaux systems, [8, 9, 10, 11, 28, 43, 61, 79]. For instance in [9] and [69], different modal logics were obtained simply by adjoining to the labelled natural deduction system for the base modal logic K, the properties of the relation that characterizes the specific modal logic in question. Observe that the labelled methodology has the advantage of being applied to a wide class of logics and, the advantage that in general the deduction systems obtained from it are intuitive and natural to understand.

The idea of incorporating semantic information in a logic system traces back to the pioneer work of Prior [60] in late 1960's, where the difficulties of incorporating semantic entities in tense logic are discussed. These entities are a new sort of atomic symbol, like propositional variables, that may appear inside formulae. Semantically they have the value *true* precisely at one point in a frame.

Labelled deduction was also analyzed by Gabbay as a methodology both for the theoretical study of logics and for the development of logical systems suitable for the needs of specific applications [43]. This proposal was the starting point for a wide research programme encompassing the development of labelled systems for modal logic [61, 76, 9, 69], temporal logic [46], sub-structural logics [28, 21], quantified modal logics [79, 70, 5, 10], as well as, the general development of labelled deduction system for non-classical logics [11]. Recently the approach was generalized by the investigation of deduction systems labelled by truth values for propositional based logics [62, 76, 55, 7].

Closely related to labelled deduction systems are hybrid logic systems [4, 3, 18, 17, 19], which follow the work of Prior [60], and so are characterized by the presence of a special type of atomic formulae, the *nominals*, which must be true at exactly one world in any model, that are treated as proposition variables.

1.3 Aims

The main aims of this thesis are

- the definition of labelled deduction and labelled semantics for first-order based logics, taking into account fibring, extending our previous work on labelled deduction for propositional based logics [61], and getting inspiration from a similar work for the semantics of non-labelled first-order based logics [74],
- the study of the fibring of labelled first-order based logics,
- the study of preservation results for soundness, completeness, entailment and consequence, for the fibring of labelled first-order based logic systems.

1.4 Overview

This thesis is organized in seven chapters.

In Chapter 1 we introduce the dissertation and briefly describe how it is organized. Moreover we made a general overview of its roots and of its aims. We conclude the chapter by presenting a claim of contributions of this dissertation.

In Chapter 2 we develop the labelled deduction and labelled semantics for first-order based logics. Labelled semantics gets inspiration from a similar work for non-labelled logic systems, [74]. Labelled deduction substantially re-formulates our previous work on labelled deduction for propositional based logics [61], and extends it to the first-order based case.

We start the chapter by defining what can be the signature and the language of a lfob logic system. Note that the signature allows the specification of label functions, and that there are some connectives common to the language of any labelled first-order based logic system. After that we begin the definition of the semantic framework. We introduce interpretation structures with the purpose of representing the information of the models of labelled first-order based logic systems in a form common to all logic systems in that class, and more adequate for fibring. We then define what is a labelled interpretation system for a lfob logic system and, concluding the section, we define satisfaction and entailment.

The deduction part begins with the definition of provisos, substitutions and rules. Since we consider schematic rules, due to fibring, we need substitutions in order to replace the schema variables by admissible formulae or terms. Moreover, rules can be constrained, i.e., there are rules that can only be applied to formulae or terms satisfying some condition, or in certain cases dependent of the conditions of the deduction. So, we need provisos.

Provisos represent conditions on substitutions, and so, are maps on substitutions that indicate if a substitution should be accepted or not. We identify and define two types of provisos: the formula provisos and the deduction provisos. After that we define what is a deduction system. Note that there are rules common to any lfob deduction system. In the sequel we define deduction and the associated consequence operator. We prove that it is extensive, monotonic and idempotent. Additionally we show also that the consequence operator is closed for substitutions. Ending the deduction section we define what is a derived rule and we prove that if there is a proof that uses derived rules then we can also construct a proof for the same deduction without derived rules. After that, lfob logic systems are defined and a soundness theorem established. Finally, ending the chapter, we establish the criteria for a lfob logic system to be a labelled presentation for a certain logic.

In Chapter 3 we study sufficient conditions for completeness. We prove that rich lfob logic systems are complete, and as a corollary that full and appropriate lfob logic systems are complete. A lfob logic system is full when it has all the structures that satisfy its rules, and is appropriate whenever every consistent set can be extended to an appropriate set consistent with respect to the same formula. The importance of the extension to an appropriate set is due to the fact that, independently of the lfob logic system, the canonical structure over an appropriate set satisfy the rules of that system. Note that in a more specific system the appropriateness condition may be equivalent to more specific and known conditions, for example over maximal consistent sets. Moreover observe that the canonical structure is always well defined over an appropriate set since we prove that this is the case for canonical sets, and since appropriate sets are canonical by definition. After obtaining these results we present three counter-examples showing that not all lfob logic systems are appropriate.

In Chapter 4 we investigate under which conditions a lfob logic system is appropriate, motivated by the completeness results of Chapter 3. We restrict our study to a general class of lfob logic systems, that we call connected lfob logic systems. We identify and study a subclass of connected lfob logic systems having a classical negation, local or non-local, and prove that they are appropriate. We then study other conditions not implying the presence of any kind of negation, that could be useful to obtain appropriateness results. As a result we identify the class of connected lfob logic systems with locality. We prove that connected lfob logic systems with locality and without disjunction and implication are appropriate. Note that those appropriate systems may have connectives like universal and existential, constrained or not, quantifiers and modalities, conjunction and any kind of negation, although that negation play no role in the construction. The restriction to logics without disjunction and implication came from the fact that we were not able to prove the appropriateness conditions for the rules $\to_I$ and $\wedge_E$. Nevertheless, observe that it does not imply that there is not a construction of an appropriate set satisfying those rules, in the context of a connected lfob deduction system with locality. Only that we were not able to do it. One path that may allow, using our construction, to build an appropriate set, not relying on any kind of negation, satisfying the appropriate conditions for implication and disjunction, is the consideration of more general forms of the rules, or even of the connectives, see Remark 4.4.8. This is a very interesting topic that we intend to pursue in the future and that may move us closer to hybrid logic. For now we concentrate in the study of the fibring of first-order based logics endowed with labelled deduction.

In Chapter 5 we illustrate the results obtained in the previous chapters by presenting lfob logic systems for first-order modal logic with decreasing domains, for the $\wedge$ fragment of first-order modal logic T, for the relevance logic R and for the $\wedge$-$\Rightarrow$ fragment of basic relevance logic B. For each of these examples we show that the lfob logic system proposed is sound, complete and is indeed a labelled presentation for that logic.

In Chapter 6 we define the fibring of a suitable pair of lfob logic systems. After fibring has been introduced we study preservation results. We show that soundness is preserved by fibring of suitable lfob logic systems, and we show also the preservation of completeness under some conditions. Moreover we show that entailment and consequence are preserved by fibring. Finally, ending the chapter, we illustrate the results obtained by studying the fibring of lfob logic systems for first-order modal logic with decreasing domains and the $\wedge$-$\Rightarrow$ fragment of basic relevance logic B. We show first that they constitute a suitable pair and then, relying on the preservation results, we show that the fibring is sound and complete, capitalizing on the fact that the components are sound and complete.

In Chapter 7 we make some final remarks and outline the guidelines for related future work.

1.5 Claim of contributions

The contributions obtained in the scope of this thesis that we would like to stress are:

- the development of labelled deduction for first-order based logics, taking into account fibring, extending our previous work on labelled deduction for propositional based logics [61] after substantial re-formulation,
- the definition of a labelled semantics for first-order based logics, taking into account fibring, by getting inspiration from a similar work for the semantics of non-labelled first-order based logics [74],
- the establishment of general completeness theorems involving rich lfob logic systems and full and appropriate lfob logic systems, and identification of the properties of canonical sets and appropriate sets,
- the identification and study of the class of connected lfob logic systems and of two subclasses of it: the class of connected lfob logic systems with a classical negation, local or non-local, and the class of connected lfob logic systems with locality, and the establishment of appropriateness results,
- the development of sound and complete labelled presentations for first-order modal logic with decreasing domains, $\wedge$ fragment of first-order modal logic T, relevance logic R, and $\wedge$-$\Rightarrow$ fragment of basic relevance logic B,
- the definition of fibring of lfob logic systems and the study of preservation issues for soundness, completeness, entailment and consequence, with the achievement of nice completeness preservation results.

Chapter 2

Language, Semantics and Deduction

In this chapter, we propose a semantics and a deduction framework for presenting labelled first-order based logics.

The labelled semantics framework we propose gets inspiration from a similar work for non-labelled first-order based logic systems [74]. Besides the differences between our approach and the approach in [74], motivated by the semantic treatment of labels, we would like to stress the following distinct points. In [74], the interpretation of a modal connective in a point, for a tuple of truth values, depends only on the information in those truth values associated to the assignment at that point. In our approach, the interpretation of a modal connective in a point, for a tuple of truth values, depends on all the information of the truth values. Another distinct point is that we abstract from a usual model for the logic, a class of semantic structures, and in [74] it is only a structure. This is because there are cases where several structures should be associated with a usual model. For instance, we may want to associate to a model for modal logic, the class of all the structures differing only on the set of individuals, but having the same modal structure.

The labelled deduction we develop for first-order based logics substantially re-formulate our previous work on labelled deduction for propositional based ones [61], and extends it to the first-order based case. Differently from [61], where judgements were considered the basic unit of reasoning, we center our labelled approach in labelled schema formulae. Moreover, we use provisos. We identified two types of provisos, formulae provisos and deduction provisos. Formulae provisos express the restrictions over formulae, present in the application of a rule. Deduction provisos are used to define the constraint related to the deduction that may appear in a rule, like for instance that some variables are fresh. In general all the treatment of deduction was re-formulated leading to a clear and intuitive notion of deduction.

There is some work reported in the literature on labelled deduction and semantics for first-order based logic. In [10, 79] the authors gave labelled proof systems for a large class of first-order modal logics. In [70] the author presents labelled natural deduction systems for several first-order modal logics. In this dissertation we develop a labelled deduction theory suitable for fibring that makes explicit many notions underlying those works, that generalizes some aspects of those works, and that are different from them also in many aspects. For instance, differing from the mentioned works, we (a) consider schematic rules,

(b) represent constraints in the rules by provisos, and (c) consider also general semantic structures, besides the usual models. A generalization of those works that we would like to stress is the possibility of our lfob logic systems to have n-ary quantifiers and modalities, that may additionally be constrained. Finally as an example of an issue that we make explicit we mention the definition of derived rule, and the proof that what we deduce with derived rules can be deduced in that system without them.

The chapter is organized as follows: we start by defining what can be the signature and the language of a lfob logic system and then we develop the labelled semantics and deduction framework to represent lfob logic systems. Then we prove a soundness theorem and finally, we define when a lfob logic system is a labelled presentation for a certain logic. Along the chapter we use first-order modal logic with decreasing domains to illustrate our concepts and results.

2.1 Language

We assume given once and for all five disjoint denumerable sets: X (the set of (*quantification*) *variables*), Ξ_t (the set of *term schema variables*), Ξ_s (the set of *finite set of assumptions schema variables*), Ξ_l (the set of *label schema variables*) and Ξ_f (the set of *formula schema variables*). We also assume as fixed the *equality symbol* $=$, the *same world symbol* $\equiv_\omega$, the *same assignment symbol* $\equiv_\alpha$, and the *global equality symbol* $=_g$. Throughout this thesis we will denote the elements of X by *variables* except when there may be ambiguity, and we denote by *schema variable* a element of $\Xi_t \cup \Xi_s \cup \Xi_l \cup \Xi_f$ when it is not relevant to distinguish the kind of schema variable.

In the following we will use, either by itself or sub-scripted or super-scripted, the Greek letter ν to refer to label schema variables, the Greek letter ξ to refer to formula schema variables, the Greek letter θ to refer to term schema variables, and the Greek letter ϑ to refer to a finite set of assumptions schema variables.

Definition 2.1.1 A *lfob signature* Σ is a tuple $\langle F^l, S, F, P, C, Q, O\rangle$ where:

- $F^l = \{F^l_k\}_{k\in I\!N}$ is a family of sets (of *label function symbols*),
- $S = \{S_k\}_{k\in I\!N^+}$ is a family of sets (of *relation symbols*),
- $F = \{F_k\}_{k\in I\!N}$ is a family of sets (of *function symbols*),
- $P = \{P_k\}_{k\in I\!N}$ is a family of sets (of *predicate symbols*),
- $C = \{C_k\}_{k\in I\!N}$ is a family of sets (of *connectives*),
- $Q = \{Q_k\}_{k\in I\!N^+}$ is a family of sets (of *quantifiers connectives*),
- $O = \{O_k\}_{k\in I\!N^+}$ is a family of sets (of *modalities connectives*).

In the following we denote by lSig the class of all lfob signatures. Given a lfob signature Σ, we can establish the family of sets of generators used in the definition of the lfob language over Σ. We provide additionally a set E of label schema variables representing an enrichment of the lfob signature with new label schema variables. Note that, in the following, when defining the sets of generators, we assume that all unions are of disjoints sets. Let S denote the set $\{\iota, \tau, \phi, \varrho\}$, where ι, τ, ϕ and ϱ are the (meta) sorts of labels, terms, formulae, and labelled formulae, respectively. So, we define the family $G = \{G_{\vec{s}s}\}_{\vec{s} \in S^*, s \in S}$ of sets of *generators* as follows:

- $G_{\epsilon\iota} = F_0^l \cup \Xi_l \cup E$,
- $G_{\iota^k\iota} = F_k^l$, for $k > 0$,
- $G_{\epsilon\tau} = F_0 \cup X \cup \Xi_t$,
- $G_{\tau^k\tau} = F_k$, for $k > 0$,
- $G_{\tau^2\phi} = \{=\} \cup P_2$,
- $G_{\tau^k\phi} = P_k$, for k not in $\{0, 2\}$,
- $G_{\epsilon\phi} = P_0 \cup C_0 \cup \Xi_f$,
- $G_{\phi^k\phi} = C_k \cup \{q_x \mid q \text{ is in } Q_k \text{ and } x \text{ is in } X\} \cup O_k$, for $k > 0$,
- $G_{\iota\phi\varrho} = \{:\}$,
- $G_{\iota\tau\iota\tau\varrho} = \{=_g\}$,
- $G_{\iota^2\varrho} = \{\equiv_\omega, \equiv_\alpha\} \cup S_2$,
- $G_{\iota^k\varrho} = S_k$, for k not in $\{0, 2\}$,
- all other sets are empty.

Consider the S-sorted free algebra induced by G. We denote by $T_{\text{lab},E}(\Sigma, \Xi_l)$ the carrier of sort ι and will refer to its elements as *label schema terms*, by $T(\Sigma, X, \Xi_t)$ the carrier of sort τ and refer to its elements as *schema terms*, by $L_{\text{fob}}(\Sigma, X, \Xi_t, \Xi_f)$ the carrier of sort ϕ and refer to its elements as *schema formulae*, and by $L_E(\Sigma, X, \Xi_l, \Xi_t, \Xi_f)$ the carrier of sort ϱ and refer to its elements as *labelled schema formulae.* Furthermore, we denote by $T(\Sigma, X)$, $L_{\text{fob}}(\Sigma, X)$, and $L_E(\Sigma, X, \Xi_l)$, respectively, the sets of schema terms, schema formulae and labelled schema formulae written without formula and term schema variables and refer to its elements by *terms*, *formulae* and *labelled formulae*, respectively. Note that labelled formulae may have label schema variables. We denote a labelled formula without label schema variables by *labelled non-schema formula.* Moreover, we denote by $T_{\text{lab}}(\Sigma)$ the set of label schema terms that do not have label schema variables and refer to its elements by *label terms.*

In the sequel, we denote by *relation* or by *relation symbol*, besides the elements of S_k for $k \geq 1$, also $\equiv_\omega$ or $\equiv_\alpha$. In the following, either by itself or sub-scripted or sup-scripted, we will use the letter v to refer to label (schema) terms, the letter t to refer to (schema) terms, the letters φ, γ or ϕ to refer to (schema) formulae, the letters Γ or Φ to refer to sets of (schema) formulae, the letters ψ or η to refer to labelled (schema) formulae and the letter Ψ to refer to sets of labelled (schema) formulae.

In order to simplify the presentation, and when there is no ambiguity, we may denote by $T_{\text{lab},E}$ the set $T_{\text{lab},E}(\Sigma, \Xi_l)$, by T the set $T(\Sigma, X, \Xi_t)$, by L_{fob} the set $L_{\text{fob}}(\Sigma, X, \Xi_t, \Xi_f)$, and by L_E the set $L_E(\Sigma, X, \Xi_l, \Xi_t, \Xi_f)$. We omit the subscript E when it is the $\emptyset$.

In the following we refer to (schema) terms without quantification variables by *ground (schema) terms*, to (schema) formulae with all quantification variables bounded by *closed (schema) formulae*, to labelled (schema) formulae built using the constructors in $S \cup \{\equiv_\omega, \equiv_\alpha\}$ by *relational (schema) formulae*, to a labelled (schema) formula which is not a relational (schema) formula by *labelled non-relational (schema) formula*, and to the pairs composed by a label (schema) term and a (schema) term by *labelled (schema) terms.*

Example 2.1.2 *Signature for first-order modal logic with decreasing domains.* Consider the signature Σ_{FOMdD} defined as follows:

- $F^l_k = \emptyset$ for $k \geq 0$,
- $S_2 = \{R_\Box\} \cup \{\equiv_x \mid x \in X\}$, and $S_k = \emptyset$ for $k \geq 3$ and $k = 1$,
- $\langle F, P\rangle$ is a first-order alphabet, and ε is in P_1,
- $C_0 = \{\bot\}$, $C_1 = \{\neg\}$, $C_2 = \{\wedge\}$, and $C_k = \emptyset$ for $k \geq 3$,
- $Q_1 = \{\forall\}$, $O_1 = \{\Box\}$, and $Q_k = O_k = \emptyset$ for $k \geq 2$.

In the context of the lfob signature for first-order modal logic with decreasing domains, the labelled schema formula $v{:}t \neq t'$ is defined as the abbreviation of $v{:}\neg(t = t')$, $v{:}\varphi_1 \to \varphi_2$ as the abbreviation of $v{:}\neg(\varphi_1 \wedge \neg\varphi_2)$, $v{:}\exists_x\varphi$ as the abbreviation of $v{:}\neg\forall_x\neg\varphi$, and $v{:}\Diamond\varphi$ as the abbreviation of $v{:}\neg\Box\neg\varphi$. We decided to consider in the signature the connectives $\wedge$ and $\neg$ because it is our intention to fibre first-order modal logic with decreasing domains with the $\wedge$-$\Rightarrow$ fragment of basic relevance logic B, introduced in Example 5.4.1, sharing the connective $\wedge$.

2.2 Semantics

We start the section by introducing semantic structures for lfob logic systems, based on [74], where is given a semantics for first-order based logic endowed with Hilbert systems. These structures have the purpose of interpreting and give a meaning to the syntactic entities of any lfob logic system, in a form adequate for fibring. Note that the fact that lfob logic systems have a homogeneous semantics is very important for fibring. Nevertheless, we have always to guarantee that the typical semantics for the logic and the semantics we provide are equivalent, i.e., they define an equivalent entailment. After that we define what is a labelled interpretation system for a lfob logic system and, conclude the section by defining satisfaction and entailment.

Definition 2.2.1 A *lfob structure* over a lfob signature Σ is a tuple $\langle U, A, W, \alpha, \omega, D, \mathcal{E}, \mathcal{B}, [.]\rangle$ with the following components

- U is a non-empty set, (of *points*),
- A is a non-empty set, (of *assignments*),
- W is a non-empty set, (of *worlds*),
- D is a non-empty set, (of *individuals*),

- $\alpha : U \to A$,
- $\omega : U \to W$,
- $\mathcal{E}$ contained in D^U is a set, (of *individual concepts*),
- $\mathcal{B}$ contained in 2^U is a set, (of *truth values*),
- $[.]$ is the *interpretation map*, defined by the following clauses
  - $[x]$ is $\{[x]_a\}_{a \in A}$ where $[x]_a$ is in D, for each x in X,
  - $[f]$ is $\{[f]_w\}_{w \in W}$ where $[f]_w : D^k \to D$, for each f in F_k,
  - $[p]$ is $\{[p]_w\}_{w \in W}$ where $[p]_w : D^k \to 2$, for each p in P_k,
  - $[c]$ is $\{[c]_{wa}\}_{w \in W, a \in A}$ where $[c]_{wa} : (\mathcal{B}_{wa})^k \to \mathcal{B}_{wa}$, for each c in C_k,
  - $[q_x]$ is $\{[q_x]_w\}_{w \in W}$ where $[q_x]_w : (\mathcal{B}_w)^k \to \mathcal{B}_w$, for each q in Q_k and x in X,
  - $[o] : \mathcal{B}^k \to \mathcal{B}$, for each o in O_k,
  - $[f^l] : U^k \to U$, for each f^l in F^l_k,
  - $[r]$ is contained in U^k, for each r in S_k,

  where

$$\begin{array}{llcll}
\mathcal{B}_a & = \{b \cap U_a \mid b \in \mathcal{B}\} & \text{and} & U_a & = \{u \in U \mid \alpha(u) = a\} \\
\mathcal{B}_w & = \{b \cap U_w \mid b \in \mathcal{B}\} & \text{and} & U_w & = \{u \in U \mid \omega(u) = w\} \\
\mathcal{B}_{wa} & = \{b \cap U_{wa} \mid b \in \mathcal{B}\} & \text{and} & U_{wa} & = U_w \cap U_a
\end{array}$$

- the sets $\mathcal{E}$ and $\mathcal{B}$ considered above and the interpretation map $[.]$ are assumed to be such that the following functions are well defined
  - $\widehat{x} : \ \to \mathcal{E}$ by $\widehat{x}(u) = [x]_{\alpha(u)}$,
  - $\widehat{f} : \mathcal{E}^k \to \mathcal{E}$ by $\widehat{f}(e_1, \ldots, e_k)(u) = [f]_{\omega(u)}(e_1(u), \ldots, e_k(u))$,
  - $\widehat{p} : \mathcal{E}^k \to \mathcal{B}$ by $\widehat{p}(e_1, \ldots, e_k)(u) = [p]_{\omega(u)}(e_1(u), \ldots, e_k(u))$,
  - $\widehat{=} : \mathcal{E}^2 \to \mathcal{B}$ by $\widehat{=}(e_1, e_2)(u) = 1$ iff $e_1(u) = e_2(u)$,
  - $\widehat{c} : \mathcal{B}^k \to \mathcal{B}$ by $\widehat{c}(b_1, \ldots, b_k)(u) = [c]_{\omega(u)\alpha(u)}(b_1 \cap U_{\omega(u)\alpha(u)}, \ldots, b_k \cap U_{\omega(u)\alpha(u)})(u)$,
  - $\widehat{q_x} : \mathcal{B}^k \to \mathcal{B}$ by $\widehat{q_x}(b_1, \ldots, b_k)(u) = [q_x]_{\omega(u)}(b_1 \cap U_{\omega(u)}, \ldots, b_k \cap U_{\omega(u)})(u)$,
  - $\widehat{o} : \mathcal{B}^k \to \mathcal{B}$ by $\widehat{o}(b_1, \ldots, b_k) = [o](b_1, \ldots, b_k)$.

In the following we denote a lfob structure over a lfob signature Σ simply by Σ structure and represent the class of all Σ structures by $\text{lStr}(\Sigma)$.

Definition 2.2.2 A *lfob interpretation system* is a tuple $\langle \Sigma, M, \check{\cdot} \rangle$ where Σ is a lfob signature, M is a class (of *models*) and $\check{\cdot}$ is a map that associates to each element of M a non-empty class of Σ structures.

With the purpose of illustrating the concepts introduced so far, we now present a labelled semantics for first-order modal logic with decreasing domains, inspired by [10, 79, 49, 12, 48, 39, 51].

Example 2.2.3 *First-order modal logic with decreasing domains lfob interpretation system.* Consider the lfob interpretation system $\langle \Sigma, M, \breve{\cdot} \rangle$, denoted by $\mathcal{I}_{\mathrm{FOMdD}}$, where Σ is Σ_{FOMdD} defined in Example 2.1.2, and M is the class of all tuples of the form

$$\langle W, R^{\circ}_{\Box}, D, \_^{F}, \_^{P} \rangle$$

where

- D and W are non-empty sets,
- $R^{\circ}_{\Box}$ is contained in $W \times W$,
- $\_^{F}$ is a family $\{\_^{F}{}_{k,w}\}_{k \in I\!N, w \in W}$ where $\_^{F}{}_{k,w} : F_k \to D^k \to D$,
- $\_^{P}$ is a family $\{\_^{P}{}_{k,w}\}_{k \in I\!N, w \in W}$ where $\_^{P}{}_{k,w} : P_k \to D^k \to 2$, such that
  - $\varepsilon^{P}_{1,w_2}(d) = 1$ implies $\varepsilon^{P}_{1,w_1}(d) = 1$ if $w_1 R^{\circ}_{\Box} w_2$, for any w_1, w_2 in W and d in D,

and $\breve{\cdot}$ maps each relational model $\langle W, R^{\circ}_{\Box}, D, \_^{F}, \_^{P} \rangle$ denoted by m, to a singleton with the structure induced by m, denoted by s_m, i.e., the structure $\langle U, A, W, \alpha, \omega, D, \mathcal{E}, \mathcal{B}, [.] \rangle$ where

- $A = D^X$, $\quad U = W \times A$, $\quad \alpha(\langle w, a \rangle) = a$, $\quad \omega(\langle w, a \rangle) = w$,
- $\mathcal{E}$ is D^U, $\quad \mathcal{B}$ is 2^U,

and

- $[x]_a = a(x)$ for x in X,
- $[f]_w = f^{F}_{k,w}$ for f in F_k,
- $[p]_w = p^{P}_{k,w}$ for p in P_k,
- $[\bot]_{\omega(u)\alpha(u)}(u) = 0$,
- $[\neg]_{\omega(u)\alpha(u)}(b)(u) = 1$ iff $b(u) = 0$,
- $[\wedge]_{\omega(u)\alpha(u)}(b_1, b_2)(u) = 1$ iff $b_1(u) = 1$ and $b_2(u) = 1$,
- $[R_{\Box}] = \{\langle u, u_1 \rangle \mid \alpha(u) = \alpha(u_1)$ and $\omega(u) R^{\circ}_{\Box} \omega(u_1)\}$,
- $[\Box](b)(u) = 1$ iff for all u_1, if $\langle u, u_1 \rangle$ is in $[R_{\Box}]$ then u_1 is in b,
- $[\equiv_x] = \{\langle u, u_1 \rangle \mid \omega(u) = \omega(u_1)$, and $\alpha(u)$ and $\alpha(u_1)$ are x co-equivalent$\}$,
- $[\forall_x]_{\omega(u)}(b)(u) = 1$ iff for all u_1, if $\langle u, u_1 \rangle \in [\equiv_x]$ and $\alpha(u_1)(x) \in [\varepsilon]_{\omega(u_1)}$ then $u_1 \in b$.

In order to give a denotation, a meaning, for schema variables, in the context of a lfob structure, we need an additional operator that we call *schema assignments*. Moreover, we will need to define the concept of schema assignments be co-equivalent, in order to semantically capture the deductive notion of fresh schema variables.

Definition 2.2.4 A *schema assignment* α over the Σ structure $\langle U, A, W, \alpha, \omega, D, \mathcal{E}, \mathcal{B}, [.]\rangle$ denoted by s and over a additional set of label schema variables E, is a tuple $\langle \alpha_l, \alpha_t, \alpha_f\rangle$ where

- $\alpha_l : \Xi_l \cup E \to U$,
- $\alpha_t : \Xi_t \to \mathcal{E}$,
- $\alpha_f : \Xi_f \to \mathcal{B}$.

Definition 2.2.5 Schema assignments α and α' over a Σ structure s and an additional set of label schema variables E, are Υ *co-equivalent*, where Υ is contained in $\Xi_l \cup E$, iff

- $\alpha_t = \alpha'_t$,
- $\alpha_f = \alpha'_f$,
- $\nu\alpha_l = \nu\alpha'_l$ for ν in $(\Xi_l \cup E) \setminus \Upsilon$.

So, we are now ready to define the interpretation of a label schema term, a schema term and a schema formula, in the context of a lfob structure and of a schema assignment over that structure.

Definition 2.2.6 Given a Σ structure $\langle U, A, W, \alpha, \omega, D, \mathcal{E}, \mathcal{B}, [.]\rangle$ denoted by s, and a schema assignment α over s and E, we define the maps

- $[\![.]\!]^{\iota s}_{\alpha}$ from $T_{\mathrm{lab},E}$ to U, inductively by
  - $[\![\nu]\!]^{\iota s}_{\alpha} = \alpha_l(\nu)$, for ν in $\Xi_l \cup E$, and
  - $[\![f^l(v_1, \ldots, v_k)]\!]^{\iota s}_{\alpha} = [f^l]([\![v_1]\!]^{\iota s}_{\alpha}, \ldots, [\![v_k]\!]^{\iota s}_{\alpha})$, for f^l in F^l_k,
- $[\![.]\!]^{\tau s}_{\alpha}$ from T to $\mathcal{E}$, inductively by
  - $[\![x]\!]^{\tau s}_{\alpha} = \widehat{x}$, for x in X,
  - $[\![\theta]\!]^{\tau s}_{\alpha} = \alpha_t(\theta)$, for θ in Ξ_t,
  - $[\![f(t_1, \ldots, t_k)]\!]^{\tau s}_{\alpha} = \widehat{f}([\![t_1]\!]^{\tau s}_{\alpha}, \ldots, [\![t_k]\!]^{\tau s}_{\alpha})$, for f in F_k,
- $[\![.]\!]^{\phi s}_{\alpha}$ from L_E to $\mathcal{B}$, inductively, in the same way as $[\![.]\!]^{\tau s}_{\alpha}$, using α_f, $\widehat{=}$, $\widehat{p}$, $\widehat{c}$, $\widehat{q_x}$ and $\widehat{o}$, and taking into account $[\![.]\!]^{\tau s}_{\alpha}$.

In the sequel, when there is no ambiguity, we denote by $[\![.]\!]^{s}_{\alpha}$ the maps $[\![.]\!]^{\iota s}_{\alpha}$, $[\![.]\!]^{\tau s}_{\alpha}$ and $[\![.]\!]^{\phi s}_{\alpha}$, as well as the tuple $\langle [\![.]\!]^{\iota s}_{\alpha}, [\![.]\!]^{\tau s}_{\alpha}, [\![.]\!]^{\phi s}_{\alpha}\rangle$.

Finally, we define when a labelled schema formula is satisfied by a lfob structure and a schema assignment.

Definition 2.2.7 Given a Σ structure $\langle U, A, W, \alpha, \omega, D, \mathcal{E}, \mathcal{B}, [.]\rangle$ denoted by s, and a schema assignment α over s and E we define the relation $\Vdash$ as follows

- $s, \alpha \Vdash v{:}\varphi$ iff $[\![\varphi]\!]^{s}_{\alpha}([\![v]\!]^{s}_{\alpha}) = 1$ for a label schema term v and a schema formula φ,
- $s, \alpha \Vdash r\, v_1 \ldots v_k$ iff $([\![v_1]\!]^{s}_{\alpha}, \ldots, [\![v_k]\!]^{s}_{\alpha}) \in [r]$ for label schema terms $v_1, \ldots, v_k$ and $r \in S_k$,

- $s, \alpha \Vdash v_1 \equiv_\omega v_2$ iff $\omega(\llbracket v_1 \rrbracket^s_\alpha) = \omega(\llbracket v_2 \rrbracket^s_\alpha)$ for label schema terms v_1 and v_2,
- $s, \alpha \Vdash v_1 \equiv_\alpha v_2$ iff $\alpha(\llbracket v_1 \rrbracket^s_\alpha) = \alpha(\llbracket v_2 \rrbracket^s_\alpha)$ for label schema terms v_1 and v_2,
- $s, \alpha \Vdash v_1{:}t_1 =_g v_2{:}t_2$ iff $\llbracket t_1 \rrbracket^s_\alpha(\llbracket v_1 \rrbracket^s_\alpha) = \llbracket t_2 \rrbracket^s_\alpha(\llbracket v_2 \rrbracket^s_\alpha)$ for label schema terms v_1 and v_2, and schema terms t_1 and t_2.

Definition 2.2.8 Given a lfob interpretation system $\langle \Sigma, M, \check{\cdot} \rangle$ the *entailment relation* $\models$ over sets of labelled schema formulae and labelled schema formulae, is defined as follows

$$\Psi \models \eta \qquad \text{iff} \qquad s, \alpha \Vdash \Psi \quad \text{implies} \quad s, \alpha \Vdash \eta$$

for any Σ structure s in $\check{m}$, model m in M, and schema assignment α over s.

We say a labelled schema formula is *valid* whenever it is entailed by the empty set of labelled schema formulae. With the purpose of illustrating the concepts of interpretation, satisfaction, entailment, and validity, we now show that the labelled schema formula $v{:}\forall_x \varphi \to \varphi'$ where φ' is obtained by substituting all the free occurrences of x in φ by a term t, and such that no variable in t is captured by a quantifier, is not valid in our semantics for first-order modal logic with decreasing domains, given in Example 2.2.3. This happens since in this case the universal quantification only guarantees that a formula holds for terms existing at that world. Recall that the connective $\to$ is defined as an abbreviation, i.e., $v{:}\varphi_1 \to \varphi_2$ is the abbreviation of $v{:}\neg(\varphi_1 \wedge \neg\varphi_2)$.

Example 2.2.9 *The labelled schema formula $v{:}\forall_x \varphi \to \varphi'$ where φ' is obtained by substituting all the free occurrences of x in φ by a term t, and such that no variable in t is captured by a quantifier, is not valid in $\mathcal{I}_{\text{FOMdD}}$.* Consider a model m equal to $\langle W, R^\circ_\Box, D, \_^F, \_^P \rangle$ in $\mathcal{I}_{\text{FOMdD}}$ such that

- W is $\{w\}$,
- $R^\circ_\Box$ is $\{\langle w, w \rangle\}$,
- D is $\{d\}$,
- $\varepsilon^P_{1,w}(d) = 0$,
- $p^P_{1,w}(d) = 0$.

Let s_m denote the structure induced by m, as defined in $\mathcal{I}_{\text{FOMdD}}$, u a point in s_m and α a schema assignment over s_m. Then,

$$\llbracket p(x) \rrbracket^{s_m}_\alpha(u_1) = 1$$

for all u_1 such that $\omega(u) = \omega(u_1)$, $\alpha(u_1)$ is x co-equivalent with $\alpha(u)$ and $\alpha(u_1)(x)$ is in $\varepsilon^P_{1,\omega(u_1)}$. To see why this happens note that i. U is a singleton with the pair $\langle w, 1 \rangle$ where 1 is the only element in D^X, and so u is $\langle w, 1 \rangle$, and ii. $\varepsilon^P_{1,w}(d) = 0$, and so there is no u_1 in those conditions, which means it is vacuously true. So,

$$\llbracket \forall_x p(x) \rrbracket^{s_m}_\alpha(u) = 1.$$

Nevertheless $\llbracket p(x) \rrbracket^{s_m}_\alpha(u) = 0$ since $p^P_{1,w}(d) = 0$. Therefore we can conclude that $\llbracket \forall_x p(x) \to p(x) \rrbracket^{s_m}_\alpha(u) = 0$, i.e., $s_m, \alpha \not\Vdash \nu{:}\forall_x p(x) \to p(x)$, as we wanted to show.

Similarly to the preceding example we can show that the Converse Barcan formula $v{:}\Box\forall_x \varphi \to \forall_x \Box\varphi$ is not valid in $\mathcal{I}_{\text{FOMdD}}$. This happens because $\mathcal{I}_{\text{FOMdD}}$ has decreasing domains.

2.3 Deduction

We start by defining provisos, substitutions and rules. Since we consider schematic rules, due to fibring, we need substitutions in order to replace the schema variables by admissible formulae or terms. Moreover, rules can be constrained, i.e., there are rules that can only be applied to formulae or terms satisfying some condition, or in certain cases dependent of the conditions of the deduction. So, we need provisos. Provisos represent conditions on substitutions, and so, are maps on substitutions that indicate if a substitution should be accepted or not. We identify and define two types of provisos: the formula provisos and the deduction provisos. After that we define what is a deduction system. Note that there are rules common to any lfob logic system. In the sequel we define deduction and the associated consequence operator. We prove that it is extensive, monotonic and idempotent. Additionally we show also that the consequence operator is closed for substitutions. Ending the deduction section we rigorously define what is a derived rule and we prove that if there is a proof that uses derived rules then we can also construct a proof for the same deduction without derived rules.

2.3.1 Provisos and rules

We start by defining what is a schema substitution and what is a substitution. Those operators are used to provide instantiations, when doing a deduction, of the labelled schema formulae that appear in the rules of the lfob deduction system. Schema substitutions differ from substitutions in the fact that the result may have formula or term schema variables.

Definition 2.3.1 A Σ *schema substitution* σ over E and within $L_{E'}$, where E and E' are additional sets of label schema variables, is a tuple $\langle \sigma_l, \sigma_t, \sigma_f, \sigma_s \rangle$ where

- $\sigma_l : \Xi_l \cup E \to T_{\mathrm{lab},E'}$,
- $\sigma_t : \Xi_t \to T$,
- $\sigma_f : \Xi_f \to L_{\mathrm{fob}}$,
- $\sigma_s : \Xi_s \to \wp_{\mathrm{fin}} L_{E'}$.

A Σ *substitution* ρ over E and within $L_{E'}$ is a tuple $\langle \rho_l, \rho_t, \rho_f, \rho_s \rangle$ where

- $\rho_l : \Xi_l \cup E \to T_{\mathrm{lab},E'}$,
- $\rho_t : \Xi_t \to T(\Sigma, X)$,
- $\rho_f : \Xi_f \to L(\Sigma, X)$,
- $\rho_s : \Xi_s \to \wp_{\mathrm{fin}} L_{E'}$.

In the following we will denote the set of Σ schema substitutions over E and within $L_{E'}$, for additional sets of label schema variables E and E', by $\mathrm{sSub}(\Sigma, E; E')$, and by $\mathrm{Sub}(\Sigma, E; E')$ the set of Σ substitutions over E and within $L_{E'}$. We will omit the subscripts l, t, f, s in substitutions and schema substitutions, as well as the reference to the signature and to the additional sets of label schema variables, when no ambiguity arises.

Remark 2.3.2 In the sequel, given a schema substitution σ we denote by σ_{nl} the components of σ with non-labelled results, i.e., the pair $\langle\sigma_t, \sigma_f\rangle$. Similarly for substitutions.

Definition 2.3.3 The Σ schema substitutions σ and σ' over E and within $L_{E'}$ and $L_{E''}$ respectively, are Υ *co-equivalent*, for a set Υ contained in $\Xi_l \cup E$, iff

- $\sigma_t = \sigma'_t$,
- $\sigma_f = \sigma'_f$,
- $\nu\sigma_l = \nu\sigma'_l$ for ν in $(\Xi_l \cup E) \setminus \Upsilon$.

We define the notion of *co-equivalence* for substitutions in the same way as we did for schema substitutions.

Remark 2.3.4 In the sequel we refer to the map $\mathrm{lsv} : \wp_{\mathrm{fin}} L_E \to \Xi_l \cup E$ defined inductively as follows: $\mathrm{lsv}(\emptyset) = \emptyset$ and $\mathrm{lsv}(\{\eta\} \cup \Psi) = \mathrm{lsv}_\varrho(\eta) \cup \mathrm{lsv}(\Psi)$ where $\mathrm{lsv}_\varrho : L_E \to \Xi_l \cup E$ is a map defined by case analysis as follows: $\mathrm{lsv}_\varrho(\eta)$ is $\mathrm{lsv}_\iota(v)$ whenever η is the labelled schema formula $v{:}\varphi$, $\mathrm{lsv}_\varrho(\eta)$ is $\mathrm{lsv}_\iota(v_1) \cup \ldots \cup \mathrm{lsv}_\iota(v_k)$ whenever η is the labelled schema formula $r\, v_1 \ldots v_k$ and r is a relation with arity k, and $\mathrm{lsv}_\varrho(\eta)$ is $\mathrm{lsv}_\iota(v_1) \cup \mathrm{lsv}_\iota(v_2)$ whenever η is $v_1{:}t_1 =_g v_2{:}t_2$. Finally the map $\mathrm{lsv}_\iota : T_{\mathrm{lab},E} \to \Xi_l \cup E$ is inductively defined as follows: $\mathrm{lsv}_\iota(\nu) = \{\nu\}$ for any ν in $\Xi_l \cup E$, $\mathrm{lsv}_\iota(f^l) = \emptyset$ for $f^l \in F^l_0$, and $\mathrm{lsv}_\iota(f^l(v_1, \ldots, v_k)) = \mathrm{lsv}_\iota(v_1) \cup \ldots \cup \mathrm{lsv}_\iota(v_k)$, for any f^l in F^l_k and $k \geq 1$.

When defining the rules of deduction of a lfob logic system, it is possible that a rule only applies either if some condition on the deduction is satisfied or if a labelled schema formula satisfies some condition depending of its form. Formula provisos and deduction provisos are the (rigorous) way we found to express those conditions. Formula provisos for conditions involving labelled schema formulae, and deduction provisos for conditions involving the deduction.

Definition 2.3.5 A Σ *formula proviso* is a map that given any pair composed of a Σ term substitution and a Σ formula substitution, returns a value in 2. A *formula proviso* π is a family $\{\pi_\Sigma\}_{\Sigma \in \mathrm{lSig}}$ where

- π_Σ is a Σ formula proviso,
- $\pi_\Sigma(\rho_{\mathrm{nl}}) = \pi_{\Sigma'}(\rho'_{\mathrm{nl}})$ whenever $\theta\rho_t = \theta\rho'_t$ and $\xi\rho_f = \xi\rho'_f$ for any Σ substitution ρ and Σ' substitution ρ'.

This last condition keeps the coherence of the components of the formula provisos which is of particular importance to fibring. The intuition is that the evaluation of substitutions by a formula proviso only depends on the values the substitutions gave. With the purpose of illustrating formula provisos we now show the definition of the basic formula provisos 0 and 1. So, for every lfob signature Σ and substitution ρ over Σ we set, obviously,

$$0_\Sigma(\rho_{\mathrm{nl}}) \text{ to } 0 \qquad \text{and} \qquad 1_\Sigma(\rho_{\mathrm{nl}}) \text{ to } 1.$$

In the following, when there is no ambiguity, we will denote a Σ formula proviso π_Σ simply by π. We define the Σ formula proviso $(\pi\sigma_{\mathrm{nl}})$ by

$$(\pi\sigma_{\mathrm{nl}})(\rho_{\mathrm{nl}}) = 1 \quad \text{iff} \quad \pi(\sigma_{\mathrm{nl}}\rho_{\mathrm{nl}}) = 1$$

for any Σ formula proviso π, Σ schema substitution σ, and Σ substitution ρ, and define the Σ formula proviso $\pi * \pi'$ by

$$(\pi * \pi')(\rho_{\mathrm{nl}}) = 1 \quad \text{iff} \quad \pi(\rho_{\mathrm{nl}}) = 1 \text{ and } \pi'(\rho_{\mathrm{nl}}) = 1$$

for any Σ formula provisos π and π', and Σ substitution ρ. We say

$$\pi \leq \pi'$$

iff $\pi(\rho_{\mathrm{nl}}) \leq \pi'(\rho_{\mathrm{nl}})$, for any Σ formula provisos π and π', and Σ substitution ρ.

Remark 2.3.6 Note that, for any Σ formula provisos π and π', and Σ schema substitution σ, we have that $((\pi * \pi')\sigma_{\mathrm{nl}})$ is $(\pi\sigma_{\mathrm{nl}}) * (\pi'\sigma_{\mathrm{nl}})$ since $((\pi * \pi')\sigma_{\mathrm{nl}})(\rho_{\mathrm{nl}}) = 1$ iff $(\pi * \pi')(\sigma_{\mathrm{nl}}\rho_{\mathrm{nl}}) = 1$ iff $\pi(\sigma_{\mathrm{nl}}\rho_{\mathrm{nl}}) = 1$ and $\pi'(\sigma_{\mathrm{nl}}\rho_{\mathrm{nl}}) = 1$ iff $(\pi\sigma_{\mathrm{nl}})(\rho_{\mathrm{nl}}) = 1$ and $(\pi'\sigma_{\mathrm{nl}})(\rho_{\mathrm{nl}}) = 1$ iff $((\pi\sigma_{\mathrm{nl}}) * (\pi'\sigma_{\mathrm{nl}}))(\rho_{\mathrm{nl}}) = 1$, for any Σ substitution ρ.

Observe also that, for any Σ formula proviso π, and Σ schema substitutions σ and σ', $((\pi\sigma_{\mathrm{nl}})\sigma'_{\mathrm{nl}})$ is $(\pi(\sigma_{\mathrm{nl}}\sigma'_{\mathrm{nl}}))$ since we have $((\pi\sigma_{\mathrm{nl}})\sigma'_{\mathrm{nl}})(\rho_{\mathrm{nl}}) = (\pi\sigma_{\mathrm{nl}})(\sigma'_{\mathrm{nl}}\rho_{\mathrm{nl}}) = \pi(\sigma_{\mathrm{nl}}(\sigma'_{\mathrm{nl}}\rho_{\mathrm{nl}})) = \pi((\sigma_{\mathrm{nl}}\sigma'_{\mathrm{nl}})\rho_{\mathrm{nl}}) = (\pi(\sigma_{\mathrm{nl}}\sigma'_{\mathrm{nl}}))(\rho_{\mathrm{nl}})$, for any Σ substitution ρ.

If $(\pi\sigma_{\mathrm{nl}})$ is the Σ formula proviso 1 then $((\pi\sigma_{\mathrm{nl}})\sigma'_{\mathrm{nl}})$ is also the Σ formula proviso 1, for any Σ formula proviso π, and Σ schema substitutions σ and σ'. To see this observe that $((\pi\sigma_{\mathrm{nl}})\sigma'_{\mathrm{nl}})(\rho_{\mathrm{nl}}) = (\pi\sigma_{\mathrm{nl}})(\sigma'_{\mathrm{nl}}\rho_{\mathrm{nl}}) = 1$, for any Σ substitution ρ.

Finally note that if σ and σ' are Σ schema substitutions with $\sigma_t = \sigma'_t$ and $\sigma_f = \sigma'_f$ then for any Σ formula proviso π we have that $(\pi\sigma_{\mathrm{nl}}) = (\pi\sigma'_{\mathrm{nl}})$.

When there is ambiguity, we will use explicitly the words formula proviso and value to distinguish between the (Σ) formula proviso 1 and the value 1. Similarly for the (Σ) formula proviso or value 0.

Definition 2.3.7 A Σ *deduction proviso* within L_E is a map that given any Σ schema substitution within L_E returns a value in 2. A *deduction proviso* π is a family $\{\pi_{\Sigma;E}\}_{\Sigma \in \mathrm{lSig}, E \in \mathrm{Set}}$ such that

- $\pi_{\Sigma;E}$ is a Σ deduction proviso within L_E,
- $\pi_{\Sigma;E}(\sigma) = \pi_{\Sigma';E'}(\sigma')$ whenever $\vartheta\sigma_s = \vartheta\sigma'_s$, $\theta\sigma_t = \theta\sigma'_t$, $\xi\sigma_f = \xi\sigma'_f$ and $\nu\sigma_l = \nu\sigma'_l$ for any Σ schema substitution σ and Σ' schema substitution σ'.

Remark 2.3.8 To illustrate also deduction provisos we now show the definition of the deduction proviso $\mathrm{fresh}(\Upsilon, \langle\vartheta, \Psi, \eta\rangle)$ denoted by fresh deduction proviso or simply *fresh proviso*. So, for any lfob signature Σ and set E of label schema variables we define $\mathrm{fresh}(\Upsilon, \langle\vartheta, \Psi, \eta\rangle)_{\Sigma;E}(\sigma) = 1$ iff

- $\nu\sigma$ is a label schema variable not in $(\mathrm{lsv}(\Psi \cup \{\eta\}) \setminus \{\nu\})\sigma$, for any ν in Υ,
- $\Upsilon\sigma \cap \mathrm{lsv}(\vartheta\sigma \setminus \Psi\sigma) = \emptyset$.

In the following, when there is no ambiguity, we will denote a Σ deduction proviso $\pi_{\Sigma;E}$ within L_E simply by π.

Definition 2.3.9 A *rule* is a tuple $\langle\{\langle\vartheta_1,\Psi_1,\eta_1\rangle,\dots,\langle\vartheta_k,\Psi_k,\eta_k\rangle\},\eta,P_f,P_d\rangle$, written

$$\frac{\Psi_1\,/\,\eta_1\dots\Psi_k\,/\,\eta_k}{\eta}\ r;\ P_f;\ P_d$$

defined over a lfob signature Σ, where $\vartheta_1,\dots,\vartheta_k$ are in Ξ_s, $\Psi_1,\dots,\Psi_k$ are finite sets of labelled schema formulae, $\eta_1,\dots,\eta_k$ and η are labelled schema formulae, and P_f and P_d are finite sets of formula and deduction provisos, respectively, such that if $\mathrm{fresh}(\Upsilon^{\cdot},\langle\vartheta^{\cdot},\Psi^{\cdot},\eta^{\cdot}\rangle)$ is in P_d, then

- $\langle\vartheta^{\cdot},\Psi^{\cdot},\eta^{\cdot}\rangle$ is equal to $\langle\vartheta_j,\Psi_j,\eta_j\rangle$, for some j in $1,\dots,k$;
- $\mathrm{lsv}(\eta)$ is contained in $\mathrm{lsv}(\Psi^{\cdot}\cup\{\eta^{\cdot}\})\setminus\Upsilon^{\cdot}$;
- $\Upsilon^{\cdot}\cap\mathrm{lsv}(\Psi_i\cup\{\eta_i\})=\emptyset$, for $i=1,\dots,k$, $i\neq j$;
- $\Upsilon^{\cdot}$ is non-empty and is properly contained in $\mathrm{lsv}(\Psi^{\cdot}\cup\{\eta^{\cdot}\})$.

In the following, P_f and P_d are omitted if they are the empty set. Moreover, the 1 formula proviso in P_f and in P_d is not presented, the schema variables $\vartheta_1,\dots,\vartheta_k$ are also omitted, Ψ_i for i in $\{1,\dots,k\}$ is omitted whenever it is the empty set and in general we omit brackets when presenting sets and omit delimiters when presenting tuples. In the sequel, for each rule $\langle\{\langle\vartheta_1,\Psi_1,\eta_1\rangle,\dots,\langle\vartheta_k,\Psi_k,\eta_k\rangle\},\eta,P_f,P_d\rangle$, we denote by *fresh variables* the label schema variables present in the union of all sets Υ for all deduction provisos $\mathrm{fresh}(\Upsilon,\langle\vartheta_j,\Psi_j,\eta_j\rangle)$ in P_d. Note that η has no fresh variables.

2.3.2 Deduction

So, we are now able to introduce labelled first-order based deduction systems.

Definition 2.3.10 A *lfob deduction system* is a tuple $\langle\Sigma,R\rangle$ where Σ is a lfob signature and R is a set of rules containing

$$\frac{\nu'\equiv_\omega\nu}{\nu\equiv_\omega\nu'}\ \equiv_{\omega s}\qquad\frac{\nu\equiv_\omega\nu'\quad\nu'\equiv_\omega\nu''}{\nu\equiv_\omega\nu''}\ \equiv_{\omega t}\qquad\frac{}{\nu\equiv_\omega\nu}\ \equiv_{\omega r}$$

$$\frac{\nu'\equiv_\alpha\nu}{\nu\equiv_\alpha\nu'}\ \equiv_{\alpha s}\qquad\frac{\nu\equiv_\alpha\nu'\quad\nu'\equiv_\alpha\nu''}{\nu\equiv_\alpha\nu''}\ \equiv_{\alpha t}\qquad\frac{}{\nu\equiv_\alpha\nu}\ \equiv_{\alpha r}$$

$$\frac{\nu{:}\theta=_g\nu'{:}\theta'}{\nu'{:}\theta'=_g\nu{:}\theta}\ =_{g_s}\qquad\frac{\nu{:}\theta=_g\nu'{:}\theta'\quad\nu'{:}\theta'=_g\nu''{:}\theta''}{\nu{:}\theta=_g\nu''{:}\theta''}\ =_{g_t}\qquad\frac{}{\nu{:}\theta=_g\nu{:}\theta}\ =_{g_r}$$

$$\frac{\nu{:}\theta=\theta'}{\nu{:}\theta=_g\nu{:}\theta'}\ =_E\qquad\frac{\nu{:}\theta=_g\nu{:}\theta'}{\nu{:}\theta=\theta'}\ =_I$$

$$\frac{\nu{:}\theta_1=_g\nu'{:}\theta'_1\quad\dots\quad\nu{:}\theta_k=_g\nu'{:}\theta'_k\quad\nu\equiv_\omega\nu'}{\nu{:}f(\theta_1,\dots,\theta_k)=_g\nu'{:}f(\theta'_1,\dots,\theta'_k)}\ =_{g_f}$$

$$\frac{\nu{:}\theta_1 =_g \nu'{:}\theta'_1 \quad \dots \quad \nu{:}\theta_k =_g \nu'{:}\theta'_k \quad \nu{:}p(\theta_1,\dots,\theta_k) \quad \nu \equiv_\omega \nu'}{\nu'{:}p(\theta'_1,\dots,\theta'_k)} =_{g_p} \qquad \frac{\nu \equiv_\alpha \nu'}{\nu{:}x =_g \nu'{:}x} \equiv_{\alpha_g}^{x}$$

for every x in X, k in $\mathbb{N}_0$, p in P_k and f in F_k, and such that for each rule r in R only fresh provisos are in the deduction provisos of r.

Note that for each rule $\langle\{\langle\vartheta_1,\Psi_1,\eta_1\rangle,\dots,\langle\vartheta_k,\Psi_k,\eta_k\rangle\},\eta,P_f,P_d\rangle$ in a lfob deduction system, if k is 0 then P_d is the empty set.

Observe also that the rules presented in Definition 2.3.10 are part of the rules of any lfob deduction system and that basically, they characterize the symbols common to all lfob signatures, as should be expected. So, for instance, the rules impose that the relations $\equiv_\omega$, $\equiv_\alpha$, and $=_g$ are equivalence relations, and they impose also that when labels are related by some of that relations, then some kinds of information are inherited from one label to another.

From now on, when there is no ambiguity, and when there is no need to stress the fact that we are working with lfob deduction systems, and in order to lighten the presentation, we may omit the designation first-order based when referring to a lfob deduction system. In order to illustrate this concept we now present a lfob deduction system for first-order modal logic with decreasing domains.

Example 2.3.11 *First-order modal logic with decreasing domains lfob deduction system.* Consider the lfob deduction system $\langle\Sigma_{\mathrm{FOMdD}},R\rangle$, in the sequel denoted by $\mathcal{D}_{\mathrm{FOMdD}}$, where Σ_{FOMdD} is the signature introduced in Example 2.1.2, and R, besides the rules specified in Definition 2.3.10 common to all lfob deduction systems, contains

$$\frac{\nu{:}\xi_1 \qquad \nu{:}\xi_2}{\nu{:}\xi_1 \wedge \xi_2} \wedge_I \qquad\qquad \frac{\nu{:}\xi_1 \wedge \xi_2}{\nu{:}\xi_1} \wedge_E^1 \qquad\qquad \frac{\nu{:}\xi_1 \wedge \xi_2}{\nu{:}\xi_2} \wedge_E^2$$

$$\frac{\nu{:}\xi \,/\, \nu'{:}\bot}{\nu{:}\neg\xi} \neg_I \qquad\qquad \frac{\nu{:}\neg\xi \qquad \nu{:}\xi}{\nu'{:}\bot} \neg_E \qquad\qquad \frac{\nu{:}\neg\xi \,/\, \nu'{:}\bot}{\nu{:}\xi} \neg_c$$

$$\frac{\nu \equiv_x \nu', \nu'{:}\varepsilon(x) \,/\, \nu'{:}\xi}{\nu{:}\forall_x\xi} \forall_{xI};\ \mathrm{fresh}(\nu',\langle\vartheta_1,\{\nu \equiv_x \nu',\nu'{:}\varepsilon(x)\},\nu'{:}\xi\rangle)$$

$$\frac{\nu{:}\forall_x\xi \quad \nu \equiv_x \nu' \quad \nu'{:}\varepsilon(x)}{\nu'{:}\xi} \forall_{xE} \qquad\qquad \frac{\nu{:}\forall_x\xi \quad \nu{:}\varepsilon(\theta)}{\nu{:}\xi'} {\forall_x}^{\mathrm{spc}};\ \xi' \lhd \xi^x_\theta$$

$$\frac{\nu R_\Box \nu' \quad \nu'{:}\varepsilon(x)}{\nu{:}\varepsilon(x)} dD$$

$$\frac{\nu R_\Box \nu' \,/\, \nu'{:}\xi}{\nu{:}\Box\xi} \Box_I;\ \mathrm{fresh}(\nu',\langle\vartheta_1,\{\nu R_\Box \nu'\},\nu'{:}\xi\rangle) \qquad\qquad \frac{\nu{:}\Box\xi \quad \nu R_\Box \nu'}{\nu'{:}\xi} \Box_E$$

$$\frac{\nu R_\Box \nu'}{\nu \equiv_\alpha \nu'}\ R_{\Box\alpha^{1,2}} \qquad \frac{\nu R_\Box \nu_1 \quad \nu \equiv_\omega \nu' \quad \nu_1 \equiv_\omega \nu'_1 \quad \nu' \equiv_\alpha \nu'_1}{\nu' R_\Box \nu'_1}\ R_{\Box\alpha}^{\text{gen}}$$

$$\frac{\nu \equiv_x \nu'}{\nu \equiv_\omega \nu'}\ \equiv_{x\omega^{1,2}} \qquad \frac{\nu \equiv_x \nu_1 \quad \nu \equiv_\alpha \nu' \quad \nu_1 \equiv_\alpha \nu'_1 \quad \nu' \equiv_\omega \nu'_1}{\nu' \equiv_x \nu'_1}\ \equiv_{x\omega}^{\text{gen}}$$

$$\frac{\nu \equiv_x \nu'}{\nu{:}\theta =_g \nu'{:}\theta}\ \equiv_{xg^{1,2}};\ \text{vardif}(x,\theta)$$

$$\frac{\nu_1 \equiv_\omega \nu', \nu' \equiv_\alpha \nu_2 \,/\, \nu''{:}\xi''}{\nu''{:}\xi''}\ \mathit{exh};\ \text{fresh}(\nu', \langle \vartheta_1, \{\nu_1 \equiv_\omega \nu', \nu' \equiv_\alpha \nu_2\}, \nu''{:}\xi''\rangle)$$

$$\frac{\nu \equiv_x \nu' \quad \nu' \equiv_x \nu''}{\nu \equiv_x \nu''}\ \equiv_{xt} \qquad \frac{}{\nu \equiv_x \nu}\ \equiv_{xr} \qquad \frac{\nu' \equiv_x \nu}{\nu \equiv_x \nu'}\ \equiv_{xs}$$

$$\frac{\nu \equiv_\omega \nu' \qquad \nu{:}\theta_1 =_g \nu'{:}\theta_1 \quad \ldots \quad \nu{:}\theta_k =_g \nu'{:}\theta_k \qquad \nu{:}\xi}{\nu'{:}\xi}\ \mathit{gmon}^k_{\equiv_\omega};\ \text{p-gmon}_{\equiv_\omega}(\xi,\theta_1,\ldots,\theta_k)$$

for any x in X, any k greater or equal to 0, and where the provisos $\xi' \lhd \xi^x_\theta$, vardif(x,θ), and p-gmon$_{\equiv_\omega}(\xi,\theta_1,\ldots,\theta_k)$ are such that

- vardif$(x,\theta)_{\Sigma_{\text{FOMdD}}}(\rho_{\text{nl}}) = 1$ iff $\theta\rho_t$ is a variable different of x,
- p-gmon$_{\equiv_\omega}(\xi,\theta_1,\ldots,\theta_k)_{\Sigma_{\text{FOMdD}}}(\rho_{\text{nl}}) = 1$ iff $\xi\rho_f$ is a formula whose free variables are $\theta_1\rho_t,\ldots,\theta_k\rho_t$,
- $\xi' \lhd \xi^x_{\theta\,\Sigma_{\text{FOMdD}}}(\rho_{\text{nl}}) = 1$ iff $\xi'\rho_f$ is obtained by replacing the free occurrences of x in $\xi\rho_f$ by a term $\theta\rho_t$, equal to a variable whenever some free x in $\xi\rho_f$ is in the scope of a modality, such that no variable in $\theta\rho_t$ is captured by a quantifier.

So, we can now define the key concept of deduction of a labelled schema formula in L_E from a given set of labelled schema formulae contained in L_E constrained by a Σ formula proviso.

Definition 2.3.12 In the context of a lfob deduction system $\langle\Sigma, R\rangle$ and given an additional set E of label schema variables, a labelled schema formula η in L_E is *deduced* from a set Ψ of labelled schema formulae contained in L_E, constrained by a Σ formula proviso π,

$$\Psi \vdash_{\langle\Sigma,R\rangle,E} \eta;\ \pi$$

iff there is a sequence $\langle\eta_1,\Psi_1,\pi_1\rangle,\ldots,\langle\eta_n,\Psi_n,\pi_n\rangle$ where $\eta_1,\ldots,\eta_n$ are labelled schema formulae in L_E, $\Psi_1,\ldots,\Psi_n$ are sets of labelled schema formulae contained in L_E, $\pi_1,\ldots$, π_n are Σ formula provisos, and for each $i = 1,\ldots,n$

- either $\Psi_i = \{\eta_i\}$ and π_i is 1;

- or there are rule $\langle\{\langle\vartheta'_1,\Psi'_1,\eta'_1\rangle,\dots,\langle\vartheta'_k,\Psi'_k,\eta'_k\rangle\},\eta',P'_f,P'_d\rangle$, Σ schema substitution σ within L_E, and $i_1,\dots,i_k$ in $\{1,\dots,i-1\}$, with
  - $\pi'_{\Sigma;E}(\sigma)$ is 1 for each π' in P'_d,
  - $\eta_{i_j}=\eta'_j\sigma$ and $\Psi_{i_j}=\vartheta'_j\sigma$ for each $j=1,\dots,k$,
  - $\Psi_i=\Psi_{i_1}\setminus\Psi'_1\sigma\cup\ldots\cup\Psi_{i_k}\setminus\Psi'_k\sigma$,
  - $\eta_i=\eta'\sigma$,
  - $\pi_i=\pi_{i_1}*\ldots*\pi_{i_k}**_{\pi'\in P'_f}(\pi'_\Sigma\sigma_{\mathrm{nl}})$,

such that η_n is η, $\Psi_n\subseteq\Psi$ and $\pi\leq\pi_n$.

A sequence satisfying the conditions in Definition 2.3.12, will be called a deduction sequence. A deduction sequence is said to be *sober* iff no proper subsequence is a deduction sequence for the same inference. Obviously, from any deduction sequence we can always obtain a sober one by removing superfluous steps. Due to this facility, in the sequel we will use intensively sober deduction sequences because it facilitates, makes simpler and easier to understand, the proofs involving deduction sequences. In the following, when there is no ambiguity, we will omit the reference to the lfob deduction system in $\vdash_{\mathcal{D},E}$, and will represent by $\Psi\vdash_E\eta$ the fact that η in L_E is deducible from Ψ contained in L_E with proviso 1.

With the purpose of illustrating deductions we now show that $\nu{:}x=y,\nu\equiv_\alpha\nu'\vdash\nu'{:}\Box x=y$ holds in the context of $\mathcal{D}_{\mathrm{FOMdD}}$, introduced in Example 2.3.11. So, observe that,

1	$\nu{:}x=y$	$\{\nu{:}x=y\}$	1	asp	
2	$\nu\equiv_\alpha\nu'$	$\{\nu\equiv_\alpha\nu'\}$	1	asp	
3	$\nu'R_\Box\nu''$	$\{\nu'R_\Box\nu''\}$	1	asp	
4	$\nu'\equiv_\alpha\nu''$	$\{\nu'R_\Box\nu''\}$	1	$R_{\Box\alpha^{1,2}}$	3
5	$\nu\equiv_\alpha\nu''$	$\{\nu\equiv_\alpha\nu',\nu'R_\Box\nu''\}$	1	$\equiv_{\alpha_t}$	2, 4
6	$\nu{:}x=_g\nu''{:}x$	$\{\nu\equiv_\alpha\nu',\nu'R_\Box\nu''\}$	1	$\equiv_{\alpha_g^x}$	5
7	$\nu{:}y=_g\nu''{:}y$	$\{\nu\equiv_\alpha\nu',\nu'R_\Box\nu''\}$	1	$\equiv_{\alpha_g^y}$	5
8	$\nu{:}x=_g\nu{:}y$	$\{\nu{:}x=y\}$	1	$=_E$	1
9	$\nu{:}x=_g\nu''{:}y$	$\{\nu{:}x=y,\nu\equiv_\alpha\nu',\nu'R_\Box\nu''\}$	1	$=_{g_t}$	8, 7
10	$\nu''{:}x=_g\nu{:}x$	$\{\nu\equiv_\alpha\nu',\nu'R_\Box\nu''\}$	1	$=_{g_s}$	6
11	$\nu''{:}x=_g\nu''{:}y$	$\{\nu{:}x=y,\nu\equiv_\alpha\nu',\nu'R_\Box\nu''\}$	1	$=_{g_t}$	10, 9
12	$\nu''{:}x=y$	$\{\nu{:}x=y,\nu\equiv_\alpha\nu',\nu'R_\Box\nu''\}$	1	$=_I$	11
13	$\nu'{:}\Box x=y$	$\{\nu{:}x=y,\nu\equiv_\alpha\nu'\}$	1	$\Box_I$	12

is a deduction sequence for that deduction. In order to illustrate a deduction constrained by formula provisos we now show that $\nu{:}\forall_x\varphi,\nu\equiv_\omega\nu'\vdash\nu'{:}\varphi;\ \text{p-gmon}_{\equiv_\omega}(\forall_x\varphi,\theta_1\sigma,\dots,\theta_k\sigma)$ holds in $\mathcal{D}_{\mathrm{FOMdD}}$. So, here is a sober deduction sequence for that deduction:

1	$\nu{:}\forall_x\varphi$	$\{\nu{:}\forall_x\varphi\}$	1	asp	
2	$\nu\equiv_\omega\nu'$	$\{\nu\equiv_\omega\nu'\}$	1	asp	
3	$\nu'{:}\forall_x\varphi$	$\{\nu{:}\forall_x\varphi,\nu\equiv_\omega\nu'\}$	$\text{p-gmon}_{\equiv_\omega}(\forall_x\varphi,\theta_1\sigma,\dots,\theta_k\sigma)$	$gmon^k_{\equiv_\omega}$	1, 2
4	$\nu'\equiv_x\nu'$	$\emptyset$	1	$\equiv_{xr}$	
5	$\nu'{:}\varphi$	$\{\nu{:}\forall_x\varphi,\nu\equiv_\omega\nu'\}$	$\text{p-gmon}_{\equiv_\omega}(\forall_x\varphi,\theta_1\sigma,\dots,\theta_k\sigma)$	$\forall_{xE}$	3, 4.

Now, based on the notion of deduction introduced in Definition 2.3.12, we prove that $\vdash_E$ is a consequence operator, i.e., an operator on sets of labelled schema formulae that is extensive, monotonic, idempotent and closed under substitutions.

Proposition 2.3.13 If ϖ is a sober deduction sequence for $\Psi \vdash_E \eta;\ \pi$ then ϖ is a sober deduction sequence for $\Psi', \Psi \vdash_{E,E'} \eta;\ \pi'$, for any

- sets E and E' of label schema variables,
- sets Ψ and Ψ' contained in L_E and $L_{E \cup E'}$ respectively,
- labelled schema formula η in L_E,
- Σ provisos π and π' with $\pi' \leq \pi$.

Proposition 2.3.14 If ϖ is a sober deduction sequence for $\Psi \vdash_E \eta$ ending in $\langle \eta, \Psi', 1 \rangle$, then there is a sober deduction sequence with the same length as ϖ for $\Psi\sigma \vdash_{E'} \eta\sigma$ ending in $\langle \eta\sigma, \Psi'', 1 \rangle$ where $\Psi'' \subseteq \Psi'\sigma$, for any

- sets E and E' of label schema variables,
- set Ψ of labelled schema formulae contained in L_E,
- labelled schema formula η in L_E,
- Σ schema substitution σ over E within $L_{E'}$,

in the context of a lfob deduction system $\langle \Sigma, R \rangle$.

Proof We start by briefly sketching the proof. The idea is to construct a deduction sequence for $\Psi\sigma \vdash_E \eta\sigma$ based on a deduction sequence for $\Psi \vdash_E \eta$, with the same rules applied, and where the schema substitutions are obtained by composing the schema substitutions used along the deduction for $\Psi \vdash_E \eta$ with this new schema substitution σ. So, the proof follows by induction on the length of a sober deduction sequence for $\Psi \vdash_E \eta$. We now start the proof.

Base. Let σ be a Σ schema substitution over E within $L_{E'}$ and suppose $\langle \eta, \Psi_1, 1 \rangle$ is a sober deduction sequence for $\Psi \vdash_E \eta$. So, consider two cases.

- *Either* Ψ_1 is $\{\eta\}$ and so, it is straightforward to see that $\langle \eta\sigma, \Psi_1\sigma, 1 \rangle$ is a sober deduction sequence for $\Psi\sigma \vdash_{E'} \eta\sigma$;

- *Or* there are a rule $\langle \emptyset, \eta', P'_f, P'_d \rangle$, named r, and a Σ schema substitution σ' such that $\eta'\sigma'$ is η, $\Psi_1 = \emptyset$ and $(\pi'_\Sigma \sigma'_{\mathrm{nl}})$ is the Σ proviso 1 for each π' in P'_f. So, for each π' in P'_f, $(\pi'_\Sigma(\sigma'_{\mathrm{nl}}\sigma_{\mathrm{nl}}))$ is also the 1 formula proviso, see Remark 2.3.6. Note that, by Definition 2.3.9, P'_d is $\emptyset$, because $k = 0$. Hence, $\langle \eta\sigma, \Psi_1\sigma, 1 \rangle$ is a sober deduction sequence for $\Psi\sigma \vdash_{E'} \eta\sigma$, by rule r and Σ schema substitution $\sigma \circ \sigma'$ within $L_{E'}$.

The induction hypothesis is as follows: if ϖ is a sober deduction sequence with length less than or equal to n for $\Psi \vdash_E \eta$ ending in $\langle \eta, \Psi', 1 \rangle$, then there is a sober deduction sequence with the same length as ϖ for $\Psi\sigma \vdash_{E'} \eta\sigma$ ending in $\langle \eta\sigma, \Psi'', 1 \rangle$ where Ψ'' is contained in $\Psi'\sigma$, for any sets E and E' of label schema variables, set Ψ with labelled schema formulae contained in L_E, labelled schema formula η in L_E, and Σ schema substitution σ over E

within $L_{E'}$.

Step. Let E and E' be any sets of label schema variables, Ψ a set with labelled schema formulae contained in L_E, η a labelled schema formula in L_E, and σ a Σ schema substitution over E within $L_{E'}$. Suppose there is a sober deduction sequence named ϖ, with length $n+1$, ending in $\langle \eta, \Psi_{n+1}, 1\rangle$, for $\Psi \vdash_E \eta$. Then, there are Σ schema substitution σ' within L_E, rule $\langle\{\langle \vartheta'_1, \Psi'_1, \eta'_1\rangle, \dots, \langle \vartheta'_k, \Psi'_k, \eta'_k\rangle\}, \eta', P'_f, P'_d\rangle$, named r, and $i_1, \dots, i_k$ in $\{1, \dots, n\}$, with $\eta_{i_j} = \eta'_j\sigma'$, $\Psi_{i_j} = \vartheta'_j\sigma'$ and $\pi_{i_j} = 1$ for each $j = 1, \dots, k$, $\Psi_{n+1} = \Psi_{i_1} \setminus \Psi'_1\sigma' \cup \dots \cup \Psi_{i_k} \setminus \Psi'_k\sigma'$, $(\pi'_\Sigma\sigma'_{\mathrm{nl}}) = 1$, for each π' in P'_f, and $\pi'_{\Sigma;E}(\sigma') = 1$, for each π' in P'_d, and $\eta = \eta'\sigma'$.

Denote by Υ'_j the set Υ if the deduction proviso $\mathrm{fresh}(\Upsilon, \langle \vartheta'_j, \Psi'_j, \eta'_j\rangle)$ is in P'_d, or $\emptyset$, otherwise, for each $j = 1, \dots, k$.

Consider a Σ schema substitution σ''_j over E within $L_{E'}$, $\Upsilon'_j\sigma'$ co-equivalent to σ, such that, $\nu'\sigma'\sigma''_j$ is a label schema variable not in $(\mathrm{lsv}(\Psi'_j \cup \{\eta'_j\}) \setminus \{\nu'\})\sigma'\sigma''_j$ for each ν' in Υ'_j and $\Upsilon'_j\sigma'\sigma''_j \cap \mathrm{lsv}(\vartheta'_j\sigma' \setminus \Psi'_j\sigma')\sigma = \emptyset$.

It is possible to consider a schema substitution σ''_j satisfying the above mentioned conditions because $\Upsilon'_j\sigma'$ are a set of label schema variables, the sets $\vartheta'_j\sigma' \setminus \Psi'_j\sigma'$ and Ψ'_j are finite, and $\nu'\sigma'$ is not in $(\mathrm{lsv}(\Psi'_j \cup \{\eta'_j\}) \setminus \{\nu'\})\sigma'$ for each ν' in Υ'_j, since $\pi'_\Sigma(\sigma') = 1$ for each π' in P'_d.

Denote by ϖ_j the sober deduction subsequence of ϖ, ending in $\langle \eta'_j\sigma', \vartheta'_j\sigma', 1\rangle$ for $\vartheta'_j\sigma' \vdash_E \eta'_j\sigma'$, for each $j = 1, \dots, k$. Thus, by induction hypothesis, for each $j = 1, \dots, k$, there is a sober deduction sequence, named ϖ''_j, of the same length as ϖ_j, ending in $\langle \eta'_j\sigma'\sigma''_j, \Psi''_j, 1\rangle$, for $\vartheta'_j\sigma'\sigma''_j \vdash_{E'} \eta'_j\sigma'\sigma''_j$.

Consider the Σ schema substitution σ''' within $L_{E'}$, $\cup_{j=1,\dots,k}\Upsilon'_j$ co-equivalent to $\sigma'\sigma$ such that $\vartheta'_j\sigma'''_s$ is Ψ''_j and $\nu\sigma'''$ is $\nu\sigma'\sigma''_j$ for any ν in Υ'_j and $j = 1, \dots, k$.

Note that, $\psi\sigma'''$ is $\psi\sigma'\sigma''_j$, for each ψ in $\Psi'_j \cup \{\eta'_j\}$ and j in $\{1, \dots, k\}$. This is shown by case analysis on ψ. Suppose ψ is $v{:}\gamma$. Then $\psi\sigma'''$ is $v\sigma'''{:}\gamma\sigma'''$ which is $v\sigma'''{:}\gamma\sigma'\sigma''_j$ since σ''' is co-equivalent to $\sigma'\sigma''_j$. Now we show by induction on the structure of a label schema term v that $v\sigma'''$ is $v\sigma'\sigma''_j$.

- *v is a label schema variable in Υ'_j.* Then, by definition of σ''', $v\sigma'''$ is $v\sigma'\sigma''_j$;

- *v is a label schema variable not in Υ'_j.* Then v is not in $\cup_{i=1,\dots,k}\Upsilon'_i$ because, v is in $\mathrm{lsv}(\Psi'_j \cup \{\eta'_j\})$, and by Definition 2.3.9, for each $l = 1, \dots, k$, $\Upsilon'_l \cap \cup_{i=1,\dots,k \text{ and } i \neq l}\mathrm{lsv}(\Psi'_i \cup \{\eta'_i\}) = \emptyset$. So, by definition of σ''', $v\sigma'''$ is $v\sigma'\sigma$, which is $v\sigma'\sigma''_j$ since σ''_j is $\Upsilon'_j\sigma'$ co-equivalent to σ, and $v\sigma'$ is not in $\Upsilon'_j\sigma'$ because $v'\sigma'$ is not in $(\mathrm{lsv}(\Psi'_j \cup \{\eta'_j\}) \setminus \{v'\})\sigma'$ (recall that $\pi'_\Sigma(\sigma') = 1$ for each π' in P'_d), and $v\sigma'$ is in it, for each v' in Υ'_j.

- *v is F^l_0.* Then $v\sigma'''$ is v which is $v\sigma'\sigma''_j$.

- *v is $f^l(v_1, \dots, v_k)$.* Then $f^l(v_1, \dots, v_k)\sigma'''$ is $f^l(v_1\sigma''', \dots, v_k\sigma''')$ which is, by induction hypothesis, $f^l(v_1\sigma'\sigma''_j, \dots, v_k\sigma'\sigma''_j)$, and so is $f^l(v_1, \dots, v_k)\sigma'\sigma''_j$.

- For the other possible cases of ψ the proof follows similarly.

Now we show that the sequence $\varpi''_1, \dots, \varpi''_k, \langle \eta'\sigma''', \Psi''_1 \setminus \Psi'_1\sigma''' \cup \dots \cup \Psi''_k \setminus \Psi'_k\sigma''', 1\rangle$, named ϖ''', is a sober deduction sequence with the same length as ϖ for $\Psi\sigma \vdash_{E'} \eta\sigma$ such that $\Psi''_1 \setminus \Psi'_1\sigma''' \cup \dots \cup \Psi''_k \setminus \Psi'_k\sigma'''$ is contained in $\Psi_{n+1}\sigma$, by rule r and Σ schema substitution σ''' within $L_{E'}$. To see this note that

- $\eta'_j\sigma'''$ *is* $\eta'_j\sigma'\sigma''_j$, as showed above.

- $\vartheta'_j\sigma''$ *is* Ψ''_j, by definition of σ'''.

- $\eta'\sigma'''$ *is* $\eta\sigma$. To show this note that, $\eta'\sigma'''$ is $\eta'\sigma'\sigma$, i.e., $\eta\sigma$, by definition of σ''', since $\text{lsv}(\eta') \cap \cup_{j=1,\dots,k}\Upsilon'_j = \emptyset$ by Definition 2.3.9.

- $\Psi''_1 \setminus \Psi'_1\sigma''' \cup \ldots \cup \Psi''_k \setminus \Psi'_k\sigma'''$ *is contained in* $\Psi_{n+1}\sigma$ which is $(\vartheta'_1\sigma' \setminus \Psi'_1\sigma')\sigma \cup \ldots \cup (\vartheta'_k\sigma' \setminus \Psi'_k\sigma')\sigma$, and so in $\Psi\sigma$. This happens since, for each $j = 1, \ldots, k$,

(i) $\Psi''_j \setminus \Psi'_j\sigma'''$ is contained in $\vartheta'_j\sigma'\sigma''_j \setminus \Psi'_j\sigma'\sigma''_j$, since Ψ''_j is contained in $\vartheta'_j\sigma'\sigma''_j$ and $\Psi'_j\sigma'''$ is $\Psi'_j\sigma'\sigma''_j$, as showed above,

(ii) $\vartheta'_j\sigma'\sigma''_j \setminus \Psi'_j\sigma'\sigma''_j$ is contained in $(\vartheta'_j\sigma' \setminus \Psi'_j\sigma')\sigma''_j$ because if ψ is in $\vartheta'_j\sigma'$ and $\psi\sigma''_j$ is not in $\Psi'_j\sigma'\sigma''_j$ then ψ is not in $\Psi'_j\sigma'$, as desired, and

(iii) $(\vartheta'_j\sigma' \setminus \Psi'_j\sigma')\sigma''_j$ is $(\vartheta'_j\sigma' \setminus \Psi'_j\sigma')\sigma$, by definition of σ''_j, since $\Upsilon'_j\sigma' \cap (\vartheta'_j\sigma' \setminus \Psi'_j\sigma') = \emptyset$ because $\pi'(\sigma') = 1$, for each π' in P'_d;

- $(\pi'_\Sigma\sigma'''_{\text{nl}}) = 1$ *for each* π' in P'_f. To show this, let π' be in P'_f. Note that $(\pi'_\Sigma\sigma'_{\text{nl}})$ is 1. Then, $(\pi'_\Sigma(\sigma'_{\text{nl}}\sigma_{\text{nl}}))$ is also 1, see Remark 2.3.6. Thus $(\pi'_\Sigma\sigma'''_{\text{nl}}) = 1$ since, $(\pi'_\Sigma(\sigma'\sigma)_{\text{nl}}) = (\pi'_\Sigma\sigma'''_{\text{nl}})$ because $\sigma'''_t = (\sigma'\sigma)_t$ and $\sigma'''_f = (\sigma'\sigma)_f$, see Remark 2.3.6.

- $\pi'_{\Sigma;E'}(\sigma''') = 1$ *for each* π' in P'_d. Suppose π' is $\text{fresh}(\Upsilon'_j, \langle\vartheta'_j, \Psi'_j, \eta'_j\rangle)$. Then,

– $\nu'\sigma'''$ is a label schema variable for each ν' in Υ'_j. To show this, let ν' be in Υ'_j. So, $\nu'\sigma'''$, by definition of σ''', is $\nu'\sigma'\sigma''_j$, which, by definition of σ''_j and σ', is a label schema variable.

– $\nu'\sigma'''$ is not in $(\text{lsv}(\Psi'_j \cup \{\eta'_j\}) \setminus \{\nu'\})\sigma'''$ for each ν' in Υ'_j. Let ν' be in Υ'_j. Then $\nu'\sigma'''$, by definition of σ''', is $\nu'\sigma'\sigma''_j$, which, taking into account the definition of σ''_j, is not in $(\text{lsv}(\Psi'_j \cup \{\eta'_j\}) \setminus \{\nu'\})\sigma'\sigma''_j$. So, the result follows because $(\text{lsv}(\Psi'_j \cup \{\eta'_j\}) \setminus \{\nu'\})\sigma'''$ is contained in $(\text{lsv}(\Psi'_j \cup \{\eta'_j\}) \setminus \{\nu'\})\sigma'\sigma''_j$. To see this let ν''' be in $(\text{lsv}(\Psi'_j \cup \{\eta'_j\}) \setminus \{\nu'\})\sigma'''$. Then, there is a labelled formula ψ in $\Psi'_j \cup \{\eta'_j\}$ and a label schema variable ν in ψ, distinct of ν', with $\nu\sigma''' = \nu'''$. Since, as shown above, $\psi\sigma'''$ is $\psi\sigma'\sigma''_j$ then $\nu'''(= \nu\sigma''') = \nu\sigma'\sigma''_j$, and so ν''' is in $(\text{lsv}(\Psi'_j \cup \{\eta'_j\}) \setminus \{\nu'\})\sigma'\sigma''_j$.

– $\Upsilon'_j\sigma''' \cap \text{lsv}(\vartheta'_j\sigma''' \setminus \Psi'_j\sigma''') = \emptyset$. This happens because (i) $\Upsilon'_j\sigma'''$ is $\Upsilon'_j\sigma'\sigma''_j$, by definition of σ''', (ii) $\vartheta'_j\sigma''' \setminus \Psi'_j\sigma'''$ is contained in $(\vartheta'_j\sigma' \setminus \Psi'_j\sigma')\sigma$, as shown above, and (iii) $\Upsilon'_j\sigma'\sigma''_j \cap (\vartheta'_j\sigma' \setminus \Psi'_j\sigma')\sigma = \emptyset$, by definition of σ''_j. *QED*

Now we turn our attention towards idempotence.

Proposition 2.3.15 In a lfob deduction system $\langle\Sigma, R\rangle$ we have $\Psi, \Psi' \vdash_E \eta$ if $\Psi, \eta' \vdash_E \eta$ and $\Psi' \vdash_E \eta'$, for any

- set E of label schema variables,
- sets Ψ and Ψ' of labelled schema formulae contained in L_E,
- labelled schema formulae η and η' in L_E.

Proof We start by briefly sketching the proof. The idea is to construct a deduction sequence for $\Psi, \Psi' \vdash_E \eta$ by replacing all η' hypothesis that are not discharged along a deduction for $\Psi, \eta' \vdash_E \eta$ by a deduction sequence for $\Psi' \vdash_E \eta'$. Since the deduction sequence for $\Psi' \vdash_E \eta'$ may bring new hypothesis we have to re-formulate the schema substitutions used in the proof of $\Psi, \eta' \vdash_E \eta$ in order for the deduction provisos to be satisfied with the new hypothesis. So, the proof follows by induction on the length of a sober deduction sequence for $\Psi, \eta' \vdash_E \eta$.

Let ϖ' be a sober deduction sequence for $\Psi' \vdash_E \eta'$ ending in $\langle \eta', \Psi'_{m'}, 1\rangle$. Then we show that if ϖ is a sober deduction sequence for $\Psi, \eta' \vdash_E \eta$ ending in $\langle \eta, \Psi_m, 1\rangle$, then there is a sober deduction sequence for $\Psi, \Psi' \vdash_E \eta$ ending in $\langle \eta, (\Psi_m \setminus \{\eta'\}) \cup \Phi'', 1\rangle$, where Φ'' is contained in $\Psi'_{m'}$. This is shown by induction on the length of ϖ.

Base: suppose $\langle \eta, \Psi_1, 1\rangle$ is a sober deduction sequence for $\Psi, \eta' \vdash_E \eta$. We consider two cases:

either Ψ_1 is $\{\eta\}$ and we consider two sub-cases

1. η is η'. Then by monotony (Proposition 2.3.13), ϖ' is a sober deduction sequence for $\Psi', \Psi \vdash_E \eta$. Note that $\Psi'_{m'}$ is $(\Psi_1 \setminus \{\eta'\}) \cup \Psi'_{m'}$ since $\Psi_1 \setminus \{\eta'\}$ is $\emptyset$;

2. η is distinct of η'. So, $\langle \eta, \Psi_1, 1\rangle$ is a sober deduction sequence for $\Psi', \Psi \vdash_E \eta$, since Ψ_1 is $\{\eta\}$ and is contained in $\Psi \cup \Psi'$. Note that Ψ_1 is $(\Psi_1 \setminus \{\eta'\}) \cup \emptyset$;

or there is a rule $\langle \emptyset, \eta'', P_f, P_d\rangle$, and Σ schema substitution σ within L_E such that Ψ_1 is $\emptyset$, $\eta''\sigma$ is η, $(\pi_\Sigma \sigma_{\mathrm{nl}})$ is the Σ proviso 1, for each π in P_f, and $\pi_{\Sigma;E}(\sigma)$ is 1 for each π in P_d. So, $\langle \eta, \Psi_1, 1\rangle$ is a sober deduction sequence for $\Psi', \Psi \vdash_E \eta$. Note that Ψ_1 is $(\Psi_1 \setminus \{\eta'\}) \cup \emptyset$.

The induction hypothesis is as follows: for any set Ψ of labelled schema formulae contained in L_E, and labelled schema formula η in L_E, if there is a sober deduction sequence of length less than or equal to n, ending in $\langle \eta, \Psi_m, 1\rangle$, for $\Psi, \eta' \vdash_E \eta$, then, there is a sober deduction sequence for $\Psi, \Psi' \vdash_E \eta$, ending in $\langle \eta, (\Psi_m \setminus \{\eta'\}) \cup \Phi'', 1\rangle$ where Φ'' is contained in $\Psi'_{m'}$.

Step. Let Ψ be a set of labelled schema formulae contained in L_E, η a labelled schema formula in L_E, and ϖ a sober deduction sequence with length $n+1$ for $\Psi, \eta' \vdash_E \eta$ ending in $\langle \eta, \Psi_{n+1}, 1\rangle$. Then, there are rule $\langle \{\langle \vartheta''_1, \Psi''_1, \eta''_1\rangle, \ldots, \langle \vartheta''_k, \Psi''_k, \eta''_k\rangle\}, \eta'', P''_f, P''_d\rangle$, Σ schema substitution σ within L_E, and $i_1, \ldots, i_k$ in $\{1, \ldots, n\}$, with $\eta_{i_j} = \eta''_j\sigma$, $\Psi_{i_j} = \vartheta''_j\sigma$ and $\pi_{i_j} = 1$ for each $j = 1, \ldots, k$, $\Psi_{n+1} = \Psi_{i_1} \setminus \Psi''_1\sigma \cup \ldots \cup \Psi_{i_k} \setminus \Psi''_k\sigma$, $(\pi_\Sigma \sigma_{\mathrm{nl}}) = 1$ for each π in P''_f, $\pi_{\Sigma;E}(\sigma) = 1$ for each π in P''_d, and $\eta = \eta''\sigma$.

We denote by Υ''_j the set Υ if the deduction proviso $\mathrm{fresh}(\Upsilon, \langle \vartheta''_j, \Psi''_j, \eta''_j\rangle)$ is in P''_d, or $\emptyset$, otherwise, for each $j = 1, \ldots, k$.

Consider a Σ schema substitution σ''_j within L_E, $\Upsilon''_j\sigma$ co-equivalent to id, such that, $\nu\sigma\sigma''_j$ is a label schema variable not in $(\mathrm{lsv}(\Psi''_j \cup \{\eta''_j\}) \setminus \{\nu\})\sigma\sigma''_j$ for each ν in Υ''_j, and $\Upsilon''_j\sigma\sigma''_j \cap \mathrm{lsv}(\vartheta''_j\sigma \setminus \Psi''_j\sigma)\sigma''_j = \emptyset$, and $\Upsilon''_j\sigma\sigma'' \cap \mathrm{lb}(\Psi'_{m'}) = \emptyset$. Note that $\mathrm{lsv}(\vartheta''_j\sigma \setminus \Psi''_j\sigma)\sigma''_j$ is $\mathrm{lsv}(\vartheta''_j\sigma \setminus \Psi''_j\sigma)$.

It is possible to consider a schema substitution σ''_j satisfying the above conditions because $\Upsilon''_j\sigma$ is a set of label schema variables, the sets $\Psi'_{m'}$, $\vartheta''_j\sigma \setminus \Psi''_j\sigma$ and Ψ''_j are finite, and $\nu\sigma$ is not in $(\mathrm{lsv}(\Psi''_j \cup \{\eta''_j\}) \setminus \{\nu\})\sigma$ for each ν in Υ''_j, since $\pi_{\Sigma;E}(\sigma) = 1$ for each π in P''_d.

Denote by ϖ_j the sober deduction subsequence of ϖ, ending in $\langle \eta''_j\sigma, \vartheta''_j\sigma, 1\rangle$ for $\vartheta''_j\sigma \vdash_E \eta''_j\sigma$, for each $j = 1, \ldots, k$. Thus, by Proposition 2.3.14, for each $j = 1, \ldots, k$, there is a sober deduction sequence, named ϖ''_j, with the same length as ϖ_j, ending in $\langle \eta''_j\sigma\sigma''_j, \Psi_j, 1\rangle$, for $\vartheta''_j\sigma\sigma''_j \vdash_E \eta''_j\sigma\sigma''_j$.

For each $j = 1, \ldots, k$, we define a set Φ''_j contained in $\Psi'_{m'}$, and a sober deduction sequence ϖ'''_j in the following way:

if η' is in $\Psi_j \setminus \Psi''_j\sigma\sigma''_j$, then ϖ'''_j is the sober deduction sequence for $\vartheta''_j\sigma\sigma''_j \setminus \{\eta'\}, \Psi' \vdash_E \eta_{i_j}$, which exists by the induction hypothesis applied to ϖ''_j, and Φ''_j is such that ϖ'''_j ends in $\langle \eta''_j\sigma\sigma''_j, \Psi_j \setminus \{\eta'\} \cup \Phi''_j, 1\rangle$;

otherwise we set Φ''_j to the empty set and ϖ'''_j to ϖ''_j.

Note that ϖ'''_j ends in $\langle \eta''_j\sigma\sigma''_j, \Psi_j \setminus \{\eta'\} \cup \Phi''_j, 1\rangle$ for each $j = 1, \ldots, k$.

Consider the Σ schema substitution σ''' within L_E, $\cup_{j=1,\ldots,k}\Upsilon''_j$ co-equivalent to σ such that $\sigma'''_s(\vartheta''_j)$ is $(\Psi_j \setminus \{\eta'\}) \cup \Phi''_j$ and $\nu\sigma'''$ is $\nu\sigma\sigma''_j$ for any ν in Υ''_j and $j = 1, \ldots, k$. Now we prove some useful facts concerning σ''':

1. $\psi\sigma'''$ is $\psi\sigma\sigma''_j$, for each ψ in $\Psi''_j \cup \{\eta''_j\}$ and j in $\{1, \ldots, k\}$. This is shown by case analysis. Suppose ψ is $v{:}\gamma$. Then $\psi\sigma'''$ is $v\sigma'''{:}\gamma\sigma'''$ which is $v\sigma'''{:}\gamma\sigma\sigma''_j$ since σ''' is co-equivalent to σ, and σ''_j is co-equivalent to id. Now we show by induction on the structure of a label schema term v that $v\sigma'''$ is $v\sigma\sigma''_j$.

- *v is a label schema variable in Υ''_j.* Then, by definition of σ''', $v\sigma'''$ is $v\sigma\sigma''_j$ and we are done.

- *v is a label schema variable not in Υ''_j.* Then v is not in $\cup_{i=1,\ldots,k}\Upsilon''_i$ because, v is in $\mathrm{lsv}(\Psi''_j \cup \{\eta''_j\})$, and by Definition 2.3.9, for each $l = 1, \ldots, k$, $\Upsilon''_l \cap \cup_{i=1,\ldots,k \text{ and } i \neq l}\mathrm{lsv}(\Psi''_i \cup \{\eta''_i\}) = \emptyset$. So, by definition of σ''', $v\sigma'''$ is $v\sigma$, which is $v\sigma\sigma''_j$ since σ''_j is $\Upsilon''_j\sigma$ co-equivalent to id, and $v\sigma$ is not in $\Upsilon''_j\sigma$ because $v'\sigma$ is not in $(\mathrm{lsv}(\Psi''_j \cup \{\eta''_j\}) \setminus \{v'\})\sigma$ (recall that $\pi_\Sigma(\sigma) = 1$ for each π in P''_d), and $v\sigma$ is in it, for each v' in Υ''_j.

- *v is F^l_0.* Then $v\sigma'''$ is v which is $v\sigma\sigma''_j$.

- *v is $f^l(v_1, \ldots, v_k)$.* Then $f^l(v_1, \ldots, v_k)\sigma'''$ is $f^l(v_1\sigma''', \ldots, v_k\sigma''')$ which is, by induction hypothesis, $f^l(v_1\sigma\sigma''_j, \ldots, v_k\sigma\sigma''_j)$, and so is $f^l(v_1, \ldots, v_k)\sigma\sigma''_j$.

- For the other possible cases of ψ the proof follows similarly.

2. $\Phi''_j \setminus \Psi''_j\sigma'''$ is contained in $\Psi'_{m'}$, for each j in $\{1, \ldots, k\}$. It is enough to see that Φ''_j is contained in $\Psi'_{m'}$, by definition of Φ''_j, for each j in $\{1, \ldots, k\}$.

3. $\Psi_j \setminus (\{\eta'\} \cup \Psi''_j\sigma''')$ is contained in $\vartheta''_j\sigma \setminus (\{\eta'\} \cup \Psi''_j\sigma)$, for each j in $\{1, \ldots, k\}$. Note that Ψ_j is contained in $\vartheta''_j\sigma\sigma''_j$. Let ψ be in Ψ_j and ψ'' be in $\vartheta''_j\sigma$ such that $\psi''\sigma''_j$ is ψ. Suppose $\psi''\sigma''_j$ is not in $\Psi''_j\sigma'''$, i.e., $\Psi''_j\sigma\sigma''_j$, and is different of η'. Then ψ'' is not in $\Psi''_j\sigma$. Thus ψ'' is in $\vartheta''_j\sigma \setminus \Psi''_j\sigma$. So $\Upsilon''_j\sigma \cap \mathrm{lsv}(\psi'') = \emptyset$, since $\pi_\Sigma(\sigma) = 1$ for each π in P''_d. Hence ψ, which is $\psi''\sigma''$, is ψ'' and so ψ is in $\vartheta''_j\sigma$ and not in $\Psi''_j\sigma$ and is different of η', as we wanted to show.

4. $(\mathrm{lsv}(\Psi''_j \cup \{\eta''_j\}) \setminus \{\nu\})\sigma'''$ is contained in $(\mathrm{lsv}(\Psi''_j \cup \{\eta''_j\}) \setminus \{\nu\})\sigma\sigma''_j$, for each $j = 1, \ldots, k$. To see this let ν''' be in $(\mathrm{lsv}(\Psi''_j \cup \{\eta''_j\}) \setminus \{\nu\})\sigma'''$. Then, there is a labelled formula ψ in $\Psi''_j \cup \{\eta''_j\}$ and a label schema variable ν' in ψ, distinct of ν, with $\nu'\sigma''' = \nu'''$. Since, as shown above, $\psi\sigma'''$ is $\psi\sigma\sigma''_j$ then $\nu'''(= \nu'\sigma''') = \nu'\sigma\sigma''_j$, and so ν''' is in $(\mathrm{lsv}(\Psi''_j \cup \{\eta''_j\}) \setminus \{\nu\})\sigma\sigma''_j$.

So, the sequence $\varpi'''_1, \ldots, \varpi'''_k, \langle \eta''\sigma''', ((\Psi_1 \setminus \{\eta'\}) \cup \Phi''_1) \setminus \Psi''_1\sigma''' \cup \ldots \cup ((\Psi_k \setminus \{\eta'\}) \cup \Phi''_k) \setminus \Psi''_k\sigma''', 1\rangle$ is a sober deduction sequence for $\Psi, \Psi' \vdash_E \eta$, by rule r and Σ schema substitution σ''' within L_E, since:

- *$\eta''_j\sigma'''$ is $\eta''_j\sigma\sigma''_j$*, for each $j = 1, \ldots, k$, as showed in fact 1. above.

- *$\vartheta''_j\sigma'''$ is $(\Psi_j \setminus \{\eta'\}) \cup \Phi''_j$*, for each $j = 1, \ldots, k$, by definition of σ'''.

- *$\eta''\sigma'''$ is η.* To show this note that, $\eta''\sigma'''$ is $\eta''\sigma$, by definition of σ''', since $\mathrm{lsv}(\eta'') \cap$

$\cup_{j=1,\ldots,k}\Upsilon''_j = \emptyset$ by Definition 2.3.9.

- $((\Psi_1 \setminus \{\eta'\}) \cup \Phi''_1) \setminus \Psi''_1\sigma''' \cup \ldots \cup ((\Psi_k \setminus \{\eta'\}) \cup \Phi''_k) \setminus \Psi''_k\sigma'''$ *is contained in* $\Psi \cup \Psi'$. Note first that $((\Psi_j \setminus \{\eta'\}) \cup \Phi''_j) \setminus \Psi''_j\sigma'''$, i.e., $\Psi_j \setminus (\{\eta'\} \cup \Psi''_j\sigma''') \cup (\Phi''_j \setminus \Psi''_j\sigma''')$, is contained in $(\vartheta''_j\sigma \setminus \Psi''_j\sigma) \setminus \{\eta'\} \cup (\Phi''_j \setminus \Psi''_j\sigma''')$ for each $j = 1, \ldots, k$, using fact 3. above. So, $\cup_{j=1,\ldots,k}(\Psi_j \setminus \{\eta'\} \cup \Phi''_j) \setminus \Psi''_j\sigma'''$ is contained in $\Psi_{n+1} \setminus \{\eta'\} \cup \Psi''$ where Ψ'' is contained in $\Psi'_{m'}$ and is the set $\Phi''_1 \setminus \Psi''_1\sigma''' \cup \ldots \cup \Phi''_k \setminus \Psi''_k\sigma'''$. Thus, the intended result follows, by noting that $\Psi_{n+1} \setminus \{\eta'\} \cup \Psi''$ is contained in $\Psi \cup \Psi'$.

- $(\pi_\Sigma\sigma'''_{\mathrm{nl}})$ *is* the Σ proviso 1 for each π in P'_f. To show this, let π be in P'_f. Note that $(\pi_\Sigma\sigma_{\mathrm{nl}})$ is $(\pi_\Sigma\sigma'''_{\mathrm{nl}})$ because σ''' is co-equivalent to σ, see Remark 2.3.6. Then $(\pi_\Sigma\sigma'''_{\mathrm{nl}})$ is 1, since $(\pi'_\Sigma\sigma_{\mathrm{nl}})$ is 1 for each π' in P'_f.

- $\pi_{\Sigma;E}(\sigma''') = 1$ for each π in P''_d. Let π be fresh$(\Upsilon''_j, \langle\vartheta''_j, \Psi''_j, \eta''_j\rangle)$. Then,
(a) $\nu\sigma'''$ is a label schema variable for each ν in Υ''_j. Let ν be in Υ''_j. Then $\nu\sigma'''$, by definition of σ''', is $\nu\sigma\sigma''_j$, which by definition of σ''_j is a label schema variable.

(b) $\nu\sigma'''$ is not in $(\mathrm{lsv}(\Psi''_j \cup \{\eta''_j\}) \setminus \{\nu\})\sigma'''$ for each ν in Υ''_j. Let ν be in Υ''_j. Then $\nu\sigma'''$, by definition of σ''', is $\nu\sigma\sigma''_j$, which, taking into account the definition of σ''_j, is not in $(\mathrm{lsv}(\Psi''_j \cup \{\eta''_j\}) \setminus \{\nu\})\sigma\sigma''_j$. So, the result follows because $(\mathrm{lsv}(\Psi''_j \cup \{\eta''_j\}) \setminus \{\nu\})\sigma'''$ is contained in $(\mathrm{lsv}(\Psi''_j \cup \{\eta''_j\}) \setminus \{\nu\})\sigma\sigma''_j$, as shown in fact 4. above.

(c) $\Upsilon''_j\sigma''' \cap \mathrm{lsv}(\vartheta''_j\sigma''' \setminus \Psi''_j\sigma''') = \emptyset$. This happens because (i) $\Upsilon''_j\sigma'''$ is $\Upsilon''_j\sigma\sigma''_j$, by definition of σ''', (ii) $\vartheta''_j\sigma''' \setminus \Psi''_j\sigma'''$, which is $(\Psi_j \setminus (\{\eta'\} \cup \Phi''_j)) \setminus \Psi''_j\sigma'''$, i.e., $\Psi_j \setminus (\{\eta'\} \cup \Psi''_j\sigma''') \cup \Phi''_j \setminus \Psi''_j\sigma'''$ is contained in $(\vartheta''_j\sigma \setminus \Psi''_j\sigma) \cup \Psi'_{m'}$, from the facts 2. and 3. above, and (iii) $\Upsilon''_j\sigma\sigma''_j \cap ((\vartheta''_j\sigma \setminus \Psi''_j\sigma) \cup \Psi'_{m'}) = \emptyset$, by definition of σ''_j. *QED*

2.3.3 Derived rules

Derived rules are a very important way to extend the set of rules that can be used in a deduction, with rules that do not change the consequence relation, helping in this way to simplify and clarify a deduction. In order to motivate and exemplify derived rules note that

$$\frac{\nu{:}\xi_1 \,/\, \nu{:}\xi_2}{\nu{:}\xi_1 \to \xi_2}\to_I \qquad \text{and} \qquad \frac{\nu{:}\xi_1 \to \xi_2 \qquad \nu{:}\xi_1}{\nu{:}\xi_2}\to_E$$

are, according to Definition 2.3.17 introduced below, derived rules for the implication connective in the context of the lfob deduction system $\mathcal{D}_{\mathrm{FOMdD}}$ introduced in Example 2.3.11 for first-order modal logic with decreasing domains. Before presenting the definition of derived rule we need two auxiliary notions:

Remark 2.3.16 Given a deduction sequence ϖ we define the set of positions of the elements used in ϖ to obtain the element at a position l, usE(ϖ, l), inductively on the number of rule applications to get that element, as follows:

- if the element at position l is an assumption then usE(ϖ, l) is $\{l\}$,
- else it is obtained by a rule, suppose with k premises, and previous elements at positions $i_1, \ldots, i_k$. So usE$(\varpi, l) = \{l\} \cup \cup_{j=1,\ldots,k}usE(\varpi, i_j)$.

Given a deduction sequence ϖ, and positions l_1 and l_2 in that sequence such that l_1 is in usE(ϖ, l_2) and there is only one path in ϖ from l_1 to l_2, we define the set of labelled schema formulae removed from l_1 to l_2 in ϖ, $\mathrm{remL}(\varpi, l_1, l_2)$ inductively on the number of rule applications to go from l_1 to l_2, as follows:

- if $l_1 = l_2$ then $\mathrm{remL}(\varpi, l_1, l_2) = \emptyset$,
- else the element at position l_1 is used as the j'th premise to obtain the tuple at a position l' by $\langle\{\langle\vartheta_1, \Psi_1, \eta_1\rangle, \dots, \langle\vartheta_k, \Psi_k, \eta_k\rangle\}, \eta, P_f, P_d\rangle$ and σ, and l' is in $\mathrm{usE}(\varpi, l_2)$, then $\mathrm{remL}(\varpi, l_1, l_2) = \Psi_j\sigma \cup \mathrm{remL}(\varpi, l', l_2)$.

Observe that an equivalent definition for the else part is the following:

else there is an element at a position l' used as a premise, for instance the j'th, to obtain the tuple at l_2 by the rule $\langle\{\langle\vartheta_1, \Psi_1, \eta_1\rangle, \dots, \langle\vartheta_k, \Psi_k, \eta_k\rangle\}, \eta, P_f, P_d\rangle$ and σ, such that l_1 is in $\mathrm{usE}(\varpi, l')$. Then $\mathrm{remL}(\varpi, l_1, l_2) = \mathrm{remL}(\varpi, l_1, l') \cup \Psi_j\sigma$.

Definition 2.3.17 In the context of a lfob deduction system $\langle\Sigma, R\rangle$ the tuple

$$\langle\{\langle\vartheta_1, \Psi_1, \eta_1\rangle, \dots, \langle\vartheta_k, \Psi_k, \eta_k\}, \eta, \{\pi\}, P_d\rangle$$

is a *derived rule* whenever (i) it is a rule, see Definition 2.3.9, (ii) if there is, for each $i = 1, \dots, k$, a sober deduction sequence ϖ_i ending in $\langle\eta_i, \Psi_i, 1\rangle$ then there is a sober deduction sequence $\varpi_1 \dots \varpi_k\varpi$ ending in $\langle\eta, \emptyset, \pi_\Sigma\rangle$, (iii) $\pi_{\Sigma'}$, for Σ' distinct of Σ, is $\pi^1_{\Sigma'} * \dots * \pi^n_{\Sigma'}$, where $\pi^1, \dots, \pi^n$ are the formula provisos appearing in the rules used in ϖ, (iv) derived rules are not used in ϖ, (v) ϖ is not null, (vi) η is obtained in ϖ by application of a rule, (vii) the last element of ϖ_i is its only element referenced in ϖ, for $i = 1, \dots, k$, and if we denote by

- l_i the position of the last element of ϖ_i in $\varpi_1 \dots \varpi_k\varpi$, for each $i = 1, \dots, k$,
- l_f the position of the last element of $\varpi_1 \dots \varpi_k\varpi$,

then (viii) for each $i = 1, \dots, k$, there is only one path in ϖ from l_i to l_f, (ix) for each $i = 1, \dots, k$, the set Ψ_i is contained in $\mathrm{remL}(\varpi_1 \dots \varpi_k\varpi, l_i, l_f)$, and (x) for each $\mathrm{fresh}(\Upsilon'_{j'}, \langle\vartheta'_{j'}, \Psi'_{j'}, \eta'_{j'}\rangle)$ in ϖ at position l in $\varpi_1 \dots \varpi_k\varpi$, if we denote by

- $l_{j'}$ the position of the j''th premise used to obtain the element at l,
- σ' the schema substitution used to obtain the element at l,

then there is only one i in $1, \dots, k$, with

- $\mathrm{fresh}(\Upsilon'_{j'}\sigma', \langle\vartheta_i, \Psi_i, \eta_i\rangle)$ is in P_d,
- l_i is in $\mathrm{usE}(\varpi_1 \dots \varpi_k\varpi, l_{j'})$,
- $\Psi_i \subseteq \Psi'_{j'}\sigma' \cup \mathrm{remL}(\varpi_1 \dots \varpi_k\varpi, l_i, l_{j'})$,
- $\mathrm{lsv}(\Psi'_{j'} \cup \{\eta'_{j'}\})\sigma' \subseteq \mathrm{lsv}(\Psi_i \cup \{\eta_i\})$,
- $\mathrm{lsv}(\vartheta'_{j'}\sigma' \setminus \Psi'_{j'}\sigma')$ is contained in $\mathrm{lsv}(\Psi_i \cup \{\eta_i\}) \setminus \Upsilon'_{j'}\sigma'$.

In order to illustrate and motivate the points of this definition we now present some examples of derived rules in the context of the lfob deduction system $\mathcal{D}_{\mathrm{FOMdD}}$ introduced in Example 2.3.11 for first-order modal logic with decreasing domains. Recall that, in that context, $v{:}\exists_x\varphi$ is defined as the abbreviation of $v{:}\neg\forall_x\neg\varphi$, and $v{:}\Diamond\varphi$ as the abbreviation of $v{:}\neg\Box\neg\varphi$.

Example 2.3.18 In the context of the first-order modal logic with decreasing domains lfob deduction system $\mathcal{D}_{\text{FOMdD}}$ of Example 2.3.11,

$$\frac{\nu':\xi \quad \nu R_{\Box}\nu'}{\nu:\Diamond\xi}\ \Diamond_I$$

is a derived rule for the introduction of $\Diamond$. To see this Observe that (i) it is a rule, (ii) if there are sober deduction sequences ending in $\langle \nu':\xi, \emptyset, 1\rangle$ and $\langle \nu R_{\Box}\nu', \emptyset, 1\rangle$ then, there is a sober deduction sequence

$\vdots$	$\vdots$	$\vdots$	$\vdots$		
i	$\nu':\xi$	$\emptyset$	1		
$\vdots$	$\vdots$	$\vdots$	$\vdots$		
j	$\nu R_{\Box}\nu'$	$\emptyset$	1		
$j+1$	$\nu:\Box\neg\xi$	$\{\nu:\Box\neg\xi\}$	1	asp	
$j+2$	$\nu':\neg\xi$	$\{\nu:\Box\neg\xi\}$	1	$\Box_E$	$j+1, j$
$j+3$	$\nu':\bot$	$\{\nu:\Box\neg\xi\}$	1	$\neg_E$	$j+2, i$
$j+4$	$\nu:\neg\Box\neg\xi$	$\emptyset$	1	$\neg_I$	$j+3$

ending in $\langle \nu:\Diamond\xi, \emptyset, 1\rangle$, made of all the other sober deduction sequences. The other points of the definition of derived rule, Definition 2.3.17, are trivially fulfilled. Note that the set of fresh variables of the rule should be $\emptyset$ since no fresh proviso appeared in the specific deduction sequence.

Example 2.3.19 In the context of the first-order modal logic with decreasing domains lfob deduction system $\mathcal{D}_{\text{FOMdD}}$ of Example 2.3.11,

$$\frac{\nu:\Diamond\xi \quad \nu R_{\Box}\nu', \nu':\xi \,/\, \nu'':\xi''}{\nu'':\xi''}\ \Diamond_E;\ P_d$$

where P_d is a singleton with $\text{fresh}(\nu', \langle \vartheta_2, \{\nu R_{\Box}\nu', \nu':\xi\}, \nu'':\xi''\rangle)$, is a derived rule. To see this observe that all conditions in the definition of derived rule, Definition 2.3.17, are satisfied: (i) it is a rule, (ii) if there are sober deduction sequences ending in $\langle \nu:\Diamond\xi, \emptyset, 1\rangle$, i.e., $\langle \nu:\neg\Box\neg\xi, \emptyset, 1\rangle$, and $\langle \nu'':\xi'', \{\nu R_{\Box}\nu', \nu':\xi\}, 1\rangle$, then there is a sober deduction sequence

$\vdots$	$\vdots$	$\vdots$	$\vdots$		
i	$\nu'':\xi''$	$\{\nu R_{\Box}\nu', \nu':\xi\}$	1		
$\vdots$	$\vdots$	$\vdots$	$\vdots$		
j	$\nu:\neg\Box\neg\xi$	$\emptyset$	1		
$j+1$	$\nu'':\neg\xi''$	$\{\nu'':\neg\xi''\}$	1	asp	
$j+2$	$\nu':\bot$	$\{\nu'':\neg\xi'', \nu R_{\Box}\nu', \nu':\xi\}$	1	$\neg_E$	$j+1, i$
$j+3$	$\nu':\neg\xi$	$\{\nu'':\neg\xi'', \nu R_{\Box}\nu'\}$	1	$\neg_I$	$j+2$
$j+4$	$\nu:\Box\neg\xi$	$\{\nu'':\neg\xi''\}$	1	$\Box_I$	$j+3$
$j+5$	$\nu:\bot$	$\{\nu'':\neg\xi''\}$	1	$\neg_E$	$j+4, j$
$j+6$	$\nu'':\xi''$	$\emptyset$	1	$\neg_c$	$j+5$

ending in $\langle \nu'':\xi'', \emptyset, 1\rangle$, made of all the other sober deduction sequences, (iii) π is 1, (iv) no derived rules were used, (v) the deduction sequence is not null, (vi) the last element of the deduction sequence was obtained by application of a rule, (vii) $\langle \nu{:}\neg\Box\neg\xi, \emptyset, 1\rangle$ and $\langle \nu''{:}\xi'', \{\nu R_\Box \nu', \nu'{:}\xi\}, 1\rangle$ are the only elements of the assumed sequences that were referenced, (viii) there is only one path from $\langle \nu{:}\neg\Box\neg\xi, \emptyset, 1\rangle$ to $\langle \nu''{:}\xi'', \emptyset, 1\rangle$ and from $\langle \nu''{:}\xi'', \{\nu R_\Box \nu', \nu'{:}\xi\}, 1\rangle$ to $\langle \nu''{:}\xi'', \emptyset, 1\rangle$, (ix) the set $\{\nu R_\Box \nu', \nu'{:}\xi\}$ is contained in the set of labelled schema formulae removed in the deduction sequence from $\langle \nu''{:}\xi'', \{\nu R_\Box \nu', \nu'{:}\xi\}, 1\rangle$ to $\langle \nu''{:}\xi'', \emptyset, 1\rangle$, which is the set $\{\nu'{:}\neg\bot, \nu'{:}\xi, \nu R_\Box \nu', \nu''{:}\neg\xi''\}$, and (x) the deduction proviso fresh$(\nu', \langle \vartheta_1, \{\nu R_\Box \nu'\}, \nu'{:}\xi\rangle)$ appeared in the deduction sequence in the application of rule $\Box_I$ with a schema substitution σ' with $\nu\sigma'$ is ν, $\nu'\sigma'$ is ν', $\xi\sigma'$ is $\neg\xi$ and we have that a) fresh$(\nu', \langle \vartheta_2, \{\nu R_\Box \nu', \nu'{:}\xi\}, \nu''{:}\xi''\rangle)$ is in P_d, b) $\langle \nu''{:}\xi'', \{\nu R_\Box \nu', \nu'{:}\xi\}, 1\rangle$ was used in the sequence until the point where the fresh proviso appeared, c) $\{\nu R_\Box \nu', \nu'{:}\xi\}$ is contained in $\{\nu R_\Box \nu'\} \cup \{\nu'{:}\neg\bot, \nu'{:}\xi\}$ where $\{\nu'{:}\neg\bot, \nu'{:}\xi\}$ is the set of labelled schema formulae removed in the sequence until (but not including) the point where the fresh proviso appeared, d) the set of label schema variables of the original fresh proviso, $\{\nu, \nu'\}$, is contained in the set of label schema variables of the new fresh proviso $\{\nu, \nu', \nu''\}$, and finally e) the set of label schema variables of the hypothesis (without the discharged ones), at the point originating the fresh proviso, $\{\nu''\}$, is contained in the set of not fresh label schema variables appearing in the fresh proviso of the derived rule, i.e., $\{\nu'', \nu\}$.

Note that the derived rules for the introduction and elimination of the $\exists$ connective are similar to the derived rules for the introduction and elimination of the $\Diamond$ connective. And so are their justifications. But, given a deduction sequence employing derived rules, for a certain deduction $\Psi \vdash_E \eta;\ \pi$, is it possible to obtain another deduction sequence for that deduction but without derived rules? It is the goal of next proposition to show that it is indeed possible, and so that derived rules are only a useful but not necessary tool, i.e., all we can do with derived rules we can do without them.

Proposition 2.3.20 If there is a sober deduction sequence involving derived rules for $\Psi \vdash_E \eta;\ \pi$ then, there is also a sober deduction sequence not involving derived rules for that deduction.

Proof We start by briefly sketching the proof. The idea is to show that for every sober deduction sequence for $\Psi \vdash_E \eta;\ \pi$ involving N applications of derived rules it is possible to construct a sober deduction sequence for $\Psi \vdash_E \eta;\ \pi$ with $N-1$ applications of derived rules. Thus, in order to obtain a deduction sequence for $\Psi \vdash_E \eta;\ \pi$ without applications of derived rules we only have to apply that procedure repeatedly. So, given a deduction sequence ϖ for $\Psi \vdash_E \eta;\ \pi$ with a application of a derived rule, the deduction sequence without that application of the derived rule is $\varpi_1 \ldots \varpi_k \varpi'' \varpi''^{\cdot}$, where for each $j = 1, \ldots, k$, ϖ_j is the deduction sequence leading to the element of ϖ corresponding to the k premise of the derived rule, ϖ'' is the sequence based on the deduction sequence ϖ', of Definition 2.3.17, for that derived rule, and finally $\varpi''^{\cdot}$ is a sequence based on $\varpi^{\cdot}$ where $\varpi^{\cdot}$ is ϖ without the ϖ_j's and the application of the derived rule. Note that it is necessary to consider ϖ'' since ϖ' does not cope with the additional hypothesis that may be present in ϖ in the moment of the application of the derived rule. Moreover, it is necessary to consider $\varpi''^{\cdot}$ since $\varpi^{\cdot}$ is defined assuming hypothesis that may be discharged along ϖ''. So, we start to define the sequences ϖ'' and $\varpi''^{\cdot}$ and at the same time we define some functions needed along the proof. Then we interrupt the proof for showing some useful lemmas and after the lemmas proved we resume the proof and we have all the necessary information to conclude. We now start the proof.

In the context of a lfob deduction system $\langle \Sigma, R\rangle$ let ϖ be a sober deduction sequence for $\Psi \vdash_E \eta;\ \pi$, and assume that the tuple at step $n+1$ was obtained by the derived rule $\langle\{\langle\vartheta'_1, \Psi'_1, \eta'_1\rangle, \dots, \langle\vartheta'_k, \Psi'_k, \eta'_k\rangle\}, \eta', \{\pi'\}, P'_d\rangle$ named r, and that no derived rules were used before.

Then, there are schema substitution σ' within L_E, and $i'_1, \dots, i'_k$ in $\{1, \dots, n\}$, with $\eta_{i'_j} = \eta'_j\sigma'$ and $\Psi_{i'_j} = \vartheta'_j\sigma'$ for each $j = 1, \dots, k$, $\pi_{\Sigma;E}(\sigma') = 1$ for each π in P'_d, $\Psi_{n+1} = \Psi_{i'_1} \setminus \Psi'_1\sigma' \cup \dots \cup \Psi_{i'_k} \setminus \Psi'_k\sigma'$, $\pi_{n+1} = \pi_{i'_1} * \dots * \pi_{i'_k} * (\pi'_\Sigma{\sigma'}_{\mathrm{nl}})$, and $\eta_{n+1} = \eta'\sigma'$.

Denote by ϖ_j, for each $j = 1, \dots, k$, the sober deduction sub-sequence of ϖ ending in $\langle\eta'_j\sigma', {\vartheta'}_j\sigma', \pi_{i'_j}\rangle$.

Note that, since r is a derived rule, if there is, for each $j = 1, \dots, k$, a sober deduction sequence ϖ'_j ending in $\langle\eta'_j, {\Psi'}_j, 1\rangle$, then there is a sober deduction sequence $\varpi'_1 \dots \varpi'_k\varpi'$ ending in $\langle\eta', \emptyset, \pi'\rangle$.

To simplify the presentation we define a partial map $\cdot^{\rightarrow}$ that given a non-zero natural transforms it to a corresponding position in the sequence $\varpi'_1 \dots \varpi'_k\varpi'$. In order to define that map denote by l_i the position of the last element of ϖ_i in $\varpi_1 \dots \varpi_k$ for each $i = 1, \dots, k$, by $l_{1''}$ the position $l_k + 1$, by $l_{i'}$ the position of the last element of ϖ'_i in $\varpi'_1 \dots \varpi'_k\varpi'$ for each $i = 1, \dots, k$, and by $l_{1'}$ the position of the first element of ϖ' in the sequence $\varpi'_1 \dots \varpi'_k\varpi'$. So, $\cdot^{\rightarrow}$ is defined as follows: $l_i^{\rightarrow}$ is $l_{i'}$ for each $i = 1, \dots, k$, otherwise $l^{\rightarrow}$ is $l_{1'} + (l - l_{1''})$ whenever $l \geq l_{1''}$, otherwise $l^{\rightarrow}$ is not defined.

Consider the sequence ϖ'', with the same length, suppose m, as ϖ', defined as follows: for each $l = 1, \dots, m$, let the l'th element $\langle\eta_{l_{1''}+l-1}, \Psi_{l_{1''}+l-1}, \pi_{l_{1''}+l-1}\rangle$ of ϖ'' at position $l_{1''} + l - 1$ in $\varpi_1 \dots \varpi_k\varpi''$ be:

$\langle\dot\eta\sigma', \{\dot\eta\sigma'\}, 1\rangle$ whenever the element of ϖ' at position $l_{1'} + l - 1$ is $\langle\dot\eta, \{\dot\eta\}, 1\rangle$, i.e., *an assumption* (σ' is the schema substitution used in the derived rule);

$\langle\eta''\sigma''\sigma', \Psi_{i''_1}\setminus\Psi''_1\sigma''\sigma'\cup\dots\cup\Psi_{i''_{k''}}\setminus\Psi''_{k''}\sigma''\sigma', \pi_{i''_1}*\dots*\pi_{i''_{k''}}**_{\pi\in P''_f}(\pi_\Sigma\sigma''_{\mathrm{nl}}\sigma'_{\mathrm{nl}})\rangle$ whenever the element of ϖ' at position $l_{1'}+l-1$ *is obtained by rule* $\langle\{\langle\vartheta''_1, \Psi''_1, \eta''_1\rangle, \dots, \langle\vartheta''_{k''}, \Psi''_{k''}, \eta''_{k''}\rangle\}, \eta'', P''_f, P''_d\rangle$, schema substitution σ'', and premises located in ${i''_1}^{\rightarrow}, \dots, {i''_{k''}}^{\rightarrow}$ within $\{1, \dots, l_{1'} + l - 2\}$. Note that $\langle\eta_{i''_{j''}}, \Psi_{i''_{j''}}, \pi_{i''_{j''}}\rangle$ is an element already defined in ϖ'', for each $j'' = 1, \dots, k''$.

Denote by $\varpi^{\cdot}$ the sequence ϖ without the sober deduction sequences $\varpi_1 \dots \varpi_k$ and without the element t_r obtained by the derived rule that have being considered.

Note that if an element in $\varpi^{\cdot}$ is obtained in ϖ by a rule, then the premises appear before that element in the sequence $\varpi_1 \dots \varpi_k t_r \varpi^{\cdot}$.

Define the partial map $\cdot^{\triangleright}$ between natural numbers as follows: (i) the position in ϖ of an element in ϖ_i, for some $i = 1, \dots, k$, is mapped in the position of that element in the sequence $\varpi_1 \dots \varpi_k$, (ii) the position in ϖ of t_r is mapped to $l_{1''} + m - 1$, (iii) the position in ϖ of the l'th element of $\varpi^{\cdot}$ is mapped to $l_{1''} + m - 1 + l$, and (iv) the mapping of other natural number is undefined.

Consider the sequence $\varpi''^{\cdot}$ with the same length, suppose $m^{\cdot}$, as $\varpi^{\cdot}$ defined as follows:

for each $l = 1, \ldots, m^{\cdot}$, let the l'th element $\langle \eta_{l_{1''}+m-1+l}, \Psi_{l_{1''}+m-1+l}, \pi_{l_{1''}+m-1+l} \rangle$ of $\varpi^{\prime\prime\cdot}$ at position $l_{1''} + m - 1 + l$ in $\varpi_1 \ldots \varpi_k \varpi'' \varpi^{\prime\prime\cdot}$ be:

$\langle \dot{\eta}, \{\dot{\eta}\}, 1 \rangle$ whenever the l'th element of $\varpi^{\cdot}$ is the *assumption* $\langle \dot{\eta}, \{\dot{\eta}\}, 1 \rangle$;

$\langle \eta'' \sigma'', \Psi_{i_1^{\prime\prime\triangleright}} \setminus \Psi_1'' \sigma'' \cup \ldots \cup \Psi_{i_{k''}^{\prime\prime\triangleright}} \setminus \Psi_{k''}'' \sigma'', \pi_{i_1^{\prime\prime\triangleright}} * \ldots * \pi_{i_{k''}^{\prime\prime\triangleright}} * *_{\pi \in P_f''} (\pi_\Sigma \sigma_{\mathrm{nl}}'') \rangle$ whenever the l'th element of $\varpi^{\cdot}$ *is in* ϖ *obtained by rule* $\langle \{\langle \vartheta_1'', \Psi_1'', \eta_1'' \rangle, \ldots, \langle \vartheta_{k''}'', \Psi_{k''}'', \eta_{k''}'' \rangle\}, \eta'', P_f'', P_d'' \rangle$, schema substitution σ'', and previous premises at positions $i_1'', \ldots, i_{k''}''$ in ϖ. Note that, as referred above, the premises appear before that element in $\varpi_1 \ldots \varpi_k t_r \varpi^{\cdot}$. So $i_1^{\prime\prime\triangleright}, \ldots, i_{k''}^{\prime\prime\triangleright}$ are less than $l_{1''} + m - 1 + l$, and so they are previously defined elements in the sequence $\varpi_1 \ldots \varpi_k \varpi'' \varpi^{\prime\prime\cdot}$.

We now prove some useful lemmas.

Lemma 2.3.21 If $\langle \eta^{\cdot}, \Psi^{\cdot}, \pi^{\cdot} \rangle$ is an element either in ϖ'' at position l in $\{1, \ldots, m\}$ or in the last position of ϖ_i for some $i = 1, \ldots, k$, and the corresponding element in $\varpi_1' \ldots \varpi_k' \varpi'$ is the j'''th premise of the rule $\langle \{\langle \vartheta_1'', \Psi_1'', \eta_1'' \rangle, \ldots, \langle \vartheta_{k''}'', \Psi_{k''}'', \eta_{k''}'' \rangle\}, \eta'', P_f'', P_d'' \rangle$, using the schema substitution σ'', then $\eta^{\cdot}$ is $\eta_{j''}'' \sigma'' \sigma'$.

Proof The proof follows by induction on the number of rules applied in ϖ'' in order to obtain $\langle \eta^{\cdot}, \Psi^{\cdot}, \pi^{\cdot} \rangle$:

0 *rules* applied in ϖ'' in order to obtain $\langle \eta^{\cdot}, \Psi^{\cdot}, \pi^{\cdot} \rangle$. Then it is either (i) an assumption, $\langle \dot{\eta} \sigma', \{\dot{\eta} \sigma'\}, 1 \rangle$, or (ii) the last element of ϖ_j, i.e., $\langle \eta_j' \sigma', \Psi_{i_j'}, \pi_{i_j'} \rangle$, for some $j = 1, \ldots, k$. Suppose the corresponding element in ϖ' is used to obtain a posterior element by a rule $\langle \{\langle \vartheta_1'', \Psi_1'', \eta_1'' \rangle, \ldots, \langle \vartheta_{k''}'', \Psi_{k''}'', \eta_{k''}'' \rangle\}, \eta'', P_f'', P_d'' \rangle$, and a schema substitution σ'', and it corresponds to the j'''th premise of the rule. Then, if (i) holds then the corresponding element in ϖ' is $\langle \dot{\eta}, \{\dot{\eta}\}, 1 \rangle$, and $\dot{\eta}$ is $\eta_{j''}'' \sigma''$. So $\eta^{\cdot}$, i.e., $\dot{\eta} \sigma'$, is $\eta_{j''}'' \sigma'' \sigma'$, as desired; if (ii) holds then the corresponding element in $\varpi_1', \ldots, \varpi_k', \varpi'$ is $\langle \eta_j', \Psi_j', 1 \rangle$, and η_j' is $\eta_{j''}'' \sigma''$. So $\eta^{\cdot}$, i.e., $\eta_j' \sigma'$, is $\eta_{j''}'' \sigma'' \sigma'$, as desired.

$n + 1$ *rules* applied in ϖ'' to obtain $\langle \eta^{\cdot}, \Psi^{\cdot}, \pi^{\cdot} \rangle$. Suppose it is at the l'th position in ϖ'', with l in $\{1, \ldots, m - 1\}$. Let $\langle \{\langle \vartheta_1'', \Psi_1'', \eta_1'' \rangle, \ldots, \langle \vartheta_{k''}'', \Psi_{k''}'', \eta_{k''}'' \rangle\}, \eta'', P_f'', P_d'' \rangle$, σ'' and $i_1^{\prime\prime\rightarrow}, \ldots, i_{k''}^{\prime\prime\rightarrow}$ in $\{1, \ldots, l_{1'} + l - 2\}$, be the rule, the schema substitution, and the positions of the premises, respectively, to obtain at position $l_{1'} + l - 1$ in ϖ' the corresponding element. So, by definition, $\langle \eta^{\cdot}, \Psi^{\cdot}, \pi^{\cdot} \rangle$ is the tuple $\langle \eta'' \sigma'' \sigma', \Psi_{i_1''} \setminus \Psi_1'' \sigma'' \sigma' \cup \ldots \cup \Psi_{i_{k''}''} \setminus \Psi_{k''}'' \sigma'' \sigma', \pi_{i_1''} * \ldots * \pi_{i_{k''}''} * *_{\pi \in P_f''} (\pi_\Sigma \sigma_{\mathrm{nl}}'' \sigma_{\mathrm{nl}}') \rangle$. Suppose the corresponding element in ϖ' is also the j'''th premise by the rule $\langle \{\langle \vartheta_1^{\prime\prime\cdot}, \Psi_1^{\prime\prime\cdot}, \eta_1^{\prime\prime\cdot} \rangle, \ldots, \langle \vartheta_{k^{\prime\prime\cdot}}^{\prime\prime\cdot}, \Psi_{k^{\prime\prime\cdot}}^{\prime\prime\cdot}, \eta_{k^{\prime\prime\cdot}}^{\prime\prime\cdot} \}, \eta^{\prime\prime\cdot}, P_f^{\prime\prime\cdot}, P_d^{\prime\prime\cdot} \rangle$ and schema substitution $\sigma^{\prime\prime\cdot}$, considered to obtain a posterior element. Then, $\eta'' \sigma''$ is equal to $\eta_{j''}^{\prime\prime\cdot} \sigma^{\prime\prime\cdot}$. So $\eta^{\cdot}$ which is $\eta'' \sigma'' \sigma'$, is $\eta_{j''}^{\prime\prime\cdot} \sigma^{\prime\prime\cdot} \sigma'$, as we desired. *QED*

Lemma 2.3.22 The sequence $\varpi_1 \ldots \varpi_k \varpi''$ is a sober deduction sequence.

Proof We show first that $\varpi_1 \ldots \varpi_k \varpi''$ is a deduction sequence. Observe that ϖ_i is a sober deduction sequence for each $i = 1, \ldots, k$. So, we have only to show that each element in ϖ'' is defined in such a way that the sequence leading to it is a deduction sequence. The proof follows by induction on the number of rules applied in ϖ'' in order to obtain an element:

0 *rules* applied in ϖ'' in order to obtain $\langle \eta^{\cdot}, \Psi^{\cdot}, \pi^{\cdot} \rangle$. Suppose it is at the l'th position in

ϖ'', with l in $\{1, \ldots, m-1\}$. Then the element at position l in ϖ' is an assumption, and so by definition $\langle \eta^{\cdot}, \Psi^{\cdot}, \pi^{\cdot} \rangle$ is also a well defined assumption.

$n+1$ *rules* applied in order to obtain $\langle \eta^{\cdot}, \Psi^{\cdot}, \pi^{\cdot} \rangle$. Suppose it is at the l'th position in ϖ'', with l in $\{1, \ldots, m\}$. Let $\langle \{\langle \vartheta''_1, \Psi''_1, \eta''_1 \rangle, \ldots, \langle \vartheta''_{k''}, \Psi''_{k''}, \eta''_{k''} \rangle\}, \eta'', P''_f, P''_d \rangle$, σ'' and $i''_1{}^{\rightarrow}, \ldots, i''_{k''}{}^{\rightarrow}$ in $\{1, \ldots, l_{1'} + l - 2\}$, be the rule, the schema substitution, and the positions of the premises, respectively, in order to obtain the corresponding element at position $l_{1'} + l - 1$ in ϖ'. So, by definition, $\langle \eta^{\cdot}, \Psi^{\cdot}, \pi^{\cdot} \rangle$ is the tuple $\langle \eta''\sigma''\sigma', \Psi_{i''_1} \setminus \Psi''_1\sigma''\sigma' \cup \ldots \cup \Psi_{i''_{k''}} \setminus \Psi''_{k''}\sigma''\sigma', \pi_{i''_1} * \ldots * \pi_{i''_{k''}} * *_{\pi \in P''_f}(\pi_\Sigma \sigma''_{\mathrm{nl}} \sigma'_{\mathrm{nl}}) \rangle$. Consider the schema substitution $\sigma'^{\cdot}$ within L_E defined as follows: $\sigma''_f \sigma'_f = \sigma'^{\cdot}_f$, $\sigma''_t \sigma'_t = \sigma'^{\cdot}_t$, $\sigma''_l \sigma'_l = \sigma'^{\cdot}_l$, and $\vartheta''_j \sigma'^{\cdot}_s = \Psi_{i''_j}$ for each $j = 1, \ldots, k''$. So $\langle \eta^{\cdot}, \Psi^{\cdot}, \pi^{\cdot} \rangle$, the schema substitution $\sigma'^{\cdot}$, and the tuples at positions $i''_1, \ldots, i''_{k''}$ in ϖ'', provide a good instantiation for rule r, i.e., the following facts hold:

- $\eta_{i''_j}$ *is* $\eta''_j \sigma''\sigma'$, for each $j = 1, \ldots, k''$. This result holds by Lemma 2.3.21.

- $\Psi_{i''_j}$ *is* $\vartheta''_j \sigma'^{\cdot}$, for each $j = 1, \ldots, k''$. By definition of $\sigma'^{\cdot}$.

- $\pi_{\Sigma;E}(\sigma'^{\cdot}) = 1$ for each π in P''_d. Suppose fresh$(\Upsilon''_{j''}, \langle \vartheta''_{j''}, \Psi''_{j''}, \eta''_{j''} \rangle)$, named π'', is in P''_d, for some j'' in $1, \ldots, k''$. So, by definition of derived rule, Definition 2.3.17, there is fresh$(\Upsilon''_{j''}\sigma'^{\cdot}, \langle \vartheta'_{j'}, \Psi'_{j'}, \eta'_{j'} \rangle)$ in P'_d, named π', for some j' in $1, \ldots, k$. Together they satisfy the appropriate conditions of Definition 2.3.17. The desired result follows because of:

1.1. $\nu''\sigma'^{\cdot}$ is a label schema variable not in $(\mathrm{lsv}(\Psi''_{j''} \cup \{\eta''_{j''}\}) \setminus \{\nu''\})\sigma'^{\cdot}$, for any ν'' in $\Upsilon''_{j''}$. Note that:

i) $\nu''\sigma''\sigma'$ is a label schema variable not in $(\mathrm{lsv}(\Psi'_{j'} \cup \{\eta'_{j'}\}) \setminus \{\nu''\sigma''\})\sigma'$ for each ν'' in $\Upsilon''_{j''}$, since $\pi'_{\Sigma;E}(\sigma') = 1$.

ii) $\nu''\sigma''$ is a label schema variable not in $(\mathrm{lsv}(\Psi''_{j''} \cup \{\eta''_{j''}\}) \setminus \{\nu''\})\sigma''$ for each ν'' in $\Upsilon''_{j''}$, since $\pi''_{\Sigma;\emptyset}(\sigma'') = 1$.

iii) $\mathrm{lsv}(\Psi''_{j''} \cup \{\eta''_{j''}\})\sigma'' \subseteq \mathrm{lsv}(\Psi'_{j'} \cup \{\eta'_{j'}\})$, by Definition 2.3.17.

iv) $(\mathrm{lsv}(\Psi''_{j''} \cup \{\eta''_{j''}\}) \setminus \{\nu''\})\sigma'' \subseteq (\mathrm{lsv}(\Psi''_{j''} \cup \{\eta''_{j''}\})\sigma'' \setminus \{\nu''\}\sigma'')$. Let ψ be a labelled schema formula in $\Psi''_{j''} \cup \{\eta''_{j''}\}$ containing a label schema variable ν distinct of ν''. Since $\nu''\sigma''$ is not in $(\mathrm{lsv}(\Psi''_{j''} \cup \{\eta''_{j''}\}) \setminus \{\nu''\})\sigma''$ due to ii), then it is distinct of $\nu\sigma''$. Therefore $\nu\sigma''$ is in $\mathrm{lsv}(\Psi''_{j''} \cup \{\eta''_{j''}\})\sigma'' \setminus \{\nu''\}\sigma''$.

Now the proof of 1.1. Let ν'' be in $\Upsilon''_{j''}$. Then $\nu''\sigma'^{\cdot}$, which is $\nu''\sigma''\sigma'$, is a label schema variable not in $(\mathrm{lsv}(\Psi'_{j'} \cup \{\eta'_{j'}\}) \setminus \{\nu''\sigma''\})\sigma'$, by i). Note that $\mathrm{lsv}(\Psi''_{j''} \cup \{\eta''_{j''}\})\sigma'' \subseteq \mathrm{lsv}(\Psi'_{j'} \cup \{\eta'_{j'}\})$, by iii). So $\nu''\sigma''\sigma'$ is not in $(\mathrm{lsv}(\Psi''_{j''} \cup \{\eta''_{j''}\})\sigma'' \setminus \{\nu''\}\sigma'')\sigma'$, and since by (iv), $(\mathrm{lsv}(\Psi''_{j''} \cup \{\eta''_{j''}\}) \setminus \{\nu''\})\sigma''\sigma' \subseteq (\mathrm{lsv}(\Psi''_{j''} \cup \{\eta''_{j''}\})\sigma'' \setminus \{\nu''\}\sigma'')\sigma'$, then $\nu''\sigma''\sigma'$ is not in $(\mathrm{lsv}(\Psi''_{j''} \cup \{\eta''_{j''}\}) \setminus \{\nu''\})\sigma''\sigma'$, as we wanted to show.

1.2. $\Upsilon''_{j''}\sigma'^{\cdot} \cap \mathrm{lsv}(\vartheta''_{j''}\sigma'^{\cdot} \setminus \Psi''_{j''}\sigma'^{\cdot}) = \emptyset$, i.e., $\Upsilon''_{j''}\sigma''\sigma' \cap \mathrm{lsv}(\Psi_{i''_{j''}} \setminus \Psi''_{j''}\sigma''\sigma') = \emptyset$. Note that:

i) $\Upsilon''_{j''}\sigma''\sigma' \cap \mathrm{lsv}(\vartheta'_{j'}\sigma' \setminus \Psi'_{j'}\sigma') = \emptyset$, since $\pi'_{\Sigma;E}(\sigma') = 1$.

ii) $\Psi'_{j'}\sigma' \subseteq \Psi''_{j''}\sigma''\sigma' \cup \mathrm{remL}(\varpi'_1 \ldots \varpi'_k \varpi', i'_{j'}{}^{\rightarrow}, i''_{j''}{}^{\rightarrow})\sigma'$. To see this note that $\Psi'_{j'} \subseteq \Psi''_{j''}\sigma'' \cup \mathrm{remL}(\varpi'_1 \ldots \varpi'_k \varpi', i'_{j'}{}^{\rightarrow}, i''_{j''}{}^{\rightarrow})$, by Definition 2.3.17. Hence $\Psi'_{j'}\sigma' \subseteq \Psi''_{j''}\sigma''\sigma' \cup \mathrm{remL}(\varpi'_1 \ldots \varpi'_k \varpi', i'_{j'}{}^{\rightarrow}, i''_{j''}{}^{\rightarrow})\sigma'$.

iii) $\mathrm{lsv}(\vartheta''_{j''}\sigma''\sigma' \setminus \Psi''_{j''}\sigma''\sigma')$ is contained in $\mathrm{lsv}(\vartheta''_{j''}\sigma'' \setminus \Psi''_{j''}\sigma'')\sigma'$. Let ψ' be in $\vartheta''_{j''}\sigma''\sigma'$ but not in $\Psi''_{j''}\sigma''\sigma'$. Then ψ' is $\psi\sigma'$ where ψ is in $\vartheta''_{j''}\sigma''$. Note that ψ is not in $\Psi''_{j''}\sigma''$. So ψ is in $\vartheta''_{j''}\sigma'' \setminus \Psi''_{j''}\sigma''$. Thus if ν' is in $\mathrm{lsv}(\vartheta''_{j''}\sigma''\sigma' \setminus \Psi''_{j''}\sigma''\sigma')$ then it is in $\mathrm{lsv}((\vartheta''_{j''}\sigma'' \setminus \Psi''_{j''}\sigma'')\sigma')$, and so ν' is in $\mathrm{lsv}(\vartheta''_{j''}\sigma'' \setminus \Psi''_{j''}\sigma'')\sigma'$.

iv) $\mathrm{lsv}((\vartheta'_{j'}\sigma' \setminus \mathrm{remL}(\varpi'_1 \ldots \varpi'_k\varpi', i'_{j'}{}^{\rightarrow}, i''_{j''}{}^{\rightarrow})\sigma') \setminus \Psi''_{j''}\sigma''\sigma')$ is contained in $\mathrm{lsv}(\vartheta'_{j'} \setminus \Psi'_{j'}\sigma')$. To see this observe that $(\vartheta'_{j'}\sigma' \setminus \mathrm{remL}(\varpi'_1 \ldots \varpi'_k\varpi', i'_{j'}{}^{\rightarrow}, i''_{j''}{}^{\rightarrow})\sigma') \setminus \Psi''_{j''}\sigma''\sigma'$ is equal to the set $\vartheta'_{j'}\sigma' \setminus (\mathrm{remL}(\varpi'_1 \ldots \varpi'_k\varpi', i'_{j'}{}^{\rightarrow}, i''_{j''}{}^{\rightarrow})\sigma' \cup \Psi''_{j''}\sigma''\sigma')$, which, taking into account ii) is contained in $\vartheta'_{j'}\sigma' \setminus \Psi'_{j'}\sigma'$.

v) $(\mathrm{lsv}(\Psi'_{j'} \cup \{\eta'_{j'}\}) \setminus \Upsilon''_{j''}\sigma'')\sigma'$ is contained in $\mathrm{lsv}(\Psi'_{j'} \cup \{\eta'_{j'}\})\sigma' \setminus \Upsilon''_{j''}\sigma''\sigma'$. First we show that if ν' is in $\Upsilon''_{j''}\sigma''\sigma'$ then ν' is not in $(\mathrm{lsv}(\Psi'_{j'} \cup \{\eta'_{j'}\}) \setminus \Upsilon''_{j''}\sigma'')\sigma'$. Let ν' be in $\Upsilon''_{j''}\sigma''\sigma'$. Then, there is ν in $\Upsilon''_{j''}\sigma''$ with $\nu\sigma' = \nu'$ and such that $\nu\sigma'$ is not in $(\mathrm{lsv}(\Psi'_{j'} \cup \{\eta'_{j'}\}) \setminus \{\nu\})\sigma'$. Since $\mathrm{lsv}(\Psi'_{j'} \cup \{\eta'_{j'}\}) \setminus \Upsilon''_{j''}\sigma''$ is contained in $\mathrm{lsv}(\Psi'_{j'} \cup \{\eta'_{j'}\}) \setminus \{\nu\}$ we have that $\nu\sigma'$, i.e., ν' is not in $(\mathrm{lsv}(\Psi'_{j'} \cup \{\eta'_{j'}\}) \setminus \Upsilon''_{j''}\sigma'')\sigma'$. Now the proof of v). Let ν' be in $(\mathrm{lsv}(\Psi'_{j'} \cup \{\eta'_{j'}\}) \setminus \Upsilon''_{j''}\sigma'')\sigma'$. Then obviously it is in $\mathrm{lsv}(\Psi'_{j'} \cup \{\eta'_{j'})\sigma'$, and by the previous fact it is not in $\Upsilon''_{j''}\sigma''\sigma'$, as we wanted to show.

vi) for any element $\langle \eta_l, \Psi_l, \pi_l \rangle$ at a position l, either in ϖ'' or the last position of ϖ_i for some $i = 1, \ldots, k$, we have that Ψ_l is contained in $\Psi_{l\rightarrow}\sigma' \cup \cup_{i'_{j^{\cdot}}{}^{\rightarrow} \in \mathrm{usE}(\varpi'_1 \ldots \varpi'_k\varpi', l^{\rightarrow})}(\vartheta'_{j^{\cdot}}\sigma' \setminus \mathrm{remL}(\varpi'_1 \ldots \varpi'_k\varpi', i'_{j^{\cdot}}{}^{\rightarrow}, l^{\rightarrow})\sigma')$, where $\langle \eta_{l\rightarrow}, \Psi_{l\rightarrow}, \pi_{l\rightarrow} \rangle$ is the element at position $l^{\rightarrow}$ in $\varpi'_1 \ldots \varpi'_k\varpi'$. The proof follows by induction on the length of the longest path in ϖ' that arrives to the element at position $l^{\rightarrow}$:

0 *rules.* Then we face two cases (i) l is the last position of the sequence $\varpi_{\check{\jmath}}$ for some $\check{\jmath} = 1, \ldots, k$, or (ii) the element at l is an assumption, i.e., $\langle \eta_{l\rightarrow}\sigma', \{\eta_{l\rightarrow}\sigma'\}, 1 \rangle$, where $\langle \eta_{l\rightarrow}, \{\eta_{l\rightarrow}\}, 1 \rangle$ is the element at position $l^{\rightarrow}$ in the sequence $\varpi'_1 \ldots \varpi'_k\ \varpi'$. If the case (i) holds then Ψ_l is $\vartheta'_{\check{\jmath}}\sigma'$ and $\check{\jmath}$ is the only value in $1, \ldots, k$ with $i'_{\check{\jmath}}{}^{\rightarrow}$ in $\mathrm{usE}(\varpi'_1 \ldots \varpi'_k\varpi', l^{\rightarrow})$. Note that $i'_{\check{\jmath}}$ is l. So $\cup_{i'_{j^{\cdot}}{}^{\rightarrow} \in \mathrm{usE}(\varpi'_1 \ldots \varpi'_k\varpi', l^{\rightarrow})}(\ \vartheta'_{j^{\cdot}}\sigma' \setminus \mathrm{remL}(\varpi'_1 \ldots \varpi'_k\varpi', i'_{j^{\cdot}}{}^{\rightarrow}, l^{\rightarrow})\sigma') = \vartheta'_{\check{\jmath}}\sigma'$ since we have $\mathrm{remL}(\varpi'_1 \ldots \varpi'_k\varpi',\ i'_{j^{\cdot}}{}^{\rightarrow}, l^{\rightarrow}) = \emptyset$. Hence the result follows straightforwardly. If the case (ii) holds then $\cup_{i'_{j^{\cdot}}{}^{\rightarrow} \in \mathrm{usE}(\varpi'_1 \ldots \varpi'_k\varpi', l^{\rightarrow})}(\vartheta'_{j^{\cdot}}\sigma' \setminus \mathrm{remL}(\varpi'_1 \ldots \varpi'_k\varpi'', i'_{j^{\cdot}}{}^{\rightarrow}, l^{\rightarrow})) = \emptyset$ since $i'_{j^{\cdot}}{}^{\rightarrow}$ is not in $\mathrm{usE}(\varpi'_1 \ldots \varpi'_k\varpi', l^{\rightarrow})$ for each $j^{\cdot} = 1, \ldots, k$. Note that $\Psi'_{l\rightarrow}$ is $\{\eta\rightarrow\}$, by definition of ϖ''. So the result follows straightforwardly.

$n+1$ *rules.* Then, the element at position $l^{\rightarrow}$ is obtained by rule $\langle \{\langle \vartheta^{\prime\prime\cdot}_1, \Psi^{\prime\prime\cdot}_1, \eta^{\prime\prime\cdot}_1 \rangle, \ldots, \langle \vartheta^{\prime\prime\cdot}_{k^{\prime\prime\cdot}}, \Psi^{\prime\prime\cdot}_{k^{\prime\prime\cdot}}, \eta^{\prime\prime\cdot}_{k^{\prime\prime\cdot}} \rangle\}, \eta^{\prime\prime\cdot}, P^{\prime\prime\cdot}_f, P^{\prime\prime\cdot}_d \rangle$, schema substitution $\check{\sigma}$, and premises at positions $i^{\prime\prime\cdot}_1{}^{\rightarrow}, \ldots, i^{\prime\prime\cdot}_{k^{\prime\prime\cdot}}{}^{\rightarrow}$. So, Ψ_l, is contained in $\Psi_{i^{\prime\prime\cdot}_1} \setminus \Psi^{\prime\prime\cdot}_1\check{\sigma}\sigma' \cup \ldots \cup \Psi_{i^{\prime\prime\cdot}_{k^{\prime\prime\cdot}}} \setminus \Psi^{\prime\prime\cdot}_{k^{\prime\prime\cdot}}\check{\sigma}\sigma'$ and by IH in $\vartheta^{\prime\prime\cdot}_1\check{\sigma}\sigma' \setminus \Psi^{\prime\prime\cdot}_1\check{\sigma}\sigma' \cup \ldots \cup \vartheta^{\prime\prime\cdot}_{k^{\prime\prime\cdot}}\check{\sigma}\sigma' \setminus \Psi^{\prime\prime\cdot}_{k^{\prime\prime\cdot}}\check{\sigma}\sigma' \cup \cup_{i'_{j^{\cdot}}{}^{\rightarrow} \in \mathrm{usE}(\varpi'_1 \ldots \varpi'_k\varpi', i^{\prime\prime\cdot}_1{}^{\rightarrow})}(\vartheta'_{j^{\cdot}}\sigma' \setminus (\mathrm{remL}(\varpi'_1 \ldots \varpi'_k\varpi', i'_{j^{\cdot}}{}^{\rightarrow}, i^{\prime\prime\cdot}_1{}^{\rightarrow})\sigma' \cup \Psi^{\prime\prime\cdot}_1\check{\sigma}\sigma')) \cup \ldots \cup \cup_{i'_{j^{\cdot}}{}^{\rightarrow} \in \mathrm{usE}(\varpi'_1 \ldots \varpi'_k\varpi', i^{\prime\prime\cdot}_{k^{\prime\prime\cdot}}{}^{\rightarrow})}(\vartheta'_{j^{\cdot}}\sigma' \setminus (\mathrm{remL}(\varpi'_1 \ldots \varpi'_k\varpi', i'_{j^{\cdot}}{}^{\rightarrow},\ i^{\prime\prime\cdot}_k{}^{\rightarrow})\sigma' \cup \Psi^{\prime\prime\cdot}_{k^{\prime\prime\cdot}}\check{\sigma}\sigma'))$ contained in $(\vartheta^{\prime\prime\cdot}_1\check{\sigma} \setminus \Psi^{\prime\prime\cdot}_1\check{\sigma})\sigma' \cup \ldots \cup (\vartheta^{\prime\prime\cdot}_{k^{\prime\prime\cdot}}\check{\sigma} \setminus \Psi^{\prime\prime\cdot}_{k^{\prime\prime\cdot}}\check{\sigma})\sigma' \cup \cup_{i'_{j^{\cdot}}{}^{\rightarrow} \in \mathrm{usE}(\varpi'_1 \ldots \varpi'_k\varpi', l^{\rightarrow})}(\vartheta'_{j^{\cdot}}\sigma' \setminus \mathrm{remL}(\varpi'_1 \ldots \varpi'_k\varpi', i'_{j^{\cdot}}{}^{\rightarrow}, l^{\rightarrow})\sigma')$, i.e., contained in $\Psi_{l\rightarrow}\sigma' \cup \cup_{i'_{j^{\cdot}}{}^{\rightarrow} \in \mathrm{usE}(\varpi'_1 \ldots \varpi'_k\varpi', l^{\rightarrow})}(\vartheta'_{j^{\cdot}}\sigma' \setminus \mathrm{remL}(\varpi'_1 \ldots \varpi'_k\varpi', i'_{j^{\cdot}}{}^{\rightarrow}, l^{\rightarrow})\sigma')$. So, we need only to check, for each $j^{\prime\prime\cdot} = 1, \ldots, k^{\prime\prime\cdot}$ that $\vartheta^{\prime\prime\cdot}_{j^{\prime\prime\cdot}}\check{\sigma}\sigma' \setminus \Psi^{\prime\prime\cdot}_{j^{\prime\prime\cdot}}\check{\sigma}\sigma'$ is contained in $(\vartheta^{\prime\prime\cdot}_{j^{\prime\prime\cdot}}\check{\sigma} \setminus \Psi^{\prime\prime\cdot}_{j^{\prime\prime\cdot}}\check{\sigma})\sigma'$. Let ψ' be in $\vartheta^{\prime\prime\cdot}_{j^{\prime\prime\cdot}}\check{\sigma}\sigma' \setminus \Psi^{\prime\prime\cdot}_{j^{\prime\prime\cdot}}\check{\sigma}\sigma'$. Then ψ' is in $\vartheta^{\prime\prime\cdot}_{j^{\prime\prime\cdot}}\check{\sigma}\sigma'$ but not in $\Psi^{\prime\prime\cdot}_{j^{\prime\prime\cdot}}\check{\sigma}\sigma'$. So ψ' is $\psi\sigma'$ and ψ is in $\vartheta^{\prime\prime\cdot}_{j^{\prime\prime\cdot}}\check{\sigma}$. Note that ψ is not in $\Psi^{\prime\prime\cdot}_{j^{\prime\prime\cdot}}\check{\sigma}$. So ψ is in $\vartheta^{\prime\prime\cdot}_{j^{\prime\prime\cdot}}\check{\sigma} \setminus \Psi^{\prime\prime\cdot}_{j^{\prime\prime\cdot}}\check{\sigma}$. Hence ψ' which is $\psi\sigma'$ is in $(\vartheta^{\prime\prime\cdot}_{j^{\prime\prime\cdot}}\check{\sigma} \setminus \Psi^{\prime\prime\cdot}_{j^{\prime\prime\cdot}}\check{\sigma})\sigma'$.

vii) $\Psi_{i''_{j''}}$ is contained in $\vartheta''_{j''}\sigma''\sigma' \cup (\vartheta'_{j'}\sigma' \setminus \mathrm{remL}(\varpi'_1 \ldots \varpi'_k\varpi', i'_{j'}{}^{\rightarrow}, i''_{j''}{}^{\rightarrow})\sigma')$. It is straightforward to see that the intended result is just a special case of vi) since, by Definition 2.3.17, only $i'_{j'}{}^{\rightarrow}$ is in $\mathrm{usE}(\varpi'_1 \ldots \varpi'_k\varpi', i''_{j''}{}^{\rightarrow})$.

viii) $\Upsilon''_{j''}\sigma''\sigma' \cap \mathrm{lsv}(\vartheta''_{j''}\sigma'' \setminus \Psi''_{j''}\sigma'')\sigma' = \emptyset$. To see this note that, by Definition 2.3.17, $\mathrm{lsv}(\vartheta''_{j''}\sigma'' \setminus \Psi''_{j''}\sigma'')\sigma'$ is contained in $(\mathrm{lsv}(\Psi'_{j'} \cup \{\eta'_{j'}\}) \setminus \Upsilon''_{j''}\sigma'')\sigma'$. So, the result follows by v).

Now the proof of 1.2.. Note first, by vii), that, $\Psi_{i''_{j''}} \setminus \Psi''_{j''}\sigma''\sigma'$ is contained in the union of $\vartheta''_{j''}\sigma''\sigma' \setminus \Psi''_{j''}\sigma''\sigma'$ with $(\vartheta'_{j'}\sigma' \setminus \mathrm{remL}(\varpi'_1 \ldots \varpi'_k\varpi', i'_{j'}{}^{\rightarrow}, i''_{j''}{}^{\rightarrow})\sigma') \setminus \Psi''_{j''}\sigma''\sigma'$. So, by iii) and iv), we have that $\Upsilon''_{j''}\sigma''\sigma' \cap \mathrm{lsv}(\Psi_{i''_{j''}} \setminus \Psi''_{j''}\sigma''\sigma')$ is contained in the union of $\Upsilon''_{j''}\sigma''\sigma' \cap \mathrm{lsv}(\vartheta''_{j''}\sigma'' \setminus \Psi''_{j''}\sigma'')\sigma'$ with $\Upsilon''_{j''}\sigma''\sigma' \cap \mathrm{lsv}(\vartheta'_{j'} \setminus \Psi'_{j'}\sigma')$, which, by viii) and i) is $\emptyset$, as we wanted to show.

We now show that $\varpi_1 \ldots \varpi_k\varpi''$ is a sober deduction sequence. This happens since $\varpi_1 \ldots \varpi_k\varpi''$ is a deduction sequence by Lemma 2.3.22, ϖ_i is a sober deduction sequence, for each $i = 1, \ldots, k$, and since:

- any element of ϖ'', and the last element of ϖ_j, for each $j = 1, \ldots, k$, are referenced later in the sequence ϖ''. This happens since, if this would not be the case then, the last element of the sequence ϖ'_j, for some $j = 1, \ldots, k$, or some element in ϖ', would not be referenced later in the sequence ϖ', which contradicts the fact that $\varpi'_1 \ldots \varpi'_k\varpi'$ is a sober deduction sequence.

- there is no proper subsequence $\varpi''_{.}$ of ϖ'' such that $\varpi_1 \ldots \varpi_k\varpi''_{.}$ is a sober deduction sequence because if this was the case then there would be a proper subsequence $\varpi'_{.}$ of ϖ' such that $\varpi'_1 \ldots \varpi'_k\varpi'_{.}$ is a sober deduction sequence. *QED*

Lemma 2.3.23 The last element of ϖ'', at position $l_{1''} + m - 1$, is such that $\eta_{l_{1''}+m-1}$ is $\eta'\sigma'$, $\Psi_{l_{1''}+m-1} \subseteq \Psi_{i'_1} \setminus \Psi'_1\sigma' \cup \ldots \cup \Psi_{i'_k} \setminus \Psi'_k\sigma'$, and $\pi_{l_{1''}+m-1}$ is $\pi_{i'_1} * \ldots * \pi_{i'_k} * (\pi'_\Sigma\sigma'_{\mathrm{nl}})$.

Proof To see this happen, note that:

a) for any element $\langle \eta_l, \Psi_l, \pi_l \rangle$ at a position l, either in ϖ'' or the last position of the sequence ϖ_i for some $i = 1, \ldots, k$, we have that π_l is $(\pi_{l^{\rightarrow}}\sigma'_{\mathrm{nl}}) * *_{i'_{j'}{}^{\rightarrow} \in \mathrm{usE}(\varpi'_1 \ldots \varpi'_k\varpi', l^{\rightarrow})}\pi_{i'_{j'}}$, where $\langle \eta_{l^{\rightarrow}}, \Psi_{l^{\rightarrow}}, \pi_{l^{\rightarrow}} \rangle$ is the element at position $l^{\rightarrow}$ in $\varpi'_1 \ldots \varpi'_k\varpi'$. The proof follows by induction on the length of the longest path in ϖ' that arrives to the element at position $l^{\rightarrow}$:

0 *rules.* Then we face two cases (i) l is the last position of the sequence $\varpi_{\check{\jmath}}$ for some $\check{\jmath} = 1, \ldots, k$, or (ii) the element at l is an assumption, i.e., $\langle \eta_{l^{\rightarrow}}\sigma', \{\eta_{l^{\rightarrow}}\sigma'\}, 1\rangle$, where $\langle \eta_{l^{\rightarrow}}, \{\eta_{l^{\rightarrow}}\}, 1\rangle$ is the element at position $l^{\rightarrow}$ in the sequence $\varpi'_1 \ldots \varpi'_k\ \varpi'$. If the case (i) holds then π_l is $\pi_{i'_{\check{\jmath}}}$, $\pi_{l^{\rightarrow}}$ is the proviso 1 and so $(\pi_{l^{\rightarrow}}\sigma'_{\mathrm{nl}})$ is also the proviso 1, and $*_{i'_{j'}{}^{\rightarrow} \in \mathrm{usE}(\varpi'_1 \ldots \varpi'_k\varpi', l^{\rightarrow})}\pi_{i'_{j'}}$ is $\pi_{i'_{\check{\jmath}}}$. So the result follows straightforwardly. If the case (ii) holds then π_l is the proviso 1, $(\pi_{l^{\rightarrow}}\sigma'_{\mathrm{nl}})$ is the proviso 1, and $*_{i'_{j'}{}^{\rightarrow} \in \mathrm{usE}(\varpi'_1 \ldots \varpi'_k\varpi', l^{\rightarrow})}\pi_{i'_{j'}}$ is also the proviso 1 since $i'_{j'}$ is not in $\mathrm{usE}(\varpi'_1 \ldots \varpi'_k\varpi', l^{\rightarrow})$ for each $j' = 1, \ldots, k$. So the result follows straightforwardly.

$n + 1$ *rules.* Then, the element at position $l^{\rightarrow}$ is obtained by rule $\langle\{\langle \vartheta^{\prime\prime\cdot}_1, \Psi^{\prime\prime\cdot}_1, \eta^{\prime\prime\cdot}_1\rangle, \ldots, \langle \vartheta^{\prime\prime\cdot}_{k^{\prime\prime\cdot}}, \Psi^{\prime\prime\cdot}_{k^{\prime\prime\cdot}}, \eta^{\prime\prime\cdot}_{k^{\prime\prime\cdot}}\rangle\}, \eta^{\prime\prime\cdot}, P^{\prime\prime\cdot}_f, P^{\prime\prime\cdot}_d\rangle$, schema substitution $\check{\sigma}$, and premises at positions $i^{\prime\prime\cdot}_1{}^{\rightarrow}, \ldots, i^{\prime\prime\cdot}_{k^{\prime\prime\cdot}}{}^{\rightarrow}$. So, π_l which, by definition of ϖ'', is $\pi_{i''_1} * \ldots * \pi_{i''_{k^{\prime\prime\cdot}}} * *_{\pi \in P^{\prime\prime\cdot}_f}(\pi_\Sigma\check{\sigma}_{\mathrm{nl}}\sigma'_{\mathrm{nl}})$ is, by IH, $(\pi_{i''_1{}^{\rightarrow}}\sigma'_{\mathrm{nl}}) * *_{i'_{j'}{}^{\rightarrow} \in \mathrm{usE}(\varpi'_1 \ldots \varpi'_k\varpi', i''_1{}^{\rightarrow})}\ \pi_{i'_{j'}} * \ldots * (\pi_{i''_{k^{\prime\prime\cdot}}{}^{\rightarrow}}\sigma'_{\mathrm{nl}}) * *_{i'_{j'}{}^{\rightarrow} \in \mathrm{usE}(\varpi'_1 \ldots \varpi'_k\varpi', i''_{k^{\prime\prime\cdot}}{}^{\rightarrow})}\pi_{i'_{j'}} * *_{\pi \in P^{\prime\prime\cdot}_f}(\pi_\Sigma\check{\sigma}_{\mathrm{nl}}\sigma'_{\mathrm{nl}})$ which by Remark 2.3.6 is $((\pi_{i''_1{}^{\rightarrow}} * \ldots * \pi_{i''_{k^{\prime\prime\cdot}}{}^{\rightarrow}} * *_{\pi \in P^{\prime\prime\cdot}_f}(\pi_\Sigma\check{\sigma}_{\mathrm{nl}}))\sigma'_{\mathrm{nl}}) * *_{i'_{j'}{}^{\rightarrow} \in \mathrm{usE}(\varpi'_1 \ldots \varpi'_k\varpi', l^{\rightarrow})}\pi_{i'_{j'}}$ which, by definition of deduction, is, as we wanted to show, $(\pi_{l^{\rightarrow}}\sigma'_{\mathrm{nl}}) * *_{i'_{j'}{}^{\rightarrow} \in \mathrm{usE}(\varpi'_1 \ldots \varpi'_k\varpi', l^{\rightarrow})}\pi_{i'_{j'}}$.

b) for any element $\langle \eta_l, \Psi_l, \pi_l \rangle$ at a position l, either in ϖ'' or the last position of a sequence

ϖ_i for i in $1, \ldots, k$, we have that Ψ_l is contained in $\Psi_{l^{\rightarrow}}\sigma' \cup \bigcup_{i'_{j^{\cdot}}{}^{\rightarrow} \in \mathrm{usE}(\varpi'_1 \ldots \varpi'_k\varpi', l^{\rightarrow})}(\Psi_{i'_{j^{\cdot}}} \setminus \mathrm{remL}(\varpi'_1 \ldots \varpi'_k\varpi', i'_{j^{\cdot}}{}^{\rightarrow}, l^{\rightarrow})\sigma')$, where $\langle \eta_{l^{\rightarrow}}, \Psi_{l^{\rightarrow}}, \pi_{l^{\rightarrow}} \rangle$ is the element at position $l^{\rightarrow}$ in $\varpi'_1 \ldots \varpi'_k\varpi'$. This fact is proved in item vi) of the above proof of Lemma 2.3.22.

c) $\eta_{l_{1''}+m}$ is $\eta'\sigma'$. Note that by Definition 2.3.17 η' is obtained in $\varpi'_1 \ldots \varpi'_k\varpi'$ by application of a rule. Hence, by definition of ϖ'', $\eta_{l_{1''}+m}$ is $\eta'\sigma'$.

d) $\Psi_{l_{1''}+m}$ is contained in $\Psi_{i'_1} \setminus \Psi'_1\sigma' \cup \ldots \cup \Psi_{i'_k} \setminus \Psi'_k\sigma'$. Recall that $\Psi_{l_{1''}+m^{\rightarrow}}$ is $\emptyset$ and for each $j' = 1, \ldots, k$ that $i'_{j'}{}^{\rightarrow}$ is in $\mathrm{usE}(\varpi'_1 \ldots \varpi'_k\varpi', l_{1''} + m^{\rightarrow})$. So by item vi) of the above proof of 1.2., $\Psi_{l_{1''}+m}$ is contained in $\Psi_{i'_1} \setminus \mathrm{remL}(\varpi'_1 \ldots \varpi'_k\varpi', i'_1{}^{\rightarrow}, l_{1''} + m^{\rightarrow})\sigma' \cup \ldots \cup \Psi_{i'_k} \setminus \mathrm{remL}(\varpi'_1 \ldots \varpi'_k\varpi', i'_k{}^{\rightarrow}, l_{1''} + m^{\rightarrow})\sigma'$. Note that by Definition 2.3.17 $\Psi'_{j'}\sigma'$ is contained in $\mathrm{remL}(\varpi'_1 \ldots \varpi'_k\varpi', i'_{j'}{}^{\rightarrow}, l_{1''} + m^{\rightarrow})\sigma'$ and so $\Psi_{i'_{j'}} \setminus \mathrm{remL}(\varpi'_1 \ldots \varpi'_k\varpi', i'_{j'}{}^{\rightarrow}, l_{1''} + m^{\rightarrow})\sigma'$ is contained in $\Psi_{i'_{j'}} \setminus \Psi'_{j'}\sigma'$ for each $j' = 1, \ldots, k$. So $\Psi_{l_{1''}+m}$ is contained in $\Psi_{i'_1} \setminus \Psi'_1\sigma' \cup \ldots \cup \Psi_{i'_k} \setminus \Psi'_k\sigma'$, as we wanted to show.

e) $\pi_{l_{1''}+m}$ is $\pi_{i'_1} * \ldots \pi_{i'_k} * (\pi'_{\Sigma}\sigma'_{\mathrm{nl}})$. Recall that $\pi_{l_{1''}+m^{\rightarrow}}$ is π'_{Σ} and, for each $j' = 1, \ldots, k$ that $i'_{j'}{}^{\rightarrow}$ is in $\mathrm{usE}(\varpi'_1 \ldots \varpi'_k\varpi', l_{1''} + m^{\rightarrow})$. So by fact a) above, $\pi_{l_{1''}+m}$ is $\pi_{i'_1} * \ldots * \pi_{i'_k} * (\pi'_{\Sigma}\sigma'_{\mathrm{nl}})$, as we wanted to show. *QED*

Lemma 2.3.24 If $\langle \eta^{\cdot}, \Psi^{\cdot}, \pi^{\cdot} \rangle$ is an element at position l in ϖ and $\langle \eta^{\prime\cdot}, \Psi^{\prime\cdot}, \pi^{\prime\cdot} \rangle$ is the corresponding element at position $l^{\triangleright}$ in $\varpi_1 \ldots \varpi_k\varpi''\varpi''^{\cdot}$ then $\eta^{\prime\cdot}$ is equal to $\eta^{\cdot}$, $\Psi^{\prime\cdot}$ is contained in $\Psi^{\cdot}$, and $\pi^{\prime\cdot}$ is equal to $\pi^{\cdot}$.

Proof To prove this we consider three cases for $\langle \eta^{\cdot}, \Psi^{\cdot}, \pi^{\cdot} \rangle$ in ϖ:

- $\langle \eta^{\cdot}, \Psi^{\cdot}, \pi^{\cdot} \rangle$ *is in* ϖ_i, for some $i = 1, \ldots, k$. Then $\langle \eta^{\prime\cdot}, \Psi^{\prime\cdot}, \pi^{\prime\cdot} \rangle$ is equal to $\langle \eta^{\cdot}, \Psi^{\cdot}, \pi^{\cdot} \rangle$, by definition of $\cdot^{\triangleright}$. So, the intended result is immediate.

- $\langle \eta^{\cdot}, \Psi^{\cdot}, \pi^{\cdot} \rangle$ *is* t_r, i.e. is $\langle \eta'\sigma', \Psi_{i'_1} \setminus \Psi'_1\sigma' \cup \ldots \cup \Psi_{i'_k} \setminus \Psi'_k\sigma', \pi_{i'_1} * \ldots * \pi_{i'_k} * (\pi'_{\Sigma}\sigma'_{\mathrm{nl}}) \rangle$. Then $\langle \eta^{\prime\cdot}, \Psi^{\prime\cdot}, \pi^{\prime\cdot} \rangle$ is the last element of the sequence ϖ'', and so according to 3. $\eta^{\prime\cdot}$ is $\eta'\sigma'$ and so $\eta^{\cdot}$, $\Psi^{\prime\cdot}$ is contained in $\Psi_{i'_1} \setminus \Psi'_1\sigma' \cup \ldots \cup \Psi_{i'_k} \setminus \Psi'_k\sigma'$ and so in $\Psi^{\cdot}$, and $\pi^{\prime\cdot}$ is $\pi_{i'_1} * \ldots * \pi_{i'_k} * (\pi'_{\Sigma}\sigma'_{\mathrm{nl}})$ and so is $\pi^{\cdot}$.

- $\langle \eta^{\cdot}, \Psi^{\cdot}, \pi^{\cdot} \rangle$ *is* an element in $\varpi^{\cdot}$. The proof follows by induction on the number of rules applied in $\varpi^{\cdot}$ in order to obtain $\langle \eta^{\cdot}, \Psi^{\cdot}, \pi^{\cdot} \rangle$:

0 *rules* applied in $\varpi^{\cdot}$ in order to obtain $\langle \eta^{\cdot}, \Psi^{\cdot}, \pi^{\cdot} \rangle$. So it is an assumption. Then the corresponding element $\langle \eta^{\prime\cdot}, \Psi^{\prime\cdot}, \pi^{\prime\cdot} \rangle$ at position $l^{\triangleright}$ is equal to $\langle \eta^{\cdot}, \Psi^{\cdot}, \pi^{\cdot} \rangle$ and so the result follows straightforwardly.

$n + 1$ *rules* applied in order to obtain $\langle \eta^{\cdot}, \Psi^{\cdot}, \pi^{\cdot} \rangle$. Suppose it is at the l''th position in $\varpi^{\cdot}$, with l' in $\{1, \ldots, m^{\cdot}\}$. Let $\langle \{\langle \vartheta''_1, \Psi''_1, \eta''_1 \rangle, \ldots, \langle \vartheta''_{k''}, \Psi''_{k''}, \eta''_{k''} \rangle\}, \eta'', P''_f, P''_d \rangle$, σ'' and $i''_1, \ldots, i''_{k''}$, be the rule, the schema substitution, and the position of the premises in ϖ, respectively, used to obtain $\langle \eta^{\cdot}, \Psi^{\cdot}, \pi^{\cdot} \rangle$. In the following we will denote the premise in ϖ at position i''_j by $\langle \eta^{\cdot}_{i''_j}, \Psi^{\cdot}_{i''_j}, \pi^{\cdot}_{i''_j} \rangle$, and the corresponding element in $\varpi_1 \ldots \varpi_k\varpi''\varpi''^{\cdot}$ at position $i''^{\triangleright}_j$ by $\langle \eta_{i''^{\triangleright}_j}, \Psi_{i''^{\triangleright}_j}, \pi_{i''^{\triangleright}_j} \rangle$, for each $j = 1, \ldots, k''$. So $\langle \eta^{\cdot}, \Psi^{\cdot}, \pi^{\cdot} \rangle$ is $\langle \eta''\sigma'', \Psi^{\cdot}_{i''_1} \setminus \Psi''_1\sigma'' \cup \ldots \cup \Psi^{\cdot}_{i''_{k''}} \setminus \Psi''_{k''}\sigma'', \pi^{\cdot}_{i''_1} * \ldots * \pi^{\cdot}_{i''_{k''}} * *_{\pi \in P''_f}(\pi_{\Sigma}\sigma''_{\mathrm{nl}}) \rangle$, and $\langle \eta^{\prime\cdot}, \Psi^{\prime\cdot}, \pi^{\prime\cdot} \rangle$ is by definition of $\varpi''^{\cdot}$ $\langle \eta''\sigma'', \Psi_{i''^{\triangleright}_1} \setminus \Psi''_1\sigma'' \cup \ldots \cup \Psi_{i''_{k''}{}^{\triangleright}} \setminus \Psi''_{k''}\sigma'', \pi_{i''^{\triangleright}_1} * \ldots * \pi_{i''_{k''}{}^{\triangleright}} * *_{\pi \in P''_f}(\pi_{\Sigma}\sigma''_{\mathrm{nl}}) \rangle$. Note that, for each $j = 1, \ldots, k''$, since each premise at i''_j appear in ϖ either in ϖ_i for some $i = 1, \ldots, k$, or is t_r, or is an element of $\varpi^{\cdot}$ previous to l, then we have that $\eta_{i''^{\triangleright}_j}$ is $\eta^{\cdot}_{i''_j}$, $\Psi_{i''^{\triangleright}_j}$ is contained in $\Psi^{\cdot}_{i''_j}$, and $\pi_{i''^{\triangleright}_j}$ is $\pi^{\cdot}_{i''_j}$. Therefore $\Psi^{\prime\cdot}$ which is $\Psi_{i''^{\triangleright}_1} \setminus \Psi''_1\sigma'' \cup \ldots \cup \Psi_{i''_{k''}{}^{\triangleright}} \setminus \Psi''_{k''}\sigma''$ is contained in

$\Psi^{\cdot}_{i''_1} \setminus \Psi''_1\sigma'' \cup \ldots \cup \Psi^{\cdot}_{i''_{k''}} \setminus \Psi''_{k''}\sigma''$ i.e. in $\Psi^{\cdot}$, and $\pi^{\prime\cdot}$ which is $\pi_{i^{\prime\prime\triangleright}_1} * \ldots * \pi_{i''^{\triangleright}_{k''}} * *_{\pi \in P''_f}(\pi_\Sigma \sigma''_{\mathrm{nl}}))$ is $\pi^{\cdot}_{i''_1} * \ldots * \pi^{\cdot}_{i''_{k''}} * *_{\pi \in P''_f}(\pi_\Sigma \sigma''_{\mathrm{nl}})$ i.e. is $\pi^{\cdot}$. *QED*

Lemma 2.3.25 The sequence $\varpi_1 \ldots \varpi_k \varpi'' \varpi^{\prime\prime\cdot}$ is a sober deduction sequence having as its last element $\langle \eta, \Psi', \pi \rangle$, where Ψ' is contained in Ψ.

Proof We show first that $\varpi_1 \ldots \varpi_k \varpi'' \varpi^{\prime\prime\cdot}$ is a deduction sequence. Since $\varpi_1 \ldots \varpi_k \varpi''$ is a deduction sequence by Lemma 2.3.22, we only have to show that each element in $\varpi^{\prime\prime\cdot}$ is defined in such a way that the sequence leading to it is a deduction sequence. Note that the elements in $\varpi^{\prime\prime\cdot}$ are in a one to one correspondence with the elements in $\varpi^{\cdot}$. The proof follows by induction on the number of rules applied in $\varpi^{\cdot}$ in order to obtain an element $\langle \eta^{\cdot}, \Psi^{\cdot}, \pi^{\cdot} \rangle$:

0 *rules* applied in $\varpi^{\cdot}$ in order to obtain $\langle \eta^{\cdot}, \Psi^{\cdot}, \pi^{\cdot} \rangle$. Then it is an assumption and the corresponding element in $\varpi^{\prime\prime\cdot}$ is also a well defined assumption.

$n+1$ *rules* applied in $\varpi^{\cdot}$ in order to obtain $\langle \eta^{\cdot}, \Psi^{\cdot}, \pi^{\cdot} \rangle$. Suppose it is at the l'th position in $\varpi^{\cdot}$, with l in $\{1, \ldots, m^{\cdot}\}$. The corresponding element in $\varpi^{\prime\prime\cdot}$, at position $l_{1''} + m - 1 + l$ in $\varpi_1 \ldots \varpi_k \varpi'' \varpi^{\prime\prime\cdot}$, will be denoted by $\langle \eta_{l_{1''}+m-1+l}, \Psi_{l_{1''}+m-1+l}, \pi_{l_{1''}+m-1+l} \rangle$. Let $\langle \{ \langle \vartheta''_1, \Psi''_1, \eta''_1 \rangle, \ldots, \langle \vartheta''_{k''}, \Psi''_{k''}, \eta''_{k''} \rangle \}, \eta'', P''_f, P''_d \rangle$ named r'', σ'' and $i''_1, \ldots, i''_{k''}$, be the rule, the schema substitution, and the positions in ϖ of the premises, respectively, in order to obtain $\langle \eta^{\cdot}, \Psi^{\cdot}, \pi^{\cdot} \rangle$. In the following we will denote the premise in ϖ at position i''_j by $\langle \eta^{\cdot}_{i''_j}, \Psi^{\cdot}_{i''_j}, \pi^{\cdot}_{i''_j} \rangle$, and the corresponding element in $\varpi_1 \ldots \varpi_k \varpi'' \varpi^{\prime\prime\cdot}$ at position $i^{\prime\prime\triangleright}_j$ by $\langle \eta_{i^{\prime\prime\triangleright}_j}, \Psi_{i^{\prime\prime\triangleright}_j}, \pi_{i^{\prime\prime\triangleright}_j} \rangle$, for each $j = 1, \ldots, k''$. Consider the schema substitution $\sigma^{\prime\cdot}$ defined as follows: $\sigma''_f = \sigma^{\prime\cdot}_f$, $\sigma''_t = \sigma^{\prime\cdot}_t$, $\sigma''_l = \sigma^{\prime\cdot}_l$, and $\vartheta''_j \sigma^{\prime\cdot}_s = \Psi_{i^{\prime\prime\triangleright}_j}$ for each $j = 1, \ldots, k''$. Note that $\langle \eta_{l_{1''}+m-1+l}, \Psi_{l_{1''}+m-1+l}, \pi_{l_{1''}+m-1+l} \rangle$ by definition of $\varpi^{\prime\prime\cdot}$ is $\langle \eta''\sigma'', \Psi_{i^{\prime\prime\triangleright}_1} \setminus \Psi''_1\sigma'' \cup \ldots \cup \Psi_{i''^{\triangleright}_{k''}} \setminus \Psi''_{k''}\sigma'', \pi_{i^{\prime\prime\triangleright}_1} * \ldots * \pi_{i''^{\triangleright}_{k''}} * *_{\pi \in P''_f}(\pi_\Sigma \sigma''_{\mathrm{nl}}) \rangle$. So $\langle \eta_{l_{1''}+m-1+l}, \Psi_{l_{1''}+m-1+l}, \pi_{l_{1''}+m-1+l} \rangle$, the schema substitution $\sigma^{\prime\cdot}$, and the tuples at positions $i^{\prime\prime\triangleright}_1, \ldots, i''^{\triangleright}_{k''}$, are a good instantiation for rule r'' in $\varpi_1 \ldots \varpi_k \varpi'' \varpi^{\prime\prime\cdot}$. This happens because the following facts hold:

- $\eta_{i^{\prime\prime\triangleright}_j}$ *is* $\eta''_j \sigma^{\prime\cdot}$, for each $j = 1, \ldots, k''$. Note that $\eta^{\cdot}_{i''_j}$ is equal to $\eta_{i^{\prime\prime\triangleright}_j}$, by Lemma 2.3.24. So the result follows because $\eta^{\cdot}_{i''_j}$ is $\eta''_j \sigma''$, and $\eta''_j \sigma''$ is $\eta''_j \sigma^{\prime\cdot}$.

- $\Psi_{i^{\prime\prime\triangleright}_j}$ *is* $\vartheta''_j \sigma^{\prime\cdot}$, for each $j = 1, \ldots, k''$. By definition of $\sigma^{\prime\cdot}$.

- $\pi_\Sigma(\sigma^{\prime\cdot}) = 1$ for each π in P''_d. Let fresh$(\Upsilon''_{j''}, \langle \vartheta''_{j''}, \Psi''_{j''}, \eta''_{j''} \rangle)$, named π'', be in P''_d, for some j'' in $1, \ldots, k''$. Then:

1. $\nu''\sigma^{\prime\cdot}$ is a label schema variable not in $(\mathrm{lsv}(\Psi''_{j''} \cup \{\eta''_{j''}\}) \setminus \{\nu''\})\sigma^{\prime\cdot}$, for any ν'' in $\Upsilon''_{j''}$. To see this note that $\nu''\sigma^{\prime\cdot}$ is $\nu''\sigma''$ which is a label schema variables not in $(\mathrm{lsv}(\Psi''_{j''} \cup \{\eta''_{j''}\}) \setminus \{\nu''\})\sigma''$ which is equal to $(\mathrm{lsv}(\Psi''_{j''} \cup \{\eta''_{j''}\}) \setminus \{\nu''\})\sigma^{\prime\cdot}$.

2. $\Upsilon''_{j''}\sigma^{\prime\cdot} \cap \mathrm{lsv}(\vartheta''_{j''}\sigma^{\prime\cdot} \setminus \Psi''_{j''}\sigma^{\prime\cdot}) = \emptyset$. This happens because $\Upsilon''_{j''}\sigma^{\prime\cdot}$ is $\Upsilon''_{j''}\sigma''$, $\Psi''_{j''}\sigma^{\prime\cdot}$ is $\Psi''_{j''}\sigma''$, and $\vartheta''_{j''}\sigma^{\prime\cdot}$ is contained in $\vartheta''_{j''}\sigma^{\prime\cdot}$ by Lemma 2.3.24. So $\Upsilon''_{j''}\sigma^{\prime\cdot} \cap \mathrm{lsv}(\vartheta''_{j''}\sigma^{\prime\cdot} \setminus \Psi''_{j''}\sigma^{\prime\cdot})$ is contained in $\Upsilon''_{j''}\sigma'' \cap \mathrm{lsv}(\vartheta''_{j''}\sigma'' \setminus \Psi''_{j''}\sigma'')$ which is $\emptyset$ since $\pi''(\sigma'')$ is 1.

The last element of $\varpi_1 \ldots \varpi_k \varpi'' \varpi^{\prime\prime\cdot}$ is $\langle \eta, \Psi', \pi \rangle$, where Ψ' is contained in Ψ. Note that the corresponding element in ϖ is also the last element. So the result follows by Lemma 2.3.24.

$\varpi_1 \dots \varpi_k \varpi'' \varpi''^{\cdot}$ is a sober deduction sequence. This follows straightforwardly. Note that it is a deduction sequence as showed in the first part of this proof. *QED*

Conclusion of the proof of Proposition 2.3.20 It is possible to conclude because by Lemma 2.3.25, $\varpi_1 \dots \varpi_k \varpi'' \varpi''^{\cdot}$ is a sober deduction sequence for $\Psi \vdash \eta;\ \pi$, with $N-1$ applications of derived rules, as we wanted to show. Then, in order to obtain a deduction sequence for $\Psi \vdash \eta;\ \pi$ with no applications of derived rules we only have to apply this method repeatedly until there are no application of derived rules. *QED*

2.4 Logic systems

A lfob logic system is the entity that puts together deduction and semantics.

Definition 2.4.1 A *lfob logic system* $\mathcal{L}$ is a tuple $\langle \Sigma, R, M, \breve{\cdot} \rangle$ where $\langle \Sigma, R \rangle$ is a lfob deduction system and $\langle \Sigma, M, \breve{\cdot} \rangle$ is a lfob interpretation system.

From now on, in a try to lighten the presentation, when referring to the lfob logic system $\mathcal{L}$, we assume we are referring to a lfob logic system whose components are named by the tuple $\langle \Sigma, R, M, \breve{\cdot} \rangle$.

Definition 2.4.2 A lfob logic system $\mathcal{L}$ is said to be

- *sound* iff $\Psi \vdash \eta$ implies $\Psi \vDash \eta$,
- *complete* iff $\Psi \vDash \eta$ implies $\Psi \vdash \eta$,

for every set of labelled schema formulae Ψ and labelled schema formula η.

2.4.1 Soundness

The main goal of this section is to show a soundness theorem saying that a lfob logic system is sound whenever all its structures satisfy all its rules. So we need to define when a structure satisfies a rule.

Definition 2.4.3 A Σ structure s *satisfies* rule $\langle \{\langle \vartheta_1, \Psi_1, \eta_1 \rangle, \dots, \langle \vartheta_k, \Psi_k, \eta_k \rangle\}, \eta, P_f, P_d \rangle$ iff, for every

- set E of label schema variables,
- σ within L_E such that
  - $(\pi_\Sigma \sigma_{\mathrm{nl}})$ is 1 for each π in P_f,
  - $\pi_{\Sigma;E}(\sigma)$ is 1 for each π in P_d,
- α over s and E,

if, for each $j = 1, \dots, k$ and for any α' over s and E $\Upsilon_j \sigma$ co-equivalent to α

$$s, \alpha' \Vdash \Psi_j \sigma \quad \text{implies} \quad s, \alpha' \Vdash \eta_j \sigma,$$

then

$$s, \alpha \Vdash \eta \sigma,$$

where Υ_j is

- Υ, if $\text{fresh}(\Upsilon, \langle \vartheta_j, \Psi_j, \eta_j \rangle)$ is in P_d, or
- $\emptyset$, otherwise.

Before being able to present the proof of the soundness theorem we need to prove a useful proposition used throughout the thesis, and an auxiliary lemma. The proposition establishes general properties about the relationship between interpretation and schema assignments, and the lemma is the base of the soundness theorem but considering only one single structure.

Proposition 2.4.4 We have, for any schema assignments α' and α over a Σ structure s,

1. $[\![v]\!]^s_{\alpha'} = [\![v]\!]^s_{\alpha}$, whenever $\nu\alpha' = \nu\alpha$ for any ν in $\text{lsv}(v)$,
2. $[\![t]\!]^s_{\alpha'} = [\![t]\!]^s_{\alpha}$, whenever $\theta\alpha' = \theta\alpha$ for any term schema variable θ in t,
3. $[\![\varphi]\!]^s_{\alpha'} = [\![\varphi]\!]^s_{\alpha}$, whenever $\theta\alpha' = \theta\alpha$ and $\xi\alpha' = \xi\alpha$ for any term and formula schema variable θ and ξ, respectively, in φ,
4. $s, \alpha \Vdash \eta$ iff $s, \alpha' \Vdash \eta$, whenever $\nu\alpha' = \nu\alpha$, $\theta\alpha' = \theta\alpha$ and $\xi\alpha' = \xi\alpha$ for any label schema variable ν, term schema variable θ and formula schema variable ξ in η,

for any label schema term v, schema term t, schema formula φ, and labelled schema formula η.

Proof 1. $[\![v]\!]^s_{\alpha'} = [\![v]\!]^s_{\alpha}$ whenever $\nu\alpha' = \nu\alpha$ for any ν in $\text{lsv}(v)$. The proof follows by induction on the structure of a label schema term v:
If v *is in* Ξ_l, then $[\![v]\!]^s_{\alpha'} = v\alpha' = v\alpha = [\![v]\!]^s_{\alpha}$.
If v *is in* F^l_0, then $[\![v]\!]^s_{\alpha'} = [v] = [\![v]\!]^s_{\alpha}$.
If v *is* $f^l(v_1, \dots, v_k)$, then $[\![v]\!]^s_{\alpha'} = [f^l]([\![v_1]\!]^s_{\alpha'}, \dots, [\![v_k]\!]^s_{\alpha'}) = [f^l]([\![v_1]\!]^s_{\alpha}, \dots, [\![v_k]\!]^s_{\alpha}) = [\![v]\!]^s_{\alpha}$.

2. $[\![t]\!]^s_{\alpha'} = [\![t]\!]^s_{\alpha}$ whenever $\theta\alpha' = \theta\alpha$ for any term schema variable θ in t. The proof follows by induction on the structure of a schema term t:
If t *is in* $F_0 \cup X$, then $[\![t]\!]^s_{\alpha'} = \widehat{t} = [\![t]\!]^s_{\alpha}$.
If t *is in* Ξ_t, then $[\![t]\!]^s_{\alpha'} = \alpha'_t(t) = \alpha_t(t) = [\![t]\!]^s_{\alpha}$;
If t *is* $f(t_1, \dots, t_k)$, then $[\![f(t_1, \dots, t_k)]\!]^s_{\alpha'}(u) = [f]_{\omega(u)}([\![t_1]\!]^s_{\alpha'}(u), \dots, [\![t_k]\!]^s_{\alpha'}(u))$ which is, by induction hypothesis, $[f]_{\omega(u)}([\![t_1]\!]^s_{\alpha}(u), \dots, [\![t_k]\!]^s_{\alpha}(u)) = [\![f(t_1, \dots, t_k)]\!]^s_{\alpha}(u)$.

3. $[\![\varphi]\!]^s_{\alpha'} = [\![\varphi]\!]^s_{\alpha}$, whenever $\xi\alpha' = \xi\alpha$ for any formula schema variable ξ in φ. The proof follows by induction on the structure of a schema formula φ:
If φ *is in* Ξ_f, then $[\![\varphi]\!]^s_{\alpha'} = \alpha'_f(\varphi) = \alpha_f(\varphi) = [\![\varphi]\!]^s_{\alpha}$.
If φ *is in* $P_0 \cup C_0$, then $[\![\varphi]\!]^s_{\alpha'} = \widehat{\varphi} = [\![\varphi]\!]^s_{\alpha}$.
If φ *is* $p(t_1, \dots, t_k)$, then $[\![p(t_1, \dots, t_k)]\!]^s_{\alpha'}(u) = [p]_{\omega(u)}([\![t_1]\!]^s_{\alpha'}(u), \dots, [\![t_k]\!]^s_{\alpha'}(u))$ which is $[p]_{\omega(u)}([\![t_1]\!]^s_{\alpha}(u), \dots, [\![t_k]\!]^s_{\alpha}(u)) = [\![p(t_1, \dots, t_k)]\!]^s_{\alpha}(u)$.
If φ *is* $c(\varphi_1, \dots, \varphi_k)$, then $[\![c(\varphi_1, \dots, \varphi_k)]\!]^s_{\alpha'}(u) = [c]_{\omega(u)\alpha(u)}([\![\varphi_1]\!]^s_{\alpha'} \cap U_{\omega(u)\alpha(u)}, \dots, [\![\varphi_k]\!]^s_{\alpha'} \cap U_{\omega(u)\alpha(u)})(u)$ which is equal to $[c]_{\omega(u)\alpha(u)}([\![\varphi_1]\!]^s_{\alpha} \cap U_{\omega(u)\alpha(u)}, \dots, [\![\varphi_k]\!]^s_{\alpha} \cap U_{\omega(u)\alpha(u)})(u) = [\![c(\varphi_1, \dots, \varphi_k)]\!]^s_{\alpha}(u)$.

If φ *is either* $q_x(\varphi_1, \ldots, \varphi_k)$ *or* $o(\varphi_1, \ldots, \varphi_k)$ with $q \in Q_k$ and $o \in O_k$. This case is similar to the preceding one.

4. $s, \alpha \Vdash \eta$ iff $s, \alpha' \Vdash \eta$, whenever $\nu\alpha' = \nu\alpha$, $\theta\alpha' = \theta\alpha$ and $\xi\alpha' = \xi\alpha$ for any label schema variable ν, term schema variable θ and formula schema variable ξ in η. The proof follows by case analysis on η:

If η *is* $r\, v_1, \ldots, v_k$, where r is a relation symbol, then $s, \alpha' \vdash \eta$ iff $\langle \llbracket v_1 \rrbracket^s_{\alpha'}, \ldots, \llbracket v_k \rrbracket^s_{\alpha'} \rangle$ is in $[r]$ iff (by item 1. above) $\langle \llbracket v_1 \rrbracket^s_{\alpha}, \ldots, \llbracket v_k \rrbracket^s_{\alpha} \rangle$ is in $[r]$ iff $s, \alpha \vdash \eta$.

If η *is* $v{:}t =_g v'{:}t'$, then $s, \alpha' \vdash \eta$ iff $\llbracket t \rrbracket^s_{\alpha'}(\llbracket v \rrbracket^s_{\alpha'}) = \llbracket t' \rrbracket^s_{\alpha'}(\llbracket v' \rrbracket^s_{\alpha'})$ iff (by items 1. and 2.) $\llbracket t \rrbracket^s_{\alpha}(\llbracket v \rrbracket^s_{\alpha}) = \llbracket t' \rrbracket^s_{\alpha}(\llbracket v' \rrbracket^s_{\alpha})$ iff $s, \alpha \vdash \eta$;

η *is* $v{:}\varphi$. So $s, \alpha' \vdash \eta$ iff $\llbracket v \rrbracket^s_{\alpha'} \in \llbracket \varphi \rrbracket^s_{\alpha'}$ iff (by items 1. and 3.) $\llbracket v \rrbracket^s_{\alpha} \in \llbracket \varphi \rrbracket^s_{\alpha}$ iff $s, \alpha \vdash \eta$. *QED*

Lemma 2.4.5 Given a lfob deduction system $\langle \Sigma, R \rangle$, a Σ structure s, a set E of label schema variables, a schema assignment α over s and E, a set Ψ within L_E and η in L_E, we have

$$s, \alpha \Vdash \eta$$

whenever (i) $\Psi \vdash_E \eta$, (ii) $s, \alpha \Vdash \Psi$, and (iii) s satisfies all rules of R.

Proof We show by induction on the length of a sober deduction sequence for $\Psi \vdash_E \eta$, that, for any Σ structure s, schema assignment α over s and E, set Ψ within L_E and η in L_E, if $\Psi \vdash_E \eta$ and $s, \alpha \Vdash \Psi$ then $s, \alpha \Vdash \eta$.

Base. Assume that $\langle \eta, \Psi_1, 1 \rangle$ is a sober deduction sequence for $\Psi \vdash_E \eta$ and that $s, \alpha \Vdash \Psi$. Then we can consider two cases:

Either Ψ_1 is $\{\eta\}$ and so $s, \alpha \Vdash \eta$ since $\Psi_1 \subseteq \Psi$.

Or there are rule $\langle \emptyset, \eta', P'_f, P'_d \rangle$, named r, and schema substitution σ within L_E such that $\eta'\sigma$ is η, $\Psi_1 = \emptyset$, $(\pi_\Sigma \sigma_{\mathrm{nl}}) = 1$ for each π in P'_f, and $\pi_{\Sigma;E}(\sigma)$ is 1 for each π in P'_d. So, $s, \alpha \Vdash \eta'\sigma$ since s satisfies r, see Definition 2.4.3.

Assume as induction hypothesis that, for any Σ structure s, schema assignment α over s and E, set Ψ within L_E and η in L_E, if $s, \alpha \Vdash \Psi$, and $\Psi \vdash_E \eta$ has a sober deduction sequence of length less than or equal to n then $s, \alpha \Vdash \eta$.

Step. Let s be a Σ structure, α a schema assignment over s and E, Ψ a set within L_E, and η a labelled schema formula in L_E, such that there is a sober deduction sequence $\langle \eta_1, \Psi_1, \pi_1 \rangle \ldots \langle \eta, \Psi_{n+1}, 1 \rangle$, named ϖ, for $\Psi \vdash_E \eta$, with length $n+1$, and $s, \alpha \Vdash \Psi$. Then there are rule $\langle \{\langle \vartheta'_1, \Psi'_1, \eta'_1 \rangle, \ldots, \langle \vartheta'_k, \Psi'_k, \eta'_k \rangle\}, \eta', P'_f, P'_d \rangle$, named r, schema substitution σ within L_E, and $i_1, \ldots, i_k$ in $\{1, \ldots, n\}$, with $\eta_{i_j} = \eta'_j\sigma$, $\Psi_{i_j} = \vartheta'_j\sigma$ and $\pi_{i_j} = 1$ for each $j = 1, \ldots, k$, $\Psi_{n+1} = \Psi_{i_1} \setminus \Psi'_1\sigma \cup \ldots \cup \Psi_{i_k} \setminus \Psi'_k\sigma$, $(\pi'_\Sigma \sigma_{\mathrm{nl}}) = 1$ for each π' in P'_f, $\pi'_{\Sigma;E}(\sigma) = 1$ for each π' in P'_d, and $\eta = \eta'\sigma$.

Note that, there is a sober deduction sequence of length less than or equal to n for $\vartheta'_j\sigma \vdash_E \eta'_j\sigma$. So, by induction hypothesis, for any Σ structure s and schema assignment α' over s and E, if $s, \alpha' \Vdash \vartheta'_j\sigma$ then $s, \alpha' \Vdash \eta'_j\sigma$, for each $j = 1, \ldots, k$.

Hence we can conclude $s, \alpha \Vdash \eta'\sigma$, i.e., $s, \alpha \Vdash \eta$, by using the fact that s satisfies r, see Definition 2.4.3. We can use this fact because:

i. σ is such that $(\pi_\Sigma \sigma_{\mathrm{nl}})$ is 1 for each π in P'_f, and $\pi_{\Sigma;E}(\sigma)$ is 1 for each π in P'_d;

ii. for each $j = 1, \ldots, k$, we have $s, \alpha' \Vdash \eta'_j\sigma$ whenever $s, \alpha' \Vdash \Psi'_j\sigma$, for every α' over s and E, $\Upsilon'_j\sigma$ co-equivalent to α, where Υ'_j is Υ' whenever the fresh deduction proviso $\mathrm{fresh}(\Upsilon', \langle \vartheta'_j, \Psi'_j, \eta'_j \rangle)$ is in P'_d, and is $\emptyset$ otherwise. Now the proof: let j be in $\{1, \ldots, k\}$, and α' be a schema assignment over s and E, $\Upsilon'_j\sigma$ co-equivalent to α. Suppose $s, \alpha' \Vdash \Psi'_j\sigma$. Note that $s, \alpha \Vdash \vartheta'_j\sigma \setminus \Psi'_j\sigma$ since $s, \alpha \Vdash \Psi$. Observe also that $\nu\alpha = \nu\alpha'$, for each ν in $\mathrm{lsv}(\vartheta'_j\sigma \setminus \Psi'_j\sigma)$, since $\pi_{\Sigma;E}(\sigma)$ is 1 for each π in P'_d, and so $\Upsilon'_j\sigma \cap \mathrm{lsv}(\vartheta'_j\sigma \setminus \Psi'_j\sigma) = \emptyset$. Then, by Proposition 2.4.4 we can conclude that $s, \alpha' \Vdash \vartheta'_j\sigma \setminus \Psi'_j\sigma$, and so $s, \alpha' \Vdash \vartheta'_j\sigma$. Then by the induction hypothesis we have $s, \alpha' \Vdash \eta'_j\sigma$ as desired. *QED*

Now we prove the soundness theorem.

Theorem 2.4.6 A lfob logic system is sound whenever all its structures satisfy all its rules.

Proof Let $\langle \Sigma, R, M, \breve{\cdot} \rangle$ be a logic system where, for any model m in M, and any structure s in $\breve{m}$, we have that s satisfies r, for any rule r in R. Suppose $\Psi \vdash_E \eta$ and let m be a model in M, s a structure in $\breve{m}$ and α a schema assignment over s and E with $s, \alpha \Vdash \Psi$. Then, by Lemma 2.4.5, we have that $s, \alpha \Vdash \eta$. Therefore, taking into account Definition 2.2.8 of entailment we conclude $\Psi \vDash \eta$ as we wanted to show. *QED*

Before ending this section we show a very useful proposition proving that every structure over a lfob signature satisfies the rules common to all lfob deduction systems (rules introduced in Definition 2.3.10).

Proposition 2.4.7 Any lfob structure satisfies the rules that are common to all lfob deduction systems (rules introduced in Definition 2.3.10).

Proof Let Σ be a lfob signature, r a rule $\langle \{\langle \vartheta_1, \Psi_1, \eta_1 \rangle, \ldots, \langle \vartheta_k, \Psi_k, \eta_k \rangle\}, \eta, P_f, P_d \rangle$ mentioned in Definition 2.3.10, s a Σ structure, σ a Σ schema substitution within L_E such that $(\pi_\Sigma \sigma_{\mathrm{nl}})$ is 1 for each π in P_f, and $\pi_{\Sigma;E}(\sigma)$ is 1 for each π in P_d, and α a schema assignment over s and E. Thus, suppose r is

- $\equiv_{\omega_s}$ and $s, \alpha \Vdash \nu'\sigma \equiv_\omega \nu\sigma$. Then $\omega(\llbracket \nu'\sigma \rrbracket^s_\alpha) = \omega(\llbracket \nu\sigma \rrbracket^s_\alpha)$. So, $s, \alpha \Vdash \nu\sigma \equiv_\omega \nu'\sigma$.
- $\equiv_{\omega_t}$, $\equiv_{\omega_r}$, $\equiv_{\alpha_s}$, $\equiv_{\alpha_t}$ or $\equiv_{\alpha_r}$. The proof is similar to the previous one.

- $=_{g_s}$ and $s, \alpha \Vdash \nu\sigma{:}\theta\sigma =_g \nu'\sigma{:}\theta'\sigma$. Then $\llbracket \theta\sigma \rrbracket^s_\alpha(\llbracket \nu\sigma \rrbracket^s_\alpha) = \llbracket \theta'\sigma \rrbracket^s_\alpha(\llbracket \nu'\sigma \rrbracket^s_\alpha)$, and so $s, \alpha \Vdash \nu'\sigma{:}\theta'\sigma =_g \nu\sigma{:}\theta\sigma$.

- $=_{g_t}$ or $=_{g_r}$. The proof is similar to the previous one.

- $=_E$ and $s, \alpha \Vdash \nu\sigma{:}\theta\sigma = \theta'\sigma$. Then $\llbracket \theta\sigma \rrbracket^s_\alpha(\llbracket \nu\sigma \rrbracket^s_\alpha) = \llbracket \theta'\sigma \rrbracket^s_\alpha(\llbracket \nu\sigma \rrbracket^s_\alpha)$. So, $s, \alpha \Vdash \nu\sigma{:}\theta\sigma =_g \nu\sigma{:}\theta'\sigma$.

- $=_I$. The proof is similar to the previous one.

- $=_{g_f}$, for f in F_k, and $s, \alpha \Vdash \nu\sigma{:}\theta_i\sigma =_g \nu'\sigma{:}\theta'_i\sigma$, for each $i = 1, \ldots, k$, and $s, \alpha \Vdash \nu\sigma \equiv_\omega \nu'\sigma$. Then $\llbracket \theta_i\sigma \rrbracket^s_\alpha(\llbracket \nu\sigma \rrbracket^s_\alpha) = \llbracket \theta'_i\sigma \rrbracket^s_\alpha(\llbracket \nu'\sigma \rrbracket^s_\alpha)$, for each $i = 1, \ldots, k$, and $\omega(\llbracket \nu\sigma \rrbracket^s_\alpha) = \omega(\llbracket \nu'\sigma \rrbracket^s_\alpha)$. So $\llbracket f(\theta_1\sigma, \ldots, \theta_k\sigma) \rrbracket^s_\alpha(\llbracket \nu\sigma \rrbracket^s_\alpha) = [f]_{\omega(\llbracket \nu\sigma \rrbracket^s_\alpha)}(\llbracket \theta_1\sigma \rrbracket^s(\llbracket \nu\sigma \rrbracket^s_\alpha), \ldots, \llbracket \theta_k\sigma \rrbracket^s_\alpha(\llbracket \nu\sigma \rrbracket^s_\alpha))$ is equal to $[f]_{\omega(\llbracket \nu'\sigma \rrbracket^s_\alpha)}(\llbracket \theta'_1\sigma \rrbracket^s_\alpha(\llbracket \nu'\sigma \rrbracket^s_\alpha), \ldots, \llbracket \theta'_k\sigma \rrbracket^s_\alpha(\llbracket \nu'\sigma \rrbracket^s_\alpha)) = \llbracket f(\theta'_1\sigma, \ldots, \theta'_k\sigma) \rrbracket^s_\alpha(\llbracket \nu'\sigma \rrbracket^s_\alpha)$. Therefore $s, \alpha \Vdash \nu\sigma{:}f(\theta_1\sigma, \ldots, \theta_k\sigma) =_g \nu'\sigma{:}f(\theta'_1\sigma, \ldots, \theta'_k\sigma)$.

- $=_{g_p}$, for p in P_k, and suppose $s, \alpha \Vdash \nu\sigma{:}p(\theta_1\sigma, \ldots, \theta_k\sigma)$, $s, \alpha \Vdash \nu\sigma{:}\theta_i\sigma =_g \nu'\sigma{:}\theta'_i\sigma$ for each $i = 1, \ldots, k$, and $s, \alpha \Vdash \nu\sigma \equiv_\omega \nu'\sigma$. Therefore $\llbracket \theta_i\sigma \rrbracket^s_\alpha(\llbracket \nu\sigma \rrbracket^s_\alpha) = \llbracket \theta'_i\sigma \rrbracket^s_\alpha(\llbracket \nu'\sigma \rrbracket^s_\alpha)$, for each $i = 1, \ldots, k$, $\llbracket p(\theta_1\sigma, \ldots, \theta_k\sigma) \rrbracket^s_\alpha(\llbracket \nu\sigma \rrbracket^s_\alpha) = 1$, and $\omega(\llbracket \nu\sigma \rrbracket^s_\alpha) = \omega(\llbracket \nu'\sigma \rrbracket^s_\alpha)$. Then, we have that $[p]_{\omega(\llbracket \nu\sigma \rrbracket^s_\alpha)}(\llbracket \theta_1\sigma \rrbracket^s_\alpha(\llbracket \nu\sigma \rrbracket^s_\alpha), \ldots, \llbracket \theta_k\sigma \rrbracket^s_\alpha(\llbracket \nu\sigma \rrbracket^s_\alpha))$, is equal as we expected to $[p]_{\omega(\llbracket \nu'\sigma \rrbracket^s_\alpha)}(\llbracket \theta'_1\sigma \rrbracket^s_\alpha(\llbracket \nu'\sigma \rrbracket^s_\alpha), \ldots, \llbracket \theta'_k\sigma \rrbracket^s_\alpha(\llbracket \nu'\sigma \rrbracket^s_\alpha))$. So $s, \alpha \Vdash \nu'\sigma{:}p(\theta'_1\sigma, \ldots, \theta'_k\sigma)$.

- $\equiv_{\alpha^x_g}$, for x in X, and $s, \alpha \Vdash \nu\sigma \equiv_\alpha \nu'\sigma$. Then $\alpha(\llbracket \nu\sigma \rrbracket^s_\alpha) = \alpha(\llbracket \nu'\sigma \rrbracket^s_\alpha)$. So, $[x]_{\alpha(\llbracket \nu\sigma \rrbracket^s_\alpha)} = [x]_{\alpha(\llbracket \nu'\sigma \rrbracket^s_\alpha)}$. Therefore $s, \alpha \Vdash \nu\sigma{:}x =_g \nu'\sigma{:}x$. *QED*

2.4.2 Labelled vs Non-labelled

In this sub-section we concentrate on the conditions a lfob logic system and a non-labelled first-order based logic system should satisfy in order that they correspond to each other. When this happens we say that the lfob logic system is a labelled presentation for that logic system. The motivations for considering a labelled presentation for a logic vary, and were detailed in the Introduction, Chapter 1, and in the beginning of this chapter. We start by introducing what we mean by a non-labelled first-order based logic system.

Definition 2.4.8 Given disjoint denumerable sets $\Xi^{\cdot}_t$, $\Xi^{\cdot}_f$, and $X^{\cdot}$, a *non-labelled first-order based logic system* is a pair $\langle \Sigma^{\cdot}, \propto \rangle$ where

- $\Sigma^{\cdot}$ is a tuple $\langle F^{\cdot}, P^{\cdot}, C^{\cdot} \rangle$, where
  - $F^{\cdot}$ is a family $\{F^{\cdot}_k\}_{k \in I\!N}$ of sets,
  - $P^{\cdot}$ is a family $\{P^{\cdot}_k\}_{k \in I\!N}$ of sets,
  - $C^{\cdot}$ is a family $\{C^{\cdot}_k\}_{k \in I\!N}$ of sets,
- $\propto$ is a binary relation between subsets of $L(\Sigma^{\cdot})$ and elements of $L(\Sigma^{\cdot})$, where $L(\Sigma^{\cdot})$ is the carrier of sort ϕ in the S-sorted free algebra induced by the family $\{G_{\vec{s}s}\}_{\vec{s} \in S^*, s \in S}$, where S is $\{\tau, \phi\}$, defined as follows
  - $G_{\epsilon\tau} = F^{\cdot}_0 \cup X^{\cdot} \cup \Xi^{\cdot}_t$,
  - $G_{\tau^k\tau} = F^{\cdot}_k$, for $k > 0$,
  - $G_{\tau^k\phi} = P^{\cdot}_k$, for $k > 0$,
  - $G_{\epsilon\phi} = P^{\cdot}_0 \cup C^{\cdot}_0 \cup \Xi^{\cdot}_f$,
  - $G_{\phi^k\phi} = C^{\cdot}_k$, for $k > 0$,
  - all other sets are empty,

  such that

  - $\Phi \propto \gamma$ whenever γ is in Φ,

  - $\Phi \propto \gamma$ whenever $\{\gamma' \mid \Phi \propto \gamma'\} \propto \gamma$,
  - $\Phi \propto \gamma$ whenever $\Phi' \propto \gamma$ and Φ' is contained in Φ.

In the following we will use the designation *(non-labelled) (fob) logic system* to mean a non-labelled first-order based logic system. Similarly for signatures. Note that it is possible to associate more information to a fob logic system than the one we specified in the previous definition. Nevertheless, for our purposes, it is enough to consider that level of abstraction. So, we are now ready to rigorously say when a lfob logic system corresponds to a fob logic system.

Definition 2.4.9 A lfob logic system $\langle \Sigma, R, M, \breve{\cdot} \rangle$ is a *labelled presentation for* a fob logic system $\langle \Sigma^{\cdot}, \propto \rangle$ *with respect to* $L^{\cdot}$ iff

- $\langle \Sigma, R, M, \breve{\cdot} \rangle$ is sound and complete,
- $\Xi^{\cdot}_t$ is either $\emptyset$ or is equal to Ξ_t, similarly for $\Xi^{\cdot}_f$ and $X^{\cdot}$,
- $L^{\cdot}$ is contained in $L(\Sigma^{\cdot})$ and is non-empty,
- $F^{\cdot}$ is equal to F,
- $P^{\cdot}$ is equal to P,
- $C^{\cdot}$ is equal to $C \cup Q \cup O$,
- $\Phi \propto \gamma$ iff $\{\nu{:}\phi \mid \phi \text{ in } \Phi\} \models \nu{:}\gamma$, where ν is a label schema variable, for any set Φ contained in $L^{\cdot}$ and γ in $L^{\cdot}$.

In the following, when a lfob logic system is a labelled presentation for a logic system $\langle \Sigma^{\cdot}, \propto \rangle$ with respect to the whole language $L(\Sigma^{\cdot})$, we simply say that it is a labelled presentation for $\langle \Sigma^{\cdot}, \propto \rangle$.

Definition 2.4.10 A lfob logic system is a *labelled presentation for* a logic with respect to a set of non-labelled schema formulae when it is a labelled presentation for some fob logic system for that logic with respect to that set.

Remark 2.4.11 In the following when a lfob logic system is a labelled presentation for a non-labelled logic $\mathcal{L}^{\cdot}$ with respect to a set of non-labelled schema formulae, we say that it is a $\mathcal{L}^{\cdot}$ lfob logic system with respect to that set.

Chapter 3

Completeness

In this chapter we analyze sufficient conditions for lfob logic systems to be complete. We establish a theorem saying that rich lfob logic systems are complete. This theorem will be used to show that the lfob logic systems presented in Chapter 5 are complete. Moreover it is at the core of completeness preservation by fibring since the completeness of the lfob logic system resulting from the fibring is based on a corollary of it saying that full and appropriate lfob logic systems are complete. The reason why we need the completeness theorem and not only its corollary, is because, when proposing a lfob logic system for a logic, in Chapter 5, we do not need to assume that it contains all the structures that satisfy the rules, i.e., we do not need to assume that it is full. This is useful in the proof that it is indeed a labelled presentation for a logic, because we can select by additional properties the structures to be present in the logic system, and not only by the fact that they satisfy the rules. This additional information about the structures in the logic helps when establishing a relation between those structures and the usual models of the logic.

We now illustrate the inter-play between the completeness theorem and the completeness corollary. Suppose we want to fibre two logics. So, we begin by proposing labelled presentations for each logic and prove that they are sound, complete and are indeed equivalent to a non-labelled first-order based logic system for that logic. To prove completeness, we use the theorem for rich systems. After that, we consider the lfob logic systems obtained by the fullness closure of that lfob logic systems and by the adjustment, if necessary, of some components of the provisos over their joint signature. The need to consider the fullness closure comes from the fact that we use the corollary for full and appropriate systems in our completeness preservation theorems, and so the lfob logic systems must be full and appropriate. Note that these lfob logic systems have the same consequence relation and the same entailment as the original systems, as we show in Proposition 3.4.2, and so are equivalent to that systems. Hence they are sound, complete and are a labelled presentation for the logic in question. So, if they constitute a suitable pair, it is possible to fibre them, and then use the preservation results to conclude that the lfob logic system resulting of the fibring is also sound and complete.

The study of completeness theorems for first-order based logics has been deserving continuous attention [50, 48, 52, 12, 41, 49]. The study of completeness theorems with sufficient conditions preserved by fibring was also done in [74] but in a very different context of non-labelled logic systems endowed with Hilbert style deduction systems. In that work a completeness theorem is established for full, congruent, persistent, and uniform logic systems with equality and inequality that is the base of the completeness preservation by fibring theorem (also established there). The conditions of persistence, uniformity and presence of equality and inequality are the conditions identified by the authors to be able to construct the sets of formulae their canonical structure relies on. They correspond, in our context, to a collection of conditions that guarantees that a logic system is appropriate. Note that in Chapter 4 we identify also two distinct collections of conditions guaranteeing that a lfob logic system is appropriate. One collection is for lfob logic systems with a classical negation, local or non-local, and the other for lfob logic systems that may not have any kind of negation. The condition of congruence of the system is important for the canonical structure to be well defined and in a sense is very similar to our condition imposing that the set over which a canonical structure is defined, is canonical. Finally the fullness condition serves to guarantee the presence of the canonical structures in the system, and is equal to the condition we impose, with the same purpose, in our corollary of the completeness theorem.

The chapter is organized as follows: we start by identifying a collection of conditions for sets of labelled schema formulae that are sufficient for a structure over them be well defined. We call the sets satisfying the conditions *canonical sets*. Then we define how to construct a canonical structure over a canonical set. After that we investigate which additional conditions over canonical sets are sufficient for the canonical structures over them to satisfy the rules of the underlying deduction system. We call the sets satisfying those conditions *appropriate sets*. Then, we prove the completeness theorem for rich lfob logic systems, and its corollary for full and appropriate systems. Moreover, we show that the fullness closure of a sound and complete lfob logic system is also sound, complete, and the entailment and the consequence of both systems coincide. Next, we investigate if all lfob logic systems are appropriate. We argue that this is not the case by presenting three counter-examples, each by its own reason, and we discuss in some detail that reason. Finally, ending the chapter we show two propositions. The first proves that any consistent and deductively closed set, in any lfob deduction system allowing the extension of consistent sets to maximal ones, satisfies a property very similar to appropriateness, where the antecedent of that implication refers to all maximal consistent extensions of the set, instead of that set. The last proposition shows a property similar to appropriateness but in terms of deduction. That property of deductively closed sets will be very used in the sequel.

3.1 Canonical sets and canonical structures

The main goal of this section is to define the canonical structure over a deductively closed and E canonical set of labelled schema formulae, where E is a set of label schema variables. In order to simplify the presentation we need to introduce some notation. Given a set of label schema variables E, a set Ψ within L_E, and v in $T_{\mathrm{lab},E}$, we denote by $[v]_{\omega}^{\Psi,E}$ the equivalence class under $\equiv_{\omega}$ of v in Ψ, i.e. the set $\{v' \in T_{\mathrm{lab},E} \mid \Psi \vdash_E v \equiv_{\omega} v'\}$. We define $[v]_{\alpha}^{\Psi,E}$ and $[v]_{\omega\alpha}^{\Psi,E}$ analogously. Similarly, given a set Ψ of labelled schema formulae within L_E and a labelled schema term $v{:}t$ where v is in $T_{\mathrm{lab},E}$, we denote by $[v{:}t]_{g}^{\Psi,E}$ the equivalence class of $v{:}t$ under $=_g$, i.e. the set $\{v'{:}t' \mid \Psi \vdash_E v{:}t =_g v'{:}t'\}$, by D_{Ψ} the set $\{[v{:}t]_{g}^{\Psi,E} \mid v \text{ is in } T_{\mathrm{lab},E} \text{ and } t \text{ is in } T\}$, and finally we denote by W_{Ψ} the set $\{[v]_{\omega}^{\Psi,E} \mid v \text{ is in } T_{\mathrm{lab},E}\}$.

Definition 3.1.1 A set Ψ is E *canonical* iff (i) E is a set of label schema variables, (ii) Ψ is within L_E, and (iii) for every c in C_k and v in $T_{\mathrm{lab},E}$, if for any $j = 1, \ldots, k$, label schema term v' in $[v]^{\Psi,E}_{\omega\alpha}$, and schema formulae φ_j and φ'_j,

$$v'{:}\varphi_j \in \Psi \qquad \text{iff} \qquad v'{:}\varphi'_j \in \Psi$$

then

$$v{:}c(\varphi_1, \ldots, \varphi_k) \in \Psi \qquad \text{iff} \qquad v{:}c(\varphi'_1, \ldots, \varphi'_k) \in \Psi,$$

analogously for q in Q_k, and for o in O_k.

In the following, we will refer interchangeably to a set being E canonical and to a set being canonical over E. Intuitively, a canonical set is a set of labelled schema formulae satisfying the minimum number of conditions for being possible to define a canonical structure over it. Condition (iii) is essential in order for the denotations in the canonical structure of the connectives in C, Q or O, be well defined.

The Henkin construction

Given a lfob deduction system $\mathcal{D}$, a set E of label schema variables, and a deductively closed and E canonical set Ψ, we define the *canonical structure*

$$s_\Psi$$

over Ψ and E on $\mathcal{D}$ as the tuple $\langle U, A, W, \alpha, \omega, D, \mathcal{E}, \mathcal{B}, [.]\rangle$ as follows. We set

$$U = T_{\mathrm{lab},E}$$

The sets W and A, and the functions ω and α are defined by

$$W = W_\Psi \qquad\qquad A = \{[v]^{\Psi,E}_\alpha \mid v \text{ is in } U\}$$

$$\omega(v) = [v]^{\Psi,E}_\omega \qquad\qquad \alpha(v) = [v]^{\Psi,E}_\alpha$$

and

$$D = D_\Psi$$

Observe that $v, v' \in U_w$ iff $v \equiv_\omega v' \in \Psi$ and $v, v' \in U_a$ iff $v \equiv_\alpha v' \in \Psi$, for some assignment a and world w. The schema formula extension $|.|^\phi$ is a map that given a schema formula φ returns a function from U into $\{0, 1\}$ and the schema term extension $|.|^\tau$ is a map that given a schema term t, returns a function from U into D:

$$|\varphi|^\phi(v) = 1 \quad \text{iff} \quad v{:}\varphi \in \Psi$$

$$|t|^\tau(v) = [v{:}t]^{\Psi,E}_g$$

In the sequel, when there is no ambiguity, we denote by $|.|$, either $|.|^\phi$ or $|.|^\tau$ or the tuple $\langle \mathrm{id}_{T_{\mathrm{lab},E}}, |.|^\tau, |.|^\phi\rangle$. As far as extensions of schema formulae are concerned, we will shift freely from the functional notation to the set notation, that is, we will often write $v \in |\varphi|$ instead of $|\varphi|(v) = 1$. The sets $\mathcal{B}$ and $\mathcal{E}$ are defined by:

$$\mathcal{B} = \{|\varphi| \mid \varphi \text{ is a schema formula}\}$$

and

$$\mathcal{E} = \{|t| \mid t \text{ is a schema term}\}$$

We will use, possibly indexed, $|\varphi|$ and $|t|$ to denote elements of $\mathcal{B}$ and of $\mathcal{E}$, respectively, and set

$$|\varphi|_w = |\varphi| \cap U_w \qquad |\varphi|_a = |\varphi| \cap U_a$$

and

$$|\varphi|_{wa} = |\varphi|_w \cap |\varphi|_a.$$

Finally

$$\mathcal{B}_w = \{|\varphi|_w \mid \varphi \text{ is a schema formula}\} \qquad \mathcal{B}_a = \{|\varphi|_a \mid \varphi \text{ is a schema formula}\}$$

and

$$\mathcal{B}_{wa} = \{|\varphi|_{wa} \mid \varphi \text{ is a schema formula}\}.$$

The construction of the Σ structure is accomplished by the following definition of the interpretation map $[.]$. We first define the interpretation of the elements X, F_k and of P_k:

- for x in X, a in A and v in U_a,
$$[x]_a = [v{:}x]_g^{\Psi,E},$$
- for f in F_k, w in W and v in U_w,
$$[f]_w(|t_1|(v), \dots, |t_k|(v)) = |f(t_1, \dots, t_k)|(v),$$
- for p in P_k, w in W and v in U_w,
$$[p]_w(|t_1|(v), \dots, |t_k|(v)) = |p(t_1, \dots, t_k)|(v).$$

Observe that the schema terms considered in the result of $[f]_w$ and $[p]_w$ are the schema terms in the equivalence classes of the arguments that are labelled by a same label schema term. The others combinations of labelled schema terms appearing in the equivalence classes of the arguments that do not share the same label, are not relevant for the result. Finally note that $[f]_w$ and $[p]_w$ are only called in the process of interpreting a schema term or a schema formula for tuples of elements of D with a common label schema term (see Definition 2.2.1).

As far as the interpretation of variables is concerned, we have to show that, for every quantification variable x, every assignment a, and v, v' in U_a, $[v{:}x]_g^{\Psi,E} = [v'{:}x]_g^{\Psi,E}$. This is straightforward since $v \equiv_\alpha v'$ is in Ψ and so $v{:}x =_g v'{:}x$ is in Ψ due to rule $\equiv_{\alpha g}^{x}$ (see Definition 2.3.10) because Ψ is deductively closed. In order to show that $[f]_w$ is well defined suppose there are terms $t'_1, \dots, t'_k$ and v and v' in U_w such that $|t'_i|(v') = |t_i|(v)$, i.e. $[v'{:}t'_i]_g^{\Psi,E} = [v{:}t_i]_g^{\Psi,E}$ for each $i = 1, \dots, k$. Then $v \equiv_\omega v'$ is in Ψ and $v{:}t_i =_g v'{:}t'_i$ is in Ψ for each $i = 1, \dots, k$. So $v{:}f(t_1, \dots, t_k) =_g v'{:}f(t'_1, \dots, t'_k)$ is in Ψ by rule $=_{g_f}$ (see Definition 2.3.10) since Ψ is deductively closed. Then $[v{:}f(t_1, \dots, t_k)]_g^{\Psi,E} = [v'{:}f(t'_1, \dots, t'_k)]_g^{\Psi,E}$, i.e. $|f(t_1, \dots, t_k)|(v) = |f(t'_1, \dots, t'_k)|(v')$ as we wanted to show. To show that $[p]_w$ is well defined, we follow a procedure similar to the one of $[f]_w$ using the rule $=_{g_p}$.

The functions $[c]_{wa}$, $[q_x]_w$ and $[o]_a$ are defined by

- for every c in C_k, w in W, and any assignment a in A,
$$[c]_{wa}(|\varphi_1|_{wa}, \ldots, |\varphi_k|_{wa}) = |c(\varphi_1, \ldots, \varphi_k)|_{wa},$$
- for every q in Q_k, x in X, and any world w in W,
$$[q_x]_w(|\varphi_1|_w, \ldots, |\varphi_k|_w) = |q_x(\varphi_1, \ldots, \varphi_k)|_w,$$
- for every o in O_k,
$$[o](|\varphi_1|, \ldots, |\varphi_k|) = |o(\varphi_1, \ldots, \varphi_k)|.$$

To show that $[c]_{wa}$ is well defined suppose $|\varphi_j|_{wa} = |\varphi'_j|_{wa}$, for each $j = 1, \ldots, k$. Then $v{:}\varphi_j \in \Psi$ iff $v{:}\varphi'_j \in \Psi$, for each $j = 1, \ldots, k$ and v in U_{wa}. So, using the fact that Ψ is canonical, $v{:}c(\varphi_1, \ldots, \varphi_k) \in \Psi$ iff $v{:}c(\varphi'_1, \ldots, \varphi'_k) \in \Psi$, for each v in U_{wa}. Therefore we can conclude that $|c(\varphi_1, \ldots, \varphi_k)|_{wa} = |c(\varphi'_1, \ldots, \varphi'_k)|_{wa}$. In the same way we show that $[q_x]_w$ and $[o]$ are well defined.

Finally the relations $[r]$ and the label functions $[f^l]$ are defined by

- for every r in S_k,
$$[r] = \{\langle v_1, \ldots, v_k\rangle \mid r\, v_1 \ldots v_k \in \Psi\},$$
- and for every f^l in F^l_k,
$$[f^l](v_1, \ldots, v_k) = f^l(v_1, \ldots, v_k).$$

End of Henkin construction

We now show some useful propositions, but before, in order to lighten the presentation, we introduce an abbreviation.

Remark 3.1.2 In the context of a canonical structure s_Ψ over a E canonical set Ψ, where E is a set of label schema variables, and given a schema substitution σ over E' and within L_E, we define the schema assignment $|.| \circ \sigma$ over s_Ψ and a set E' of label schema variables, as the composition of $|.|$, i.e. $\langle \mathrm{id}_{T_{\mathrm{lab},E}}, |.|^\tau, |.|^\phi\rangle$, and $\langle \sigma_l, \sigma_t, \sigma_f\rangle$. So $(|.| \circ \sigma)_l(\nu) = (\mathrm{id}_{T_{\mathrm{lab},E}} \circ \sigma_l)(\nu) = \nu\sigma_l$, $(|.| \circ \sigma)_t(\theta) = (|.|^\tau \circ \sigma_t)(\theta) = |\theta\sigma_t|$, and $(|.| \circ \sigma)_f(\xi) = (|.|^\phi \circ \sigma_f)(\xi) = |\xi\sigma_f|$.

Proposition 3.1.3 Given a deductively closed and canonical set Ψ over a set E of label schema variables,

- $v\sigma = [\![v]\!]^{s_\Psi}_{|.|\circ\sigma}$
- $|t\sigma| = [\![t]\!]^{s_\Psi}_{|.|\circ\sigma}$
- $|\varphi\sigma| = [\![\varphi]\!]^{s_\Psi}_{|.|\circ\sigma}$
- $\eta\sigma \in \Psi \quad$ iff $\quad s_\Psi, |.| \circ \sigma \Vdash \eta$.

for every η in $L_{E'}$, where E' is a set of label schema variables, and every substitution σ over E' and within L_E.

Proof

1. $v\sigma = \llbracket v \rrbracket^{s_\Psi}_{|.|\circ\sigma}$, for every label schema term v in $T_{\mathrm{lab},E'}$. The proof follows by induction on the structure of v:

If v *is in* $\Xi_l \cup E'$ then $\llbracket v \rrbracket^{s_\Psi}_{|.|\circ\sigma} = (|.| \circ \sigma)(v) = v\sigma$.

If v *is in* F^l_0 then $\llbracket v \rrbracket^{s_\Psi}_{|.|\circ\sigma} = [v] = v\sigma$.

If v *is* $f^l(v_1, \ldots, v_k)$ then $\llbracket v \rrbracket^{s_\Psi}_{|.|\circ\sigma} = [f^l](\llbracket v_1 \rrbracket^{s_\Psi}_{|.|\circ\sigma}, \ldots, \llbracket v_k \rrbracket^{s_\Psi}_{|.|\circ\sigma}) = [f^l](v_1\sigma, \ldots, v_k\sigma) = v\sigma$.

2. $|t\sigma| = \llbracket t \rrbracket^{s_\Psi}_{|.|\circ\sigma}$, for every schema term t. The proof follows by induction on the structure of t:

If t *is in* X then $\llbracket t \rrbracket^{s_\Psi}_{|.|\circ\sigma}(v) = [t]_{\alpha(v)} = [v{:}t]^{\Psi,E}_g = |t\sigma|(v)$.

If t *is in* F_0 then $\llbracket f \rrbracket^{s_\Psi}_{|.|\circ\sigma}(v) = [f]_{\omega(v)} = |f|(v) = |f\sigma|(v)$.

If t *is in* Ξ_t then $\llbracket t \rrbracket^{s_\Psi}_{|.|\circ\sigma} = (|.| \circ \sigma)_t(t) = |t\sigma|$.

If t *is* $f(t_1, \ldots, t_k)$ then $\llbracket t \rrbracket^{s_\Psi}_{|.|\circ\sigma}(v) = [f]_{\omega(v)}(\llbracket t_1 \rrbracket^{s_\Psi}_{|.|\circ\sigma}(v), \ldots, \llbracket t_k \rrbracket^{s_\Psi}_{|.|\circ\sigma}(v)) = [f]_{\omega(v)}(|t_1\sigma|(v), \ldots, |t_k\sigma|(v)) = |t\sigma|$.

3. $|\varphi\sigma| = \llbracket \varphi \rrbracket^{s_\Psi}_{|.|\circ\sigma}$, for every schema formula φ. The proof follows by induction on the structure of a schema formula φ:

If φ *is in* Ξ_f then $\llbracket \varphi \rrbracket^{s_\Psi}_{|.|\circ\sigma} = (|.| \circ \sigma)_f(\varphi) = |\varphi\sigma|$.

If φ *is in* C_0 then $\llbracket \varphi \rrbracket^{s_\Psi}_{|.|\circ\sigma}(v) = [\varphi\sigma]_{\omega(v)\alpha(v)}(v) = |\varphi\sigma|_{\omega(v)\alpha(v)}(v) = |\varphi\sigma|(v)$.

If φ *is* $p(t_1, \ldots, t_k)$ then $\llbracket \varphi \rrbracket^{s_\Psi}_{|.|\circ\sigma}(v) = [p]_{\omega(v)}(\llbracket t_1 \rrbracket^{s_\Psi}_{|.|\circ\sigma}(v), \ldots, \llbracket t_k \rrbracket^{s_\Psi}_{|.|\circ\sigma}(v))$ which is by fact 2. above, $[p]_{\omega(v)}(|t_1\sigma|(v), \ldots, |t_k\sigma|(v)) = |p(t_1, \ldots, t_k)\sigma|(v)$.

If φ *is* $c(\varphi_1, \ldots, \varphi_k)$ then we have $\llbracket \varphi \rrbracket^{s_\Psi}_{|.|\circ\sigma}(v) = [c]_{\omega(v)\alpha(v)}(\llbracket \varphi_1 \rrbracket^{s_\Psi}_{|.|\circ\sigma} \cap U_{\omega(v)\alpha(v)}, \ldots, \llbracket \varphi_k \rrbracket^{s_\Psi}_{|.|\circ\sigma} \cap U_{\omega(v)\alpha(v)})(v)$ which is, by induction hypothesis, $[c]_{\omega(v)\alpha(v)}(|\varphi_1\sigma| \cap U_{\omega(v)\alpha(v)}, \ldots, |\varphi_k\sigma| \cap U_{\omega(v)\alpha(v)})(v) = |\varphi\sigma|_{\omega(v)\alpha(v)}(v) = |\varphi\sigma|(v)$.

If φ *is either* $q_x(\varphi_1, \ldots, \varphi_k)$ *or* $o(\varphi_1, \ldots, \varphi_k)$ with q in Q_k and o in O_k then this case is similar to the preceding one.

4. $\eta\sigma \in \Psi$ iff $s_\Psi, |.| \circ \sigma \Vdash \eta$. The proof follows by case analysis on η:

If η *is* $r\, v_1, \ldots, v_k$ then $s_\Psi, |.| \circ \sigma \Vdash \eta$ iff $(\llbracket v_1 \rrbracket^{s_\Psi}_{|.|\circ\sigma}, \ldots, \llbracket v_k \rrbracket^{s_\Psi}_{|.|\circ\sigma}) \in [r]$ iff $(v_1\sigma, \ldots, v_k\sigma) \in [r]$ iff $\eta\sigma \in \Psi$.

If η *is* $v{:}t =_g v'{:}t'$ then $s_\Psi, |.| \circ \sigma \Vdash \eta$ iff $\llbracket t \rrbracket^{s_\Psi}_{|.|\circ\sigma}(\llbracket v \rrbracket^{s_\Psi}_{|.|\circ\sigma}) = \llbracket t' \rrbracket^{s_\Psi}_{|.|\circ\sigma}(\llbracket v' \rrbracket^{s_\Psi}_{|.|\circ\sigma})$ iff $|t\sigma|(v\sigma) = |t'\sigma|(v'\sigma)$ iff $[(v{:}t)\sigma]^{\Psi,E}_g = [(v'{:}t')\sigma]^{\Psi,E}_g$ iff $(v{:}t)\sigma =_g (v'{:}t')\sigma \in \Psi$ iff $\eta\sigma \in \Psi$.

If η *is* $v{:}\varphi$ then $s_\Psi, |.| \circ \sigma \Vdash \eta$ iff $\llbracket \varphi \rrbracket^{s_\Psi}_{|.|\circ\sigma}(\llbracket v \rrbracket^{s_\Psi}_{|.|\circ\sigma}) = 1$ iff $|\varphi\sigma|(v\sigma) = 1$ iff $v\sigma{:}\varphi\sigma \in \Psi$ iff $\eta\sigma \in \Psi$. *QED*

Proposition 3.1.4 Given a canonical structure s_Ψ over a deductively closed and E canonical set Ψ we have that (i) for any schema assignments $\alpha^{\cdot}$ and α over E' within s_Ψ with $\alpha^{\cdot}$ Υ co-equivalent to α, there are schema substitutions σ and $\sigma^{\cdot}$ over E' within L_E with $\sigma^{\cdot}$ Υ co-equivalent to σ,

$$\alpha^{\cdot} \quad \text{equal to} \quad |.| \circ \sigma^{\cdot} \qquad \text{and} \qquad \alpha \quad \text{equal to} \quad |.| \circ \sigma,$$

and (ii) vice-versa, i.e., for any schema substitutions σ and $\sigma^{\cdot}$ over E' within L_E with $\sigma^{\cdot}$

Υ co-equivalent to σ, the maps $|.| \circ \sigma$ and $|.| \circ \sigma^{\cdot}$ are schema assignments over E' within s_Ψ, and $|.| \circ \sigma^{\cdot}$ is Υ co-equivalent to $|.| \circ \sigma$.

Proof To show (i) let $\alpha^{\cdot}$ and α be schema assignments over E' within L_E, such that $\alpha^{\cdot}$ is Υ co-equivalent to α. Consider the schema substitution σ with $|\xi\sigma_f| = \xi\alpha_f$, $|\theta\sigma_t| = \theta\alpha_t$ and $\sigma_l = \alpha_l$, and the schema substitution $\sigma^{\cdot}$ Υ co-equivalent to σ with $\nu\sigma^{\cdot}_l = \nu\alpha^{\cdot}_l$ for ν in Υ. Then,

- α is $|.| \circ \sigma$. Recall the definition of $|.| \circ \sigma$ in Remark 3.1.2. This happens since $\xi\alpha_f = |\xi\sigma_f| = (|.| \circ \sigma)_f(\xi)$, $\theta\alpha_t = |\theta\sigma_t| = (|.| \circ \sigma)_t(\theta)$, and $\nu\alpha_l = \nu\sigma_l = (|.| \circ \sigma)_l(\nu)$.

- $\alpha^{\cdot}$ is $|.| \circ \sigma^{\cdot}$. Recall the definition of $|.| \circ \sigma^{\cdot}$ in Remark 3.1.2. This happens since $\xi\alpha^{\cdot}_f = \xi\alpha_f = |\xi\sigma_f| = |\xi\sigma^{\cdot}_f| = (|.| \circ \sigma^{\cdot})_f(\xi)$, $\theta\alpha^{\cdot}_t = \theta\alpha_t = |\theta\sigma_t| = |\theta\sigma^{\cdot}_t| = (|.| \circ \sigma^{\cdot})_t(\theta)$, and, if ν is in Υ then $\nu\alpha^{\cdot}_l = \nu\sigma^{\cdot}_l = (|.| \circ \sigma^{\cdot})_l(\nu)$, otherwise $\nu\alpha^{\cdot}_l = \nu\alpha_l = \nu\sigma_l = \nu\sigma^{\cdot}_l = (|.| \circ \sigma^{\cdot})_l(\nu)$, as we wanted to show.

To show (ii) let σ and $\sigma^{\cdot}$ be substitutions over E' within L_E with $\sigma^{\cdot}$ Υ co-equivalent to σ. Then,

- $|.| \circ \sigma$ and $|.| \circ \sigma^{\cdot}$ are schema assignments over E' within s_Ψ. See Remark 3.1.2.

- $|.|\circ\sigma^{\cdot}$ is Υ co-equivalent to $|.|\circ\sigma$. To see this note that $(|.|\circ\sigma)_f(\xi) = |\xi\sigma_f| = |\xi\sigma^{\cdot}_f| = (|.|\circ\sigma^{\cdot})_f(\xi)$, $(|.|\circ\sigma)_t(\theta) = |\theta\sigma_t| = |\theta\sigma^{\cdot}_t| = (|.|\circ\sigma^{\cdot})_t(\theta)$, and $(|.|\circ\sigma)_l(\nu) = \nu\sigma_l = \nu\sigma^{\cdot}_l = (|.|\circ\sigma^{\cdot})_l(\nu)$ for ν not in Υ, as we wanted to show. *QED*

3.2 Appropriate sets

In this section we identify the additional sufficient condition for canonical sets such that the canonical structure over them satisfies all the rules of the underlying lfob logic system. We call the canonical sets satisfying that condition *appropriate sets.* We start the section by defining what is an appropriate set. Then we prove that a canonical structure over them satisfies all the rules of the underlying lfob logic system and we show that an appropriate set is, as was intuitively expected, deductively closed. Finally ending the section, we illustrate, in the context of the lfob deduction system $\mathcal{D}_{\text{FOMdD}}$ for first-order modal logic with decreasing domains, introduced in Example 2.3.11, what is the condition for rule $\Box_I$ that an appropriate set Ψ should satisfy.

Definition 3.2.1 A canonical set Ψ over E is E *appropriate* iff if for any $j = 1, \ldots, k$ and any ρ_j over E within L_E $\Upsilon_j\sigma$ co-equivalent to id

$$\Psi_j\sigma\rho_j \subseteq \Psi \quad \text{implies} \quad \eta_j\sigma\rho_j \in \Psi$$

then

$$\eta\sigma \in \Psi,$$

for every

- rule $\langle\{\langle\vartheta_1, \Psi_1, \eta_1\rangle, \ldots, \langle\vartheta_k, \Psi_k, \eta_k\rangle\}, \eta, P_f, P_d\rangle$ in R,
- schema substitution σ within L_E such that
  - $(\pi_\Sigma\sigma_{\text{nl}})$ is 1 for each π in P_f,

- $\pi_{\Sigma;E}(\sigma)$ is 1 for each π in P_d,

where Υ_j is

- Υ, if $\mathrm{fresh}(\Upsilon, \langle\vartheta_j, \Psi_j, \eta_j\rangle)$ is in P_d, or
- $\emptyset$, otherwise.

In the following we will refer interchangeably to a set being appropriate over E and to a set being E appropriate. The importance associated to appropriate sets stems from the fact that, as we show in Theorem 3.2.2, the canonical structure over an appropriate set satisfies all the rules of the underlying lfob deduction system. Note that in the context of a specific lfob deduction system the appropriateness conditions may be equivalent to more familiar conditions. Those conditions may involve for example maximal consistent sets.

Theorem 3.2.2 Any canonical structure over an appropriate set satisfy all the rules of the lfob deduction system over which it is based.

Proof Let $\langle\Sigma, R\rangle$ be a lfob deduction system and s_Ψ the canonical structure over an E appropriate set Ψ. Let E' be a set of label schema variables, α' a schema assignment over E' within s_Ψ, $\langle\{\langle\vartheta_1, \Psi_1, \eta_1\rangle, \ldots, \langle\vartheta_k, \Psi_k, \eta_k\rangle\}, \eta, P_f, P_d\rangle$ a rule in R, and σ a schema substitution within $L_{E'}$, such that $(\pi_\Sigma\sigma_{\mathrm{nl}})$ is 1 for each π in P_f, and $\pi_{\Sigma;E'}(\sigma)$ is 1 for each π in P_d.

Let $\ddot\sigma'$ be a schema substitution over E' within L_E with $|.|\circ\ddot\sigma'$ equal to α', which exists by Proposition 3.1.4, and let, for each $j = 1, \ldots, k$, Υ_j be Υ whenever $\mathrm{fresh}(\Upsilon, \langle\vartheta_j, \Psi_j, \eta_j\rangle)$ is in P_d and be $\emptyset$ otherwise.

Suppose, for each $j = 1, \ldots, k$, that we have $s_\Psi, \alpha'_j \Vdash \eta_j\sigma$ whenever $s_\Psi, \alpha'_j \Vdash \Psi_j\sigma$, for any α'_j over E' within s_Ψ, $\Upsilon_j\sigma$ co-equivalent to α'.

Then, by Proposition 3.1.4, this is equivalent to, for each $j = 1, \ldots, k$, if $s_\Psi, |.|\circ\ddot\sigma'_j \Vdash \Psi_j\sigma$ then $s_\Psi, |.|\circ\ddot\sigma'_j \Vdash \eta_j\sigma$, for any $\ddot\sigma'_j$ over E' within L_E, $\Upsilon_j\sigma$ co-equivalent to $\ddot\sigma'$.

But this is equivalent, by Proposition 3.1.3, and for each $j = 1, \ldots, k$, to $\eta_j\sigma\ddot\sigma'_j \in \Psi$ whenever $\Psi_j\sigma\ddot\sigma'_j \subseteq \Psi$, for any $\ddot\sigma'_j$ over E' within L_E, $\Upsilon_j\sigma$ co-equivalent to $\ddot\sigma'$ over E' within L_E.

So, by Lemma 3.2.3, there exists a schema substitution $\ddot\sigma''$ within L_E, with

1. $(\pi_\Sigma\ddot\sigma''_{\mathrm{nl}}) = 1$ for each π in P_f;
2. $\pi_{\Sigma;E}(\ddot\sigma'') = 1$ for each π in P_d;
3. $\eta\ddot\sigma''$ is $\eta\sigma\ddot\sigma'$;
4. for each $j = 1, \ldots, k$, if $\Psi_j\ddot\sigma''\rho_j \subseteq \Psi$ then $\eta_j\ddot\sigma''\rho_j \in \Psi$, for any schema substitution ρ_j over E within L_E, $\Upsilon_j\ddot\sigma''$ co-equivalent to id over E within L_E.

Hence, by fact 4 and since Ψ is an appropriate set, see Definition 3.2.1, we conclude that $\eta\ddot\sigma''$ is in Ψ, i.e., that $\eta\sigma\ddot\sigma'$ is in Ψ, by fact 3.. So, by Proposition 3.1.3 we conclude $s_\Psi, |.|\circ\ddot\sigma' \Vdash \eta\sigma$, i.e., $s_\Psi, \alpha' \Vdash \eta\sigma$, as we would like to show. *QED*

We now show a lemma needed during the proof of Theorem 3.2.2.

Lemma 3.2.3 Given an E appropriate set Ψ, a schema substitution σ within $L_{E'}$, a rule $\langle\{\langle\vartheta_1, \Psi_1, \eta_1\rangle, \ldots, \langle\vartheta_k,\ \Psi_k, \eta_k\rangle\}, \eta, P_f, P_d\rangle$, such that $(\pi_\Sigma\sigma_{\mathrm{nl}})$ is 1 for each π in P_f, and $\pi_{\Sigma;E'}(\sigma)$ is 1 for each π in P_d, and a substitution $\ddot{\sigma}'$ over E' within L_E, there is $\ddot{\sigma}''$ within L_E, with

1. $(\pi_\Sigma\ddot{\sigma}''_{\mathrm{nl}}) = 1$ for each π in P_f,
2. $\pi_{\Sigma;E}(\ddot{\sigma}'') = 1$ for each π in P_d,
3. $\eta\ddot{\sigma}''$ is $\eta\sigma\ddot{\sigma}'$,
4. for each $j = 1, \ldots, k$, $\Psi_j\ddot{\sigma}''\sigma_j \subseteq \Psi$ implies $\eta_j\ddot{\sigma}''\sigma_j \in \Psi$, for any schema substitution σ_j over E within L_E, $\Upsilon_j\ddot{\sigma}''$ co-equivalent to id over E within L_E,

whenever for each $j = 1, \ldots, k$ we have that $\Psi_j\sigma\ddot{\sigma}'_j \subseteq \Psi$ implies $\eta_j\sigma\ddot{\sigma}'_j \in \Psi$, for any $\ddot{\sigma}'_j$ over E' within L_E, $\Upsilon_j\sigma$ co-equivalent to $\ddot{\sigma}'$.

Proof We start by briefly sketching the proof. We begin by defining, for each $j = 1, \ldots, k$, the schema substitutions $\ddot{\sigma}''_j$, $\Upsilon_j\sigma$ co-equivalent to $\ddot{\sigma}'$, satisfying the j fresh proviso, if it exists. Relying on the $\ddot{\sigma}''_j$'s we define $\ddot{\sigma}''$, and after showing two useful facts, we prove that facts 1., …, 4. hold. In order to prove item 4. we have to rely on the assumption. So, for each $j = 1, \ldots, k$ we have to consider a schema substitution $\ddot{\sigma}'_j$ in the conditions of the assumption and related in such a way with $\ddot{\sigma}''$ that our result, for each $j = 1, \ldots, k$, follows by the assumption applied to $\ddot{\sigma}'_j$. We now start the proof.

Suppose for each $j = 1, \ldots, k$ that we have that $\Psi_j\sigma\ddot{\sigma}'_j \subseteq \Psi$ implies $\eta_j\sigma\ddot{\sigma}'_j \in \Psi$, for any $\ddot{\sigma}'_j$ over E' within L_E, $\Upsilon_j\sigma$ co-equivalent to $\ddot{\sigma}'$.

Consider, for each $j = 1, \ldots, k$, a schema substitution $\ddot{\sigma}''_j$ over E' within L_E, $\Upsilon_j\sigma$ co-equivalent to $\ddot{\sigma}'$, such that, $\nu\sigma\ddot{\sigma}''_j$ is a label schema variable not in $(\mathrm{lsv}(\Psi_j \cup \{\eta_j\}) \setminus \{\nu\})\sigma\ddot{\sigma}''_j$ for each ν in Υ_j, and $\Upsilon_j\sigma\ddot{\sigma}''_j \cap \mathrm{lsv}(\vartheta_j\sigma \setminus \Psi_j\sigma)\ddot{\sigma}' = \emptyset$. It is possible to consider a schema substitution $\ddot{\sigma}''_j$ satisfying these conditions because $\Upsilon_j\sigma$ is a set of label schema variables, the sets $\vartheta_j\sigma \setminus \Psi_j\sigma$ and Ψ_j are finite, and $\nu\sigma$ is not in $(\mathrm{lsv}(\Psi_j \cup \{\eta_j\}) \setminus \{\nu\})\sigma$ for each ν in Υ_j, since $\pi_{\Sigma;E'}(\sigma) = 1$ for each π in P_d.

Consider the schema substitution $\ddot{\sigma}''$ within L_E, $\cup_{j=1,\ldots,k}\Upsilon_j$ co-equivalent to $\sigma\ddot{\sigma}'$ within L_E such that $\vartheta_j\ddot{\sigma}''$ is $\vartheta_j\sigma\ddot{\sigma}''_j$ and $\nu\ddot{\sigma}''$ is $\nu\sigma\ddot{\sigma}''_j$ for any ν in Υ_j and $j = 1, \ldots, k$.

Note that,

(a) $\psi\ddot{\sigma}''$ is $\psi\sigma\ddot{\sigma}''_j$, for each ψ in $\Psi_j \cup \{\eta_j\}$ and j in $\{1, \ldots, k\}$. This is shown by case analysis:

ψ is $v{:}\gamma$. Then $\psi\ddot{\sigma}''$ is $v\ddot{\sigma}''{:}\gamma\ddot{\sigma}''$ which is $v\ddot{\sigma}''{:}\gamma\sigma\ddot{\sigma}''_j$ since $\ddot{\sigma}''$ is co-equivalent to $\sigma\ddot{\sigma}''_j$. Now we show that $v\ddot{\sigma}''$ is $v\sigma\ddot{\sigma}''_j$. The proof follows by induction on the structure of the label schema term v:

- v is in Ξ_l. Consider two cases for v, (i) if v is in Υ_j then, by definition of $\ddot{\sigma}''$, $v\ddot{\sigma}''$ is $v\sigma\ddot{\sigma}''_j$ and we are done, (ii) if v is not in Υ_j then v is not in $\cup_{i=1,\ldots,k}\Upsilon_i$ because, v is in $\mathrm{lsv}(\Psi_j \cup \{\eta_j\})$, and by Definition 2.3.10, for each $l = 1, \ldots, k$, $\Upsilon_l \cap \cup_{i=1,\ldots,k \text{ and } i \neq l}\mathrm{lsv}(\Psi_i \cup \{\eta_i\}) = \emptyset$. So, by definition of $\ddot{\sigma}''$, $v\ddot{\sigma}''$ is $v\sigma\ddot{\sigma}'$, which is $v\sigma\ddot{\sigma}''_j$ since $\ddot{\sigma}''_j$ is $\Upsilon_j\sigma$ co-equivalent to $\ddot{\sigma}'$, and $v\sigma$ is not in $\Upsilon_j\sigma$ because, for each ν in Υ_j, $\nu\sigma$ is not in $(\mathrm{lb}(\Psi_j \cup \{\eta_j\}) \setminus \{\nu\})\sigma$, and $v\sigma$ is in it.

- v is in F_0^l. Then $v\ddot{\sigma}''$ is v which is $v\sigma\ddot{\sigma}''_j$.

- v is in $f(v_1,\dots,v_k)$. Then $v\ddot{\sigma}''$ is $f(v_1\ddot{\sigma}'',\dots,v_k\ddot{\sigma}'')$, which by induction hypothesis is $f(v_1\sigma\ddot{\sigma}''_j,\dots,v_k\sigma\ddot{\sigma}''_j)$ i.e. $v\sigma\ddot{\sigma}''_j$.
- For the other possible cases of ψ the proof follows similarly.

(b) $\vartheta_j\ddot{\sigma}''\setminus\Psi_j\ddot{\sigma}''$ is contained in $(\vartheta_j\sigma\setminus\Psi_j\sigma)\ddot{\sigma}'$ for each $j=1,\dots,k$. This happens since for each $j=1,\dots,k$,

(i) $\vartheta_j\ddot{\sigma}''\setminus\Psi_j\ddot{\sigma}''$ is $\vartheta_j\sigma\ddot{\sigma}''_j\setminus\Psi_j\sigma\ddot{\sigma}''_j$, because $\vartheta_j\ddot{\sigma}''$ is $\vartheta_j\sigma\ddot{\sigma}''_j$ by definition of $\ddot{\sigma}''$, and $\Psi_j\ddot{\sigma}''$ is $\Psi_j\sigma\ddot{\sigma}''_j$, as showed above in item (a),

(ii) $\vartheta_j\sigma\ddot{\sigma}''_j\setminus\Psi_j\sigma\ddot{\sigma}''_j$ is contained in $(\vartheta_j\sigma\setminus\Psi_j\sigma)\ddot{\sigma}''_j$ because if ψ is in $\vartheta_j\sigma$ and $\psi\ddot{\sigma}''_j$ is not in $\Psi_j\sigma\ddot{\sigma}''_j$ then ψ is not in $\Psi_j\sigma$, as desired,

(iii) $(\vartheta_j\sigma\setminus\Psi_j\sigma)\ddot{\sigma}''_j$ is $(\vartheta_j\sigma\setminus\Psi_j\sigma)\ddot{\sigma}'$, by definition of $\ddot{\sigma}''_j$, since $\Upsilon_j\sigma\cap(\vartheta_j\sigma\setminus\Psi_j\sigma)=\emptyset$ because $\pi_\Sigma(\sigma)=1$, for each π in P_d;

So now we can show 1.,…,4.:

1. $(\pi_\Sigma\ddot{\sigma}''_{\mathrm{nl}})=1$ for each π in P_f. To show this, let π be in P_f. Note that $\ddot{\sigma}''_{\mathrm{nl}}$ is $(\sigma\ddot{\sigma}')_{\mathrm{nl}}$ which is $\sigma_{\mathrm{nl}}\ddot{\sigma}'_{\mathrm{nl}}$. So by Remark 2.3.6, we have that $(\pi_\Sigma\ddot{\sigma}''_{\mathrm{nl}})$ is the proviso 1 since $(\pi_\Sigma\sigma_{\mathrm{nl}})=1$.

2. $\pi_{\Sigma;E}(\ddot{\sigma}'')=1$ for each π in P_d. Suppose π is fresh$(\Upsilon_j,\langle\vartheta_j,\Psi_j,\eta_j\rangle)$. Then,

i) $\nu\ddot{\sigma}''$ is a label schema variable for each ν in Υ_j. Let ν be in Υ_j. Then $\nu\ddot{\sigma}''$, by definition of $\ddot{\sigma}''$, is $\nu\sigma\ddot{\sigma}''_j$, which by definition of $\ddot{\sigma}''_j$ is a label schema variable;

ii) $\nu\ddot{\sigma}''$ is not in $(\mathrm{lsv}(\Psi_j\cup\{\eta_j\})\setminus\{\nu\})\ddot{\sigma}''$ for each ν in Υ_j. Let ν be in Υ_j. Then $\nu\ddot{\sigma}''$, by definition of $\ddot{\sigma}''$, is $\nu\sigma\ddot{\sigma}''_j$, which, taking into account the definition of $\ddot{\sigma}''_j$, is not in $(\mathrm{lsv}(\Psi_j\cup\{\eta_j\})\setminus\{\nu\})\sigma\ddot{\sigma}''_j$. So, the result follows because $(\mathrm{lsv}(\Psi_j\cup\{\eta_j\})\setminus\{\nu\})\ddot{\sigma}''$ is contained in $(\mathrm{lsv}(\Psi_j\cup\{\eta_j\})\setminus\{\nu\})\sigma\ddot{\sigma}''_j$. To see this let ν'' be in $(\mathrm{lsv}(\Psi_j\cup\{\eta_j\})\setminus\{\nu\})\ddot{\sigma}''$. Then, there is a labelled schema formula ψ in $\Psi_j\cup\{\eta_j\}$ and a label schema variable ν' in ψ, distinct of ν, with $\nu'\ddot{\sigma}''=\nu''$. Since, as shown above, $\psi\ddot{\sigma}''$ is $\psi\sigma\ddot{\sigma}''_j$ then $\nu''(=\nu'\ddot{\sigma}'')=\nu'\sigma\ddot{\sigma}''_j$, and so ν'' is in $(\mathrm{lsv}(\Psi_j\cup\{\eta_j\})\setminus\{\nu\})\sigma\ddot{\sigma}''_j$;

iii) $\Upsilon_j\ddot{\sigma}''\cap\mathrm{lsv}(\vartheta_j\ddot{\sigma}''\setminus\Psi_j\ddot{\sigma}'')=\emptyset$. This happens because (1) $\Upsilon_j\ddot{\sigma}''$ is $\Upsilon_j\sigma\ddot{\sigma}''_j$, by definition of $\ddot{\sigma}''$, (2) $\vartheta_j\ddot{\sigma}''\setminus\Psi_j\ddot{\sigma}''$ is contained in $(\vartheta_j\sigma\setminus\Psi_j\sigma)\ddot{\sigma}'$, as shown in item (b) above, and (3) $\Upsilon_j\sigma\ddot{\sigma}''_j\cap(\vartheta_j\sigma\setminus\Psi_j\sigma)\ddot{\sigma}'=\emptyset$, by definition of $\ddot{\sigma}''_j$.

3. $\eta\ddot{\sigma}''$ is $\eta\sigma\ddot{\sigma}'$. This happens by definition of $\ddot{\sigma}''$ since $\mathrm{lsv}(\eta)\cap\cup_{j=1,\dots,k}\Upsilon_j=\emptyset$ by Definition 2.3.9;

4. for each $j=1,\dots,k$, $\Psi_j\ddot{\sigma}''\sigma_j\subseteq\Psi$ implies $\eta_j\ddot{\sigma}''\sigma_j\in\Psi$, for any σ_j over E within L_E, $\Upsilon_j\ddot{\sigma}''$ co-equivalent to id over E within L_E.

Let j be in $\{1,\dots,k\}$ and σ_j be a schema substitution over E within L_E, $\Upsilon_j\ddot{\sigma}''$ co-equivalent to id over E within L_E. Suppose $\Psi_j\ddot{\sigma}''\sigma_j$ is contained in Ψ. Then, by fact (a) above, $\Psi_j\sigma\ddot{\sigma}''_j\sigma_j$ is contained in Ψ.

Consider the substitution $\ddot{\sigma}'_j$ over E' and within L_E, defined as $\nu\ddot{\sigma}'_j=\nu\ddot{\sigma}''_j\sigma_j$ for each ν in $\mathrm{lsv}(\Psi_j\cup\{\eta_j\})\sigma$, and $\nu\ddot{\sigma}'_j=\nu\ddot{\sigma}'$ otherwise, and $\ddot{\sigma}'_{jo}=(\ddot{\sigma}''_j\sigma_j)_o$, for o in $\{t,f\}$. Then,

(i) $\ddot{\sigma}'_j$ is $\Upsilon_j\sigma$ co-equivalent to $\ddot{\sigma}'$. Let ν be a label schema variable not in $\Upsilon_j\sigma$. Note that $\nu\ddot{\sigma}''_j$ is $\nu\ddot{\sigma}'$, by definition of $\ddot{\sigma}''_j$. So, if ν is not in $\mathrm{lsv}(\Psi_j\cup\{\eta_j\})\sigma$ then $\nu\ddot{\sigma}'_j$ is $\nu\ddot{\sigma}'$, by definition of $\ddot{\sigma}'_j$, and we are done. Otherwise, $\nu\ddot{\sigma}'_j$ is $\nu\ddot{\sigma}''_j\sigma_j$ and there is a label schema variable ν_0 in $\mathrm{lsv}(\Psi_j \cup \{\eta_j\})$ with $\nu_0\sigma = \nu$ and ν_0 is not in Υ_j. Then, $\nu_0\ddot{\sigma}''$ is not in $\Upsilon_j\ddot{\sigma}''$, since $\nu^{\cdot}\sigma''$ is not in $(\mathrm{lsv}(\Psi_j \cup \{\eta_j\}) \setminus \{\nu^{\cdot}\})\ddot{\sigma}'' = \emptyset$, for each $\nu^{\cdot}$ in Υ_j, because $\pi_{\Sigma;E}(\ddot{\sigma}'')$ is 1 for each π in P_d by item 2. above, and ν_0 is in $(\mathrm{lsv}(\Psi_j \cup \{\eta_j\}) \setminus \Upsilon_j)$. So $\nu_0\ddot{\sigma}''\sigma_j$ is $\nu_0\ddot{\sigma}''$, by definition of σ_j. Note that $\nu_0\sigma\ddot{\sigma}''_j$ is $\nu_0\ddot{\sigma}''$ by fact (a). Therefore $\nu\ddot{\sigma}'_j = \nu\ddot{\sigma}''_j\sigma_j = \nu_0\sigma\ddot{\sigma}''_j\sigma_j = \nu_0\ddot{\sigma}''\sigma_j = \nu_0\ddot{\sigma}'' = \nu_0\sigma\ddot{\sigma}''_j = \nu\ddot{\sigma}''_j = \nu\ddot{\sigma}'$.

(ii) $\Psi_j\sigma\ddot{\sigma}'_j$ is $\Psi_j\sigma\ddot{\sigma}''_j\sigma_j$ and $\eta_j\sigma\ddot{\sigma}'_j$ is $\eta_j\sigma\ddot{\sigma}''_j\sigma_j$. By definition of $\ddot{\sigma}'_j$.

So, $\Psi_j\sigma\ddot{\sigma}'_j$ is contained in Ψ. Therefore $\eta_j\sigma\ddot{\sigma}'_j$ is in Ψ using the assumption, and so $\eta_j\sigma\ddot{\sigma}''_j\sigma_j$, i.e. $\eta_j\ddot{\sigma}''\sigma_j$, by fact (a) above, is in Ψ, as we wanted to show. *QED*

A canonical structure is defined over deductively closed and canonical sets. In the next proposition we show that an appropriate set, besides being by definition canonical, is also deductively closed.

Proposition 3.2.4 Any appropriate set is deductively closed.

Proof We show that, for any set E of label schema variables, and for any appropriate set of labelled schema formulae Ψ within L_E and labelled schema formula η in L_E, by induction on the length of a sober deduction sequence for $\Psi \vdash_E \eta$ that η is in Ψ.

Base of induction. Let E be a set of label schema variables, Ψ a set of labelled schema formulae within L_E, and η a labelled schema formula in L_E, and suppose that $\langle \eta, \Psi_1, 1\rangle$ is a sober deduction sequence for $\Psi \vdash_E \eta$. Consider two cases:

- *either* Ψ_1 is $\{\eta\}$ and so, η is in Ψ by definition of deduction, see Definition 2.3.12;

- *or* there are rule $\langle \emptyset, \eta', P'_f, P'_d\rangle$, named r, and Σ schema substitution σ within L_E such that $\eta'\sigma$ is η, $\Psi_1 = \emptyset$ and $(\pi'_\Sigma\sigma_{\mathrm{nl}})$ is the Σ proviso 1 for each π' in P'_f. Then, $\eta'\sigma$, i.e., η is in Ψ, by definition of appropriate set, Definition 3.2.1, as we want to show.

Assume as induction hypothesis that, for any set E of label schema variables, any appropriate set Ψ of labelled schema formulae and labelled schema formula η, if ϖ is a sober deduction sequence with length less than or equal to n for $\Psi \vdash_E \eta$ then η is in Ψ. Let E be a set of label schema variables, Ψ an appropriate set with labelled schema formulae contained in L_E, η a labelled schema formula in L_E, and suppose there is a sober deduction sequence $\langle \eta_1, \Psi_1, 1\rangle \ldots \langle \eta, \Psi_{n+1}, 1\rangle$, named ϖ, with length $n+1$, for $\Psi \vdash_E \eta$. Then, there are Σ schema substitution σ within L_E, rule $\langle\{\langle\vartheta'_1, \Psi'_1, \eta'_1\rangle, \ldots, \langle\vartheta'_k, \Psi'_k, \eta'_k\rangle\}, \eta', P'_f, P'_d\rangle$, named r, and $i_1, \ldots, i_k$ in $\{1, \ldots, n\}$, with $\eta_{i_j} = \eta'_j\sigma$, $\Psi_{i_j} = \vartheta'_j\sigma$ and $\pi_{i_j} = 1$ for each $j = 1, \ldots, k$, $\Psi_{n+1} = \Psi_{i_1} \setminus \Psi'_1\sigma \cup \ldots \cup \Psi_{i_k} \setminus \Psi'_k\sigma$, $(\pi'_\Sigma\sigma_{\mathrm{nl}}) = 1$, for each π' in P'_f, and $\pi'_{\Sigma;E}(\sigma) = 1$, for each π' in P'_d, and $\eta = \eta'\sigma$.

Denote, for each $j = 1, \ldots, k$, by Υ'_j the set Υ if the deduction proviso $\mathrm{fresh}(\Upsilon, \langle\vartheta'_j, \Psi'_j, \eta'_j\rangle)$ is in P'_d, or $\emptyset$, otherwise, and denote by ϖ_j the sober deduction subsequence of ϖ, ending in $\langle\eta'_j\sigma, \vartheta'_j\sigma, 1\rangle$ for $\vartheta'_j\sigma \vdash_E \eta'_j\sigma$, for each $j = 1, \ldots, k$.

We now use the fact that Ψ is an appropriate set, see Definition 3.2.1, to show that $\eta'\sigma$, i.e., η, is in Ψ. So, we prove that for any $j = 1, \ldots, k$ and any ρ_j over E within L_E $\Upsilon'_j\sigma$ co-equivalent to id

$$\Psi'_j\sigma\rho_j \subseteq \Psi \quad \text{implies} \quad \eta'_j\sigma\rho_j \in \Psi.$$

Note that σ satisfies the conditions on the definition of appropriate set. So, let j be in $\{1,\ldots,k\}$, and ρ_j be a schema substitution over E within L_E $\Upsilon'_j\sigma$ co-equivalent to id. Suppose $\Psi'_j\sigma\rho_j$ is contained in Ψ. Then, by Proposition 2.3.14, there is a sober deduction sequence ϖ_{ρ_j} with the same length of ϖ_j for $\vartheta'_j\sigma\rho_j \vdash_E \eta'_j\sigma\rho_j$. We now show that $\vartheta'_j\sigma\rho_j$ is contained in Ψ. So, let ψ be in $\vartheta'_j\sigma\rho_j$. Then, consider two cases:

- ψ is in $\Psi'_j\sigma\rho_j$. Immediate using the hypothesis that $\Psi'_j\sigma\rho_j \subseteq \Psi$.

- ψ is in $\vartheta'_j\sigma\rho_j \setminus \Psi'_j\sigma\rho_j$. Then ψ is in $\vartheta'_j\sigma \setminus \Psi'_j\sigma$, as we just show, and so is in Ψ. Thus, we have to show that:

i. $\vartheta'_j\sigma\rho_j \setminus \Psi'_j\sigma\rho_j$ is contained in $(\vartheta'_j\sigma \setminus \Psi'_j\sigma)\rho_j$. Let ψ be in $\vartheta'_j\sigma$ such that $\psi\rho_j$ is not in $\Psi'_j\sigma\rho_j$. Then ψ is not in $\Psi'_j\sigma$, as desired.

ii. $(\vartheta'_j\sigma\setminus\Psi'_j\sigma)\rho_j = \vartheta'_j\sigma\setminus\Psi'_j\sigma$. The result follows immediately because $\Upsilon'_j\sigma\cap(\vartheta'_j\sigma\setminus\Psi'_j\sigma) = \emptyset$ and ρ_j is $\Upsilon'_j\sigma$ co-equivalent to id.

So, $\vartheta'_j\sigma\rho_j$ is contained in Ψ. Then, by monotony, Proposition 2.3.13, the sequence ϖ_{ρ_j} is a sober deduction sequence for $\Psi \vdash_E \eta'_j\sigma\rho_j$. Note that ϖ_{ρ_j} has length less than $n+1$. Hence, by induction hypothesis, $\eta'_j\sigma\rho_j$ is in Ψ as we wanted to show.

So, by definition of appropriate set, see Definition 3.2.1, $\eta'\sigma$, i.e., η, is in Ψ. *QED*

In order to exemplify the form of the conditions that an appropriate set should satisfy, suppose we are in the context of the lfob deduction system $\mathcal{D}_{\text{FOMdD}}$ for first order modal logic with decreasing domains introduced in Example 2.3.11 and suppose that Ψ is an appropriate set. Then, due to the rule $\Box_I$, we have that, for any label schema variable v and schema formula φ, if

$$\text{for any label schema variable } v',\ vR_\Box v' \text{ is in } \Psi \text{ implies } v'{:}\varphi \text{ is in } \Psi$$

then

$$v{:}\Box\varphi \text{ is in } \Psi.$$

Since Ψ satisfies also the appropriateness condition associated to rule $\Box_E$, the previous condition is in fact a equivalence.

3.3 Sufficient conditions for completeness

In this section we establish a completeness theorem stating that rich lfob logic systems are complete. After that we prove a corollary of the completeness theorem saying that full and appropriate systems are also complete. But before we introduce some necessary definitions.

Definition 3.3.1 A set Ψ is $v{:}\varphi$ *consistent* iff $\Psi \not\vdash_E v{:}\varphi$ where E is a set of label schema variables such that Ψ is contained in L_E and $v{:}\varphi$ is in L_E, and it is *consistent* if it is $v{:}\varphi$ consistent for some labelled schema formula $v{:}\varphi$.

Definition 3.3.2 A lfob logic system is *rich* iff for every $v{:}\varphi$ consistent set Ψ_0 within L, there exist a model m, a set E of label schema variables, and an E appropriate set Ψ extending Ψ_0 and $v{:}\varphi$ consistent, with s_Ψ in $\breve{m}$.

Theorem 3.3.3 Every rich lfob logic system is complete.

Proof Let $\mathcal{L}$ be a rich lfob logic system, Ψ_0 a set of labelled schema formulae within L and $v{:}\varphi$ a labelled schema formula in L. Assume $\Psi_0 \not\vdash v{:}\varphi$. Then, there exist a model m and an appropriate set Ψ extending Ψ_0 and $v{:}\varphi$ consistent, with s_Ψ in $\breve{m}$. Consider the identity schema substitution σ_{id} over E and within L_E. Then, (i) $s_\Psi, |.| \circ \sigma_{id} \Vdash \Psi_0$ by Proposition 3.1.3, since $\Psi_0\sigma_{id}$ i.e. Ψ_0 is contained in Ψ, and (ii) $s_\Psi, |.| \circ \sigma_{id} \nVdash v{:}\varphi$ by Proposition 3.1.3, since $(v{:}\varphi)\sigma_{id}$ i.e. $v{:}\varphi$ is not in Ψ. Therefore, $\Psi_0 \not\models v{:}\varphi$. *QED*

We now prove a corollary of the completeness theorem stating that every full and appropriate lfob logic system is complete. This corollary applies to lfob logic systems having the property that every consistent set can be extended to an appropriate set consistent with respect to the same formula. We call a lfob logic system having this property an appropriate lfob logic system. Motivated by this corollary, in Chapter 4, we study the class of connected lfob logic systems with a classical negation, local or non-local, and the class of connected lfob logic systems with locality and investigate appropriateness results for that systems.

Definition 3.3.4 A lfob deduction system is said to be *appropriate* iff for every $v{:}\varphi$ consistent set Ψ_0 within L, there exists a $v{:}\varphi$ consistent and E appropriate set containing Ψ_0 for some set E of label schema variables.

Definition 3.3.5 A lfob logic system is *appropriate* iff its underlying lfob deduction system is appropriate.

Definition 3.3.6 A lfob logic system $\langle \Sigma, R, M, \breve{\cdot} \rangle$ is *full* iff for every Σ structure s satisfying all rules of R there is a model m in M with s in $\breve{m}$.

Corollary 3.3.7 Every full and appropriate lfob logic system is complete.

To prove the previous corollary it is enough to show that every full and appropriate lfob logic system is rich, because then the corollary follows immediately by Theorem 3.3.3. So we now prove that every full and appropriate lfob logic system is rich.

Lemma 3.3.8 Every full and appropriate lfob logic system is rich.

Proof Let $\mathcal{L}$ be a full and appropriate lfob logic system and Ψ_0 a $v{:}\varphi$ consistent set. Consider an appropriate set Ψ extending Ψ_0 and $v{:}\varphi$ consistent, which exists since $\mathcal{L}$ is appropriate. Note that s_Ψ satisfy all the rules of $\mathcal{L}$, by Theorem 3.2.2. Hence, since $\mathcal{L}$ is full, there is a model m in M such that s_Ψ is in $\breve{m}$, as we wanted to show. *QED*

3.4 Fullness closure

Fullness closure is an operation on lfob logic systems that associate to a lfob logic system another lfob logic system with the same deduction system but with the interpretation system enriched with all structures that satisfy its rules. Fullness closure is very useful since it is a property preserved by fibring, it is one of the conditions of the completeness corollary, Corollary 3.3.7, and the fullness closure of a sound and complete lfob logic system is a lfob logic system with the same entailment and consequence as the original system, as we show in Proposition 3.4.2. So, we can prove the completeness of a lfob logic system using the fact that it is rich, without creating any problem, when we want to fibre that lfob logic system with another system, for the preservation of completeness by fibring, which uses the fact that both systems are full and appropriate.

Definition 3.4.1 The *fullness closure* of a lfob logic system $\langle \Sigma, R, M, \breve{\cdot} \rangle$ is the lfob logic system $\langle \Sigma, R, M', \hat{\cdot} \rangle$ defined as follows:

- M' is the extension of M with all the Σ structures s such that
  - s satisfies all the rules in R,
  - there is no model m in M with s in $\breve{m}$,
- $\hat{m}'$ is
  - $\breve{m}'$ whenever m' is in M,
  - $\{m'\}$ whenever m' is not in M.

Observe that, by definition of fullness closure, the consequence relation of a lfob logic system coincides with the consequence relation of the lfob logic system resulting from its fullness closure. Below we show when the entailments coincide.

Proposition 3.4.2 The fullness closure of a sound and complete lfob logic system is also a sound and complete lfob logic system, and their entailments coincide.

Proof Let $\mathcal{L}$, equal to $\langle \Sigma, R, M, \breve{\cdot} \rangle$, be a sound and complete lfob logic system, and $\langle \Sigma, R, M', \hat{\cdot} \rangle$ the lfob logic system obtained by the fullness closure of $\mathcal{L}$. Denote by $\vdash$ and $\vDash$ the consequence and the entailment relations, respectively, of $\mathcal{L}$, and by $\vdash'$ and $\vDash'$ the consequence and entailment relations, respectively, of $\langle \Sigma, R, M', \hat{\cdot} \rangle$.

$\vDash$ *is contained in* $\vDash'$. Note that it is only enough to show that $\vdash'$ is contained in $\vDash'$, i.e. the soundness of $\langle \Sigma, R, M', \breve{\cdot} \rangle$, because $\vDash$ is equal to $\vdash'$ since (i) $\vDash$ is equal to $\vdash$, recall that $\mathcal{L}$ is sound and complete, and (ii) $\vdash$ coincides with $\vdash'$, as is straightforward to see. Suppose $\Psi \vdash' \eta$ and let m' be a model in M', s a Σ structure in $\hat{m}'$, and α an assignment over s with $s, \alpha \Vdash \Psi$. According to the definition of fullness closure, Definition 3.4.1, we can consider two cases for m':

1. m' is not in M. Then $s, \alpha \Vdash \eta$ as we want, according to Lemma 2.4.5, since (i) $\Psi \vdash \eta$, (ii) $s, \alpha \Vdash \Psi$, and (iii) s satisfies all rules of R, by definition of fullness closure, Definition 3.4.1.

2. m' is in M. Then $s, \alpha \Vdash \eta$ as we want, by definition of entailment, Definition 2.2.8, since s is in $\breve{m}'$ and $\Psi \vDash \eta$.

$\vDash'$ *is contained in* $\vDash$. Suppose $\Psi \vDash' \eta$, and let m be a model in M, s a Σ structure in $\breve{m}$, and α a assignment over s with $s, \alpha \Vdash \Psi$. Recall that by definition of fullness closure, Definition 3.4.1, m is in M', and s is in $\hat{m}$ because $\hat{m}$ is $\breve{m}$. So we have as we want $s, \alpha \Vdash \eta$, by definition of entailment, Definition 2.2.8, since $\Psi \vDash' \eta$.

$\langle \Sigma, R, M', \hat{\cdot} \rangle$ is sound and complete. It is only enough to see that $\vDash'$ is equal to $\vDash$, as we just proved, which coincides with $\vdash$ since $\mathcal{L}$ is sound and complete, and $\vdash$ is equal to $\vdash'$ as is straightforward to see. *QED*

Now we show two propositions very useful when studying the fibring of lfob logic systems in Chapter 6.

Proposition 3.4.3 Given two lfob logic systems with the same signature Σ, differing at most in some components of provisos over signatures distinct of Σ, we have that the lfob logic systems resulting from their fullness closure differ exactly in that different components of the provisos.

Proof Note that, according to the definition of fullness closure, Definition 3.4.1, the only possible difference from a lfob logic system and its fullness closure, is in its interpretation system. Moreover, recall that the satisfaction of a rule by a Σ structure only depends of the Σ component of the provisos present in the rule, Definition 2.4.3. Let, $\mathcal{L}_1$ and $\mathcal{L}_2$ be lfob logic systems with the same signature Σ differing at most in some components of provisos over signatures distinct of Σ. Thus, the lfob logic systems resulting from their fullness closure differ also at most in that component of the provisos, because they have the same the interpretation system, since they have the same signature, the same rules, and the same Σ component of the provisos over their common signature. *QED*

In the next corollary we show that the operations fullness closure and change of components of provisos (over signatures different from the signature of the lfob logic system) are interchangeable.

Corollary 3.4.4 For any lfob logic system $\mathcal{L}$, denoting by fc the map that receives a lfob logic system and gives its fullness closure, we have that

$$\mathrm{fc}(g(\mathcal{L})) = g(\mathrm{fc}(\mathcal{L}))$$

for any map g that receives a lfob logic system with signature Σ and returns a lfob logic system equal to the one received, except at most in certain components over signatures different from Σ of the provisos present in the rules, and such that the provisos in $g(\mathcal{L}_1)$ and $g(\mathcal{L}_2)$ are equal whenever $\mathcal{L}_1$ and $\mathcal{L}_2$ have the same deduction system.

Proof Denote by Σ the signature of $\mathcal{L}$. Note that,

0. $\mathrm{fc}(g(\mathcal{L}))$ and $\mathrm{fc}(\mathcal{L})$ have the same interpretation system. This happens by Proposition 3.4.3, since $g(\mathcal{L})$ and $\mathcal{L}$ differ at most in signature (different from their common signature) components of some provisos present in their common rules, according to the conditions imposed in the definition of g.

1. $g(\mathcal{L})$ and $g(\mathrm{fc}(\mathcal{L}))$ have the same deduction system. This happens by the conditions imposed in the definition of g, since $\mathcal{L}$ and $\mathrm{fc}(\mathcal{L})$ have the same deduction system by definition of fullness closure, Definition 3.4.1.

Then, as we wanted to show,

- $\mathrm{fc}(g(\mathcal{L}))$ and $g(\mathrm{fc}(\mathcal{L}))$ have the same deduction system. Note that $\mathrm{fc}(g(\mathcal{L}))$ and $g(\mathcal{L})$ have the same deduction system by definition of fullness closure, Definition 3.4.1. So the result follows by 1.

- $\mathrm{fc}(g(\mathcal{L}))$ and $g(\mathrm{fc}(\mathcal{L}))$ have the same interpretation system. Note that $\mathrm{fc}(\mathcal{L})$ and $g(\mathrm{fc}(\mathcal{L}))$ have the same interpretation system by the conditions imposed in the definition of g. So the result follows by 0. *QED*

3.5 Non appropriate deduction systems

Given the completeness corollary, Corollary 3.3.7, there is a question that immediately raises: Are all lfob deduction systems appropriate? The answer is negative, and we present three simple lfob deduction systems that are not appropriate, each by a different reason, that we try to clarify.

Remark 3.5.1 Note that, when there are lfob deduction systems which are not appropriate, that means that, in the context of that lfob deduction system, it is not possible, in all situations where a set of labelled schema formulae do not deduce a labelled schema formula, to construct a canonical structure that semantically do the same, i.e., that satisfy the set and do not satisfy the formula. And the reason is because it is not possible to construct the appropriate set for this pair (set, formula) over which that canonical structure should be based. So, it is not possible to show the completeness of that system by the canonical structure construction.

3.5.1 Positive system with existential and universal connectives

Example 3.5.2 *Positive lfob deduction system with an existential and an universal connectives, and a disjunction.* Consider the lfob deduction system $\langle \Sigma, R \rangle$, in the sequel denoted by $\mathcal{D}_{\mathrm{EU}^+}$, where Σ is

- $F^l_k = \emptyset$ for $k \geq 0$;
- $S_2 = \{R_{\tilde{\Box}}\}$, and $S_k = \emptyset$ for $k \geq 3$ and $k = 1$;
- $F_k = \emptyset$ for $k \geq 0$;
- P_0 is a non-empty set of propositional symbols, and $P_k = \emptyset$ for $k \geq 1$;
- $C_0 = \{\top\}$, $C_2 = \{\vee\}$, and $C_k = \emptyset$ for $k \geq 3$ and $k = 1$;
- $Q_k = \emptyset$ for $k \geq 1$;
- $O_1 = \{\tilde{\Diamond}, \tilde{\Box}\}$, and $O_k = \emptyset$ for $k \geq 2$,

and R, besides the rules specified in Definition 2.3.10 common to all lfob deduction systems, contains:

$$\frac{\nu R_{\tilde{\Box}} \nu' \,/\, \nu' {:} \xi}{\nu {:} \tilde{\Box} \xi}\ \tilde{\Box}_I;\ \text{fresh}(\nu', \langle \vartheta_1, \{\nu R_{\tilde{\Box}} \nu'\}, \nu' {:} \xi \rangle) \qquad \frac{\nu {:} \tilde{\Box} \xi \quad \nu R_{\tilde{\Box}} \nu'}{\nu' {:} \xi}\ \tilde{\Box}_E$$

$$\frac{\nu {:} \tilde{\Diamond} \xi \quad \nu R_{\tilde{\Box}} \nu', \nu' {:} \xi \,/\, \nu'' {:} \xi''}{\nu'' {:} \xi''}\ \tilde{\Diamond}_E;\ \text{fresh}(\nu', \langle \vartheta_2, \{\nu R_{\tilde{\Box}} \nu', \nu' {:} \xi\}, \nu'' {:} \xi'' \rangle)$$

$$\frac{\nu R_{\tilde{\Box}} \nu' \quad \nu' {:} \xi}{\nu {:} \tilde{\Diamond} \xi}\ \tilde{\Diamond}_I \qquad \frac{}{\nu {:} \top}\ \top_I$$

$$\frac{\nu{:}\xi_1 \vee \xi_2 \quad \nu{:}\xi_1 \,/\, \nu''{:}\xi'' \quad \nu{:}\xi_2 \,/\, \nu''{:}\xi''}{\nu''{:}\xi''} \vee_E \qquad \frac{\nu{:}\xi_1}{\nu{:}\xi_1 \vee \xi_2} \vee_I^1 \qquad \frac{\nu{:}\xi_2}{\nu{:}\xi_1 \vee \xi_2} \vee_I^2.$$

In order to prove that $\mathcal{D}_{\mathrm{EU}^+}$ is not an appropriate lfob deduction system we need to consider a set and a labelled schema formula not deducible from that set, and then we have to show that we can not extend that set to an appropriate set for some set of label schema variables, and consistent for that labelled schema formula.

Conjecture 3.5.3 In the context of $\mathcal{D}_{\mathrm{EU}^+}$ we have that $\nvdash \nu{:}(\tilde{\Diamond}\top) \vee (\tilde{\Box}p)$, for p in P_0.

Suppose $\mathcal{D}_{\mathrm{EU}^+}$ is an appropriate lfob deduction system. Then, there exists a set E of label schema variables, and a set Ψ deductively closed, $\nu{:}(\tilde{\Diamond}\top) \vee (\tilde{\Box}p)$ consistent, and appropriate over E, extending $\emptyset$. So, since Ψ is an appropriate set, we have:

1. for every v in $T_{\mathrm{lab},E}$ we have (by $\top_I$)

$$v{:}\top \in \Psi,$$

2. for every v in $T_{\mathrm{lab},E}$ and schema formula φ, we have that

$$vR_{\tilde{\Box}}v' \in \Psi \quad \text{implies} \quad v'{:}\varphi \in \Psi, \quad \text{for any } v' \text{ in } T_{\mathrm{lab},E},$$

if (by $\tilde{\Box}_E$) and only if (by $\tilde{\Box}_I$)

$$v{:}\tilde{\Box}\varphi \in \Psi,$$

3. for every v in $T_{\mathrm{lab},E}$ and schema formula φ, we have

$$vR_{\tilde{\Box}}v' \in \Psi \quad \text{and} \quad v'{:}\varphi \in \Psi, \quad \text{for some } v' \text{ in } T_{\mathrm{lab},E},$$

if (by $\tilde{\Diamond}_E$) and only if (by $\tilde{\Diamond}_I$)

$$v{:}\tilde{\Diamond}\varphi \in \Psi.$$

Observe, in order to justify the if part of 3., that it is equivalent to the strict transposition of the $\tilde{\Diamond}_E$ rule, i.e. to, if $v{:}\tilde{\Diamond}\varphi \in \Psi$ and for any v' if $vR_{\tilde{\Box}}v' \in \Psi$ and $v'{:}\varphi \in \Psi$ then $v''{:}\varphi'' \in \Psi$, then $v''{:}\varphi'' \in \Psi$.

Note that $\nu{:}(\tilde{\Diamond}\top) \notin \Psi$ and $\nu{:}(\tilde{\Box}p) \notin \Psi$ because otherwise Ψ will not be $\nu{:}(\tilde{\Diamond}\top) \vee (\tilde{\Box}p)$ consistent by rule $\vee^1$ or $\vee^2$. So, using the above facts about Ψ, we can conclude, since $\nu{:}(\tilde{\Diamond}\top) \notin \Psi$, by 3., that there is no v' in $T_{\mathrm{lab},E}$ with $\nu R_{\tilde{\Box}}v' \in \Psi$ and $v'{:}\top \in \Psi$, which is equivalent by 1., to no exists v' in $T_{\mathrm{lab},E}$ with $\nu R_{\tilde{\Box}}v' \in \Psi$. And, since $\nu{:}(\tilde{\Box}p) \notin \Psi$, by 2., that exists v' in $T_{\mathrm{lab},E}$ with $\nu R_{\tilde{\Box}}v'$ and $v'{:}p \notin \Psi$. Which is a contradiction.

Therefore, there is no set Ψ consistent with respect to $\nu{:}(\tilde{\Diamond}\top) \vee (\tilde{\Box}p)$, and appropriate over a set E of label schema variables, extending $\emptyset$. Hence $\mathcal{D}_{\mathrm{EU}^+}$ is not an appropriate lfob deduction system.

The point to realize about this example is that, for certain lfob deduction systems, the fact of adding a relational schema formula to a set, may cause the deduction of labelled schema formulae not deduced before. And this fact cause that it is not possible to extend a set of labelled schema formulae in order to be appropriate, as it was the case with the example presented.

3.5.2 Positive system with a docked universal connective

The point that we want to show with the next example is that lfob deduction systems with rules allowing the inheritance of schema formulae between worlds connected by relations associated with universal connectives may not be appropriate. The reason why this is a problem is because when extending a set in order to be appropriate, it may be necessary to add new labels, link them to old ones by a relation, and assume certain formulae hold in that new labels. But if these formulae are inherited by the old labels they may cause deductions not previously possible and not wanted. This may imply that it is not possible to extend that set, in order to be appropriate and at the same time consistent with respect to a formula.

Example 3.5.4 *Positive lfob deduction system with a binary universal connective docked at an assignment, and with a disjunction.* Consider the lfob deduction system $\langle \Sigma, R \rangle$, in the sequel denoted by $\mathcal{D}_{\mathrm{bU}^{+}_{\downarrow}}$, where Σ is

- $F^l_k = \emptyset$ for $k \geq 0$;
- $S_3 = \{R_{\Rightarrow}\}$, and $S_k = \emptyset$ for $k \geq 4$ and $k = 2$ and $k = 1$;
- $F_k = \emptyset$ for $k \geq 0$;
- $P_k = \emptyset$ for $k \geq 0$;
- $C_2 = \{\vee\}$, and $C_k = \emptyset$ for $k \geq 3$ and $k = 1$ and $k = 0$;
- $Q_k = \emptyset$ for $k \geq 1$;
- $O_2 = \{\Rightarrow\}$, and $O_k = \emptyset$ for $k \geq 3$ and $k = 1$,

and R, besides the rules specified in Definition 2.3.10 common to all lfob deduction systems, contains:

$$\frac{R_{\Rightarrow}\nu\nu_1\nu_2, \nu_1{:}\xi_1 \,/\, \nu_2{:}\xi_2}{\nu{:}\xi_1 \Rightarrow \xi_2} \Rightarrow_I; \text{ fresh}(\{\nu_1, \nu_2\}, \langle \vartheta_1, \{R_{\Rightarrow}\nu\nu_1\nu_2, \nu_1{:}\xi_1\}, \nu_2{:}\xi_2 \rangle)$$

$$\frac{\nu{:}\xi_1 \Rightarrow \xi_2 \qquad \nu_1{:}\xi_1 \qquad R_{\Rightarrow}\nu\nu_1\nu_2}{\nu_2{:}\xi_2} \Rightarrow_E$$

$$\frac{R_{\Rightarrow}\nu_1\nu_2\nu_3 \quad \nu_1 \equiv_\omega \nu_1' \quad \nu_2 \equiv_\omega \nu_2' \quad \nu_3 \equiv_\omega \nu_3' \quad \nu_1' \equiv_\alpha \nu_2' \quad \nu_2' \equiv_\alpha \nu_3'}{R_{\Rightarrow}\nu_1'\nu_2'\nu_3'} R_{\Rightarrow}{}^{\mathrm{gen}}_{\alpha}$$

$$\frac{R_{\Rightarrow}\nu_1\nu_2\nu_3}{\nu_1 \equiv_\alpha \nu_2} R_{\Rightarrow\alpha^{1,2}} \qquad\qquad \frac{R_{\Rightarrow}\nu_1\nu_2\nu_3}{\nu_2 \equiv_\alpha \nu_3} R_{\Rightarrow\alpha^{2,3}}$$

$$\frac{\nu{:}\xi_1 \vee \xi_2 \quad \nu{:}\xi_1 \,/\, \nu''{:}\xi'' \quad \nu{:}\xi_2 \,/\, \nu''{:}\xi''}{\nu''{:}\xi''} \vee_E \qquad\qquad \frac{\nu{:}\xi_1}{\nu{:}\xi_1 \vee \xi_2} \vee^1_I \qquad \frac{\nu{:}\xi_2}{\nu{:}\xi_1 \vee \xi_2} \vee^2_I.$$

Similarly to $\mathcal{D}_{\mathrm{EU}^+}$ we now show that $\mathcal{D}_{\mathrm{bU}^+_\downarrow}$ is also not an appropriate lfob deduction system.

Conjecture 3.5.5 In the context of $\mathcal{D}_{\mathrm{bU}^+_\downarrow}$ we have that $\not\vdash \nu{:}((x = y) \Rightarrow (x = z)) \vee (x = y)$, for x, y and z in X.

Suppose $\mathcal{D}_{\mathrm{bU}^+_\downarrow}$ is an appropriate lfob deduction system. Then, there exists a set E of label schema variables, and a set Ψ, $\nu{:}((x = y) \Rightarrow (x = z)) \vee (x = y)$ consistent, and appropriate over E, extending $\emptyset$. So, since Ψ is an appropriate set:
1. for every v in $T_{\mathrm{lab},E}$ and schema formulae φ_1 and φ_2, we have that

$$R_\Rightarrow v v_1 v_2 \in \Psi \text{ and } v_1{:}\varphi_1 \in \Psi \quad \text{implies} \quad v_2{:}\varphi_2 \in \Psi, \quad \text{for any } v_1, v_2 \text{ in } T_{\mathrm{lab},E},$$

if (by $\Rightarrow_E$) and only if (by $\Rightarrow_I$)

$$v{:}\varphi_1 \Rightarrow \varphi_2 \in \Psi.$$

Observe that $\nu{:}(x = y) \Rightarrow (x = z) \notin \Psi$ and $\nu{:}(x = y) \notin \Psi$. So, using 1., we can conclude, since $\nu{:}(x = y) \Rightarrow (x = z) \notin \Psi$, that there are v_1 and v_2 in $T_{\mathrm{lab},E}$ with $R_\Rightarrow \nu v_1 v_2 \in \Psi$, $v_1{:}x = y \in \Psi$, and $v_2{:}x = z \notin \Psi$. So, $\nu{:}x = y \in \Psi$ since Ψ is deductively closed, by rules $R_{\Rightarrow\alpha^{1,2}}$, $\equiv_{\alpha g}^{x}$, $\equiv_{\alpha g}^{y}$, $=_I$, $=_E$, $=_{g_s}$, and $=_{g_t}$. But this is a contradiction.

Therefore, there is no set Ψ deductively closed, $\nu{:}((x = y) \Rightarrow (x = z)) \vee (x = y)$ consistent, and appropriate over a set E of label schema variables, extending $\emptyset$. Hence $\mathcal{D}_{\mathrm{bU}^+_\downarrow}$ is not an appropriate lfob deduction system.

3.5.3 System with a negation and non-reflexive relation

Finally, with the next example, we want to address the case of lfob deduction systems, having rules that allow from a special labelled schema formula, like $v{:}\perp$, the deduction of other specific schema formulae independently of the relations between the labels. Lfob deduction systems with this property may not be appropriate because, when extending a set in order to be appropriate, it may be obligatory to add to that set those special formulae. But this addition may allow not wanted deductions, as the deduction of the formula with respect to which it should be consistent. This is the case that we want to show. So now we present a lfob deduction system with this kind of rules that is not appropriate.

Example 3.5.6 *Lfob deduction system with a binary universal connective, a disjunction, and a non-local negation connective.* Consider the lfob deduction system $\langle \Sigma, R \rangle$, in the sequel denoted by $\mathcal{D}_{\mathrm{bUnN}}$, where Σ is

- $F^l_1 = \{*\}$, and $F^l_k = \emptyset$ for $k \geq 2$ and $k = 0$;
- $S_3 = \{R_\Rightarrow\}$, and $S_k = \emptyset$ for $k \geq 4$ and $k = 2$ and $k = 1$;
- $F_k = \emptyset$ for $k \geq 0$;
- P_0 is a non-empty set of propositional symbols, and $P_k = \emptyset$ for $k \geq 1$;
- $C_0 = \{\perp\}$, $C_2 = \{\vee\}$, and $C_k = \emptyset$ for $k \geq 3$ and $k = 1$;

- $Q_k = \emptyset$ for $k \geq 1$;
- $O_1 = \{-\}$, $O_2 = \{\Rightarrow\}$, and $O_k = \emptyset$ for $k \geq 3$,

and R, besides the rules specified in Definition 2.3.10 common to all lfob deduction systems, contains:

$$\frac{R_{\Rightarrow}\nu\nu_1\nu_2, \nu_1{:}\xi_1 \,/\, \nu_2{:}\xi_2}{\nu{:}\xi_1 \Rightarrow \xi_2} \Rightarrow_I; \ \text{fresh}(\{\nu_1, \nu_2\}, \langle \vartheta_1, \{R_{\Rightarrow}\nu\nu_1\nu_2, \nu_1{:}\xi_1\}, \nu_2{:}\xi_2\rangle)$$

$$\frac{\nu{:}\xi_1 \Rightarrow \xi_2 \qquad \nu_1{:}\xi_1 \qquad R_{\Rightarrow}\nu\nu_1\nu_2}{\nu_2{:}\xi_2} \Rightarrow_E$$

$$\frac{\nu^*{:}\xi \,/\, \nu'{:}\bot}{\nu{:}-\xi} -_I \qquad\qquad \frac{\nu{:}-\xi \qquad \nu^*{:}\xi}{\nu'{:}\bot} -_E$$

$$\frac{\nu{:}\xi_1 \vee \xi_2 \quad \nu{:}\xi_1 \,/\, \nu''{:}\xi'' \quad \nu{:}\xi_2 \,/\, \nu''{:}\xi''}{\nu''{:}\xi''} \vee_E \qquad \frac{\nu{:}\xi_1}{\nu{:}\xi_1 \vee \xi_2} \vee_I^1 \qquad \frac{\nu{:}\xi_2}{\nu{:}\xi_1 \vee \xi_2} \vee_I^2.$$

The next conjecture offers us a labelled schema formula, and a set, that we will show it is not possible to extend in order to be appropriate, and consistent with respect to that formula.

Conjecture 3.5.7 In the context of $\mathcal{D}_{\text{bUnN}}$ we have that $\not\vdash \nu{:}(\bot \Rightarrow p) \vee -p$ for p in P_0.

Suppose $\mathcal{D}_{\text{bUnN}}$ is an appropriate lfob deduction system. Then, there exist a set E of label schema variables and a set Ψ consistent with respect to $\nu{:}(\bot \Rightarrow p) \vee -p$, and appropriate over E, extending $\emptyset$. So, since Ψ is an appropriate set, we have:
1. for every v in $T_{\text{lab},E}$ and schema formulae φ_1 and φ_2, we have that

$$R_{\Rightarrow}vv_1v_2 \in \Psi \text{ and } v_1{:}\varphi_1 \in \Psi \quad \text{implies} \quad v_2{:}\varphi_2 \in \Psi, \quad \text{for any } v_1, v_2 \text{ in } T_{\text{lab},E},$$

if (by $\Rightarrow_E$) and only if (by $\Rightarrow_I$)

$$v{:}\varphi_1 \Rightarrow \varphi_2 \in \Psi.$$

Observe that $\nu{:}\bot \Rightarrow p \notin \Psi$ and $\nu{:}-p \notin \Psi$. So, using the above facts about Ψ we can conclude, since $\nu{:}\bot \Rightarrow p \notin \Psi$, by 1., that exists v_1 and v_2 in $T_{\text{lab},E}$ with $R_{\Rightarrow}\nu v_1 v_2 \in \Psi$, $v_1{:}\bot \in \Psi$, and $v_2{:}p \notin \Psi$. So, $\nu{:}-p \in \Psi$, since Ψ is deductively closed, by rule $-_I$. But this is a contradiction.

Therefore, there is no set Ψ consistent with respect to $\nu{:}(\bot \Rightarrow p) \vee -p$, and appropriate over a set E of label schema variables, extending $\emptyset$. Hence $\mathcal{D}_{\text{bUnN}}$ is not an appropriate lfob deduction system.

3.5.4 General results

We end this chapter showing two propositions. In the first it is shown that any consistent and deductively closed set satisfies a condition very similar to appropriateness, with the difference that the antecedent of the implication refers to all maximal consistent extensions of the set, instead of it. This property is satisfied in the context of any lfob deduction system in which a $v{:}\varphi$ consistent set for a labelled formula $v{:}\varphi$ can be extended to a maximal $v{:}\varphi$ consistent set. Note that the appropriate lfob deduction systems are precisely those where it is sufficient to impose the satisfaction of the antecedent of that implication for the set, instead of for all its maximal consistent extensions, in order to conclude the consequent. Before we need to introduce the notion of maximal consistent set.

Definition 3.5.8 A set Ψ is a *maximal $v{:}\varphi$ consistent set* if it is $v{:}\varphi$ consistent, and (i) its subset of relational schema formulae is deductively closed, and (ii) no proper extension with labelled non relational schema formulae is $v{:}\varphi$ consistent.

Proposition 3.5.9 Let $\mathcal{D}$ be a lfob deduction system in which every $v{:}\varphi$ consistent set can be extended to a maximal $v{:}\varphi$ consistent set. Then, for every

- consistent and deductively closed set Ψ within L_E,
- $\langle\{\langle\vartheta_1, \Psi_1, \eta_1\rangle, \ldots, \langle\vartheta_k, \Psi_k, \eta_k\rangle\}, \eta, P_f, P_d\rangle$ in R,
- σ within L_E with $(\pi_\Sigma \sigma_{\mathrm{nl}}) = 1$ for each π in P_f, and $\pi_{\Sigma;E}(\sigma) = 1$ for each π in P_d,

if for any $j = 1, \ldots, k$, E' disjoint of E, maximal consistent extension $\Psi^\bullet$ of Ψ within $L_{E\cup E'}$, and ρ_j over E within $L_{E\cup E'}$ $\Upsilon_j\sigma$ co-equivalent to id over E within L_E,

$$\Psi_j\sigma\rho_j \subseteq \Psi^\bullet \quad \text{implies} \quad \eta_j\sigma\rho_j \in \Psi^\bullet$$

then

$$\eta\sigma \in \Psi$$

Proof We start by briefly sketching the proof. We show the contrapositive. We assume $\eta\sigma$ is not in Ψ. Then we conclude $\Psi \not\vdash_E \eta\sigma$ and so by Proposition 3.5.10, that there is i with $\Psi, \Psi_i\sigma' \not\vdash_{E,E'} \eta_i\sigma'$ for a certain schema substitution σ' co-equivalent to σ. Thus, we define a schema substitution ρ_i in order for $\sigma\rho_i$ be equal to σ'. So $\Psi, \Psi_i\sigma\rho_i \not\vdash_{E,E'} \eta_i\sigma\rho_i$. Hence we can conclude since exists a maximal $\eta_i\sigma\rho_i$ consistent set extending $\Psi \cup \Psi_i\sigma\rho_i$. We now start the proof.

Let E be a set of label schema variables, Ψ a deductively closed set contained in L_E, $\langle\{\langle\vartheta_1, \Psi_1, \eta_1\rangle, \ldots, \langle\vartheta_k, \Psi_k, \eta_k\rangle\}, \eta, P_f, P_d\rangle$ a rule in R, and σ be a schema substitution within L_E such that $(\pi_\Sigma \sigma_{\mathrm{nl}})$ is 1 for each π in P_f, and $\pi_{\Sigma;E}(\sigma)$ is 1 for each π in P_d.

Suppose $\eta\sigma$ is not in Ψ. Then, since Ψ is deductively closed we have that $\Psi \not\vdash_E \eta\sigma$. Then, by Proposition 3.5.10, there is i in $1, \ldots, k$, set E' of label schema variables disjoint of E with cardinality greater than $\cup_{j=1,\ldots,k}\Upsilon_j$, and schema substitution σ' within $L_{E\cup E'}$ with (i) σ' is $\cup_{j=1,\ldots,k}\Upsilon_j$ co-equivalent to σ, (ii) $\eta\sigma'$ is equal to $\eta\sigma$, (iii) $\cup_{j=1,\ldots,k}\Upsilon_j\sigma'$ is contained in E', (iv) $(\pi_\Sigma \sigma'_{\mathrm{nl}})$ is 1 for each π in P_f, and (v) $\pi_{\Sigma;E\cup E'}(\sigma')$ is 1 for each π in P_d, such that $\Psi, \Psi_i\sigma' \not\vdash_{E,E'} \eta_i\sigma'$.

Note first that $\nu\sigma$ is distinct of $\nu'\sigma$ for each ν and ν' in Υ_i because $\pi_{\Sigma;E}(\sigma) = 1$ for each

π in P_d.

Consider the schema substitution ρ_i over E within $L_{E\cup E'}$, $\Upsilon_i\sigma$ co-equivalent to id over E within L_E, defined as follows: $\nu\sigma\rho_i$ is $\nu\sigma'$ for each ν in Υ_i. Then, $\Psi, \Psi_i\sigma\rho_i \nvdash_{E,E'} \eta_i\sigma\rho_i$, because $\psi\sigma'$ is $\psi\sigma\rho_i$, for each ψ in $\Psi_i \cup \{\eta_i\}$. The proof follows by case analysis on ψ. If ψ is

- $v{:}\varphi$, then $\psi\sigma'$ is $v\sigma'{:}\varphi\sigma'$ which is $v\sigma'{:}\varphi\sigma\rho_i$ since σ' is co-equivalent to σ and ρ_i is co-equivalent to id. Now we show that $v\sigma'$ is $v\sigma\rho_i$ by induction on the structure of the label schema term v. If v is

– *a label schema variable in* Υ_i then $v\sigma'$ is $v\sigma\rho_i$ by definition of ρ_i.

– *a label schema variable not in* Υ_i. Then, v is also not in $\cup_{j=1,\ldots,k \text{ and } j\neq i}\Upsilon_j$ by Definition 2.3.9. So, $v\sigma'$ is $v\sigma$. Since v is in $\mathrm{lsv}(\Psi_i \cup \{\eta_i\}) \setminus \Upsilon_i$, then $v\sigma$ is not in $\Upsilon_i\sigma$. Therefore the result follows because $v\sigma\rho_i$ is $v\sigma$.

– *in* F^l_0. Then $v\sigma'$ is v which is $v\sigma\rho_i$.

– $f^l(v_1, \ldots, v_k)$. Then $v\sigma'$ is $f^l(v_1\sigma', \ldots, v_k\sigma')$ which is by induction hypothesis equal to $f^l(v_1\sigma\rho_i, \ldots, v_k\sigma\rho_i)$ so $v\sigma\rho_i$.

- For the other cases of ψ the proof follows similarly.

Denote by $\Psi^\bullet$ the $\eta_i\sigma\rho_i$ m.c.e. of $\Psi \cup \{\Psi_i\sigma\rho_i\}$.

Therefore, as we wanted to show, there is a j in $1, \ldots, k$, a maximal consistent extension of Ψ within $L_{E\cup E'}$ $\Psi^\bullet$, a set E' of label schema variables, and a schema substitution ρ_j over E within $L_{E\cup E'}$ $\Upsilon_j\sigma$ co-equivalent to id over E within L_E, with $\Psi_j\sigma\rho_j \subseteq \Psi^\bullet$ and $\eta_j\sigma\rho_j \in \Psi^\bullet$. *QED*

The next proposition will be often referred in the sequel and presents a property of deductively closed sets that is satisfied in any lfob deduction system, and is very similar to appropriateness.

Proposition 3.5.10 In the context of a lfob deduction system $\langle\Sigma, R\rangle$, we have $\Psi \vdash_E \eta\sigma$ whenever, for any $j = 1, \ldots, k$, we have $\Psi, \Psi_j\sigma' \vdash_{E,E'} \eta_j\sigma'$, for every

- rule $\langle\{\langle\vartheta_1, \Psi_1, \eta_1\rangle, \ldots, \langle\vartheta_k, \Psi_k, \eta_k\rangle\}, \eta, P_f, P_d\rangle$ in R,
- set E of label schema variables,
- set Ψ within L_E deductively closed,
- schema substitution σ within L_E such that
  - $(\pi_\Sigma\sigma_{\mathrm{nl}})$ is 1 for each π in P_f,
  - $\pi_{\Sigma;E\cup E'}(\sigma)$ is 1 for each π in P_d,
- set E' of label schema variables disjoint of E with cardinality greater than $\cup_{j=1,\ldots,k}\Upsilon_j$,
- σ' within $L_{E\cup E'}$ such that
  - σ' is $\cup_{j=1,\ldots,k}\Upsilon_j$ co-equivalent to σ,
  - $\eta\sigma'$ is equal to $\eta\sigma$,

- $\cup_{j=1,\dots,k}\Upsilon_j\sigma'$ is contained in E',
- $(\pi_\Sigma\sigma'_{\mathrm{nl}})$ is 1 for each π in P_f,
- $\pi_{\Sigma;E\cup E'}(\sigma')$ is 1 for each π in P_d.

Proof We start by briefly sketching the proof. We construct a deduction sequence for $\Psi \vdash_E \eta\sigma$ based on deduction sequences ϖ'_j for $\Psi, \Psi_j\sigma'\rho' \vdash_E \eta_j\sigma'\rho'$ for each $j = 1, \dots, k$. These deduction sequences exist, by Proposition 2.3.14, since $\Psi, \Psi_j\sigma' \vdash_{E,E'} \eta_j\sigma'$ for each $j = 1, \dots, k$ by assumption. The objective of the ρ' is only to rename the fresh variables so that they are only in E, and so transform a deduction using label schema variables in $E \cup E'$ in a deduction in E. So, based on $\sigma'\rho'$ we define the schema substitution $\sigma'^{\cdot}$ that will be used in the instantiation of the rule and show that the ϖ'_j's together with a suitable instance of the conclusion of the rule make a well formed deduction. The only reason to use $\sigma'^{\cdot}$ and not $\sigma'\rho'$ was that we have to say what is the instance of ϑ_j for each premise j. We now start the proof.

Let $\langle\{\langle\vartheta_1, \Psi_1, \eta_1\rangle, \dots, \langle\vartheta_k, \Psi_k, \eta_k\rangle\}, \eta, P_f, P_d\rangle$ be a rule in R, E a set of label schema variables, Ψ a set deductively closed within L_E, σ a schema substitution within L_E such that (i) $(\pi_\Sigma\sigma_{\mathrm{nl}})$ is 1 for each π in P_f, and (ii) $\pi_{\Sigma;E\cup E'}(\sigma)$ is 1 for each π in P_d, E' is a set of label schema variables disjoint of E with cardinality greater than $\cup_{j=1,\dots,k}\Upsilon_j$, σ' is a schema substitution within $L_{E\cup E'}$ such that (i) σ' is $\cup_{j=1,\dots,k}\Upsilon_j$ co-equivalent to σ, (ii) $\eta\sigma'$ is equal to $\eta\sigma$, (iii) $\cup_{j=1,\dots,k}\Upsilon_j\sigma'$ is contained in E', (iv) $(\pi_\Sigma\sigma'_{\mathrm{nl}})$ is 1 for each π in P_f, (v) $\pi_{\Sigma;E\cup E'}(\sigma')$ is 1 for each π in P_d.

Assume for any $j = 1, \dots, k$, that $\Psi, \Psi_j\sigma' \vdash_{E,E'} \eta_j\sigma'$. Therefore, for each $j = 1, \dots, k$ there is a sober deduction sequence ϖ_j ending in $\langle\eta_j\sigma', \Psi'_j, 1\rangle$ for $\Psi, \Psi_j\sigma' \vdash_{E,E'} \eta_j\sigma'$.

Consider a schema substitution ρ' over $E \cup E'$ within L_E, $\cup_{j=1,\dots,k}\Upsilon_j\sigma'$ co-equivalent to the schema substitution id over E within L_E defined such that $\nu\sigma'\rho'$ is a label schema variable not in $(\mathrm{lsv}(\Psi_j \cup \{\eta_j\}) \setminus \{\nu\})\sigma'\rho'$ for each ν in Υ_j, and $\Upsilon_j\sigma'\rho' \cap \mathrm{lsv}(\Psi'_j \setminus \Psi_j\sigma')\rho' = \emptyset$, for any $j = 1, \dots, k$.

Observe that $\mathrm{lsv}(\Psi'_j \setminus \Psi_j\sigma')\rho'$ is $\mathrm{lsv}(\Psi'_j \setminus \Psi_j\sigma')$, and that $\Upsilon_j\sigma' \cap \mathrm{lsv}(\Psi'_j \setminus \Psi_j\sigma') = \emptyset$ because $\Upsilon_j\sigma'$ is contained in E' and $\mathrm{lsv}(\Psi'_j \setminus \Psi_j\sigma')$ is contained in E. That's why $\Upsilon_j\sigma'$ is contained in E'. Note also that it is possible to find a schema substitution satisfying these conditions since $\Psi_j \cup \{\eta_j\}$ and Ψ'_j are finite sets, for each $j = 1, \dots, k$. Note also that the fact that $\Upsilon_j\sigma'$ and $\Upsilon_i\sigma'$ are not disjoint, for distinct i and j in $1, \dots, k$, is irrelevant to find such a schema substitution.

Consider the sober deduction sequence ϖ'_j for $\Psi\rho', \Psi_j\sigma'\rho' \vdash_E \eta_j\sigma'\rho'$ ending in $\langle\eta_j\sigma'\rho', \Psi''_j, 1\rangle$ where Ψ''_j is contained in $\Psi'_j\rho'$, which exists by Proposition 2.3.14, for each $j = 1, \dots, k$. Note that $\Psi\rho'$ is Ψ, and so how important it is to consider the set E' of new label schema variables.

Consider the schema substitution $\sigma'^{\cdot}$ within L_E such that $\sigma'^{\cdot}_f = \sigma'_f\rho'_f$, $\sigma'^{\cdot}_t = \sigma'_t\rho'_t$, $\sigma'^{\cdot}_l = \sigma'_l\rho'_l$, and $\vartheta_j\sigma'^{\cdot}_s = \Psi''_j$ for each $j = 1, \dots, k$.

Then, the sequence $\varpi'_1 \dots \varpi'_k\langle\eta\sigma, \Psi''_1 \setminus \Psi_1\sigma'^{\cdot} \cup \dots \cup \Psi''_k \setminus \Psi_k\sigma'^{\cdot}, 1\rangle$ is a sober deduction sequence for $\Psi \vdash_E \eta\sigma$. To see this note that

- $\eta_j\sigma'\rho'$ is $\eta_j\sigma'^{\cdot}$. By definition of $\sigma'^{\cdot}$.

- $\eta\sigma$ is $\eta\sigma'^{\cdot}$. Note first that $\mathrm{lsv}(\eta\sigma') \cap \cup_{j=1,\dots,k}\Upsilon_j\sigma' = \emptyset$, since $\mathrm{lsv}(\eta)$ is contained in $\mathrm{lsv}(\Psi_j \cup \{\eta_j\}) \setminus \Upsilon_j$ for each $j = 1, \dots, k$ where Υ_j is not empty by Definition 2.3.9, and since $\pi_{\Sigma;E\cup E'}(\sigma') = 1$ for each π in P_d. So, $\eta\sigma'\rho'$ is $\eta\sigma'$. Observe also that $\eta\sigma'^{\cdot}$ is $\eta\sigma'\rho'$. Hence the result follows because $\eta\sigma'$ is $\eta\sigma$ by definition of σ' because $\mathrm{lsv}(\eta)$ is contained in $\mathrm{lsv}(\Psi_j \cup \{\eta_j\}) \setminus \Upsilon_j$ for each $j = 1, \ldots, k$ where Υ_j is not empty, taking into account Definition 2.3.9.

- $\Psi''_1 \setminus \Psi_1\sigma'^{\cdot} \cup \ldots \cup \Psi''_k \setminus \Psi_k\sigma'^{\cdot}$ is contained in Ψ. This happens since Ψ''_j is contained in $\Psi \cup \Psi_j\sigma'\rho'$ i.e. in $\Psi \cup \Psi_j\sigma'^{\cdot}$, for each $j = 1, \ldots, k$.

- $(\pi_\Sigma\sigma'^{\cdot}_{\mathrm{nl}}) = 1$, for each π in P_f. Observe that $\sigma'^{\cdot}_{\mathrm{nl}}$ is $\sigma'_{\mathrm{nl}}\rho'_{\mathrm{nl}}$ which is σ'_{nl}. So, the result follows since $(\pi_\Sigma\sigma'_{\mathrm{nl}}) = 1$, for each π in P_f.

- $\pi_{\Sigma;E}(\sigma'^{\cdot}) = 1$, for each π in P_d. To see this happens, let π be $\mathrm{fresh}(\Upsilon_j, \langle\vartheta_j, \Psi_j, \eta_j\rangle)$. Then

– $\nu\sigma'^{\cdot}$ is a label schema variable not in $(\mathrm{lsv}(\Psi_j \cup \{\eta_j\}) \setminus \{\nu\})\sigma'^{\cdot}$, for each ν in Υ_j. Let ν be in Υ_j. The result follows immediately because, (i) by definition of $\sigma'^{\cdot}$, $\nu\sigma'^{\cdot}$ is $\nu\sigma'\rho'$, and $(\mathrm{lsv}(\Psi_j \cup \{\eta_j\}) \setminus \{\nu\})\sigma'^{\cdot}$ is $(\mathrm{lsv}(\Psi_j \cup \{\eta_j\}) \setminus \{\nu\})\sigma'\rho'$, and (ii) by definition of ρ' we have that $\nu\sigma'\rho'$ is a label schema variable not in $(\mathrm{lsv}(\Psi_j \cup \{\eta_j\}) \setminus \{\nu\})\sigma'\rho'$.

– $\Upsilon_j\sigma'^{\cdot} \cap \mathrm{lsv}(\Psi''_j \setminus \Psi_j\sigma'^{\cdot}) = \emptyset$. To see this happens note that (i) $\Upsilon_j\sigma'^{\cdot}$ is $\Upsilon_j\sigma'\rho'$ and $\Psi_j\sigma'^{\cdot}$ is $\Psi_j\sigma'\rho'$, by definition of $\sigma'^{\cdot}$, (ii) $\Upsilon_j\sigma'\rho' \cap \mathrm{lsv}(\Psi'_j \setminus \Psi_j\sigma')\rho' = \emptyset$ by definition of σ', (iii) Ψ''_j is contained in $\Psi'_j\rho'$ by definition of ϖ'_j, and (iv) $\Psi'_j\rho' \setminus \Psi_j\sigma'\rho'$ is contained in $(\Psi'_j \setminus \Psi_j\sigma')\rho'$ because if ψ is in Ψ'_j and $\psi\rho'$ is not in $\Psi_j\sigma'\rho'$ then ψ is not in $\Psi_j\sigma'$, as desired. So, by (iii) and (iv), $\Upsilon_j\sigma'\rho' \cap \mathrm{lsv}(\Psi''_j \setminus \Psi_j\sigma'\rho')$ is contained in $\Upsilon_j\sigma'\rho' \cap \mathrm{lsv}(\Psi'_j \setminus \Psi_j\sigma')\rho'$, which is $\emptyset$ by (ii). *QED*

Chapter 4

Connectedness

In the previous chapter we realized, as expected, that not all lfob logic systems are appropriate, although any consistent and deductively closed set satisfies a condition similar to appropriateness. So, the next step is to find interesting classes of lfob logic systems that are appropriate, i.e., classes of lfob logic systems for which there exist a construction that allows us to extend any $v{:}\varphi$ consistent set to a $v{:}\varphi$ consistent and appropriate set. Moreover, we are interested in classes characterized by properties that are preserved, under suitable conditions, by fibring. We restrict our study to a general class of lfob logic systems that we call connected lfob logic systems, which is rather large and comprises lfob logic systems for relevance logics, modal logics, and first-order based logics, among others. We begin by investigating the class of connected lfob logic systems with classical negation, local or non-local, Definition 4.3.3, and prove that they are appropriate. After that, we study the possibility of constructing appropriate sets from consistent ones, in the context of connected lfob logic systems in which we do not assume any kind of negation. As a result we identify the class of connected lfob logic systems with locality, and prove that connected lfob logic systems with locality and without disjunction and implication are appropriate. Note that those appropriate systems may have connectives like universal and existential, constrained or not, quantifiers and modalities, conjunction and any kind of negation, although negation plays no role in the construction. The restriction to logics without disjunction and implication came from the fact that we were not able to prove the appropriateness conditions for the rules $\rightarrow_I$ and $\wedge_E$ in the context of lfob logic systems with locality. Nevertheless, observe that it does not imply that there is not a construction of an appropriate set satisfying those rules, in the context of a clfob deduction system with locality. Only that we were not able to do it. If we are not relying on the presence of any kind of negation, in order for the set resulting from our appropriateness construction satisfy these appropriate conditions for implication and disjunction, we think we have to consider more general forms of the rules, or even of the connectives, see Remark 4.4.8. This is a very interesting topic that we intend to pursue in the future and that may move us closer to hybrid logic and perhaps far from labelled deduction. For now we concentrate on the study of the fibring of first-order based logics endowed with labelled deduction.

The study of sufficient conditions on a lfob deduction system allowing the extension, in its context, of any consistent set to a set over which the canonical structures could be based, and that may be preserved under suitable requirements by fibring, has been deserving attention by the fibring community [61, 80, 74, 25, 27]. In [74] this study was undertaken in the context of non-labelled first-order based logics endowed with Hilbert-style deduction systems. In [74], the conditions of persistence, uniformity and presence of equality and inequality were proved by the authors to be sufficient for being able to construct the sets of formulae in which the canonical structures can be based on. Although in that work this study was not singled out and separated from the whole proof of completeness.

The chapter is organized as follows. We start by introducing connected lfob logic systems. Then we study a subclass of lfob logic systems with a classical negation and prove that these lfob logic systems are appropriate. Finally, we study under which conditions connected lfob logic systems are appropriate, when we do not assume any kind of negation. We call the clfob logic systems satisfying these conditions clfob logic systems with locality, and prove that clfob logic systems with locality and without disjunction and implication are appropriate.

4.1 General definition

Before proposing the class of connected lfob logic systems we introduce some auxiliary definitions.

Definition 4.1.1 Given a lfob deduction system $\langle \Sigma, R\rangle$, we say a connective c in Σ is

○ a *universal connective*, denoted by c_u, whenever

- there are rules

  $$\frac{r_u\nu\nu_1\ldots\nu_n,\ \nu_1{:}\xi_1,\ \ldots,\ \nu_{n-1}{:}\xi_{n-1}\,/\,\nu_n{:}\xi_n}{\nu{:}c_u\xi_1\ldots\xi_n}\ c_{uI};\ P_d$$

  and
  $$\frac{\nu{:}c_u\xi_1\ldots\xi_n \qquad r_u\nu\nu_1\ldots\nu_n \qquad \nu_1{:}\xi_1 \qquad \ldots \qquad \nu_{n-1}{:}\xi_{n-1}}{\nu_n{:}\xi_n}\ c_{uE}$$

  in R, called, respectively, *introduction* and *elimination* rule for c_u, where

  – r_u is in S,

  – $P_d = \{\text{fresh}(\{\nu_1,\ldots,\nu_n\}, \langle\vartheta_1, \{r_u\nu\nu_1\ldots\nu_n, \nu_1{:}\xi_1,\ldots,\nu_{n-1}{:}\xi_{n-1}\}, \nu_n{:}\xi_n\rangle)\}$,

- c_u is in O_n or is q_x for some q in Q_n and variable x,

○ a *existential connective*, denoted by c_e, whenever

- there are rules

  $$\frac{r_e\nu\nu_1\ldots\nu_n \qquad \nu_1{:}\xi_1 \qquad \ldots \qquad \nu_n{:}\xi_n}{\nu{:}c_e\xi_1\ldots\xi_n}\ c_{eI}$$

  and
  $$\frac{\nu{:}c_e\xi_1\ldots\xi_n \qquad r_e\nu\nu_1\ldots\nu_n,\ \nu_1{:}\xi_1,\ \ldots,\ \nu_n{:}\xi_n\,/\,\nu'{:}\xi'}{\nu'{:}\xi'}\ c_{eE};\ P_d$$

  in R, called, respectively, *introduction* and *elimination* rule for c_e, where

  – r_e is in S,

- P_d is $\{\text{fresh}(\{\nu_1,\ldots,\nu_n\},\langle\vartheta_2,\{r_e\nu\nu_1\ldots\nu_n,\nu_1{:}\xi_1,\ldots,\nu_n{:}\xi_n\},\nu'{:}\xi'\rangle)\}$,

- c_e is in O_n or is q_x for some q in Q_n and variable x,

○ a *constrained universal connective*, denoted by c_{u^+}, whenever

- there are rules
$$\frac{r_{u^+}\nu\nu_1\ldots\nu_n,\eta_1,\ldots,\eta_{2n-1}\,/\,\nu_n{:}\xi_n}{\nu{:}c_{u^+}\xi_1\ldots\xi_n}\,{c_{u^+}}_I;\ P_d$$
and
$$\frac{\nu{:}c_{u^+}\xi_1\ldots\xi_n \qquad r_{u^+}\nu\nu_1\ldots\nu_n \qquad \eta_1 \qquad \ldots \qquad \eta_{2n-1}}{\nu_n{:}\xi_n}\,{c_{u^+}}_E$$
in R, called, respectively, *introduction* and *elimination* rule for c_{u^+}, where
  - $\{\eta_1,\ldots,\eta_{2n-1}\}$ is the set $\{\nu_1{:}\xi_1,\ldots,\nu_{n-1}{:}\xi_{n-1},\nu_1{:}\gamma_1,\ldots,\nu_n{:}\gamma_n\}$,
  - $\gamma_1,\ldots,\gamma_n$ are formulae, called the constraint formulae of c_{u^+},
  - P_d is $\{\text{fresh}(\{\nu_1,\ldots,\nu_n\},\langle\vartheta_1,\{r_{u^+}\nu\nu_1\ldots\nu_n,\eta_1,\ldots,\eta_{2n-1}\},\nu_n{:}\xi_n\rangle)\}$,
- c_{u^+} is in O_n or is q_x for some q in Q_n and variable x,

○ a *constrained existential connective*, denoted by c_{e^+}, whenever

- there are rules
$$\frac{r_{e^+}\nu\nu_1\ldots\nu_n \qquad \eta_1 \qquad \ldots \qquad \eta_{2n}}{\nu{:}c_{e^+}\xi_1\ldots\xi_n}\,{c_{e^+}}_I$$
and
$$\frac{\nu{:}c_{e^+}\xi_1\ldots\xi_n \qquad r_{e^+}\nu\nu_1\ldots\nu_n,\eta_1,\ldots,\eta_{2n}\,/\,\nu'{:}\xi'}{\nu'{:}\xi'}\,{c_{e^+}}_E;\ P_d$$
in R, called, respectively, *introduction* and *elimination* rule for c_{e^+}, where
  - $\{\eta_1,\ldots,\eta_{2n}\}$ is the set $\{\nu_1{:}\xi_1,\ldots,\nu_n{:}\xi_n,\nu_1{:}\gamma_1,\ldots,\nu_n{:}\gamma_n\}$,
  - $\gamma_1,\ldots,\gamma_n$ are formulae, called the constraint formulae of c_{e^+},
  - P_d is $\{\text{fresh}(\{\nu_1,\ldots,\nu_n\},\langle\vartheta_1,\{r_{e^+}\nu\nu_1\ldots\nu_n,\eta_1,\ldots,\eta_{2n}\},\nu'{:}\xi'\rangle)\}$,
- c_{e^+} is in O_n or is q_x for some q in Q_n and variable x,

○ the *non-local negation connective*, denoted by $-$, whenever

- there are rules
$$\frac{\nu^*{:}\xi\,/\,\nu'{:}\bot}{\nu{:}-\xi}\,-_I \qquad \text{and} \qquad \frac{\nu{:}-\xi \qquad \nu^*{:}\xi}{\nu'{:}\bot}\,-_E$$
in R, called, respectively, *introduction* and *elimination* rule for $-$, where
  - $\bot$ is a connective in C_0,
  - * is in F^l_1,

- $-$ is in O_1,

○ the *local negation connective*, denoted by $\neg$, whenever

- there are rules

$$\frac{\nu{:}\xi \,/\, \nu'{:}\bot}{\nu{:}\neg\xi}\ \neg_I \qquad \text{and} \qquad \frac{\nu{:}\neg\xi \qquad \nu{:}\xi}{\nu'{:}\bot}\ \neg_E$$

  in R, called, respectively, *introduction* and *elimination* rule for $\neg$, where

  - $\bot$ is a connective in C_0,

- $\neg$ is in C_1,

○ the *incoherence connective*, denoted by $\bot$, whenever there is a local or a non-local negation connective in Σ whose associated rules use that incoherence connective $\bot$,

○ the *disjunction connective*, denoted by $\vee$, whenever

- there are rules

$$\frac{\nu{:}\xi_1 \vee \xi_2 \quad \nu{:}\xi_1 \,/\, \nu''{:}\xi'' \quad \nu{:}\xi_2 \,/\, \nu''{:}\xi''}{\nu''{:}\xi''}\ \vee_E \qquad \frac{\nu{:}\xi_1}{\nu{:}\xi_1 \vee \xi_2}\ \vee_I^1 \qquad \frac{\nu{:}\xi_2}{\nu{:}\xi_1 \vee \xi_2}\ \vee_I^2$$

  in R, called, respectively, *introduction* and *elimination* rule for $\vee$,

- $\vee$ is in C_2,

○ the *conjunction connective*, denoted by $\wedge$, whenever

- there are rules

$$\frac{\nu{:}\xi_1 \qquad \nu{:}\xi_2}{\nu{:}\xi_1 \wedge \xi_2}\ \wedge_I \qquad \frac{\nu{:}\xi_1 \wedge \xi_2}{\nu{:}\xi_1}\ \wedge_E^1 \qquad \frac{\nu{:}\xi_1 \wedge \xi_2}{\nu{:}\xi_2}\ \wedge_E^2$$

  in R, called, respectively, *introduction* and *elimination* rule for $\wedge$,

- $\wedge$ is in C_2,

○ the *top connective*, denoted by $\top$, whenever

- there is rule

$$\frac{}{\nu{:}\top}\ \top_I$$

  in R, called the *introduction* rule for $\top$,

- $\top$ is in C_0,

∘ the *implication connective*, denoted by →, whenever

- there are rules

$$\frac{\nu{:}\xi_1 \,/\, \nu{:}\xi_2}{\nu{:}\xi_1 \to \xi_2} \to_I \qquad \text{and} \qquad \frac{\nu{:}\xi_1 \to \xi_2 \qquad \nu{:}\xi_1}{\nu{:}\xi_2} \to_E$$

  in R, called, respectively, *introduction* and *elimination* rule for →,

- → is in C_2.

We are now ready to introduce connected lfob logic systems.

Definition 4.1.2 A *connected* lfob, clfob, deduction system $\langle \Sigma, R\rangle$ is a lfob deduction system where

each connective in C, Q or O, is either

- a universal connective, or
- an existential connective, or
- a constrained universal connective, or
- a constrained existential connective, or
- the non local negation, or
- the local negation, or
- the incoherence, or
- the disjunction, or
- the conjunction, or
- the top, or
- the implication,

and each specific rule, not common to all lfob deduction systems (see Definition 2.3.10), $\langle\{\langle\vartheta_1, \Psi_1, \eta_1\rangle, \ldots, \langle\vartheta_k, \Psi_k, \eta_k\rangle\}, \eta, P_f, P_d\rangle$ of R, named r, is such that

- either r is an introduction or an elimination rule for some connective in C, Q or O,

- or r is

$$\frac{\nu{:}-\xi \,/\, \nu'{:}\bot}{\nu^*{:}\xi} -_c$$

  called *non-local classical negation rule for* −, where − is the non-local negation in Σ (this rule may also be denoted by $\bot_E$ when there is no ambiguity),

- or r is

$$\frac{\nu{:}\neg\xi \,/\, \nu'{:}\bot}{\nu{:}\xi}\ \neg_c$$

called *local classical negation rule for* $\neg$, where $\neg$ is the local negation in Σ (this rule may also be denoted by $\bot_E$ when there is no ambiguity),

- or r is

$$\frac{\nu'{:}\bot}{\nu{:}\xi}\ c_i$$

called *intuitionistic negation rule for* c, where c is a negation in Σ,

- or r is

$$\frac{\nu{:}c_u\,\xi}{\nu{:}\xi'}\ {c_u}^{\mathrm{spc}};\ P_f$$

called *specification rule for* c_u, where c_u is a universal connective either in O_1 or equal to q_x for some q in Q_1 and variable x, and P_f has a proviso relating ξ' and ξ,

- or r is

$$\frac{\nu{:}c_{u^+}\,\xi \qquad \nu{:}\gamma}{\nu{:}\xi'}\ {c_{u^+}}^{\mathrm{spc}};\ P_f$$

called *constraint specification rule for* c_{u^+}, where c_{u^+} is a constraint universal connective either in O_1 or equal to q_x for some q in Q_1 and variable x, γ is the constraint formula for c_{u^+}, and P_f has a proviso relating ξ' and ξ,

- or r is

$$\frac{s\,\nu_1\ldots\nu_n}{\nu_i{:}\theta =_g \nu_j{:}\theta}\ s_{g^{i,j}};\ P_f,$$

called *term inheritance rule over* s *from* i *to* j where s is in S and P_f has a proviso over θ,

- or r is

$$\frac{s\,v_1\ldots v_n \qquad \nu_i{:}\varphi}{\nu_{i'}{:}\varphi}\ ;\ P_f$$

called *schema formula inheritance rule over* s *from* i *to* i', where P_f is either the empty set or a set with a proviso over φ,

- or r is

$$\frac{\nu \equiv_\omega \nu' \qquad \nu{:}\theta_1 =_g \nu'{:}\theta_1 \quad \ldots \quad \nu{:}\theta_k =_g \nu'{:}\theta_k \qquad \nu{:}\xi}{\nu'{:}\xi}\ gmon^k_{\equiv_\omega};\ P_f$$

called *general formula inheritance rule over* $\equiv_\omega$, where k is greater than or equal to 0 and P_f is the singleton with the proviso $\text{p-gmon}(\xi, \theta_1, \ldots, \theta_k)$ defined as follows, $\text{p-gmon}(\xi, \theta_1, \ldots, \theta_k)_\Sigma(\rho_{\mathrm{nl}}) = 1$ iff $\xi\rho_f$ is a formula whose free variables are

$\theta_1\rho_t, \ldots, \theta_k\rho_t$,

- or r is
$$\frac{s\,\nu_1 \ldots \nu_n}{\nu_i \equiv_\omega \nu_j}\ s_{\omega^{i,j}}$$
called *i, j docked rule in $\equiv_\omega$ for s*, (similarly for $\equiv_\alpha$), where s is in S,

- or r is
$$\frac{}{\nu_i \equiv_\omega f^l(\nu_1, \ldots, \nu_n)}\ f^l{}_{\equiv_\omega{}^i}$$
called *i label docked rule in $\equiv_\omega$ for f^l*, (similarly for $\equiv_\alpha$),

- or r is
$$\frac{\eta_1 \qquad \ldots \qquad \eta_k}{\eta}$$
called *relational rule for s*, where η and $\eta_1, \ldots, \eta_k$ are relational schema formulae over s, s is in S, and if $k \neq 0$ then $\mathrm{lsv}(\eta) \subseteq \mathrm{lsv}(\cup_{j=1,\ldots,k}\{\eta_j\})$,

- or r is
$$\frac{s\,\nu_1 \ldots \nu_n \qquad \nu_1 \equiv_\omega \nu'_1 \quad \ldots \quad \nu_n \equiv_\omega \nu'_n \qquad \nu'_1 \equiv_\alpha \nu'_2 \quad \ldots \quad \nu'_{n-1} \equiv_\alpha \nu'_n}{s\,\nu'_1 \ldots \nu'_n}\ s^{\mathrm{gen}}_\alpha$$
called *generalization rule for s over* $\equiv_\alpha$, where s is in S, (similarly for $\equiv_\omega$),

- or r is
$$\frac{s\,\nu_1 \ldots \nu_n \qquad \nu_1 \equiv_\omega \nu'_1 \quad \ldots \quad \nu_n \equiv_\omega \nu'_n}{s\,\nu'_1 \ldots \nu'_n}\ s^{\mathrm{lgen}}_\alpha$$
called *laxed generalization rule for s over* $\equiv_\alpha$, where s is in S, (similarly for $\equiv_\omega$),

- or r is
$$\frac{\nu_1 \equiv_\omega \nu'_1 \quad \ldots \quad \nu_k \equiv_\omega \nu'_k}{f(\nu_1, \ldots, \nu_k) \equiv_\omega f(\nu'_1, \ldots, \nu'_k)}\ f^{\mathrm{lgen}}_\alpha$$
called *laxed generalization rule for f over* $\equiv_\alpha$, where f is in F^l_k, (similarly for $\equiv_\omega$),

- or r is
$$\frac{\nu_1 \equiv_\omega \nu', \nu' \equiv_\alpha \nu_2 \,/\, \nu'':\xi''}{\nu'':\xi''}\ \mathit{exh};\ \mathrm{fresh}(\nu', \langle \vartheta_1, \{\nu_1 \equiv_\omega \nu', \nu' \equiv_\alpha \nu_2\}, \nu'':\xi''\rangle)$$
called the *exhaustiveness rule*,

- or r is

  $$\frac{}{\nu_1{:}\theta_1 =_g \nu_2{:}\theta_2}\ \textit{1 ind}$$

  called the *1 individual rule*,

- or r is

  $$\frac{}{\nu_1 \equiv_\alpha \nu_2}\ \textit{1 assg}$$

  called the *1 assignment rule*,

and, for each relational symbol s appearing in an introduction or elimination rule for a connective in Q there is a generalization rule for s over $\equiv_\omega$ and docked rules for all its components over $\equiv_\omega$,

and, for each relational symbol s appearing in an introduction or elimination rule for a connective in O,

- there is a laxed generalization rule for s over $\equiv_\alpha$ iff there are no docked rules for s over $\equiv_\alpha$,
- there is a generalization rule for s over $\equiv_\alpha$ iff there are docked rules for s for all its components over $\equiv_\alpha$,

and, there is a docked rule in $\equiv_\omega$ for all components of a label function symbol appearing in a relational rule for a relation which appears in an introduction or elimination rule for a connective in Q.

Definition 4.1.3 A *connected* lfob, clfob, logic system is a lfob logic system constituted by a connected lfob deduction system.

Note that the class of connected lfob logic systems is quite large and encompasses lfob logic systems for logics like, relevance logic R (see Example 5.3.1), basic relevance logic B, some positive fragments of B (see Example 5.4.1), first-order modal logic with constant, increasing, decreasing or varying domains (see Example 5.1.1), some positive fragments of first-order modal logic (see Example 5.2.1), and modal logics, among others. Observe that it is straightforward to show that in the context of a connected lfob logic system, the deduction of relational schema formulae involves only relational schema formulae.

4.2 Strong appropriate sets

There are cases, as we see in Chapter 5 in the example for the $\wedge$ fragment of first-order modal logic T and in the example for first-order modal logic with decreasing domains, that we need to consider canonical structures over more specific appropriate sets. This happens because in order to guarantee that a labelled presentation corresponds to a specific logic, we have in certain cases to show that all its structures satisfy certain properties. One such property of the canonical structure is that its set of points can be seen as forming a grid, i.e., any two points in the canonical structure are either in the same world, or in the same assignment, or there is another point in the same world of one and in the same assignment of the other. This condition is satisfied by canonical structures over strong appropriate sets, i.e. sets such that there exists a label schema term v with $v_1 \equiv_\omega v$ and $v \equiv_\alpha v_2$ in the set, for every label schema terms v_1 and v_2 in the set. Note that this condition may not be satisfied by all canonical structures over appropriate sets. This condition is very natural when dealing with first-order modal logics. We did not impose it in appropriate sets since we want that the completeness results apply also to other types of logic in which this property is not essential, like propositional based logics, as relevance logics and modal logics.

Definition 4.2.1 An E appropriate set Ψ is *strong* whenever there is a label schema term v in $T_{\text{lab},E}$ with $v_1 \equiv_\omega v$ and $v \equiv_\alpha v_2$ in Ψ for any label schema terms v_1 and v_2 in $T_{\text{lab},E}$.

Definition 4.2.2 A lfob deduction system is said to be *strong appropriate* iff for every $v{:}\varphi$ consistent set Ψ_0 within L, there exists a $v{:}\varphi$ consistent and E strong appropriate set containing Ψ_0. A lfob logic system is *strong appropriate* iff its underlying lfob deduction system is strong appropriate.

4.3 Connected logic systems with a classical negation

The main goal of this section is to prove that any clfob logic system with a classical negation is appropriate, or strong appropriate when rule *exh* is in the lfob logic system. Before showing that, we introduce the notion of a clfob deduction system with a classical negation and prove some useful propositions.

Definition 4.3.1 A clfob deduction system $\langle \Sigma, R \rangle$ with a *non-local classical negation* $-$ is a clfob deduction system where

- R contains the rules

  $$-_c \qquad\qquad \frac{}{r\,0\nu\nu^{**}}\;{**_i} \qquad\qquad \frac{}{r\,0\nu^{**}\nu}\;{**_c} \qquad\qquad \frac{r\,0\nu_1\nu_2 \qquad \nu_1{:}\xi}{\nu_2{:}\xi}\;mon_{r\,0}$$

  where 0 is in F^l_0, $*$ is in F^l_1, r is in S_3, and

- $-$ is a non-local negation in $\langle \Sigma, R \rangle$.

Definition 4.3.2 A clfob deduction system $\langle \Sigma, R \rangle$ with a *local classical negation* $\neg$ is a clfob deduction system where R contains the rule $\neg_c$, and $\neg$ is a local negation in $\langle \Sigma, R \rangle$.

Finally,

Definition 4.3.3 A clfob deduction system *with a classical negation* is a clfob deduction system with either a non-local classical negation or a local classical negation. A clfob logic system *with a classical negation* is a lfob logic system composed of a clfob deduction system with a classical negation.

We now show two useful properties involving negation in the context of a clfob deduction system with a local classical negation.

Proposition 4.3.4 In a clfob deduction system *with a local classical negation* $\neg$ we have

$$\frac{\Psi_1, v{:}\varphi \vdash_E v'{:}\varphi' \qquad \text{and} \qquad \Psi_2, v{:}\neg\varphi \vdash_E v'{:}\varphi'}{\Psi_1, \Psi_2 \vdash_E v'{:}\varphi'}$$

and

$$\frac{\Psi_1 \vdash_E v{:}\varphi \qquad \text{and} \qquad \Psi_2 \vdash_E v{:}\neg\varphi}{\Psi_1, \Psi_2 \vdash_E v'{:}\varphi'}$$

for each set E of label schema variables, set Ψ of labelled schema formulae within L_E, and labelled schema formulae $v{:}\varphi$, $v{:}\neg\varphi$ and $v'{:}\varphi'$ in L_E.

Proof Suppose ϖ_1 ending in $\langle v'{:}\varphi', \Psi'_1, 1\rangle$ is a sober deduction sequence for $\Psi_1, v{:}\varphi \vdash_E v'{:}\varphi'$ and ϖ_2 ending in $\langle v'{:}\varphi', \Psi'_2, 1\rangle$ is a sober deduction sequence for $\Psi_2, v{:}\neg\varphi \vdash_E v'{:}\varphi'$. Then,

1	$v'{:}\neg\varphi'$	$\{v'{:}\neg\varphi'\}$	1	asp	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	ϖ_1	
i	$v'{:}\varphi'$	Ψ'_1	1		
$i+1$	$v'{:}\bot$	$\Psi'_1 \cup \{v'{:}\neg\varphi'\}$	1	$\neg_E$	$1, i$
$i+2$	$v{:}\neg\varphi$	$(\Psi'_1 \setminus \{v{:}\varphi\}) \cup \{v'{:}\neg\varphi'\}$	1	$\neg_I$	$i+1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	ϖ_2	
j	$v'{:}\varphi'$	Ψ'_2	1		
$j+1$	$v'{:}\bot$	$\Psi'_2 \cup \{v'{:}\neg\varphi'\}$	1	$\neg_E$	$1, j$
$j+2$	$v{:}\varphi$	$(\Psi'_2 \setminus \{v{:}\neg\varphi\}) \cup \{v'{:}\neg\varphi'\}$	1	$\neg_c$	$j+1$
$j+3$	$v{:}\bot$	$(\Psi'_1 \setminus \{v{:}\varphi\}) \cup (\Psi'_2 \setminus \{v{:}\neg\varphi\}) \cup \{v'{:}\neg\varphi'\}$	1	$\neg_E$	$j+2, i+2$
$j+4$	$v'{:}\varphi'$	$(\Psi'_1 \setminus \{v{:}\varphi\}) \cup (\Psi'_2 \setminus \{v{:}\neg\varphi\})$	1	$\neg_c$	$j+3$

is a deduction sequence for $\Psi_1, \Psi_2 \vdash_E v'{:}\varphi'$ as we wanted to show.
For the other meta-theorem suppose $\Psi_1 \vdash_E v{:}\varphi$ and $\Psi_2 \vdash_E v{:}\neg\varphi$. Then, it is immediate to show that there is a deduction sequence for $\Psi_1, \Psi_2 \vdash_E v'{:}\varphi'$. *QED*

Similarly to the preceding proposition, we now show two useful properties involving negation in the context of a clfob deduction system with a non-local classical negation.

Proposition 4.3.5 In a clfob deduction system *with a non-local classical negation* $-$ we have

$$\frac{\Psi_1, v{:}\varphi \vdash_E v'{:}\varphi' \qquad \text{and} \qquad \Psi_2, v^*{:}-\varphi \vdash_E v'{:}\varphi'}{\Psi_1, \Psi_2 \vdash_E v'{:}\varphi'}$$

and

$$\frac{\Psi_1 \vdash_E v{:}\varphi \qquad \text{and} \qquad \Psi_2 \vdash_E v^*{:}-\varphi}{\Psi_1, \Psi_2 \vdash_E v'{:}\varphi'}$$

for each set E of label schema variables, set Ψ of labelled schema formulae within L_E, and labelled schema formulae $v{:}\varphi$, $v^*{:}-\varphi$ and $v'{:}\varphi'$ in L_E.

Proof Suppose $\Psi_1, v{:}\varphi \vdash_E v'{:}\varphi'$ and that ϖ_2 ending in $\langle v'{:}\varphi', \Psi'_2, 1\rangle$ is a sober deduction sequence for $\Psi_2, v^*{:}-\varphi \vdash_E v'{:}\varphi'$. Let also ϖ_1 ending in $\langle v'{:}\varphi', \Psi'_1, 1\rangle$ be a sober deduction sequence for $\Psi_1, v^{**}{:}\varphi \vdash_E v'{:}\varphi'$ which exists by idempotence, Proposition 2.3.15, since $v^{**}{:}\varphi \vdash_E v{:}\varphi$. Then,

$$\begin{array}{llllll}
1 & v'^{*}{:}-\varphi' & \{v'^{*}{:}-\varphi'\} & 1 & \text{asp} & \\
\vdots & \vdots & \vdots & \vdots & \varpi_1 & \\
i & v'{:}\varphi' & \Psi'_1 & 1 & & \\
i+1 & r\,0\,v'v'^{**} & \emptyset & 1 & **_i & \\
i+2 & v'^{**}{:}\varphi' & \Psi'_1 & 1 & \mathit{mon}_{r\,0} & i+1, i \\
i+3 & v'{:}\bot & \Psi'_1 \cup \{v'^{*}{:}-\varphi'\} & 1 & -_E & i+2, 1 \\
i+4 & v^{*}{:}-\varphi & (\Psi'_1 \setminus \{v^{**}{:}\varphi\}) \cup \{v'^{*}{:}-\varphi'\} & 1 & -_I & i+3 \\
\vdots & \vdots & \vdots & \vdots & \varpi_2 & \\
j & v'{:}\varphi' & \Psi'_2 & 1 & & \\
j+1 & r\,0\,v'v'^{**} & \emptyset & 1 & **_i & \\
j+2 & v'^{**}{:}\varphi' & \Psi'_2 & 1 & \mathit{mon}_{r\,0} & j+1, j \\
j+3 & v'{:}\bot & \Psi'_2 \cup \{v'^{*}{:}-\varphi'\} & 1 & -_E & j+2, 1 \\
j+4 & v^{**}{:}\varphi & (\Psi'_2 \setminus \{v^{*}{:}-\varphi\}) \cup \{v'^{*}{:}-\varphi'\} & 1 & -_c & j+3 \\
j+5 & v{:}\bot & \Psi'' & 1 & -_E & j+4, i+4 \\
j+6 & v'^{**}{:}\varphi' & (\Psi'_1 \setminus \{v^{**}{:}\varphi\}) \cup (\Psi'_2 \setminus \{v^{*}{:}-\varphi\}) & 1 & -_c & j+5 \\
j+7 & r\,0\,v'^{**}v' & \emptyset & 1 & **_c & \\
j+8 & v'{:}\varphi' & (\Psi'_1 \setminus \{v^{**}{:}\varphi\}) \cup (\Psi'_2 \setminus \{v^{*}{:}-\varphi\}) & 1 & \mathit{mon}_{r\,0} & j+7, j+6
\end{array}$$

is a deduction sequence for $\Psi_1, \Psi_2 \vdash_E v'{:}\varphi'$ as we wanted to show. Note that Ψ'' is $(\Psi'_1 \setminus \{v^{**}{:}\varphi\}) \cup (\Psi'_2 \setminus \{v^{*}{:}-\varphi\}) \cup \{v'^{*}{:}-\varphi'\}$. For the other meta-theorem suppose ϖ_1 ending in $\langle v{:}\varphi, \Psi'_1, 1\rangle$ is a deduction sequence for $\Psi_1 \vdash_E v{:}\varphi$ and ϖ_2 ending in $\langle v^{*}{:}-\varphi, \Psi'_2, 1\rangle$ is a deduction sequence for $\Psi_2 \vdash_E v^{*}{:}-\varphi$. Then,

$$\begin{array}{llllll}
\vdots & \vdots & \vdots & \vdots & \varpi_1 & \\
i & v{:}\varphi & \Psi'_1 & 1 & & \\
i+1 & r\,0\,vv^{**} & \emptyset & 1 & **_i & \\
i+2 & v^{**}{:}\varphi & \Psi'_1 & 1 & \mathit{mon}_{r\,0} & i+1, i \\
\vdots & \vdots & \vdots & \vdots & \varpi_2 & \\
j & v^{*}{:}-\varphi & \Psi'_2 & 1 & & \\
j+1 & v{:}\bot & \Psi'_1 \cup \Psi'_2 & 1 & -_E & i+2, j \\
j+2 & v'^{**}{:}\varphi' & (\Psi'_1 \cup \Psi'_2) \setminus \{v'^{*}{:}-\varphi'\} & 1 & -_c & j+1 \\
j+3 & r\,0\,v'^{**}v' & \emptyset & 1 & **_c & \\
j+4 & v'{:}\varphi' & (\Psi'_1 \cup \Psi'_2) \setminus \{v'^{*}{:}-\varphi'\} & 1 & \mathit{mon}_{r\,0} & j+2, j+3
\end{array}$$

is a deduction sequence for $\Psi_1, \Psi_2 \vdash_E v'{:}\varphi'$ for any $v'{:}\varphi'$ in L_E. *QED*

The next two propositions say that we can define a connective that behaves like a (constrained) existential connective, from a (constrained) universal connective, because we can prove, in deduction terms, its introduction and elimination rules. This will be important in the proof of Proposition 4.3.8, in order to guarantee that the addition of witnesses for the presence in that set of labelled schema formulae whose main connective is the negation of a (constrained) universal connective does not imply non wanted deductions.

Proposition 4.3.6 Let $\mathcal{D}$ be a clfob deduction system with the local classical negation connective $\neg$. Then, we have $\Psi \vdash_E v'{:}\varphi'$ whenever

$$\Psi \vdash_E v{:}\neg c\,\varphi_1 \dots \varphi_n \qquad \text{and} \qquad \Psi, \Psi' \vdash_E v'{:}\varphi'$$

and c in Q_n or in O_n is either a universal connective, and Ψ' is $\{r\,v\nu_1 \dots \nu_n, \nu_1{:}\varphi_1, \dots, \nu_{n-1}{:}\varphi_{n-1}, \nu_n{:}\neg\varphi_n\}$, or a constrained universal connective, and Ψ' is $\{r\,v\nu_1 \dots \nu_n, \nu_1{:}\varphi_1, \dots, \nu_{n-1}{:}\varphi_{n-1}, \nu_n{:}\neg\varphi_n, \nu_1{:}\gamma_1, \dots, \nu_n{:}\gamma_n\}$, for any set of label schema variables E and label schema variables $\nu_1, \dots, \nu_n$ in E but not in $\mathrm{lsv}(\Psi) \cup \mathrm{lsv}(\{v, v'\})$.

Proof Let E be a set of label schema variables, Ψ a set within L_E, c in Q_n or in O_n a constrained universal connective, and Ψ' the set $\{r\,v\nu_1 \dots \nu_n, \nu_1{:}\varphi_1, \dots, \nu_{n-1}{:}\varphi_{n-1}, \nu_n{:}\neg\varphi_n, \nu_1{:}\gamma_1, \dots, \nu_n{:}\gamma_n\}$, such that $\nu_1, \dots, \nu_n$ are in E but not in $\mathrm{lsv}(\Psi) \cup \mathrm{lsv}(\{v, v'\})$. Suppose $\Psi \vdash_E v{:}\neg c\,\varphi_1 \dots \varphi_n$ and $\Psi, \Psi' \vdash_E v'{:}\varphi'$. Then, from $\Psi, \Psi' \vdash_E v'{:}\varphi'$ and $v'{:}\neg\varphi' \vdash_E v'{:}\neg\varphi'$ we conclude $\Psi, \Psi', v'{:}\neg\varphi' \vdash_E \nu_n{:}\varphi_n$ by Proposition 4.3.4 since $\neg$ is a local classical negation. And, from this deduction and from the fact that $\nu_n{:}\varphi_n \vdash_{\nu_n} \nu_n{:}\varphi_n$ we have that $\Psi, \Psi' \setminus \{\nu_n{:}\neg\varphi_n\}, v'{:}\neg\varphi' \vdash_E \nu_n{:}\varphi_n$. So, by Proposition 3.5.10, we obtain $\Psi, v'{:}\neg\varphi' \vdash_E v{:}c\,\varphi_1 \dots \varphi_n$. Moreover, from this deduction and from the assumption $\Psi \vdash_E v{:}\neg c\,\varphi_1 \dots \varphi_n$ we conclude $\Psi, v'{:}\neg\varphi' \vdash_E v'{:}\varphi'$ by Proposition 4.3.4 since $\neg$ is a local classical negation. Finally from this deduction and from the fact that $v'{:}\varphi' \vdash_E v'{:}\varphi'$ we conclude also by Proposition 4.3.4 that $\Psi \vdash_E v'{:}\varphi'$, as wanted.

The proof follows in a similar way whenever c is a universal connective and Ψ' is $\{r\,v\nu_1 \dots \nu_n, \nu_1{:}\varphi_1, \dots, \nu_{n-1}{:}\varphi_{n-1}, \nu_n{:}\neg\varphi_n\}$. *QED*

Similarly to the preceding proposition, we now show that we can define from a (constrained) universal connective, as an abbreviation, another connective that behaves similarly to a (constrained) existential connective, in the context of clfob deduction systems with a non-local classical negation.

Proposition 4.3.7 Let $\mathcal{D}$ be a clfob deduction system with the non-local classical negation connective $-$. Then, we have $\Psi \vdash_E v'{:}\varphi'$ whenever

$$\Psi \vdash_E v^*{:}-c\,\varphi_1 \dots \varphi_n \qquad \text{and} \qquad \Psi, \Psi' \vdash_E v'{:}\varphi'$$

and c in Q_n or in O_n is either a universal connective, and Ψ' is $\{r\,v\nu_1 \dots \nu_n, \nu_1{:}\varphi_1, \dots, \nu_{n-1}{:}\varphi_{n-1}, \nu_n^*{:}-\varphi_n\}$, or a constrained universal connective, and Ψ' is $\{r\,v\nu_1 \dots \nu_n, \nu_1{:}\varphi_1, \dots, \nu_{n-1}{:}\varphi_{n-1}, \nu_n^*{:}-\varphi_n, \nu_1{:}\gamma_1, \dots, \nu_n{:}\gamma_n\}$, for any set of label schema variables E and label schema variables $\nu_1, \dots, \nu_n$ in E but not in $\mathrm{lsv}(\Psi) \cup \mathrm{lsv}(\{v, v'\})$.

Proof Let E be a set of label schema variables, Ψ a set within L_E, c in Q_n or in O_n a constrained universal connective, and Ψ' the set $\{r\,v\nu_1 \dots \nu_n, \nu_1{:}\varphi_1, \dots, \nu_{n-1}{:}\varphi_{n-1}, \nu_n^*{:}-\varphi_n, \nu_1{:}\gamma_1, \dots, \nu_n{:}\gamma_n\}$, such that $\nu_1, \dots, \nu_n$ are in E but not in $\mathrm{lsv}(\Psi) \cup \mathrm{lsv}(\{v, v'\})$. Suppose $\Psi \vdash_E v^*{:}-c\,\varphi_1 \dots \varphi_n$ and $\Psi, \Psi' \vdash_E v'{:}\varphi'$. Then, from $\Psi, \Psi' \vdash_E v'{:}\varphi'$ and $v'^*{:}-\varphi' \vdash_E v'^*{:}-\varphi'$ we conclude $\Psi, \Psi', v'^*{:}-\varphi' \vdash_E \nu_n{:}\varphi_n$ by Proposition 4.3.5 since $-$ is a non-local classical negation. And, from this deduction and from the fact that $\nu_n{:}\varphi_n \vdash_{\nu_n} \nu_n{:}\varphi_n$ we have that $\Psi, \Psi' \setminus \{\nu_n^*{:}-\varphi_n\}, v'^*{:}-\varphi' \vdash_E \nu_n{:}\varphi_n$. So, by Proposition 3.5.10, we obtain $\Psi, v'^*{:}-\varphi' \vdash_E v{:}c\,\varphi_1 \dots \varphi_n$. Moreover, from this deduction and from the assumption

$\Psi \vdash_E v^*{:}{-c}\,\varphi_1 \ldots \varphi_n$ we conclude $\Psi, v'^*{:}{-\varphi'} \vdash_E v'{:}\varphi'$ by Proposition 4.3.5 since $-$ is a non-local classical negation. Finally from this deduction and from the fact that $v'{:}\varphi' \vdash_E v'{:}\varphi'$ we conclude also Proposition 4.3.5 that $\Psi \vdash_E v'{:}\varphi'$, as wanted.

The proof follows in a similar way whenever c is a universal connective and Ψ' is $\{r\, v\nu_1 \ldots \nu_n,\, \nu_1{:}\varphi_1,\, \ldots,\, \nu_{n-1}{:}\varphi_{n-1}, \nu_n^*{:}{-\varphi_n}\}$. *QED*

Proposition 4.3.8 Let $\mathcal{D}$ be a clfob deduction system with a classical negation connective. Then, for each $v'{:}\varphi'$ consistent set Ψ° contained in L there are a set E of label schema variables, and a set Ψ within L_E, with

1. Ψ is $v'{:}\varphi'$ consistent,
2. $v{:}\varphi \notin \Psi$ iff $\Psi, v{:}\varphi \vdash_E v'{:}\varphi'$ for any $v{:}\varphi$ in L_E,
3. Ψ extends Ψ°,
4. Ψ is deductively closed,
5. if *exh* is in $\mathcal{D}$ then for any v and $v^{\cdot}$ in $T_{\mathrm{lab},E}$ there is v_1 in $T_{\mathrm{lab},E}$ with $v \equiv_\omega v_1$ and $v_1 \equiv_\alpha v^{\cdot}$ in Ψ,
6. $v{:}\varphi \notin \Psi$ iff $v{:}\neg\varphi \in \Psi$, if $\neg$ is a local classical negation in $\mathcal{D}$,
7. $v{:}\varphi \notin \Psi$ iff $v^*{:}{-\varphi} \in \Psi$, if $-$ is a non-local classical negation in $\mathcal{D}$,
8. $v{:}c_u\varphi_1 \ldots \varphi_n \in \Psi$ iff for any $v_1, \ldots, v_n$ in $T_{\mathrm{lab},E}$, we have $v_n{:}\varphi_n \in \Psi$ whenever
$$r_u v v_1 \ldots v_n \in \Psi \quad \text{and} \quad v_1{:}\varphi_1 \in \Psi \quad \text{and} \quad \ldots \quad \text{and} \quad v_{n-1}{:}\varphi_{n-1} \in \Psi,$$
if c_u is a universal connective in $\mathcal{D}$,
9. $v{:}c_{u^+}\varphi_1 \ldots \varphi_n \in \Psi$ iff for any $v_1, \ldots, v_n$ in $T_{\mathrm{lab},E}$, we have $v_n{:}\varphi_n \in \Psi$ whenever
$$r_{u^+} v v_1 \ldots v_n \in \Psi \quad \text{and} \quad v_1{:}\varphi_1 \in \Psi \quad \text{and} \quad \ldots \quad \text{and} \quad v_{n-1}{:}\varphi_{n-1} \in \Psi$$
$$\text{and} \quad v_1{:}\gamma_1 \in \Psi \quad \text{and} \quad \ldots \quad \text{and} \quad v_n{:}\gamma_n \in \Psi,$$
if c_{u^+} is a constrained universal connective in $\mathcal{D}$, and $\gamma_1, \ldots, \gamma_n$ are the constraint formulae of c_{u^+},
10. $v{:}c_e\varphi_1 \ldots \varphi_n \in \Psi$ iff there are $v_1, \ldots, v_n$ in $T_{\mathrm{lab},E}$, with
$$r_e v v_1 \ldots v_n \in \Psi \quad \text{and} \quad v_1{:}\varphi_1 \in \Psi \text{ and } \ldots \text{ and } v_n{:}\varphi_n \in \Psi,$$
if c_e is an existential connective in $\mathcal{D}$,
11. $v{:}c_{e^+}\varphi_1 \ldots \varphi_n \in \Psi$ iff there are $v_1, \ldots, v_n$ in $T_{\mathrm{lab},E}$ with
$$r_{e^+} v v_1 \ldots v_n \in \Psi \quad \text{and} \quad v_1{:}\varphi_1 \in \Psi \quad \text{and} \quad \ldots \quad \text{and} \quad v_n{:}\varphi_n \in \Psi,$$
$$\text{and} \quad v_1{:}\gamma_1 \in \Psi \quad \text{and} \quad \ldots \quad \text{and} \quad v_n{:}\gamma_n \in \Psi,$$
if c_{e^+} is a constrained universal connective in $\mathcal{D}$, and $\gamma_1, \ldots, \gamma_n$ are the constraint formulae of c_{e^+},
12. $v{:}{-\varphi} \in \Psi$ iff $v^*{:}\varphi \in \Psi$ implies $v'{:}\bot \in \Psi$ for some v', if $-$ is a non-local negation in $\mathcal{D}$,

13. $\upsilon{:}\neg\varphi \in \Psi$ iff $\upsilon{:}\varphi \in \Psi$ implies $\upsilon'{:}\bot \in \Psi$ for some υ', if $\neg$ is a local negation in $\mathcal{D}$,

14. $\upsilon{:}\varphi_1 \wedge \varphi_2 \in \Psi$ iff $\upsilon{:}\varphi_1 \in \Psi$ and $\upsilon{:}\varphi_2 \in \Psi$, if $\wedge$ is a conjunction in $\mathcal{D}$,

15. $\upsilon{:}\varphi_1 \vee \varphi_2 \in \Psi$ iff $\upsilon{:}\varphi_1 \in \Psi$ or $\upsilon{:}\varphi_2 \in \Psi$, if $\vee$ is a disjunction in $\mathcal{D}$,

16. $\upsilon{:}\varphi_1 \rightarrow \varphi_2 \in \Psi$ iff $\upsilon{:}\varphi_1 \in \Psi$ implies $\upsilon{:}\varphi_2 \in \Psi$, if $\rightarrow$ is a implication in $\mathcal{D}$,

17. and, if for each $j = 1, \dots, k$, and ρ_j over E within L_E, $\Upsilon_j\sigma$ co-equivalent to id over E within L_E

$$\Psi_j\sigma\rho_j \subseteq \Psi \quad \text{implies} \quad \eta_j\sigma\rho_j \in \Psi$$

then

$$\eta\sigma \in \Psi$$

for every

- $\langle\{\langle\vartheta_1, \Psi_1, \eta_1\rangle, \dots, \langle\vartheta_k, \Psi_k, \eta_k\rangle\}, \eta, P_f, P_d\rangle$ in R,
- σ within L_E such that $(\pi_\Sigma\sigma_{\text{nl}})$ is 1 for each π in P_f, and $\pi_{\Sigma;E}(\sigma)$ is 1 for each π in P_d.

Proof Let Ψ° be a $\upsilon'{:}\varphi'$ consistent set in L and E an infinite set of label schema variables disjoint of Ξ_l. Consider an enumeration $\upsilon_1{:}\varphi_1, \upsilon_2{:}\varphi_2, \dots$ of all label schema terms $\upsilon_1, \upsilon_2, \dots$ in $T_{\text{lab},E}$ and schema formulae $\varphi_1, \varphi_2, \dots$ in $L_{\text{fob},E}$. We inductively define the sequence $\Psi_0 \subseteq \Psi_1 \subseteq \dots$ as follows:

Base: Ψ_0 is Ψ°.

Step: Ψ_{k+1} is defined after we first define an auxiliary set Ψ_{k^+}:

- If $\Psi_k, \upsilon_{k+1}{:}\varphi_{k+1} \vdash_E \upsilon'{:}\varphi'$ then Ψ_{k^+} is Ψ_k.

- If $\Psi_k, \upsilon_{k+1}{:}\varphi_{k+1} \not\vdash_E \upsilon'{:}\varphi'$ then Ψ_{k^+} is $\Psi_k \cup \{\upsilon_{k+1}{:}\varphi_{k+1}\} \cup \Psi^{\cdot}$ where $\Psi^{\cdot}$ is defined below according to the following cases and $\nu_1, \dots, \nu_n$ are label schema variables in E but not in $\text{lsv}(\Psi_k) \cup \text{lsv}(\{\upsilon_{k+1}{:}\varphi_{k+1}, \upsilon'{:}\varphi'\})$:

- If φ_{k+1} is $c_{e^+}\varphi_1 \dots \varphi_n$ then

  - $\Psi^{\cdot}$ is $\{r_{e^+}\upsilon_{k+1}\nu_1 \dots \nu_n, \nu_1{:}\varphi_1, \dots, \nu_n{:}\varphi_n, \nu_1{:}\gamma_1, \dots, \nu_n{:}\gamma_n\}$,

  where $\gamma_1, \dots, \gamma_n$ are the constraint formulae of c_{e^+}.

- If φ_{k+1} is $\neg c_{u^+}\varphi_1 \dots \varphi_n$ and $\neg$ is a local classical negation then

  - $\Psi^{\cdot}$ is $\{r_{u^+}\upsilon_{k+1}\nu_1 \dots \nu_n, \nu_1{:}\varphi_1, \dots, \nu_n{:}\neg\varphi_n, \nu_1{:}\gamma_1, \dots, \nu_n{:}\gamma_n\}$,

  where $\gamma_1, \dots, \gamma_n$ are the constraint formulae of c_{u^+}.

- If $\upsilon_{k+1}{:}\varphi_{k+1}$ is $\upsilon^*{:}{-}c_{u^+}\varphi_1 \dots \varphi_n$, for some label schema term υ, and $-$ is a non-local classical negation then

  - $\Psi^{\cdot}$ is $\{r_{u^+}\upsilon\nu_1 \dots \nu_n, \nu_1{:}\varphi_1, \dots, \nu_n^*{:}{-}\varphi_n, \nu_1{:}\gamma_1, \dots, \nu_n{:}\gamma_n\}$,

  where $\gamma_1, \dots, \gamma_n$ are the constraint formulae of c_{u^+}.

- If φ_{k+1} is $c_e\varphi_1 \dots \varphi_n$ then

  - $\Psi^{\cdot}$ is $\{r_e v_{k+1} \nu_1 \dots \nu_n, \nu_1{:}\varphi_1, \dots, \nu_n{:}\varphi_n\}$.

- If φ_{k+1} is $\neg c_u \varphi_1 \dots \varphi_n$ and $\neg$ is a local classical negation then
  - $\Psi^{\cdot}$ is $\{r_u v_{k+1} \nu_1 \dots \nu_n, \nu_1{:}\varphi_1, \dots, \nu_{n-1}{:}\varphi_{n-1}, \nu_n{:}\neg\varphi_n\}$.
- If $v_{k+1}{:}\varphi_{k+1}$ is $v^*{:}{-c_u}\varphi_1 \dots \varphi_n$, for some label schema term v, and $-$ is a non-local classical negation then
  - $\Psi^{\cdot}$ is $\{r_u v \nu_1 \dots \nu_n, \nu_1{:}\varphi_1, \dots, \nu_{n-1}{:}\varphi_{n-1}, \nu_n^*{:}{-\varphi_n}\}$.
- Otherwise $\Psi^{\cdot}$ is $\emptyset$.

So, Ψ_{k+1} is Ψ_{k^+} if *exh* is not in $\mathcal{D}$, otherwise Ψ_{k+1} is Ψ_{k^+,m_k} where Ψ_{k^+,m_k} is defined as follows: let $\langle v_1, v^{\cdot}_1\rangle, \dots, \langle v_{m_k}, v^{\cdot}_{m_k}\rangle$ be a finite enumeration of all distinct pairs where the first component is a label schema term in $v_1{:}\varphi_1, \dots, v_k{:}\varphi_k$ and the second component is a label schema term in $v_{k+1}{:}\varphi_{k+1}$. Then Ψ_{k^+,m_k} is the last element of the sequence $\Psi_{k^+,0} \subseteq \dots \subseteq \Psi_{k^+,m_k}$ inductively defined as follows:

Base: $\Psi_{k^+,0}$ is Ψ_{k^+}.

Step: $\Psi_{k^+,i+1}$ is defined as follows, where ν, ν_1 are in E but not in $\mathrm{lsv}(\Psi_{k^+,i} \cup \{v'{:}\varphi'\})$,

- $\Psi_{k^+,i+1}$ is $\Psi_{k^+,i}$ whenever $\Psi_{k^+,i} \vdash_E v_{i+1} \equiv_\omega v, v \equiv_\alpha v^{\cdot}_{i+1}, v_{i+1} \equiv_\alpha v^{\cdot}, v^{\cdot} \equiv_\omega v^{\cdot}_{i+1}$ for some v and $v^{\cdot}$ in $T_{\mathrm{lab},E}$.
- $\Psi_{k^+,i+1}$ is $\Psi_{k^+,i} \cup \{v_{i+1} \equiv_\alpha \nu, \nu \equiv_\omega v^{\cdot}_{i+1}\}$ whenever $\Psi_{k^+,i} \vdash_E v_{i+1} \equiv_\omega v, v \equiv_\alpha v^{\cdot}_{i+1}$ for some v in $T_{\mathrm{lab},E}$ and $\Psi_{k^+,i} \nvdash_E v_{i+1} \equiv_\alpha v^{\cdot}, v^{\cdot} \equiv_\omega v^{\cdot}_{i+1}$ for any $v^{\cdot}$ in $T_{\mathrm{lab},E}$.
- $\Psi_{k^+,i+1}$ is $\Psi_{k^+,i} \cup \{v_{i+1} \equiv_\omega \nu, \nu \equiv_\alpha v^{\cdot}_{i+1}\}$ whenever $\Psi_{k^+,i} \nvdash_E v_{i+1} \equiv_\omega v, v \equiv_\alpha v^{\cdot}_{i+1}$ for any v in $T_{\mathrm{lab},E}$ and $\Psi_{k^+,i} \vdash_E v_{i+1} \equiv_\alpha v^{\cdot}, v^{\cdot} \equiv_\omega v^{\cdot}_{i+1}$ for some $v^{\cdot}$ in $T_{\mathrm{lab},E}$.
- $\Psi_{k^+,i+1}$ is $\Psi_{k^+,i} \cup \{v_{i+1} \equiv_\omega \nu, \nu \equiv_\alpha v^{\cdot}_{i+1}, v_{i+1} \equiv_\alpha \nu_1, \nu_1 \equiv_\omega v^{\cdot}_{i+1}\}$ whenever $\Psi_{k^+,i} \nvdash_E v_{i+1} \equiv_\omega v, v \equiv_\alpha v^{\cdot}_{i+1}$ and $\Psi_{k^+,i} \nvdash_E v_{i+1} \equiv_\alpha v^{\cdot}, v^{\cdot} \equiv_\omega v^{\cdot}_{i+1}$, for any v and $v^{\cdot}$ in $T_{\mathrm{lab},E}$.

End of construction

Before defining Ψ we prove a very useful fact: for every i greater or equal to 0 the set Ψ_i is such that $\Psi_i \nvdash_E v'{:}\varphi'$. The proof follows by induction:

Base: Ψ_0 is such that $\Psi_0 \nvdash_E v'{:}\varphi'$ since Ψ_0 is Ψ° and Ψ° is $v'{:}\varphi'$ consistent.

Step: assuming $\Psi_k \nvdash_E v'{:}\varphi'$ we want to show that $\Psi_{k+1} \nvdash_E v'{:}\varphi'$. According to the definition of Ψ_{k+1} we show first that $\Psi_{k^+} \nvdash_E v'{:}\varphi'$. Thus:

- Suppose $\Psi_k, v_{k+1}{:}\varphi_{k+1} \vdash_E v'{:}\varphi'$. Then Ψ_{k^+} is Ψ_k and so $\Psi_{k^+} \nvdash_E v'{:}\varphi'$ holds by the induction hypothesis.

- Suppose $\Psi_k, v_{k+1}{:}\varphi_{k+1} \nvdash_E v'{:}\varphi'$. Following the definition of Ψ_{k^+}, we divide the proof in cases depending on $v_{k+1}{:}\varphi_{k+1}$:

- φ_{k+1} is $c_{e^+}\varphi_1 \dots \varphi_n$. Suppose $\Psi_{k^+} \vdash_E v'{:}\varphi'$, i.e. $\Psi_k, v_{k+1}{:}\varphi_{k+1}, \Psi^{\cdot} \vdash_E v'{:}\varphi'$. Since $\Psi_k, v_{k+1}{:}\varphi_{k+1} \vdash_E v_{k+1}{:}\varphi_{k+1}$ we use Proposition 3.5.10 and rule ${c_{e^+}}_E$ to conclude $\Psi_k, v_{k+1}{:}\varphi_{k+1} \vdash_E v'{:}\varphi'$ which contradicts our assumption.

- φ_{k+1} is $\neg c_{u^+}\varphi_1 \dots \varphi_k$. Suppose $\Psi_{k^+} \vdash_E v'{:}\varphi'$, i.e. $\Psi_k, v_{k+1}{:}\varphi_{k+1}, \Psi^{\cdot} \vdash_E v'{:}\varphi'$. Since $\Psi_k, v_{k+1}{:}\varphi_{k+1} \vdash_E v_{k+1}{:}\varphi_{k+1}$ we use Proposition 4.3.6 to conclude $\Psi_k, v_{k+1}{:}\varphi_{k+1} \vdash_E v'{:}\varphi'$ and so obtain a contradiction.

- $v_{k+1}{:}\varphi_{k+1}$ is $v^*{:}\neg c_{u^+}\varphi_1 \dots \varphi_k$ for some label schema term v. Suppose $\Psi_{k^+} \vdash_E v'{:}\varphi'$, i.e. $\Psi_k, v_{k+1}{:}\varphi_{k+1}, \Psi^{\cdot} \vdash_E v'{:}\varphi'$. Since $\Psi_k, v_{k+1}{:}\varphi_{k+1} \vdash_E v_{k+1}{:}\varphi_{k+1}$ we use Proposition 4.3.7 to conclude $\Psi_k, v_{k+1}{:}\varphi_{k+1} \vdash_E v'{:}\varphi'$ and so obtain a contradiction.

- for the other cases the proof is either similar to the previous ones or straightforward so it is omitted.

Hence $\Psi_{k^+} \nvdash_E v'{:}\varphi'$. In order to show that $\Psi_{k+1} \nvdash_E v'{:}\varphi'$, according to the definition of Ψ_{k+1}, we have two cases to consider:

• *exh* is not in $\mathcal{D}$. Then Ψ_{k+1} is Ψ_{k^+}, and so $\Psi_{k+1} \nvdash_E v'{:}\varphi'$, as we wanted to show.

• *exh* is in $\mathcal{D}$. Then Ψ_{k+1} is Ψ_{k^+,m_k} and $\Psi_{k+1} \nvdash_E v'{:}\varphi'$ because $\Psi_{k^+,i} \nvdash_E v'{:}\varphi'$ for all $i = 1, \dots, m_k$, as we show now by induction:

- *Base:* $\Psi_{k^+,0} \nvdash_E v'{:}\varphi'$ since $\Psi_{k^+,0}$ is Ψ_{k^+} and $\Psi_{k^+} \nvdash_E v'{:}\varphi'$ as we just proved.

- *Step:* assuming $\Psi_{k^+,i} \nvdash_E v'{:}\varphi'$ we want to show that $\Psi_{k^+,i+1} \nvdash_E v'{:}\varphi'$. So, following the definition of $\Psi_{k^+,i+1}$ we divide the proof in four cases:

- If $\Psi_{k^+,i} \vdash_E v_{i+1} \equiv_\omega v, v \equiv_\alpha v^{\cdot}_{i+1}, v_{i+1} \equiv_\alpha v^{\cdot}, v^{\cdot} \equiv_\omega v^{\cdot}_{i+1}$ for some v and $v^{\cdot}$ in $T_{\text{lab},E}$ then $\Psi_{k^+,i+1}$ is $\Psi_{k^+,i}$ and so the result follows by the induction hypothesis.

- If $\Psi_{k^+,i} \vdash_E v_{i+1} \equiv_\omega v, v \equiv_\alpha v^{\cdot}_{i+1}$ for some v in $T_{\text{lab},E}$ and $\Psi_{k^+,i} \nvdash_E v_{i+1} \equiv_\alpha v^{\cdot}, v^{\cdot} \equiv_\omega v^{\cdot}_{i+1}$ for any $v^{\cdot}$ in $T_{\text{lab},E}$ then $\Psi_{k^+,i+1}$ is $\Psi_{k^+,i} \cup \{v_{i+1} \equiv_\alpha \nu, \nu \equiv_\omega v^{\cdot}_{i+1}\}$, and so if $\Psi_{k^+,i+1} \vdash_E v'{:}\varphi'$ then, by idempontence, Proposition 2.3.15 we have that $\Psi_{k^+,i}, v^{\cdot}_{i+1} \equiv_\omega \nu, \nu \equiv_\alpha v_{i+1} \vdash_E v'{:}\varphi'$, and thus by Proposition 3.5.10 and rule *exh* we would have $\Psi_{k^+,i} \vdash_E v'{:}\varphi'$, which would contradict the initial assumption. Thus $\Psi_{k^+,i+1} \nvdash_E v'{:}\varphi'$ as we wanted to show.

- If $\Psi_{k^+,i} \nvdash_E v_{i+1} \equiv_\omega v, v \equiv_\alpha v^{\cdot}_{i+1}$ for any v in $T_{\text{lab},E}$ and $\Psi_{k^+,i} \vdash_E v_{i+1} \equiv_\alpha v^{\cdot}, v^{\cdot} \equiv_\omega v^{\cdot}_{i+1}$ for some $v^{\cdot}$ in $T_{\text{lab},E}$ then the proof is similar to the preceding one so we omit it.

- If $\Psi_{k^+,i} \nvdash_E v_{i+1} \equiv_\omega v, v \equiv_\alpha v^{\cdot}_{i+1}$ and $\Psi_{k^+,i} \nvdash_E v_{i+1} \equiv_\alpha v^{\cdot}, v^{\cdot} \equiv_\omega v^{\cdot}_{i+1}$, for any v and $v^{\cdot}$ in $T_{\text{lab},E}$ then $\Psi_{k^+,i+1}$ is $\Psi_{k^+,i} \cup \{v_{i+1} \equiv_\omega \nu, \nu \equiv_\alpha v^{\cdot}_{i+1}, v_{i+1} \equiv_\alpha \nu_1, \nu_1 \equiv_\omega v^{\cdot}_{i+1}\}$, and so if $\Psi_{k^+,i+1} \vdash_E v'{:}\varphi'$ then, by Proposition 3.5.10 and rule *exh* we would have $\Psi_{k^+,i}, v_{i+1} \equiv_\alpha \nu_1, \nu_1 \equiv_\omega v^{\cdot}_{i+1} \vdash_E v'{:}\varphi'$, and thus, by idempontence, Proposition 2.3.15 we would have that $\Psi_{k^+,i}, v^{\cdot}_{i+1} \equiv_\omega \nu_1, \nu_1 \equiv_\alpha v_{i+1} \vdash_E v'{:}\varphi'$. Hence, by Proposition 3.5.10 and rule *exh* we would have $\Psi_{k^+,i} \vdash_E v'{:}\varphi'$, which would contradict the initial assumption. Thus $\Psi_{k^+,i+1} \nvdash_E v'{:}\varphi'$ as we wanted to show.

Henceforth, $\Psi_{k+1} \nvdash_E v'{:}\varphi'$ since $\Psi_{k^+,m_k} \nvdash_E v'{:}\varphi'$, as we wanted to show.
End of proof of auxiliary fact

Finally, define Ψ as follows:

$$\Psi = \bigcup_{i \geq 0} {\Psi_i}^{\vdash_E}.$$

Now we show that Ψ satisfies the desired properties above mentioned.

1. Suppose $\Psi \vdash_E v'{:}\varphi'$. Then, there is a finite deduction sequence ϖ' ending in $\langle v'{:}\varphi', \Psi', 1\rangle$ where Ψ' is a finite set contained in Ψ. Let j be the less natural such that Ψ' is contained in ${\Psi_j}^{\vdash_E}$, which exists by definition of Ψ and since Ψ' is finite. Then ${\Psi_j}^{\vdash_E} \vdash_E \Psi'$ and so, by idempotence, Proposition 2.3.15, we have ${\Psi_j}^{\vdash_E} \vdash_E v'{:}\varphi'$ since $\Psi' \vdash_E v'{:}\varphi'$. Moreover $\Psi_j \vdash_E v'{:}\varphi'$, since $\Psi_j \vdash_E {\Psi_j}^{\vdash_E}$. But this contradicts the fact above proven that $\Psi_i \not\vdash_E v'{:}\varphi'$ for each $i \geq 0$. So, $\Psi \not\vdash_E v'{:}\varphi'$.

2. *Only if* Suppose $\Psi, v{:}\varphi \not\vdash_E v'{:}\varphi'$, where $v{:}\varphi$ is a labelled schema formula. Then, by monotony, Proposition 2.3.13, we have that, $\Psi_i, v{:}\varphi \not\vdash_E v'{:}\varphi'$ for each $i \geq 0$. Suppose $v{:}\varphi$ is at the $k+1$ position in the enumeration considered above. So, by construction, $v{:}\varphi$ is in Ψ_{k+1}, and thus in Ψ.

If Suppose $v{:}\varphi$ is in Ψ. So, if $\Psi, v{:}\varphi \vdash_E v'{:}\varphi'$ then, by idempotence, Proposition 2.3.15, we have $\Psi \vdash_E v'{:}\varphi'$, which is a contradiction, by 1.. Hence $\Psi, v{:}\varphi \not\vdash_E v'{:}\varphi'$.

3. By construction of Ψ, since Ψ_0 is Ψ°.

4. Suppose $\Psi \vdash_E \eta$. Then, there is a finite deduction sequence ending in $\langle \eta, \Psi', 1\rangle$ for $\Psi \vdash_E \eta$. Note that $(\Psi_1)^{\vdash_E} \subseteq (\Psi_2)^{\vdash_E} \subseteq \ldots$ by monotony, Proposition 2.3.13, because $\Psi_1 \subseteq \Psi_2 \subseteq \ldots$. Since Ψ' is contained in Ψ and is finite then there is i with Ψ' contained in $(\Psi_i)^{\vdash_E}$. So, by idempotence, Proposition 2.3.15, ${\Psi_i}^{\vdash_E} \vdash_E \eta$, and, by the same proposition, $\Psi_i \vdash_E \eta$, because $\Psi_i \vdash_E {\Psi_i}^{\vdash_E}$. So η is in $(\Psi_i)^{\vdash_E}$, and thus, η is in Ψ, as we wanted to show.

5. Suppose *exh* occur in $\mathcal{D}$. Let $v{:}\varphi$ and $v^{\cdot}{:}\varphi^{\cdot}$ be labelled schema formulae. Suppose they occur in the enumeration at positions i and $i^{\cdot}$, respectively, and suppose without loss of generality that $i \leq i^{\cdot}$. Then, by construction of $\Psi_{i^{\cdot}}$, we have that $\Psi_{i^{\cdot}+1} \vdash_E v \equiv_\omega v_1$ and $\Psi_{i^{\cdot}+1} \vdash_E v_1 \equiv_\alpha v^{\cdot}$, for some v_1 in $T_{\mathrm{lab},E}$. So, taking into account the definition of Ψ, there is v_1 in $T_{\mathrm{lab},E}$ with $v \equiv_\omega v_1$ and $v_1 \equiv_\alpha v^{\cdot}$ in Ψ.

6. *Only if.* Suppose $v{:}\varphi$ is not in Ψ. Then, by 2., $\Psi, v{:}\varphi \vdash_E v'{:}\varphi'$. Thus, if $v{:}\neg\varphi$ is also not in Ψ then, also by 2., $\Psi, v{:}\neg\varphi \vdash_E v'{:}\varphi'$. So by Proposition 4.3.4 we would conclude $\Psi \vdash_E v'{:}\varphi'$ which contradicts 1.. Hence $v{:}\neg\varphi$ is in Ψ.

If. Suppose $v{:}\neg\varphi$ is in Ψ. So, if $v{:}\varphi$ were also in Ψ then by Proposition 4.3.4 we would conclude $\Psi \vdash_E v'{:}\varphi'$ which contradicts 1.. So $v{:}\varphi$ is not in Ψ.

7. *Only if.* Suppose $v{:}\varphi$ is not in Ψ. Then, by 2., $\Psi, v{:}\varphi \vdash_E v'{:}\varphi'$. Thus, if $v^*{:}-\varphi$ is also not in Ψ then, also by 2., $\Psi, v^*{:}-\varphi \vdash_E v'{:}\varphi'$. So by Proposition 4.3.5 we would conclude $\Psi \vdash_E v'{:}\varphi'$ which contradicts 1.. Hence $v^*{:}-\varphi$ is in Ψ.

If. Suppose $v^*{:}-\varphi$ is in Ψ. Then if $v{:}\varphi$ were also in Ψ then by Proposition 4.3.5 we would conclude $\Psi \vdash_E v'{:}\varphi'$ which contradicts 1.. So $v{:}\varphi$ is not in Ψ.

8. *Only if.* Let c_u be a universal connective in $\mathcal{D}$, and $v_1, \ldots, v_n$ label schema terms in

$T_{\text{lab},E}$. Suppose $v{:}c_u\varphi_1 \dots \varphi_n$, $r_u v v_1 \dots v_n$, and $v_1{:}\varphi_1, \dots, v_{n-1}{:}\varphi_{n-1}$ are in Ψ. Then

1	$v{:}c_u\varphi_1 \dots \varphi_n$	$\{v{:}c_u\varphi_1 \dots \varphi_n\}$	1	asp	
2	$r_u v v_1 \dots v_n$	$\{r_u v v_1 \dots v_n\}$	1	asp	
3	$v_1{:}\varphi_1$	$\{v_1{:}\varphi_1\}$	1	asp	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$n+1$	$v_{n-1}{:}\varphi_{n-1}$	$\{v_{n-1}{:}\varphi_{n-1}\}$	1	asp	
$n+2$	$v_n{:}\varphi_n$	Ψ'	1	c_{uE}	$1, 2, 3, \dots, n+1$

where Ψ' is $\{v{:}c_u\varphi_1 \dots \varphi_n, r_u v v_1 \dots v_n, v_1{:}\varphi_1, \dots, v_{n-1}{:}\varphi_{n-1}\}$, is a sober deduction sequence for $\Psi \vdash_E v_n{:}\varphi_n$. So, $v_n{:}\varphi_n$ is in Ψ since by 4., Ψ is deductively closed.

If. Suppose $v{:}c_u\varphi_1 \dots \varphi_n$ is not in Ψ. Consider two cases:

$\neg$ is a local classical negation in $\mathcal{D}$. Then $v{:}\neg c_u\varphi_1 \dots \varphi_n$ is in Ψ by 6.. So, by construction of Ψ there are label schema variables $\nu_1, \dots, \nu_n$ in E with $r_u v \nu_1 \dots \nu_n, \nu_1{:}\varphi_1, \dots, \nu_{n-1}{:}\varphi_{n-1}$ and $\nu_n{:}\neg\varphi_n$ are in Ψ. Henceforth the result follows because $\nu_n{:}\varphi_n$ is not in Ψ by 6.

$-$ is a non-local classical negation in $\mathcal{D}$. Then $v^*{:}-c_u\varphi_1 \dots \varphi_n$ is in Ψ by 7.. So, by construction of Ψ there are label schema variables $\nu_1, \dots, \nu_n$ in E with $r_u v \nu_1 \dots \nu_n, \nu_1{:}\varphi_1, \dots,$ $\nu_{n-1}{:}\varphi_{n-1}$ and $\nu_n^*{:}-\varphi_n$ in Ψ. Hence the result follows because $\nu_n{:}\varphi_n$ is not in Ψ by 7.

9. The proof is similar to 8. so we will omit it.

10. *Only if.* Suppose $v{:}c_e\varphi_1 \dots \varphi_n$ is in Ψ. The result follows because, by construction of Ψ, there are label schema variables $\nu_1, \dots, \nu_n$, such that $r_e v \nu_1 \dots \nu_n$ and $\nu_1{:}\varphi_1, \dots, \nu_n{:}\varphi_n$ are in Ψ.

If. Suppose there are $v_1, \dots, v_n$ in $T_{\text{lab},E}$ such that $r_e v v_1 \dots v_n, v_1{:}\varphi_1, \dots, v_n{:}\varphi_n$ are in Ψ. Then

1	$r_e v v_1 \dots v_n$	$\{r_e v v_1 \dots v_n\}$	1	asp	
2	$v_1{:}\varphi_1$	$\{v_1{:}\varphi_1\}$	1	asp	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$n+1$	$v_n{:}\varphi_n$	$\{v_n{:}\varphi_n\}$	1	asp	
$n+2$	$v{:}c_e\varphi_1 \dots \varphi_n$	Ψ'	1	c_{eI}	$1, 2, \dots, n+1$

where Ψ' is the set $\{r_e v v_1 \dots v_n, v_1{:}\varphi_1, \dots, v_n{:}\varphi_n\}$, is a sober deduction sequence for $\Psi \vdash_E v{:}c_e\varphi_1 \dots \varphi_n$. So, $v{:}c_e\varphi_1 \dots \varphi_n$ is in Ψ since by 4., Ψ is deductively closed.

11. The proof is similar to 10. so we will omit it.

12. *Only if.* Suppose $v{:}-\varphi$ and $v^*{:}\varphi$ are in Ψ. Let v' be a label schema term in $T_{\text{lab},E}$. Then

1	$v{:}-\varphi$	$\{v{:}-\varphi\}$	1	asp	
2	$v^*{:}\varphi$	$\{v^*{:}\varphi\}$	1	asp	
3	$v'{:}\bot$	$\{v{:}-\varphi, v^*{:}\varphi\}$	1	$-_E$	$1, 2$

is a sober deduction sequence for $\Psi \vdash_E v'{:}\bot$. So, $v'{:}\bot$ is in Ψ since by 4., Ψ is deductively closed.

If. Suppose $v{:}-\varphi$ is not in Ψ and $v^*{:}\varphi$ is in Ψ. Then, for some v', if $v'{:}\bot$ were in Ψ then $\Psi \vdash_E v{:}-\varphi$ and since Ψ is deductively closed by 4., $v{:}-\varphi$ would be in Ψ which contradicts our initial assumption. So $v'{:}\bot$ is not in Ψ for all label schema terms v'.

13. The proof is similar to 12. so we will omit it.

14. The proof follows straightforwardly.

15. *Only if.* Suppose $v{:}\varphi_1 \vee \varphi_2$ is in Ψ. Thus, if both φ_1 and φ_2 are not in Ψ, then, by 2., there are deduction sequences ϖ_1 and ϖ_2 for $\Psi, v{:}\varphi_1 \vdash v'{:}\varphi'$ and $\Psi, v{:}\varphi_2 \vdash v'{:}\varphi'$, respectively. Henceforth

$\vdots$	$\vdots$	$\vdots$	$\vdots$	ϖ_1	
i	$v'{:}\varphi'$	Ψ_1	1		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	ϖ_2	
j	$v'{:}\varphi'$	Ψ_2	1		
$j+1$	$v{:}\varphi_1 \vee \varphi_2$	$\{v{:}\varphi_1 \vee \varphi_2\}$	1	asp	
$j+2$	$v'{:}\varphi'$	$(\Psi_1 \backslash \{v{:}\varphi_1\}) \cup (\Psi_2 \backslash \{v{:}\varphi_2\}) \cup \{v{:}\varphi_1 \vee \varphi_2\}$	1	$\vee_E$	$j+1, j, i$

is a deduction sequence for $\Psi \vdash_E v'{:}\varphi'$ which contradicts 1. So, either φ_1 or φ_2 are in Ψ.

If the proof follows straightforwardly so we will omit it.

16. *Only if.* Suppose $v{:}\varphi_1 \rightarrow \varphi_2$ and $v{:}\varphi_1$ are in Ψ. Then, it is straightforward to see that $\Psi \vdash_E v{:}\varphi_2$, and so by 4., that $v{:}\varphi_2$ is in Ψ.

If Suppose $v{:}\varphi_1 \rightarrow \varphi_2$ is not in Ψ. Note that $v{:}\varphi_2$ is not in Ψ. To show this suppose it is. Then $\Psi \vdash_E v{:}\varphi_1 \rightarrow \varphi_2$ and since Ψ is deductively closed $v{:}\varphi_1 \rightarrow \varphi_2$ would be in Ψ which contradicts our initial assumption. To show that $v{:}\varphi_1$ is in Ψ suppose it is not. Then, by item 2., we have that $\Psi, v{:}\varphi_1 \vdash_E v'{:}\varphi'$. Suppose without loss of generality that the deduction system have a non-local classical negation. So, by item 7. $v'^{*}{:}-\varphi'$ is in Ψ. Hence, by Proposition 4.3.5, we have that $\Psi, v{:}\varphi_1 \vdash_E v{:}\varphi_2$. Thus $\Psi \vdash_E v{:}\varphi_1 \rightarrow \varphi_2$ and since Ψ is deductively closed, by item 4., we have that $v{:}\varphi_1 \rightarrow \varphi_2$ is in Ψ which contradicts the fact that $v{:}\varphi_1 \rightarrow \varphi_2$ is not in Ψ. Therefore $v{:}\varphi_1$ is in Ψ as we wanted to show.

17. The proof proceeds by case analysis on the possible rules in $\mathcal{D}$. Suppose the rule is:

$\neg_I$ or $\neg_E$ where $\neg$ is a local negation in $\mathcal{D}$. The result follows by 6.

$-_I$ or $-_E$ where $-$ is a non-local negation in $\mathcal{D}$. The result follows by 7.

c_{uI} or c_{uE} where c_u is a universal connective in $\mathcal{D}$. The result follows by 8.

c_{u^+I} or c_{u^+E} where c_{u^+} is a constrained universal connective in $\mathcal{D}$. The result follows by 9.

c_{eI} or c_{eE} where c_e is an existential connective in $\mathcal{D}$. The result follows by 10.

c_{e^+I} or c_{e^+E} where c_{e^+} is a constrained existential connective in $\mathcal{D}$. The result follows by 11.

$\wedge_I$ or $\wedge_E^1$ or $\wedge_E^2$ where $\wedge$ is a conjunction in $\mathcal{D}$. The result follows by 14.

$\vee_I^1$ or $\vee_I^2$ or $\vee_E$ where $\vee$ is a disjunction in $\mathcal{D}$. The result follows by 15.

$\rightarrow_I$ or $\rightarrow_E$ where $\rightarrow$ is a implication in $\mathcal{D}$. The result follows by 16.

$\top_I$. Let σ be a schema substitution within L_E. We have that $\nu\sigma{:}\top$ is in Ψ since Ψ is deductively closed, by 4., and $\Psi \vdash_E \nu\sigma{:}\top$.

exh. Let σ be a schema substitution within L_E such that $(\pi_\Sigma\sigma_{\mathrm{nl}}) = 1$ for each π in P_f

and $(\pi_{\Sigma;E}\sigma) = 1$ for each π in P_d, and suppose that for any υ' in $T_{\mathrm{lab},E}$ if $\nu_1\sigma \equiv_\omega \upsilon'$ and $\upsilon' \equiv_\alpha \nu_2\sigma$ are in Ψ then $\nu''\sigma{:}\xi''\sigma$ is in Ψ. Let $\upsilon^{\cdot}$ be a label schema term such that $\nu_1\sigma \equiv_\omega \upsilon^{\cdot}$ and $\upsilon^{\cdot} \equiv_\alpha \nu_2\sigma$ are in Ψ, which exists by 5.. Then $\nu''\sigma{:}\xi''\sigma$ is in Ψ, as we wanted to show.

$-_c$ where $-$ is a non-local negation in $\mathcal{D}$. Let σ be a schema substitution within L_E. Suppose $\nu\sigma^*{:}\xi\sigma$ is not in Ψ, and $\nu\sigma{:}{-}\xi\sigma$ is in Ψ. Then, by 2., $\Psi, \nu\sigma^*{:}\xi\sigma \vdash_E \upsilon'{:}\varphi'$. So, if $\nu'\sigma{:}\bot$ is in Ψ then it is straightforward to see that $\Psi \vdash_E \nu\sigma^*{:}\xi\sigma$, and so, by idempotence, Proposition 2.3.15, we would have $\Psi \vdash_E \upsilon'{:}\varphi'$ which contradicts 1.. Hence $\nu'\sigma{:}\bot$ is not in Ψ, as we wanted to show.

$\neg_c$ where $\neg$ is a local negation in $\mathcal{D}$. The proof is similar to $-_c$.

c_i where c is a negation in $\mathcal{D}$. The proof is similar to $\neg_c$ so we will omit it.

${c_u}^{\mathrm{spc}}$, where c_u is a universal connective in $\mathcal{D}$. Let σ be a schema substitution within L_E such that $(\pi_\Sigma\sigma_{\mathrm{nl}}) = 1$ for each π in P_f. Suppose $\nu\sigma{:}c_u\,\xi\sigma$ is in Ψ. Then, $\Psi \vdash_E \nu\sigma{:}\xi'\sigma$ by rule ${c_u}^{\mathrm{spc}}$ since σ satisfy the proviso in P_f, and the result follows since Ψ is deductively closed, see 4.

${c_{u^+}}^{\mathrm{spc}}$, where c_{u^+} is a constrained universal connective in $\mathcal{D}$. The proof is similar to ${c_u}^{\mathrm{spc}}$ so we will omit it.

a term inheritance rule. Let σ be a schema substitution within L_E such that $(\pi_\Sigma\sigma_{\mathrm{nl}}) = 1$ for each π in P_f. Suppose $s\,\nu_1\sigma \ldots \nu_n\sigma$ is in Ψ. Then, $\Psi \vdash_E \nu\sigma{:}\theta\sigma =_g \nu'\sigma{:}\theta\sigma$. So, $\nu\sigma{:}\theta\sigma =_g \nu'\sigma{:}\theta\sigma$ is in Ψ since by 4. Ψ is deductively closed.

a formula inheritance rule. Let σ be a schema substitution within L_E such that $(\pi_\Sigma\sigma_{\mathrm{nl}}) = 1$ for each π in P_f. Suppose $s\,\nu_1\sigma \ldots \nu_n\sigma$ and $\nu_i\sigma{:}\xi\sigma$ are in Ψ. Then, $\Psi \vdash_E \nu_{i'}\sigma{:}\xi\sigma$. So, $\nu_{i'}\sigma{:}\xi\sigma$ is in Ψ since by 4. Ψ is deductively closed.

a general formula inheritance rule over $\equiv_\omega$. Straightforward.

a docked rule and *label docked rule.* Straightforward.

a relational rule. Straightforward.

a generalization rule. Straightforward.

a laxed generalization rule. Straightforward.

a laxed generalization rule for a label function symbol. Straightforward.

1 individual rule. Straightforward.

1 assignment rule. Straightforward.

a rule common to all lfob according to Definition 2.3.10. Straightforward. *QED*

The next proposition shows that every lfob deduction system with a classical negation is appropriate. Moreover, if the lfob deduction system contains the rule *exh* then it is also strong appropriate. Recall the notion of appropriate lfob deduction system in Definition 3.3.4 and of strong appropriate lfob deduction system in Definition 4.2.2.

Proposition 4.3.9 In the context of a clfob deduction system with a classical negation, there is an appropriate set extending every consistent set, which is consistent with respect to the same labelled schema formula and which is strong appropriate if *exh* is a rule of that lfob deduction system.

Proof Let Ψ° be a $v'{:}\varphi'$ consistent set. Then, by Proposition 4.3.8, there is a set E of label schema variables and a set Ψ within L_E satisfying the conditions specified in that proposition. Note that:

- Ψ extends Ψ°. Proven in Proposition 4.3.8.

- Ψ is $v'{:}\varphi'$ consistent. Proven in Proposition 4.3.8.

- If *exh* is in the lfob deduction system, then Ψ satisfies the strong condition, see Definition 4.2.1, i.e., there is a label schema term v in $T_{\mathrm{lab},E}$ with $v_1 \equiv_\omega v$ and $v \equiv_\alpha v_2$ in Ψ, for any label schema terms v_1 and v_2 in $T_{\mathrm{lab},E}$.

- Ψ is canonical. To see this:

1. Let c be a connective in C, and v a label schema term in $T_{\mathrm{lab},E}$. Suppose for each $j = 1, \ldots, k$, and any v' in $T_{\mathrm{lab},E}$ such that $\Psi \vdash_E v \equiv_\omega v'$ and $\Psi \vdash_E v \equiv_\alpha v'$, that $v'{:}\varphi_j \in \Psi$ iff $v'{:}\varphi'_j \in \Psi$. Now the proof proceeds by case analysis on the possible types of connectives c can be:

– a disjunction $\vee$. Suppose $v{:}\varphi_1 \vee \varphi_2$ is in Ψ. Then by Proposition 4.3.8 either $v{:}\varphi_1$ is in Ψ or $v{:}\varphi_2$ is in Ψ. So, either $v{:}\varphi'_1$ is in Ψ or $v{:}\varphi'_2$ is in Ψ. Hence, also by Proposition 4.3.8, $v{:}\varphi'_1 \vee \varphi'_2$ is in Ψ. The proof of the other direction follows similarly.

– a conjunction $\wedge$. Suppose $v{:}\varphi_1 \wedge \varphi_2$ is in Ψ. Then by Proposition 4.3.8 $v{:}\varphi_1$ and $v{:}\varphi_2$ are in Ψ. So, $v{:}\varphi'_1$ and $v{:}\varphi'_2$ are in Ψ. Hence, also by Proposition 4.3.8, $v{:}\varphi'_1 \wedge \varphi'_2$ is in Ψ. The proof of the other direction follows similarly.

– a implication $\rightarrow$. Suppose $v{:}\varphi_1 \rightarrow \varphi_2$ is in Ψ. Then by Proposition 4.3.8 if $v{:}\varphi_1$ is in Ψ then $v{:}\varphi_2$ is also in Ψ. So, if $v{:}\varphi'_1$ is in Ψ then $v{:}\varphi'_2$ is also in Ψ. Hence, also by Proposition 4.3.8, $v{:}\varphi'_1 \rightarrow \varphi'_2$ is in Ψ. The proof of the other direction follows similarly.

– a local negation $\neg$. Suppose $v{:}\neg\varphi$ is in Ψ. Then by Proposition 4.3.8 if $v{:}\varphi$ is in Ψ then there is v' such that $v'{:}\bot$ is also in Ψ. So, if $v{:}\varphi'$ is in Ψ then there is v' such that $v'{:}\bot$ is also in Ψ. Hence, also by Proposition 4.3.8, $v{:}\neg\varphi'$ is in Ψ. The proof of the other direction follows similarly.

2. Let c be a connective in Q, and v a label schema term in $T_{\mathrm{lab},E}$. Suppose for each $j = 1, \ldots, k$, and any v' in $T_{\mathrm{lab},E}$ such that $\Psi \vdash_E v \equiv_\omega v'$, that $v'{:}\varphi_j \in \Psi$ iff $v'{:}\varphi'_j \in \Psi$. Now the proof proceeds by case analysis on the possible types of connectives c can be:

– a universal connective c_u. Suppose $v{:}c_u\varphi_1 \ldots \varphi_n$ is in Ψ. So by Proposition 4.3.8, for any $v_1, \ldots, v_n$ in $T_{\mathrm{lab},E}$, we have $v_n{:}\varphi_n \in \Psi$ whenever $r_u\, vv_1 \ldots v_n$, and $v_1{:}\varphi_1, \ldots, v_{n-1}{:}\varphi_{n-1}$ are in Ψ. Let $v'_1, \ldots, v'_n$ be label schema terms in $T_{\mathrm{lab},E}$ such that $r_u\, vv'_1 \ldots v'_n$, and $v'_1{:}\varphi'_1, \ldots, v'_{n-1}{:}\varphi'_{n-1}$ are in Ψ. Then, since there are docked rules for all components of r_u in $\equiv_\omega$, we can conclude that $\Psi \vdash_E v \equiv_\omega v'_i$ for all $i = 1, \ldots, n$. Thus, $v'_1{:}\varphi_1, \ldots, v'_{n-1}{:}\varphi_{n-1}$ are in Ψ. So, $v'_n{:}\varphi_n$ is in Ψ, and since $\Psi \vdash_E v \equiv_\omega v'_n$ we conclude that $v'_n{:}\varphi'_n$ is in Ψ. Thus $v{:}c_u\varphi'_1, \ldots \varphi'_n$ is in Ψ. The proof of the other direction follows similarly.

– an existential connective c_e. Suppose $v{:}c_e\varphi_1 \ldots \varphi_n$ is in Ψ. Then by Proposition 4.3.8, there are $v_1, \ldots, v_n$ in $T_{\mathrm{lab},E}$, such that $r_e\, vv_1 \ldots v_n, v_1{:}\varphi_1, \ldots, v_n{:}\varphi_n$ are in Ψ. Note that $\Psi \vdash_E v \equiv_\omega v_i$ for all $i = 1, \ldots, n$, since there are docked rules for all components of r_e in

$\equiv_\omega$. So $r_e\, vv_1 \ldots v_n$, and $v_1{:}\varphi'_1, \ldots, v_n{:}\varphi'_n$ are in Ψ. Henceforth $v{:}c_e\varphi'_1 \ldots \varphi'_n$ is in Ψ. The proof of the other direction follows similarly.

– a constrained universal connective c_{u^+}. Suppose $v{:}c_{u^+}\varphi_1 \ldots \varphi_n$ is in Ψ. Then by Proposition 4.3.8, for any $v_1, \ldots, v_n$ in $T_{\text{lab},E}$, we have $v_n{:}\varphi_n \in \Psi$ whenever $r_{u^+}vv_1 \ldots v_n, v_1{:}\varphi_1, \ldots, v_{n-1}{:}\varphi_{n-1}, v_1{:}\gamma_1, \ldots, v_n{:}\gamma_n$ are in Ψ. Let $v'_1, \ldots, v'_n$ be label schema terms in $T_{\text{lab},E}$ such that $r_{u^+}vv'_1 \ldots v'_n, v'_1{:}\varphi'_1, \ldots, v'_{n-1}{:}\varphi'_{n-1}, v'_1{:}\gamma_1,\ \ldots, v'_n{:}\gamma_n$ are in Ψ. Then, since there are docked rules for all components of r_{u^+} in $\equiv_\omega$, we have that $\Psi \vdash_E v \equiv_\omega v'_i$ for all $i = 1, \ldots, n$. Thus, $v'_1{:}\varphi_1, \ldots, v'_{n-1}{:}\varphi_{n-1}$ are in Ψ. So, $v'_n{:}\varphi_n$ is in Ψ, and since $\Psi \vdash_E v \equiv_\omega v'_n$ we conclude that $v'_n{:}\varphi'_n$ is in Ψ. Thus $v{:}c_{u^+}\varphi'_1, \ldots \varphi'_n$ is in Ψ. The proof of the other direction follows similarly.

– a constrained existential connective c_{e^+}. Suppose $v{:}c_{e^+}\varphi_1 \ldots \varphi_n$ is in Ψ. Then by Proposition 4.3.8, there are label schema terms $v_1, \ldots, v_n$ in $T_{\text{lab},E}$, such that $r_{e^+}vv_1 \ldots v_n, v_1{:}\varphi_1, \ldots, v_n{:}\varphi_n, v_1{:}\gamma_1, \ldots, v_n{:}\gamma_n$ are in Ψ. Note that $\Psi \vdash_E v \equiv_\omega v_i$ for all $i = 1, \ldots, n$, since there are docked rules for all components of r_{e^+} in $\equiv_\omega$. So $r_{e^+}\, vv_1 \ldots v_n$, and $v_1{:}\varphi'_1, \ldots, v_n{:}\varphi'_n$ are in Ψ. Henceforth $v{:}c_{e^+}\varphi'_1 \ldots \varphi'_n$ is in Ψ. The proof of the other direction follows similarly.

– a non-local negation $-$. Suppose $v{:}{-}\varphi$ is in Ψ. Then, by Proposition 4.3.8, if $v^*{:}\varphi$ is in Ψ then there is a label schema term v'' such that $v''{:}\bot$ is in Ψ. Suppose $v^*{:}\varphi'$ is in Ψ. Note that $\Psi \vdash_E v \equiv_\omega v^*$. Then $v^*{:}\varphi$ is in Ψ, and so there is a label schema term v'' such that $v''{:}\bot$ is in Ψ. Thus, $v{:}{-}\varphi'$ is in Ψ. The proof of the other direction follows similarly.

3. Let c be a connective in O, and v a label schema term in $T_{\text{lab},E}$. Suppose for each $j = 1, \ldots, k$, and any v' in $T_{\text{lab},E}$, that $v'{:}\varphi_j \in \Psi$ iff $v'{:}\varphi'_j \in \Psi$. Now the proof proceeds by case analysis on the possible types of connectives c can be:

– a universal connective c_u. Suppose $v{:}c_u\varphi_1 \ldots \varphi_n$ is in Ψ. So by Proposition 4.3.8, for any $v_1, \ldots, v_n$ in $T_{\text{lab},E}$, we have $v_n{:}\varphi_n \in \Psi$ whenever $r_u\, vv_1 \ldots v_n$, and $v_1{:}\varphi_1, \ldots, v_{n-1}{:}\varphi_{n-1}$ are in Ψ. So, for any $v_1, \ldots, v_n$ in $T_{\text{lab},E}$, we have $v_n{:}\varphi'_n \in \Psi$ whenever $r_u\, vv_1 \ldots v_n$, and $v_1{:}\varphi'_1, \ldots, v_{n-1}{:}\varphi'_{n-1}$ are in Ψ. Thus $v{:}c_u\varphi'_1, \ldots \varphi'_n$ is in Ψ. The proof of the other direction follows similarly.

– for an existential connective, or a constrained universal connective, or a constrained existential connective or for a non-local negation connective, the proof is similar to that for a universal connective.

So, Ψ is appropriate and if *exh* is in the lfob logic system Ψ is strong appropriate. This happens because Ψ is canonical, is deductively closed, and it satisfies the appropriateness and the strong appropriateness conditions as proven in Proposition 4.3.8. *QED*

Corollary 4.3.10 Every clfob deduction system with a classical negation is appropriate. Moreover if it contains rule *exh* it is strong appropriate.

This corollary follows straightforwardly taking into account Proposition 4.3.9. So, we omit its proof.

4.4 Connected logic systems with locality

In the last section we proved that clfob logic systems with a classical negation are appropriate. But, can we obtain appropriateness results for clfob logic systems when we assume that they do not have any kind of negation? To answer this question we now study a class of clfob logic systems that we call *clfob logic systems with locality*. We prove that clfob logic systems with locality and without disjunction and implication are appropriate. Note that those appropriate systems may have connectives like universal and existential, constrained or not, quantifiers and modalities, conjunction and any kind of negation, although that negation play no role in the construction. The restriction to logics without disjunction and implication came from the fact that we were not able to prove the appropriateness conditions for the rules $\rightarrow_I$ and $\wedge_E$ in the context of lfob logic systems with locality. It does not imply that there is not a construction of an appropriate set satisfying those rules, in the context of a clfob deduction system with locality. Only that we were not able to do it. In order to be able, using our construction, to build an appropriate set, not relying on any kind of negation, that satisfy the appropriate conditions for implication and disjunction, we think that one solution maybe the consideration of more general forms of the rules, or even of the connectives, see Remark 4.4.8. This is a very interesting topic that we want to pursue in the future. For now we concentrate in the study of the fibring of labelled first-order based logic systems.

To motivate the conditions required for a clfob logic system be with locality, suppose we are trying to construct an appropriate set that extends a certain consistent set. We can assume that the appropriate set is built inductively, similarly as if we were in the context of a lfob logic system with a classical negation, see Proposition 4.3.8. Then, if a certain formula, whose main connective is a universal connective, can not be in the appropriate set, we have to add the witnesses to its absence. Recall the counterpart in an appropriate set Ψ, of the rules of a universal connective c_u:

$v{:}c_u\varphi_1 \ldots \varphi_n \in \Psi$ iff for any $v_1, \ldots, v_n$ in $T_{\text{lab},E}$, we have $v_n{:}\varphi_n \in \Psi$ whenever

$$r_u v v_1 \ldots v_n \in \Psi \quad \text{and} \quad v_1{:}\varphi_1 \in \Psi \quad \text{and} \quad \ldots \quad \text{and} \quad v_{n-1}{:}\varphi_{n-1} \in \Psi.$$

These witness formulae are labelled by new label schema variables not appearing in the preceding sets, and those new labels are related by the relation associated to the universal connective. The problem is that when we add the witness labelled formulae to the appropriate set it may cause the deduction of non-wanted labelled schema formulae over the existing labels. But, if we are in the context of a deduction system where, from a set of labelled schema formulae over existing labels, labelled schema formulae over new labels, and a relation linking an existing label to these new labels, we only deduce labelled schema formulae over the existing labels if those formulae were deduced from the set of labelled schema formulae over the existing labels, then the addition of the witnesses do not have the problem of non-wanted deductions.

So, we investigate the conditions that should be fulfilled by a deduction system, in order for it to satisfy that property, and name a deduction system satisfying those conditions as a deduction system with locality. The name locality comes from the fact that the effects in deduction of the labelled schema formulae with new labels are localized to those labels.

General definition

Before defining a lfob logic system with locality we need to introduce what is an inheritance position of a relation. Note that our definition of an inheritance position of a relation is specific for connected lfob logic systems. It would be interesting to study it in the context of a general lfob logic system, i.e., it would be certainly interesting the study of what is an inheritance position of a relation in the context of a general lfob logic system. We leave this study to future work.

Definition 4.4.1 We say k is a s *inheritance position*, in a clfob deduction system, where s is a relation in S with arity n, whenever k either is

- equal to 1, if a (constrained) existential connective is based on s, or
- equal to n, if a (constrained) universal connective is based on s, or
- such that there is a formula inheritance rule to k over s, or
- such that there is a k docked rule in $\equiv_\omega$ or $\equiv_\alpha$ for s, or
- such that there is a term inheritance rule to k over s.

So, we are now able to define what is a clfob deduction system with locality.

Definition 4.4.2 A clfob deduction system is *with locality* whenever

– if there is a universal connective c, constrained or not, based on a relation s, where either c is not unary or s is not reflexive, then

1. there is no local or non-local negation nor any rule involving $\perp$,
2. the exhaustiveness rule, the 1 individual rule, and the 1 assignment rule are not present,
3. each relational rule, with non-empty set of premises, over a relation s in S, is such that, if the label schema term at position k in the conclusion is
    - in Ξ_l, then it is also at some s inheritance position in a premise, and this premise is the first element of a sequence made of all the premises and such that each premise has at some s inheritance position a label schema variable which is also in a previous premise in the sequence,
    - not in Ξ_l, then it involves all the label schema variables in the premises,

    for each s inheritance position k,
4. each relational rule, without premises, over a relation s in S, is such that, in the conclusion, the set of label schema variables at any s inheritance position contains all the other label schema variables in the conclusion,

– each universal connective c, constrained or not, based on a relation s, where either c is not unary or s is not reflexive, is such that 1 is not an s inheritance position.

Definition 4.4.3 A clfob logic system *with locality* is a lfob logic system composed of a clfob deduction system with locality.

(Strong) appropriateness

Differently of clfob deduction systems with a classical negation, where the exhaustiveness rule have no interference in the fact that the system is with a classical negation, in clfob deduction systems with locality, that rule may interfere. It is not always possible to consider a clfob deduction system with locality and with the exhaustiveness rule. This implies that not all clfob deduction systems with locality are strong appropriate, since this property is intimately associated to the presence in the system of the exhaustiveness rule.

The main goal of this subsection is to prove Proposition 4.4.7, where it is shown that, in the context of a clfob deduction system with locality and without disjunction and implication, it is always possible to construct an appropriate set extending a consistent set, consistent with respect to the same formula. As we discuss in Remark 4.4.8, we think that, in order to obtain a set satisfying the appropriateness conditions associated to disjunction and implication, using our construction, the rules for that connectives would have to be more general. But this would move us far from labelled deduction and near hybrid systems, which is out of the scope of the dissertation. Nevertheless, the results we obtain for clfob deduction systems with locality can be applied to a wide class of logics, to show that, these logics, or several of their fragments without disjunction and implication, are appropriate, and so, if they are also full, then they are complete.

We now show three important lemmas for proving Proposition 4.4.7. The first lemma proves that, in the presence of specific conditions, if it is possible to deduce a relational schema formula from the union of two sets of labelled schema formulae over disjoint sets of labels, then it is possible to deduce that relational schema formula from the set of labelled schema formulae over the set of labels also in the formula.

Lemma 4.4.4 In a clfob deduction system with locality, for any infinite sets E and E' of label schema variables with E' contained in E, set Ψ within $L_{E'}$, s in S_n such that there is a universal connective c constrained or not based on s where either c is not unary or s is not reflexive, v in $T_{\mathrm{lab},E'}$, $\nu_1,\dots,\nu_n$ in E but not in E', finite set $\Psi^\cdot$ within L_E, s' in $S_{n'}$ or equal to $\equiv_\omega$ or $\equiv_\alpha$, and $v'_1,\dots,v'_{n'}$ in $T_{\mathrm{lab},E}$, such that

- $v^\cdot$ is not in $T_{\mathrm{lab},E'}$ for any label schema term $v^\cdot$ in $\Psi^\cdot$,
- $\Psi, s\,v\nu_1\dots\nu_n, \Psi^\cdot \vdash_E s'v'_1\dots v'_{n'}$,
- either v'_1 or v'_2 are in $T_{\mathrm{lab},E'}$, if s' is $\equiv_\omega$ or $\equiv_\alpha$,
- there is a label schema term in $s'v'_1\dots v'_{n'}$ at a s' inheritance position that is in $T_{\mathrm{lab},E'}$, if s' is in $S_{n'}$,

we have that

$$v'_1,\dots,v'_{n'} \quad \text{are in} \quad T_{\mathrm{lab},E'} \qquad \text{and} \qquad \Psi \vdash_E s'v'_1\dots v'_{n'}.$$

Proof The idea it to prove by induction on the length of a sober deduction sequence ϖ for $\Psi, s\,v\nu_1\dots\nu_n, \Psi^\cdot \vdash_E s'v'_1\dots v'_{n'}$ that ϖ is also a sober deduction sequence for $\Psi \vdash_E s'v'_1\dots v'_{n'}$ and that $v'_1,\dots,v'_{n'}$ are in $T_{\mathrm{lab},E'}$. Base of induction: $\langle s'v'_1\dots v'_{n'}, \Psi_1, 1\rangle$ is a sober deduction sequence for $\Psi, s\,v\nu_1\dots\nu_n, \Psi^\cdot \vdash_E s'v'_1\dots v'_{n'}$. We now distinguish the case where s' is in $S_{n'}$ from the case where s' is $\equiv_\omega$ or is $\equiv_\alpha$:

- s' is in $S_{n'}$. Denote by k a s' inheritance position in $s' v'_1 \ldots v'_{n'}$ that has a label term in $T_{\mathrm{lab},E'}$. So, we can consider two cases:

– *Either* Ψ_1 is $\{s' v'_1 \ldots v'_{n'}\}$ and is contained in $\Psi \cup \{s\, v \nu_1 \ldots \nu_n\} \cup \Psi^{\cdot}$. Then:

i. $s' v'_1 \ldots v'_{n'}$ is not in $\Psi^{\cdot}$. This happens because all label schema terms in $\Psi^{\cdot}$ are not in $T_{\mathrm{lab},E'}$.

ii. $s' v'_1 \ldots v'_{n'}$ is not $s\, v \nu_1 \ldots \nu_n$. Suppose it is. Then k is not 1 by definition of a clfob deduction system with locality, see Definition 4.4.2, since c is a universal connective, constrained or not, based on s', and either c is not unary or s' is not reflexive. So, v'_k is ν_{k-1}. But this is a contradiction since v'_k is in $T_{\mathrm{lab},E'}$ and ν_{k-1} is not in $T_{\mathrm{lab},E'}$.

Hence $s' v'_1 \ldots v'_{n'}$ is in Ψ. So $v'_1, \ldots, v'_{n'}$ are in $T_{\mathrm{lab},E'}$, because Ψ is contained in $L_{E'}$. Moreover $\Psi \vdash_E s' v'_1 \ldots v'_{n'}$.

– *Or* there are a relational rule $\langle \emptyset, \eta', \emptyset, \emptyset \rangle$, and Σ schema substitution σ' such that $\eta'\sigma'$ is $s' v'_1 \ldots v'_{n'}$ and $\Psi_1 = \emptyset$. So, $\langle s' v'_1 \ldots v'_{n'}, \Psi_1, 1 \rangle$ is a sober deduction sequence for $\Psi \vdash_E s' v'_1 \ldots v'_{n'}$. Note that by definition of a clfob deduction system with locality, Definition 4.4.2, $\mathrm{lsv}(\{v'_1, \ldots, v'_{n'}\} \setminus \{v'_k\})$ is contained in $\mathrm{lsv}(\{v'_k\})$ which is contained in $\Xi_l \cup E'$. Thus $v'_1, \ldots, v'_{n'}$ are in $T_{\mathrm{lab},E'}$.

- s' is $\equiv_\omega$ or $\equiv_\alpha$. Suppose without loss of generality that s' is $\equiv_\omega$. Then, we can consider two cases:

– *Either* Ψ_1 is $\{v'_1 \equiv_\omega v'_2\}$ and is contained in $\Psi \cup \{s\, v \nu_1 \ldots \nu_n\} \cup \Psi^{\cdot}$, and so we have that:

i. $v'_1 \equiv_\omega v'_2$ is not in $\Psi^{\cdot}$. This happens because all label schema terms in $\Psi^{\cdot}$ are not in $T_{\mathrm{lab},E'}$.

ii. $v'_1 \equiv_\omega v'_2$ is not $s\, v \nu_1 \ldots \nu_n$. This happens because s is not $\equiv_\omega$ or $\equiv_\alpha$, since s is in S due to the fact that there is a universal connective, constrained or not, based on s, recall the characteristics of a universal connective in Definition 4.1.2.

Hence $v'_1 \equiv_\omega v'_2$ is in Ψ. So v'_1 and v'_2 are in $T_{\mathrm{lab},E'}$, because Ψ is contained in $L_{E'}$. Moreover $\Psi \vdash_E v'_1 \equiv_\omega v'_2$.

– *Or* there are a rule $\langle \emptyset, \eta', \emptyset, \emptyset \rangle$ named r, and Σ schema substitution σ' such that $\eta'\sigma'$ is $v'_1 \equiv_\omega v'_2$ and $\Psi_1 = \emptyset$. So $\Psi \vdash_E v'_1 \equiv_\omega v'_2$. Note that in a clfob deduction system the only rule without premises whose conclusion is a $\equiv_\omega$ relational schema formula is the rule $\equiv_{\omega r}$. Thus v'_1 and v'_2 are the same label schema term and so, as we wanted to show, are both in $T_{\mathrm{lab},E'}$.

Assume as induction hypothesis that if ϖ is a sober deduction sequence with length less than or equal to m for $\Psi, s\, v \nu_1 \ldots \nu_n, \Psi^{\cdot} \vdash_E s' v'_1 \ldots v'_{n'}$, then ϖ is also a sober deduction sequence for $\Psi \vdash_E s' v'_1 \ldots v'_{n'}$ and $v'_1, \ldots, v'_{n'}$ are in $T_{\mathrm{lab},E'}$. Suppose there is a sober deduction sequence named ϖ, with length $m+1$, ending in $\langle s' v'_1 \ldots v'_{n'}, \Psi_{m+1}, 1 \rangle$, for $\Psi, s\, v \nu_1 \ldots \nu_n, \Psi^{\cdot} \vdash_E s' v'_1 \ldots v'_{n'}$. So, there is a rule $\langle \{\langle \vartheta'_1, \emptyset, \eta'_1 \rangle, \ldots, \langle \vartheta'_{k'}, \emptyset, \eta'_{k'} \rangle\}, \eta', \emptyset, \emptyset \rangle$, named r, with non-empty set of premises, Σ schema substitution σ' within L_E, and $i_1, \ldots, i_{k'}$ in $\{1, \ldots, m\}$, with $\eta_{i_j} = \eta'_j \sigma'$, $\Psi_{i_j} = \vartheta'_j \sigma'$ and $\pi_{i_j} = 1$ for each $j = 1, \ldots, k'$, $\Psi_{m+1} = \Psi_{i_1} \cup \ldots \cup \Psi_{i_{k'}}$, and $\eta'\sigma'$ is $s' v'_1 \ldots v'_{n'}$. Denote by ϖ_j the sober deduction subsequence of ϖ ending in $\langle \eta_{i_j}, \Psi_{i_j}, 1 \rangle$ for $\Psi_{i_j} \vdash_E \eta_{i_j}$. Similarly as for the base of induction we now distinguish the case where s' is in $S_{n'}$ from the case where s' is $\equiv_\omega$ or is $\equiv_\alpha$:

- s' is in $S_{n'}$. In this case r is either a relational rule or a generalization rule or a laxed generalization rule. Denote by k a s' inheritance position in $s'v'_1 \ldots v'_{n'}$ that has a label term in $T_{\mathrm{lab},E'}$. Then, suppose:

– r is a *relational rule*. Then $\eta'_1, \ldots, \eta'_{k'}$ are s' relational schema formulae. Consider now the two different cases for the label schema term at position k in η':

— it is a label schema variable ν'_k. Then, according to the definition of a clfob deduction system with locality, Definition 4.4.2, ν'_k is also at some s' inheritance position in a premise, which is the first element of a non-empty sequence $\eta'_{i'_1}, \ldots, \eta'_{i'_{k'}}$, where $i'_1, \ldots, i'_{k'}$ are in $\{1, \ldots, k'\}$, made of all the premises and such that each premise has at some s' inheritance position a label schema variable which is also in a previous premise in the sequence. Now we show by induction on the length of that sequence that for each $j = 1, \ldots, k'$ there is a label schema term at a s' inheritance position in $\eta'_{i'_j}\sigma$ that is in $T_{\mathrm{lab},E'}$.

Base: k' is 1. Recall that ν'_k is also at some s' inheritance position in $\eta'_{i'_1}$. Thus, $\nu'_k\sigma$ which is in $T_{\mathrm{lab},E'}$, is also in $\eta'_{i'_1}\sigma$ at some s' inheritance position, and so the result follows straightforwardly.

Step: k' is $l+1$. Note that by induction hypothesis, for each $j = 1, \ldots, l$ there is a label schema term at a s' inheritance position in $\eta'_{i'_j}\sigma$ that is in $T_{\mathrm{lab},E'}$. So we only have to show that $\eta'_{i'_{l+1}}\sigma$ has a label schema term in $T_{\mathrm{lab},E'}$ at a s' inheritance position. Observe that, by the outer induction hypothesis, we have that, for each $j = 1, \ldots, m$ all the label schema terms in $\eta'_{i'_j}\sigma$ are in $T_{\mathrm{lab},E'}$. The result follows because, by definition of a clfob deduction system with locality, Definition 4.4.2, $\eta'_{i'_{l+1}}\sigma$ has at some s' inheritance position a label schema term which is also in a previous premise in the sequence, so which is in $T_{\mathrm{lab},E'}$ since all the label schema terms in previous premises are in that set.

So for each $j = 1, \ldots, k'$ there is a label schema term at a s' inheritance position in $\eta'_{i'_j}\sigma$ that is in $T_{\mathrm{lab},E'}$ and since $\Psi, s\, v\nu_1 \ldots \nu_n, \Psi^{\cdot} \vdash_E \eta'_{i'_j}\sigma$, we can apply the induction hypothesis to conclude that (i) all the label schema terms in $\eta'_{i'_j}\sigma$ are in $T_{\mathrm{lab},E'}$, and (ii) $\Psi \vdash_E \eta'_{i'_j}\sigma$. So $\Psi \vdash_E s'v'_1 \ldots v'_{n'}$. Note that $\mathrm{lsv}(\eta')$ is contained in $\mathrm{lsv}(\cup_{j=1,\ldots,k'}\eta'_j)$ by the definition of a clfob deduction system, Definition 4.1.2. Hence, as we wanted to show, and using (i), we have that $v'_1, \ldots, v'_{n'}$ are in $T_{\mathrm{lab},E'}$.

— it is not a label schema variable. Then, according to the definition of a clfob deduction system with locality, Definition 4.4.2, we have that $\mathrm{lsv}(\{\eta'_1\sigma,\ \ldots, \eta'_{k'}\sigma\})$ is contained in $\mathrm{lsv}(v'_k)$. So, since v'_k is in $T_{\mathrm{lab},E'}$, we have that $\eta'_1\sigma, \ldots, \eta'_{k'}\sigma$ are in $L_{E'}$. But this causes, (i) that $\Psi \vdash_E \eta'_j\sigma'$ for each $j = 1, \ldots, k'$, by induction hypothesis, and (ii) that $v'_1, \ldots, v'_{n'}$ are in $T_{\mathrm{lab},E'}$, as we wanted to show, because by the definition of a clfob deduction system, Definition 4.1.2 we have that $\mathrm{lsv}(\eta')$ is contained in $\mathrm{lsv}(\cup_{j=1,\ldots,k'}\eta'_j)$. With respect to the proof that $\Psi \vdash_E s'v'_1 \ldots v'_{n'}$ it follows by (i).

– r is a *laxed generalization rule.* Suppose without loss of generality that it is over $\equiv_\alpha$. Then $\{\eta_{i_1}, \ldots, \eta_{i_{k'}}\}$ is $\{s'v''_1 \ldots v''_{n'}, v'_1 \equiv_\omega v''_1, \ldots, v'_{n'} \equiv_\omega v''_{n'}\}$. Assume without loss of generality, that η_{i_j} is $v'_j \equiv_\omega v''_j$ for $j = 1, \ldots, k'-1$, and $\eta_{i_{k'}}$ is $s'v''_1 \ldots v''_{n'}$. Recall that ϖ_k is a sober deduction sequence with length less than or equal to m for $\Psi, s\, v\nu_1 \ldots \nu_n, \Psi^{\cdot} \vdash_E v'_k \equiv_\omega v''_k$ and v'_k is a label schema term in $T_{\mathrm{lab},E'}$. So, by induction hypothesis ϖ_k is a sober deduction sequence for $\Psi \vdash_E v'_k \equiv_\omega v''_k$ and v''_k is also in $T_{\mathrm{lab},E'}$. Since $\varpi_{k'}$ is a sober deduction sequence with length less than or equal to m for $\Psi, s\,v\nu_1 \ldots \nu_n, \Psi^{\cdot} \vdash_E s'v''_1 \ldots v''_{n'}$, v''_k is a label schema term in $T_{\mathrm{lab},E'}$, and k is a s' inheritance position, we can use another time the induction hypothesis, to conclude that (i) $\Psi \vdash_E s'v''_1 \ldots v''_{n'}$ and that $v''_1, \ldots, v''_{n'}$ are label schema terms in $T_{\mathrm{lab},E'}$. Therefore for all $j = 1, \ldots, n+1$ we have that ϖ_j is a sober deduction sequence with length less than or equal to m for $\Psi, s\,v\nu_1 \ldots \nu_n, \Psi^{\cdot} \vdash_E v'_k \equiv_\omega v''_k$ and v''_k is a label term in $T_{\mathrm{lab},E'}$. So, by induction hypothesis ϖ_j is a sober deduction sequence for (ii) $\Psi \vdash_E v'_k \equiv_\omega v''_j$ and (iii) v'_j is also in $T_{\mathrm{lab},E'}$. Finally we can conclude, as wanted, that, $\Psi \vdash_E s'v'_1 \ldots v'_{n'}$, by the generalization rule over $\equiv_\alpha$ and by (i) and (ii), and we can conclude that v'_j is in $T_{\mathrm{lab},E'}$ for all $j = 1, \ldots, n'$ by (iii).

– r is a *generalization rule.* We omit the proof since it is similar to the previous one.

- s' is $\equiv_\omega$ or $\equiv_\alpha$. Suppose without loss of generality that s' is $\equiv_\alpha$. The proof proceeds by case analysis on the possible rules in a clfob deduction system with locality that have in the conclusion a $\equiv_\alpha$ relational schema formula:

– r is an i, j *docked rule in* s' i.e. in $\equiv_\alpha$ *for* some relation s'' in $S_{n''}$. Then k is 1, $\eta'_1\sigma$ is $s''v''_1 \ldots v''_{n''}$ and v'_1 and v'_2 are in $\{v''_1, \ldots, v''_{n''}\}$ at positions i and j. Note that $\Psi, s\,v\nu_1 \ldots \nu_n, \Psi^{\cdot} \vdash_E s''v''_1 \ldots v''_{n''}$, and that, by definition of an inheritance position, Definition 4.4.1, i and j are s'' inheritance positions. So, there is a label schema term in $s''v''_1 \ldots v''_{n''}$ at a s'' inheritance position in $T_{\mathrm{lab},E'}$. Thus, we can use the induction hypothesis in order to conclude that $v''_1, \ldots, v''_{n''}$ are in $T_{\mathrm{lab},E'}$ and $\Psi \vdash_E s''v''_1 \ldots v''_{n''}$. Therefore $\Psi \vdash_E v'_1 \equiv_\alpha v'_2$, and v'_1 and v'_2 are in $T_{\mathrm{lab},E'}$.

– r is a *laxed generalization rule for a label function symbol* f *in* F^l_k over $\equiv_\alpha$. Then $s'v'_1 \ldots v'_{n'}$ is indeed the relational schema formula $f(v''_1, \ldots, v''_{k''}) \equiv_\omega f(v_1^{\prime\prime\cdot}, \ldots, v_{k''}^{\prime\prime\cdot})$, and the premises of the rule are $v''_1 \equiv_\omega v_1^{\prime\prime\cdot}, \ldots, v''_{k''} \equiv_\omega v_{k''}^{\prime\prime\cdot}$. Note that, by assumption, either $f(v''_1, \ldots, v''_{k''})$ or $f(v_1^{\prime\prime\cdot}, \ldots, v_{k''}^{\prime\prime\cdot})$ are in $T_{\mathrm{lab},E'}$. Suppose without loss of generality that is $f(v''_1, \ldots, v''_{k''})$. Then $v''_1, \ldots, v''_{k''}$ are also in $T_{\mathrm{lab},E'}$. Note that there is a sober deduction sequence for $\Psi, s\,v\nu_1 \ldots \nu_n, \Psi^{\cdot} \vdash_E v''_i \equiv_\omega v_i^{\prime\prime\cdot}$ for each $i = 1, \ldots, k''$ with length less than or equal to m. Then, by induction hypothesis, $\Psi \vdash_E v''_i \equiv_\omega v_i^{\prime\prime\cdot}$ and $v_i^{\prime\prime\cdot}$ is also in $T_{\mathrm{lab},E'}$, for each $i = 1, \ldots, k''$. So, applying this same rule we can conclude that $\Psi \vdash_E f(v''_1, \ldots, v''_{k''}) \equiv_\omega f(v_1^{\prime\prime\cdot}, \ldots, v_{k''}^{\prime\prime\cdot})$ and that also $f(v_1^{\prime\prime\cdot}, \ldots, v_{k''}^{\prime\prime\cdot})$ is in $T_{\mathrm{lab},E'}$.

– r is $\equiv_{\alpha s}$. Then, η_{i_1} is $v'_2 \equiv_\alpha v'_1$. So, it is straightforward to see that we can use the induction hypothesis, to conclude that $\Psi \vdash_E v'_2 \equiv_\alpha v'_1$ and that v'_2 and v'_1 are in $T_{\mathrm{lab},E'}$, as we wanted to see.

– r is $\equiv_{\alpha t}$. Then, η_{i_1} is $v'_1 \equiv_\alpha v'$, and η_{i_2} is $v' \equiv_\alpha v'_2$. Suppose that v'_1 is in $T_{\mathrm{lab},E'}$. Then, it is straightforward to see that we can use the induction hypothesis, to conclude that $\Psi \vdash_E v'_1 \equiv_\alpha v'$ and that v'_1 and v' are in $T_{\mathrm{lab},E'}$. So we can use another time the induction hypothesis to conclude that $\Psi \vdash_E v' \equiv_\alpha v'_2$ and that v' and v'_2 are in $T_{\mathrm{lab},E'}$. Similarly if v'_2 is in $T_{\mathrm{lab},E'}$. Therefore, since one of the cases happen, we can always obtain $\Psi \vdash_E v'_1 \equiv_\alpha v'_2$ and that v'_1 and v'_2 are in $T_{\mathrm{lab},E'}$. *QED*

The next lemma is similar to the preceding lemma and proves that, in the presence of specific conditions, if it is possible to deduce a labelled non-relational schema formula from the union of two sets of labelled schema formulae over disjoint sets of labels, then it is also possible to deduce that labelled non-relational schema formula from the set of labelled schema formulae over the set of labels also in the formula.

Proposition 4.4.5 In a clfob deduction system with locality, for any

- infinite sets E and E' of label schema variables such that E' is contained in E and its difference is an infinite set,
- label schema term v in $T_{\mathrm{lab},E'}$,
- set Ψ within $L_{E'}$,
- labelled non-relational schema formula η such that
  - η is in $L_{E'}$, if η is $v_1{:}\varphi$,
  - either v_1 or v_2 are in $T_{\mathrm{lab},E'}$, if η is $v_1{:}t_1 =_g v_2{:}t_2$,
- label schema variables $\nu_1, \ldots, \nu_n$ in E but not in E',
- finite set $\Psi^{\cdot}$ within L_E such that $v^{\cdot}$ is not in $T_{\mathrm{lab},E'}$ for any label schema term $v^{\cdot}$ in $\Psi^{\cdot}$,
- relation s with arity $n+1$, such that there is a universal connective c, constrained or not, based on s, where either c is not unary or s is not reflexive,

if

$$\Psi, s\, v\nu_1 \ldots \nu_n, \Psi^{\cdot} \vdash_E \eta$$

then

$$\Psi \vdash_E \eta$$

and, if η is $v_1{:}t_1 =_g v_2{:}t_2$,

$$v_1, v_2 \quad \text{are in} \quad T_{\mathrm{lab},E'}.$$

Proof The proof follows by induction on the length of a sober deduction sequence for $\Psi, s\, v\nu_1 \ldots \nu_n, \Psi^{\cdot} \vdash_E \eta$. So, assume that $\langle \eta, \Psi_1, 1 \rangle$ is a sober deduction sequence for $\Psi, s\, v\nu_1 \ldots \nu_n, \Psi^{\cdot} \vdash_E \eta$. Then we can consider two situations:

- Ψ_1 is $\{\eta\}$. So η is in $\Psi \cup \{s\, v\nu_1 \ldots \nu_n\} \cup \Psi^{\cdot}$. Note that η is not $s\, v\nu_1 \ldots \nu_n$ since η is a labelled non-relational schema formula. Observe also that η is not in $\Psi^{\cdot}$ since η has a label schema term in $T_{\mathrm{lab},E'}$ and all the label schema terms of labelled schema formulae in $\Psi^{\cdot}$ are not in $T_{\mathrm{lab},E'}$. So η is in Ψ. Therefore $\Psi \vdash_E \eta$, and η is in $L_{E'}$.

- there is a rule $\langle \emptyset, \eta', P'_f, P'_d \rangle$ named r and Σ schema substitution σ within L_E such that Ψ_1 is $\emptyset$, $\eta'\sigma$ is η, $(\pi_\Sigma \sigma_{\mathrm{nl}})$ is the Σ proviso 1, for each π in P'_f, and $\pi_{\Sigma,E}(\sigma)$ is 1 for each π in P'_d. So, $\langle \eta, \Psi_1, 1 \rangle$ is a sober deduction sequence for $\Psi \vdash_E \eta$. Note also that, if η is $v_1{:}t_1 =_g v_2{:}t_2$, then r is $=_{g_r}$, and so v_1 and v_2 are the same label schema term. Thus, they are both in $T_{\mathrm{lab},E'}$, as we wanted to show.

Assume as induction hypothesis that, for any infinite sets E and E' of label schema variables such that E' is contained in E and its difference is an infinite set, any label schema term v in $T_{\mathrm{lab},E'}$, set Ψ within $L_{E'}$, labelled non-relational schema formula η such that (i) η is in $L_{E'}$ if η is $v_1{:}\varphi$, and (ii) either v_1 or v_2 are in $T_{\mathrm{lab},E'}$ if η is $v_1{:}t_1 =_g v_2{:}t_2$, for any label schema variables $\nu_1, \ldots, \nu_n$ in E but not in E', finite set $\Psi^{\cdot}$ within L_E such that $v^{\cdot}$ is not in $T_{\text{lab},E'}$ for any label schema term $v^{\cdot}$ in $\Psi^{\cdot}$, and any relation s with arity $n+1$ such that there is a universal connective c, constrained or not, based on it, where either c is not unary or s is not reflexive, if there is a sober deduction sequence with length less than or equal to m for $\Psi, s\, v\nu_1 \ldots \nu_n, \Psi^{\cdot} \vdash_E \eta$ then $\Psi \vdash_E \eta$, and if η is $v_1{:}t_1 =_g v_2{:}t_2$, then v_1, v_2 are in $T_{\text{lab},E'}$.

Let E and E' be infinite sets of label schema variables such that E' is contained in E and its difference is an infinite set, let v be a label schema term in $T_{\text{lab},E'}$, Ψ a set within $L_{E'}$, η a labelled non-relational schema formula that (i) η is in $L_{E'}$ if η is $v_1{:}\varphi$, and (ii) either v_1 or v_2 are in $T_{\text{lab},E'}$ if η is $v_1{:}t_1 =_g v_2{:}t_2$, for any $\nu_1, \ldots, \nu_n$ label schema variables in E but not in E', $\Psi^{\cdot}$ a finite set within L_E such that $v^{\cdot}$ is not in $T_{\text{lab},E'}$ for any label schema term $v^{\cdot}$ in $\Psi^{\cdot}$, and let s be a relation with arity $n+1$ such that there is a universal connective c, constrained or not, based on it, where either c is not unary or s is not reflexive. Suppose that ϖ is a sober deduction sequence with length $m+1$, ending in $\langle \eta, \Psi_{m+1}, 1\rangle$, for $\Psi, s\, v\nu_1 \ldots \nu_n, \Psi^{\cdot} \vdash_E \eta$. So, there are Σ schema substitution σ within L_E, rule $\langle\{\langle\vartheta'_1, \Psi'_1, \eta'_1\rangle, \ldots, \langle\vartheta'_k, \Psi'_k, \eta'_k\rangle\}, \eta', P'_f, P'_d\rangle$, and $i_1, \ldots, i_k$ in $\{1, \ldots, m\}$, with $\eta_{i_j} = \eta'_j\sigma$, $\Psi_{i_j} = \vartheta'_j\sigma$ and $\pi_{i_j} = 1$ for each $j = 1, \ldots, k$, $\Psi_{m+1} = \Psi_{i_1} \setminus \Psi'_1\sigma \cup \ldots \cup \Psi_{i_k} \setminus \Psi'_k\sigma$, $(\pi_\Sigma \sigma_{\text{nl}}) = 1$ for each π in P'_f, $\pi_{\Sigma,E}(\sigma) = 1$ for each π in P'_d, and $\eta = \eta'\sigma$.

The proof now follows by case analysis on the rules that allow as conclusion a labelled non-relational schema formula:

- r is c_{uI}, where c_u is a universal connective. Then η is $v'{:}c_u\varphi_1 \ldots \varphi_{n'}$, η_{i_1} is $\nu'_{n'}\sigma{:}\varphi_{n'}$ and Ψ_{m+1} is $\Psi_{i_1} \setminus \{s_u v'\nu'_1\sigma \ldots \nu'_{n'}\sigma, \nu'_1\sigma{:}\varphi_1, \ldots, \nu'_{n'-1}\sigma{:}\varphi_{n'-1}\}$. Denote by ϖ_1 the sober deduction subsequence of ϖ ending in $\langle\nu'_{n'}\sigma{:}\varphi_{n'}, \Psi_{i_1}, 1\rangle$ for $\Psi_{i_1} \vdash_E \nu'_{n'}\sigma{:}\varphi_{n'}$. Consider the schema substitution ρ over E within L_E $\{\nu'_1, \ldots, \nu'_{n'}\}\sigma$ co-equivalent to id such that $\nu'_j\sigma\rho$ is a label schema variable not in $\text{lsv}(\Psi_{i_1} \cup \Psi^{\cdot}) \cup \text{lsv}(\{v, v'\}) \cup \{\nu_1, \ldots, \nu_n\} \cup (\{\nu'_1, \ldots, \nu'_{n'}\} \setminus \nu'_j)\sigma\rho$, for each $j = 1, \ldots, n'$. Then, by Proposition 2.3.14, there is a deduction sequence ϖ'_1 with the same length as ϖ_1 for $\Psi_{i_1}\rho \vdash_E \nu'_{n'}\sigma\rho{:}\varphi_{n'}$, and by monotony, Proposition 2.3.13, ϖ'_1 is a deduction sequence for $\Psi, s_u v'\nu'_1\sigma\rho \ldots \nu'_{n'}\sigma\rho, \nu'_1\sigma\rho{:}\varphi_1, \ldots, \nu'_{n'-1}\sigma\rho{:}\varphi_{n'-1}, s\, v\nu_1 \ldots \nu_n, \Psi^{\cdot} \vdash_E \nu'_{n'}\sigma\rho{:}\varphi_{n'}$. Denote by E'^{+} the set $E' \cup \{\nu'_1\sigma\rho, \ldots, \nu'_{n'}\sigma\rho\}$. Then,

– $\Psi \cup \{s_u v'\nu'_1\sigma\rho \ldots \nu'_{n'}\sigma\rho, \nu'_1\sigma\rho{:}\varphi_1, \ldots, \nu'_{n'-1}\sigma\rho{:}\varphi_{n'-1}\})$ is contained in $L_{E'^{+}}$.

– $\nu'_{n'}\sigma\rho{:}\varphi_{n'}$ is in $L_{E'^{+}}$.

– $v^{\cdot}$ is not in $T_{\text{lab},E'^{+}}$ for any label schema term $v^{\cdot}$ in $\Psi^{\cdot}$. This happens since for any label schema term $v^{\cdot}$ in $\Psi^{\cdot}$, by assumption, $\text{lsv}(v^{\cdot})$ is not contained in $\Xi_l \cup E'$ and $\text{lsv}(\Psi^{\cdot}) \cap \{\nu'_1\sigma\rho, \ldots, \nu'_{n'}\sigma\rho\} = \emptyset$ by definition of ρ.

– $\nu_1 \ldots, \nu_n$ are in E but not in E'^{+} by assumption and by definition of ρ.

– E'^{+} is contained in E by assumption and definition of ρ. Its difference is an infinite set because $E \setminus E'$ is an infinite set and $\{\nu'_1\sigma\rho, \ldots, \nu'_{n'}\sigma\rho\}$ is finite.

Thus, we have, $\Psi, s_u v'\nu'_1\sigma\rho \ldots \nu'_{n'}\sigma\rho, \nu'_1\sigma\rho{:}\varphi_1, \ldots, \nu'_{n'-1}\sigma\rho{:}\varphi_{n'-1} \vdash_E \nu'_{n'}\sigma\rho{:}\varphi_{n'}$, by induction hypothesis. So, by Proposition 3.5.10, we have $\Psi \vdash_E v'{:}c_u\varphi_1 \ldots \varphi_{n'}$, as we wanted to show.

- r is c_{u^+I}, where c_{u^+} is a constrained universal connective. The proof is similar to the case where r is c_{uI} and c_u is a universal connective.

- r is c_{uE}, where c_u is a universal connective. Then η is $v'_{n'}{:}\varphi_{n'}$, η_{i_1} is $s_u v' v'_1 \dots v'_{n'}$, η_{i_2} is $v'{:}c_u\varphi_1 \dots \varphi_{n'}$, and for each $j = 1, \dots, n'-1$ we have that $\eta_{i_{j+2}}$ is $v'_j{:}\varphi_j$. Note that $v'_{n'}$ is in $T_{\mathrm{lab},E'}$, and that $v'_{n'}$ is at a s_u inheritance position in $s_u v' v'_1 \dots v'_{n'}$. Then, by Lemma 4.4.4, we have (i) $\Psi \vdash_E s_u v' v'_1 \dots v'_{n'}$, and (ii) that $v', v'_1, \dots, v'_{n'}$ are in $T_{\mathrm{lab},E'}$. So, by induction hypothesis, $\Psi \vdash_E v'{:}c_u\varphi_1 \dots \varphi_{n'}$, and for each $j = 1, \dots, n'-1$ we have that $\Psi \vdash_E v'_j{:}\varphi_j$. Then, it is straightforward to see that $\Psi \vdash_E v'_{n'}{:}\varphi_{n'}$.

- r is c_{u^+E}, where c_{u^+} is a constrained universal connective. The proof is similar to the case where r is c_{uE} and c_u is a universal connective.

- r is c_{eE}, where c_e is an existential connective. Then η_{i_1} is $v'{:}c_e\varphi_1 \dots \varphi_{n'}$, η_{i_2} is η, and Ψ_{m+1} is $\Psi_{i_1} \cup \Psi_{i_2} \setminus \{s_e v' \nu'_1\sigma \dots \nu'_{n'}\sigma, \nu'_1\sigma{:}\xi_1\sigma, \dots, \nu'_{n'}\sigma{:}\xi_{n'}\sigma\}$. Denote by ϖ_2 the sober deduction subsequence of ϖ ending in $\langle \eta, \Psi_{i_2}, 1\rangle$ for $\Psi_{i_2} \vdash_E \eta$. Consider the schema substitution ρ over E within L_E $\{\nu'_1, \dots, \nu'_{n'}\}\sigma$ co-equivalent to id such that $\nu'_j\sigma\rho$ is a label schema variable not in $\Xi_l \cup E'$ and not in $\mathrm{lsv}(\Psi^{\cdot}) \cup \mathrm{lsv}(\{v'\}) \cup \{\nu_1, \dots, \nu_n\} \cup (\{\nu'_1, \dots, \nu'_{n'}\} \setminus \nu'_j)\sigma\rho$, for each $j = 1, \dots, n'$. Then, by Proposition 2.3.14, there is a deduction sequence ϖ'_2 with the same length as ϖ_2 for $\Psi_{i_2}\rho \vdash_E \eta$, and by monotony, Proposition 2.3.13, ϖ'_2 is also a deduction sequence for $\Psi, s_e v' \nu'_1\sigma\rho \dots \nu'_{n'}\sigma\rho, \nu'_1\sigma\rho{:}\varphi_1, \dots, \nu'_{n'}\sigma\rho{:}\varphi_{n'}, s\, v\nu_1 \dots \nu_n, \Psi^{\cdot} \vdash_E \eta$. For the sake of simplification the set $\{s_e v' \nu'_1\sigma\rho \dots \nu'_{n'}\sigma\rho, \nu'_1\sigma\rho{:}\varphi_1, \dots, \nu'_{n'}\sigma\rho{:}\varphi_{n'}\}$ will be denoted by Ψ_e. Consider two cases:

1. v' is in $T_{\mathrm{lab},E'}$. For the sake of simplification denote by E'^{+} the set $E' \cup \{\nu'_1\sigma\rho, \dots, \nu'_{n'}\sigma\rho\}$. Then we can apply the induction hypothesis to conclude $\Psi \vdash_E v'{:}c_e\varphi_1 \dots \varphi_{n'}$. Moreover we also have $\Psi, \Psi_e \vdash_E \eta$ by the induction hypothesis because:

– $\Psi \cup \Psi_e$ is contained in $L_{E'^{+}}$, and η is in $L_{E'^{+}}$.

– $v^{\cdot}$ is not in $T_{\mathrm{lab},E'^{+}}$ for any label schema term $v^{\cdot}$ in $\Psi^{\cdot}$. This happens since for any label schema term $v^{\cdot}$ in $\Psi^{\cdot}$, by assumption, $\mathrm{lsv}(v^{\cdot})$ is not contained in $\Xi_l \cup E'$ and $\mathrm{lsv}(\Psi^{\cdot}) \cap \{\nu'_1\sigma\rho, \dots, \nu'_{n'}\sigma\rho\} = \emptyset$ by definition of ρ.

– $\nu_1 \dots, \nu_n$ are in E but not in E'^{+} by assumption and by definition of ρ.

– E'^{+} is contained in E by assumption and definition of ρ. Its difference is an infinite set because $E \setminus E'$ is an infinite set and $\{\nu'_1\sigma\rho, \dots, \nu'_{n'}\sigma\rho\}$ is finite.

Hence, by Proposition 3.5.10, we have $\Psi \vdash_E \eta$, as we wanted to show.

2. v' is not in $T_{\mathrm{lab},E'}$. Note that for each $i = 1, \dots, n'$ the label schema variable $\nu'_i\sigma\rho$ is not in $T_{\mathrm{lab},E'}$ since $\nu'_i\sigma\rho$ is not in $\Xi_l \cup E'$ by definition of ρ. So, we can apply the induction hypothesis to $\Psi, s\, v\nu_1 \dots \nu_n, \Psi^{\cdot}, \Psi_e \vdash_E \eta$ in order to conclude, $\Psi \vdash_E \eta$ as we wanted.

- r is c_{e^+E}, where c_{e^+} is a constrained existential connective. The proof is similar to the case where r is c_{eE} and c_e is an existential connective.

- r is c_{eI} where c_e is an existential connective. Then η_{i_1} is $s_e v' v'_1 \dots v'_{n'}$, $\eta_{i_{j+1}}$ is $v'_j{:}\varphi_j$ for $j = 1, \dots, n'$, and η is $v'{:}c_e\varphi_1 \dots \varphi_{n'}$. Note that v' is in $T_{\mathrm{lab},E'}$, and that 1 is a s_e inheritance position because there is an existential connective c_e based on it. Then, by Lemma 4.4.4, we have (i) $\Psi \vdash_E s_e v' v'_1 \dots v'_{n'}$, and (ii) that $v', v'_1, \dots, v'_{n'}$ are in $T_{\mathrm{lab},E'}$. So, for each $j = 1, \dots, n'$, by induction hypothesis, we have that $\Psi \vdash_E v'_j{:}\varphi_j$. Then, it is straightforward to see that $\Psi \vdash_E v'{:}c_e\varphi_1 \ldots \varphi_{n'}$.

- r is c_{e^+I}, where c_{e^+} is a constrained existential connective. The proof is similar to the case where r is c_{eI} and c_e is an existential connective.

- r is $\rightarrow_I$. Then η is $v'{:}\varphi_1 \rightarrow \varphi_2$. So $\Psi, v'{:}\varphi_1, s\,v\nu_1 \ldots \nu_n, \Psi^{\cdot} \vdash_E v'{:}\varphi_2$. Observe that $\Psi \cup \{v'{:}\varphi_1\}$ is in $L_{E'}$, and that $v'{:}\varphi_2$ is also in $L_{E'}$. So we can use the induction hypothesis to conclude $\Psi, v'{:}\varphi_1 \vdash_E v'{:}\varphi_2$. Hence, $\Psi \vdash_E v'{:}\varphi_1 \rightarrow \varphi_2$, as we wanted to show.

- r is $\rightarrow_E$. Then η_{i_1} is $v'{:}\varphi_1 \rightarrow \varphi_2$, η_{i_2} is $v'{:}\varphi_1$, and η is $v'{:}\varphi_2$. Thus $\Psi, s\,v\nu_1 \ldots \nu_n, \Psi^{\cdot} \vdash_E v'{:}\varphi_1 \rightarrow \varphi_2$ and $\Psi, s\,v\nu_1 \ldots \nu_n, \Psi^{\cdot} \vdash_E v'{:}\varphi_1$. Then, by induction hypothesis, $\Psi \vdash_E v'{:}\varphi_1 \rightarrow \varphi_2$ and $\Psi \vdash_E v'{:}\varphi_1$. So, $\Psi \vdash_E v'{:}\varphi_2$, as we wanted to show.

- r is $\vee_E$. The proof is similar to the case where r is c_{eE} where c_e is a (constrained) existential connective.

- r is $\vee_I$ or $\wedge_I$ or $\wedge_E$. The proof is similar to the case where r is $\rightarrow_E$.

- r is $\top_I$. The proof is straightforward so it is omitted.

- r is a specification rule $c_u{}^{\mathrm{spc}}$, where c_u is a universal connective in Q_1 or in O_1. Then η_{i_1} is $v_1{:}c_u\varphi_1$, and η is $v_1{:}\varphi$. Note that v_1 is in $T_{\mathrm{lab},E'}$, and that there is a sober deduction sequence with length less than or equal to m for $\Psi, s\,v\nu_1 \ldots \nu_n, \Psi^{\cdot} \vdash_E v_1{:}c_u\varphi_1$. So, by induction hypothesis, $\Psi \vdash_E v_1{:}c_u\varphi_1$. Therefore it is straightforward to see that $\Psi \vdash_E v_1{:}\varphi$, taking into account this same rule $c_u{}^{\mathrm{spc}}$, because φ_1 and φ satisfy the proviso in P_f.

- r is a constraint specification rule $c_{u^+}{}^{\mathrm{spc}}$, where c_{u^+} is a constraint universal connective in Q_1 or in O_1. The proof is similar to the case where r is the specification rule $c_u{}^{\mathrm{spc}}$ and c_u is a universal connective in Q_1 or in O_1.

- r is a term inheritance rule to i and j over s'. Then η_{i_1} is $s'v'_1 \ldots v'_{n'}$ and η is $v'_i{:}t =_g v'_j{:}t$. Note that either v'_i or v'_j are in $T_{\mathrm{lab},E'}$, and that i and j are s' inheritance position because r is a term inheritance rule over s' to i and j. Then, by Lemma 4.4.4, we have (i) $\Psi \vdash_E s'v'_1 \ldots v'_{n'}$, and (ii) that $v'_1, \ldots, v'_{n'}$ are in $T_{\mathrm{lab},E'}$. So, both, v'_i and v'_j are in $T_{\mathrm{lab},E'}$, and it is straightforward to see that $\Psi \vdash_E v'_i{:}t =_g v'_j{:}t$.

- r is a formula inheritance rule from i to k over a relation in S. Then η_{i_1} is $s'v'_1 \ldots v'_{n'}$, η_{i_2} is $v'_i{:}\varphi$, and η is $v'_k{:}\varphi$. Note that v'_k is in $T_{\mathrm{lab},E'}$, and that k is a s' inheritance position because r is a formula inheritance rule to k based on s'. Then, by Lemma 4.4.4, we have (i) $\Psi \vdash_E s'v'_1 \ldots v'_{n'}$, and (ii) that $v'_1, \ldots, v'_{n'}$ are in $T_{\mathrm{lab},E'}$. Thus, by induction hypothesis, $\Psi \vdash_E v'_i{:}\varphi$. So, it is straightforward to see that $\Psi \vdash_E v'_k{:}\varphi$.

- r is a general formula inheritance rule over $\equiv_\omega$. Then η is $v''{:}\varphi$, η_{i_j} is $v'{:}x_j =_g v''{:}x_j$ for each $j = 1, \ldots, k-2$, $\eta_{i_{k-1}}$ is $v' \equiv_\omega v''$ and η_{i_k} is $v'{:}\varphi$ where $x_1, \ldots, x_k$ are the free variables of φ and v'' is in $T_{\mathrm{lab},E'}$. So, by Lemma 4.4.4, we have (i) $\Psi \vdash_E v' \equiv_\omega v''$, and (ii) that v' and v'' are in $T_{\mathrm{lab},E'}$. Moreover, by induction hypothesis, we have that $\Psi \vdash_E v'{:}x_j =_g v''{:}x_j$ for each $j = 1, \ldots, k-2$, and $\Psi \vdash_E v'{:}\varphi$, since v' is in $T_{\mathrm{lab},E'}$. So, by this rule r it is straightforward to see that $\Psi \vdash_E v'':\varphi$.

- r is $=_{g_t}$. Then, η_{i_1} is $v_1:t_1 =_g v':t'$ and η_{i_2} is $v':t' =_g v_2:t_2$. Recall that by assumption either v_1 or v_2 are in $T_{\text{lab},E'}$. Suppose it is v_1. Then, by induction hypothesis, $\Psi \vdash_E v_1:t_1 =_g v':t'$ and v_1 and v' are in $T_{\text{lab},E'}$. So, using the fact that v' is in $T_{\text{lab},E'}$ we can use again the induction hypothesis to conclude that $\Psi \vdash_E v':t' =_g v_2:t_2$ and that v_2 is also in $T_{\text{lab},E'}$. A similar thing happens if it is v_2 that is in $T_{\text{lab},E'}$. Hence, we have $\Psi \vdash_E v_1:t_1 =_g v_2:t_2$.

- r is $=_{g_s}$. Then, η_{i_1} is $v_2:t_2 =_g v_1:t_1$. Recall that by assumption either v_1 or v_2 are in $T_{\text{lab},E'}$. Then, by induction hypothesis, $\Psi \vdash_E v_2:t_2 =_g v_1:t_1$ and v_1 and v_2 are in $T_{\text{lab},E'}$. Hence, $\Psi \vdash_E v_1:t_1 =_g v_2:t_2$.

- r is $=_I$. Then, η_{i_1} is $v':t =_g v':t'$ and η is $v':t = t'$. Recall that by assumption v' is in $T_{\text{lab},E'}$. Then, by induction hypothesis, $\Psi \vdash_E v':t =_g v':t'$. Hence, $\Psi \vdash_E v':t = t'$ as we wanted to show.

- r is $=_E$. Then, η_{i_1} is $v':t = t'$ and η is $v':t =_g v':t'$. Recall that by assumption v' is in $T_{\text{lab},E'}$. Then, by induction hypothesis, $\Psi \vdash_E v':t = t'$. Hence, $\Psi \vdash_E v':t =_g v':t'$ as we wanted to show.

- r is $=_{g_p}$. Then, η_{i_j} is $v':t_j =_g v'':t'_j$ for each $j = 1, \ldots, k$, $\eta_{i_{k+1}}$ is $v':p(t_1, \ldots, t_k)$, $\eta_{i_{k+2}}$ is $v' \equiv_\omega v''$, and η is $v'':p(t'_1, \ldots, t'_k)$. Recall that by assumption v'' is in $T_{\text{lab},E'}$. Then, for each $j = 1, \ldots, k$, by induction hypothesis, $\Psi \vdash_E v':t_j =_g v'':t'_j$, and v' is also in $T_{\text{lab},E'}$. Considering this last fact we are in position to apply the induction hypothesis to $\eta_{i_{k+1}}$ in order to obtain $\Psi \vdash_E v':p(t_1, \ldots, t_k)$. Moreover, by Lemma 4.4.4, we obtain $\Psi \vdash_E v' \equiv_\omega v''$. So, after all these facts, it is straightforward to see that $\Psi \vdash_E v'':p(t'_1, \ldots, t'_k)$ as we wanted to show.

- r is $=_{g_f}$. Then, η_{i_j} is $v':t_j =_g v'':t'_j$ for each $j = 1, \ldots, k$, $\eta_{i_{k+1}}$ is $v' \equiv_\omega v''$, and η is $v':f(t_1, \ldots, t_k) =_g v'':f(t'_1, \ldots, t'_k)$. Recall that by assumption either v' or v'' are in $T_{\text{lab},E'}$. Suppose it is v'. Then, for each $j = 1, \ldots, k$, by induction hypothesis, $\Psi \vdash_E v':t_j =_g v'':t'_j$, and v'' is also in $T_{\text{lab},E'}$. Similarly if it is v'' that is in $T_{\text{lab},E'}$. Moreover, by Lemma 4.4.4, we obtain $\Psi \vdash_E v' \equiv_\omega v''$. So, after all these facts, it is straightforward to see that $\Psi \vdash_E v':f(t_1, \ldots, t_k) =_g v'':f(t'_1, \ldots, t'_k)$ as we wanted to show.

- r is $\equiv_{\alpha_g^x}$. Then, η_{i_1} is $v' \equiv_\alpha v''$ and η is $v':x =_g v'':x$. Recall that either v' or v'' are in $T_{\text{lab},E'}$. Then, by Lemma 4.4.4, we have $\Psi \vdash_E v' \equiv_\alpha v''$ and that v'' is also in $T_{\text{lab},E'}$. Hence the desired result follows because $\Psi \vdash_E v':x =_g v'':x$. *QED*

The next lemma is also similar to the preceding one, but contrary to that lemma, its proof is rather short, due to the condition of the reflexivity of the relation and of additional conditions on the sets of labelled schema formulae used as hypothesis.

Lemma 4.4.6 In a lfob deduction system, for any

- infinite sets E and E' of label schema variables such that E' is contained in E and its difference is an infinite set,

- label schema term v in $T_{\mathrm{lab},E'}$,
- set Ψ within $L_{E'}$,
- labelled non-relational schema formula η in $L_{E'}$,
- label schema variables $\nu_1, \ldots, \nu_n$ in E but not in E',
- relation s with arity $n+1$, reflexive,

if

$$\Psi, s\, v\nu_1 \ldots \nu_n \vdash_E \eta$$

then

$$\Psi \vdash_E \eta.$$

Proof Suppose $\Psi, s\, v\nu_1 \ldots \nu_n \vdash_E \eta$. Consider the schema substitution ρ over E within L_E $\{\nu_1, \ldots, \nu_n\}$ co-equivalent to id defined as $\nu_i\rho = v$ for each $i = 1, \ldots, n$. Then, by Proposition 2.3.14, we have that $\Psi, s\, vv \ldots v \vdash_E \eta$. So, using idempotence, Proposition 2.3.15, we have $\Psi \vdash_E \eta$, as we wanted to show, since $\vdash_E s\, v \ldots v$. *QED*

Finally, we now prove that, in the context of a clfob deduction system with locality, for any $v{:}\varphi$ consistent set there is a set that extends it, which is $v{:}\varphi$ consistent and satisfies all the conditions of an appropriate set except the conditions associated to the rules $\vee_E$ and $\rightarrow_I$. See Remark 4.4.8 and the introduction of Subsection 4.4, for a discussion of this fact. The construction of the appropriate set we consider in Proposition 4.4.7 is a generalization of the construction of a saturated maximal consistent set for first-order logic.

Proposition 4.4.7 In the context of a clfob deduction system with locality $\mathcal{D}$, for each $v'{:}\varphi'$ consistent set Ψ° within L there are a set E of label schema variables, and sets Ψ and $\overline{\Psi}$ within L_E, with

1. Ψ extends Ψ°,
2. $\overline{\Psi}$ extends $\{v'{:}\varphi'\}$,
3. $\Psi \nvdash_E \overline{v}{:}\overline{\varphi}$ for any $\overline{v}{:}\overline{\varphi}$ in $\overline{\Psi}$,
4. $v{:}\varphi \in \Psi$ iff $\Psi, v{:}\varphi \nvdash_E \overline{v}{:}\overline{\varphi}$, for any $\overline{v}{:}\overline{\varphi}$ in $\overline{\Psi}$ and $v{:}\varphi$ in L_E,
5. Ψ is deductively closed,
6. if *exh* is in $\mathcal{D}$ then for any v and $v^{\cdot}$ in $T_{\mathrm{lab},E}$ there is v_1 in $T_{\mathrm{lab},E}$ with $v \equiv_\omega v_1$ and $v_1 \equiv_\alpha v^{\cdot}$ in Ψ,
7. $v{:}c_u\varphi_1 \ldots \varphi_n \in \Psi$ iff for any $v_1, \ldots, v_n$ in $T_{\mathrm{lab},E}$, we have $v_n{:}\varphi_n \in \Psi$ whenever

$$r_u v v_1 \ldots v_n \in \Psi \quad \text{and} \quad v_1{:}\varphi_1 \in \Psi \quad \text{and} \quad \ldots \quad \text{and} \quad v_{n-1}{:}\varphi_{n-1} \in \Psi,$$

if c_u is a universal connective in $\mathcal{D}$,

8. $v{:}c_{u^+}\varphi_1 \ldots \varphi_n \in \Psi$ iff for any $v_1, \ldots, v_n$ in $T_{\mathrm{lab},E}$, we have $v_n{:}\varphi_n \in \Psi$ whenever

$$r_{u^+} v v_1 \ldots v_n \in \Psi \quad \text{and} \quad v_1{:}\varphi_1 \in \Psi \quad \text{and} \quad \ldots \quad \text{and} \quad v_{n-1}{:}\varphi_{n-1} \in \Psi$$

$$\text{and} \quad v_1{:}\gamma_1 \in \Psi \quad \text{and} \quad \ldots \quad \text{and} \quad v_n{:}\gamma_n \in \Psi,$$

if c_{u^+} is a constrained universal connective in $\mathcal{D}$, and $\gamma_1, \ldots, \gamma_n$ are the constraint formulae of c_{u^+},

9. $v{:}c_e\varphi_1 \ldots \varphi_n \in \Psi$ iff there are $v_1, \ldots, v_n$ in $T_{\mathrm{lab},E}$, with

$$r_e v v_1 \ldots v_n \in \Psi \quad \text{and} \quad v_1{:}\varphi_1 \in \Psi \text{ and } \ldots \text{ and } v_n{:}\varphi_n \in \Psi,$$

if c_e is an existential connective in $\mathcal{D}$,

10. $v{:}c_{e^+}\varphi_1 \ldots \varphi_n \in \Psi$ iff there are $v_1, \ldots, v_n$ in $T_{\mathrm{lab},E}$ with

$$r_{e^+} v v_1 \ldots v_n \in \Psi \quad \text{and} \quad v_1{:}\varphi_1 \in \Psi \quad \text{and} \quad \ldots \quad \text{and} \quad v_n{:}\varphi_n \in \Psi,$$

$$\text{and} \quad v_1{:}\gamma_1 \in \Psi \quad \text{and} \quad \ldots \quad \text{and} \quad v_n{:}\gamma_n \in \Psi,$$

if c_{e^+} is a constrained existential connective in $\mathcal{D}$, and $\gamma_1, \ldots, \gamma_n$ are the constraint formulae of c_{e^+},

11. $v{:}-\varphi \in \Psi$ iff $v^*{:}\varphi \in \Psi$ implies $v'{:}\bot \in \Psi$ for some v', if $-$ is a non-local negation in $\mathcal{D}$,

12. $v{:}\neg\varphi \in \Psi$ iff $v{:}\varphi \in \Psi$ implies $v'{:}\bot \in \Psi$ for some v', if $\neg$ is a local negation in $\mathcal{D}$,

13. $v{:}\varphi_1 \wedge \varphi_2 \in \Psi$ iff $v{:}\varphi_1 \in \Psi$ and $v{:}\varphi_2 \in \Psi$, if $\wedge$ is a conjunction in $\mathcal{D}$,

14. $v{:}\varphi_1 \vee \varphi_2 \in \Psi$ if $v{:}\varphi_1 \in \Psi$ or $v{:}\varphi_2 \in \Psi$, if $\vee$ is a disjunction in $\mathcal{D}$,

15. $v{:}\varphi_1 \rightarrow \varphi_2 \in \Psi$ only if $v{:}\varphi_1 \in \Psi$ implies $v{:}\varphi_2 \in \Psi$, if $\rightarrow$ is a implication in $\mathcal{D}$,

16. and, if for each $j = 1, \ldots, k$, and ρ_j over E within L_E, $\Upsilon_j\sigma$ co-equivalent to id over E within L_E

$$\Psi_j\sigma\rho_j \subseteq \Psi \quad \text{implies} \quad \eta_j\sigma\rho_j \in \Psi$$

then

$$\eta\sigma \in \Psi$$

for every

- $\langle\{\langle\vartheta_1, \Psi_1, \eta_1\rangle, \ldots, \langle\vartheta_k, \Psi_k, \eta_k\rangle\}, \eta, P_f, P_d\rangle$ in $\mathcal{D}$, distinct of $\vee_E$ if a disjunction $\vee$ is in $\mathcal{D}$, and distinct of $\rightarrow_I$ if the implication $\rightarrow$ is in $\mathcal{D}$,
- σ within L_E such that $(\pi_\Sigma\sigma_{\mathrm{nl}})$ is 1 for each π in P_f, and $\pi_{\Sigma;E}(\sigma)$ is 1 for each π in P_d.

Proof The proof proceeds in the following way. We first construct inductively the sets Ψ_i and $\overline{\Psi}_i$ for all $i \geq 0$, from an enumeration of labelled schema formulae and taking into account the presence or not of the exhaustive rule. Then we prove an auxiliary fact and only after that we define the sets Ψ and $\overline{\Psi}$. Finally, we show the properties enumerated in the proposition. So, let Ψ° be a $v'{:}\varphi'$ consistent set and E an infinite set of label schema variables. Consider an enumeration $v_1{:}\varphi_1, v_2{:}\varphi_2, \ldots$ of all label schema terms $v_1, v_2, \ldots$ in $T_{\mathrm{lab},E}$ and schema formulae $\varphi_1, \varphi_2, \ldots$ in $L_{\mathrm{fob},E}$. We inductively define the sequences $\Psi_0 \subseteq \Psi_1 \subseteq \ldots$, and $\overline{\Psi}_0 \subseteq \overline{\Psi}_1 \subseteq \ldots$ as follows:

Base: Ψ_0 is Ψ°, and $\overline{\Psi}_0$ is $\{v'{:}\varphi'\}$.

Step: note that Ψ_{k+1} is defined after we first define an auxiliary set Ψ_{k^+}:

- if $\Psi_k, v_{k+1}{:}\varphi_{k+1} \nvdash_E \overline{v}{:}\overline{\varphi}$ for any $\overline{v}{:}\overline{\varphi}$ in $\overline{\Psi}_k$, then

- if φ_{k+1} is $c_{e^+}\varphi_1 \dots \varphi_n$ and $\gamma_1, \dots, \gamma_n$ are the constraint formulae of c_{e^+} then
  - $\overline{\Psi}_{k+1}$ is $\overline{\Psi}_k$,
  - Ψ_{k^+} is $\Psi_k \cup \{v_{k+1}{:}\varphi_{k+1}\} \cup \{r_{e^+}v_{k+1}\nu_1 \dots \nu_n, \nu_1{:}\varphi_1, \dots, \nu_n{:}\varphi_n, \nu_1{:}\gamma_1, \dots, \nu_n{:}\gamma_n\}$, where $\nu_1, \dots, \nu_n$ are in E but not in $\mathrm{lsv}(\Psi_k \cup \overline{\Psi}_k \cup \{v_{k+1}\})$,
- if φ_{k+1} is $c_e\varphi_1 \dots \varphi_n$ then
  - $\overline{\Psi}_{k+1}$ is $\overline{\Psi}_k$,
  - Ψ_{k^+} is $\Psi_k \cup \{v_{k+1}{:}\varphi_{k+1}\} \cup \{\underline{r_e}v_{k+1}\nu_1 \dots \nu_n, \nu_1{:}\varphi_1, \dots, \nu_n{:}\varphi_n\}$, where $\nu_1, \dots, \nu_n$ are in E but not in $\mathrm{lsv}(\Psi_k \cup \overline{\Psi}_k \cup \{v_{k+1}\})$.
- otherwise
  - $\overline{\Psi}_{k+1}$ is $\overline{\Psi}_k$,
  - Ψ_{k^+} is $\Psi_k \cup \{v_{k+1}{:}\varphi_{k+1}\}$.

- else ($\Psi_k, v_{k+1}{:}\varphi_{k+1} \vdash_E \overline{v}{:}\overline{\varphi}$ for some $\overline{v}{:}\overline{\varphi}$ in $\overline{\Psi}_k$),

- if φ_{k+1} is $c_{u^+}\varphi_1 \dots \varphi_n$ and $\gamma_1, \dots, \gamma_n$ are the constraint formulae of c_{u^+} then
  - $\overline{\Psi}_{k+1}$ is $\overline{\Psi}_k \cup \{v_{k+1}{:}\varphi_{k+1}\} \cup \{\nu_n{:}\varphi_n\}$,
  - Ψ_{k^+} is $\Psi_k \cup \{r_{u^+}v_{k+1}\nu_1 \dots \nu_n, \nu_1{:}\varphi_1, \dots, \nu_{n-1}{:}\varphi_{n-1}, \nu_1{:}\gamma_1, \dots, \nu_n{:}\gamma_n\}$, where ν_1, $\dots, \nu_n$ are in E but not in $\mathrm{lsv}(\Psi_k \cup \overline{\Psi}_k \cup \{v_{k+1}\})$.
- if φ_{k+1} is $c_u\varphi_1 \dots \varphi_n$ then
  - $\overline{\Psi}_{k+1}$ is $\overline{\Psi}_k \cup \{v_{k+1}{:}\varphi_{k+1}\} \cup \{\nu_n{:}\varphi_n\}$,
  - Ψ_{k^+} is $\Psi_k \cup \{r_uv_{k+1}\nu_1 \dots \nu_n, \nu_1{:}\varphi_1, \dots, \nu_{n-1}{:}\varphi_{n-1}\}$, where $\nu_1, \dots, \nu_n$ are in E but not in $\mathrm{lsv}(\Psi_k \cup \overline{\Psi}_k \cup \{v_{k+1}\})$.
- otherwise
  - $\overline{\Psi}_{k+1}$ is $\overline{\Psi}_k \cup \{v_{k+1}{:}\varphi_{k+1}\}$,
  - Ψ_{k^+} is Ψ_k.

So, Ψ_{k+1} is Ψ_{k^+} if *exh* is not in $\mathcal{D}$, otherwise Ψ_{k+1} is Ψ_{k^+,m_k}, where Ψ_{k^+,m_k} is defined as follows: let $\langle v_1, v^{\cdot}_1\rangle, \dots, \langle v_{m_k}, v^{\cdot}_{m_k}\rangle$ be a finite enumeration of all distinct pairs where the first component is a label schema term in $v_1{:}\varphi_1, \dots, v_k{:}\varphi_k$ and the second component is a label schema term in $v_{k+1}{:}\varphi_{k+1}$. Then Ψ_{k^+,m_k} is the last element of the sequence $\Psi_{k^+,0} \subseteq \dots \subseteq \Psi_{k^+,m_k}$ inductively defined as follows:

Base: $\Psi_{k^+,0}$ is Ψ_{k^+}.

Step: $\Psi_{k^+,i+1}$ is defined as follows, where ν and ν_1 are in E but not in $\mathrm{lsv}(\Psi_{k^+,i} \cup \overline{\Psi}_{k+1})$,

- $\Psi_{k^+,i+1}$ is $\Psi_{k^+,i}$ whenever $\Psi_{k^+,i} \vdash_E v_{i+1} \equiv_\omega v, v \equiv_\alpha v^{\cdot}_{i+1}, v_{i+1} \equiv_\alpha v^{\cdot}, v^{\cdot} \equiv_\omega v^{\cdot}_{i+1}$ for some v and $v^{\cdot}$ in $T_{\mathrm{lab},E}$.

- $\Psi_{k^+,i+1}$ is $\Psi_{k^+,i} \cup \{v_{i+1} \equiv_\alpha \nu, \nu \equiv_\omega v^{\cdot}_{i+1}\}$ whenever $\Psi_{k^+,i} \vdash_E v_{i+1} \equiv_\omega v, v \equiv_\alpha v^{\cdot}_{i+1}$ for some v in $T_{\mathrm{lab},E}$ and $\Psi_{k^+,i} \not\vdash_E v_{i+1} \equiv_\alpha v^{\cdot}, v^{\cdot} \equiv_\omega v^{\cdot}_{i+1}$ for any $v^{\cdot}$ in $T_{\mathrm{lab},E}$.

- $\Psi_{k^+,i+1}$ is $\Psi_{k^+,i} \cup \{v_{i+1} \equiv_\omega \nu, \nu \equiv_\alpha v^{\cdot}_{i+1}\}$ whenever $\Psi_{k^+,i} \nvdash_E v_{i+1} \equiv_\omega v, v \equiv_\alpha v^{\cdot}_{i+1}$ for any v in $T_{\text{lab},E}$ and $\Psi_{k^+,i} \vdash_E v_{i+1} \equiv_\alpha v^{\cdot}, v^{\cdot} \equiv_\omega v^{\cdot}_{i+1}$ for some $v^{\cdot}$ in $T_{\text{lab},E}$.

- $\Psi_{k^+,i+1}$ is $\Psi_{k^+,i} \cup \{v_{i+1} \equiv_\omega \nu, \nu \equiv_\alpha v^{\cdot}_{i+1}, v_{i+1} \equiv_\alpha \nu_1, \nu_1 \equiv_\omega v^{\cdot}_{i+1}\}$ whenever $\Psi_{k^+,i} \nvdash_E v_{i+1} \equiv_\omega v, v \equiv_\alpha v^{\cdot}_{i+1}$ and $\Psi_{k^+,i} \nvdash_E v_{i+1} \equiv_\alpha v^{\cdot}, v^{\cdot} \equiv_\omega v^{\cdot}_{i+1}$, for any v and $v^{\cdot}$ in $T_{\text{lab},E}$.

End of construction

Before defining Ψ we prove a very useful fact: for every i greater than or equal to 0 the set Ψ_i is such that $\Psi_i \nvdash_E \overline{v}{:}\overline{\varphi}$ for any $\overline{v}{:}\overline{\varphi}$ in $\overline{\Psi}_i$. The proof follows by induction:

Base: recall that Ψ_0 is Ψ° and that $\overline{\Psi}_0$ is $\{v'{:}\varphi'\}$. Then the result follows because $\Psi^\circ \nvdash_E v'{:}\varphi'$.

Step: assuming $\Psi_k \nvdash_E \overline{v}{:}\overline{\varphi}$ for any $\overline{v}{:}\overline{\varphi}$ in $\overline{\Psi}_k$, we want to show that $\Psi_{k+1} \nvdash_E \overline{v}{:}\overline{\varphi}$ for any $\overline{v}{:}\overline{\varphi}$ in $\overline{\Psi}_{k+1}$. According to the definition of Ψ_{k+1} we show first that $\Psi_{k^+} \nvdash_E \overline{v}{:}\overline{\varphi}$ for any $\overline{v}{:}\overline{\varphi}$ in $\overline{\Psi}_{k+1}$. To show this, following the definition of Ψ_{k^+} and $\overline{\Psi}_{k+1}$, we divide the proof in cases:

• if $\Psi_k, v_{k+1}{:}\varphi_{k+1} \nvdash_E \overline{v}{:}\overline{\varphi}$ for any $\overline{v}{:}\overline{\varphi}$ in $\overline{\Psi}_k$, then if φ_{k+1} is:

- $c_e\varphi_1 \ldots \varphi_n$ then, for any $\overline{v}{:}\overline{\varphi}$ in $\overline{\Psi}_{k+1} = \overline{\Psi}_k$, to show $\Psi_{k^+} \nvdash_E \overline{v}{:}\overline{\varphi}$ is equivalent to show $\Psi_k, v_{k+1}{:}\varphi_{k+1}, r_e v_{k+1}\nu_1 \ldots \nu_n, \nu_1{:}\varphi_1, \ldots, \nu_n{:}\varphi_n \nvdash_E \overline{v}{:}\overline{\varphi}$. So, suppose $\Psi_k, v_{k+1}{:}\varphi_{k+1}, r_e v_{k+1} \nu_1 \ldots \nu_n, \nu_1{:}\varphi_1, \ldots, \nu_n{:}\varphi_n \vdash_E \overline{v}{:}\overline{\varphi}$ for some $\overline{v}{:}\overline{\varphi}$ in $\overline{\Psi}_k$. Since $\Psi_k, v_{k+1}{:}\varphi_{k+1} \vdash_E v_{k+1}{:}\varphi_{k+1}$ we can use Proposition 3.5.10 and rule c_{eE} to conclude $\Psi_k, v_{k+1}{:}\varphi_{k+1} \vdash_E \overline{v}{:}\overline{\varphi}$ which contradicts our assumption.

- $c_{e^+}\varphi_1 \ldots \varphi_n$ then the proof is similar to the preceding one.

- not one of the preceding cases, then the proof is straightforward so it is omitted.

• if $\Psi_k, v_{k+1}{:}\varphi_{k+1} \vdash_E \overline{v}{:}\overline{\varphi}$ for some $\overline{v}{:}\overline{\varphi}$ in $\overline{\Psi}_k$, then if φ_{k+1} is (note first that $\Psi_k \nvdash_E v_{k+1}{:}\varphi_{k+1}$, since if this is not the case then, by idempotence, Proposition 2.3.15, we would have $\Psi_k \vdash_E \overline{v}{:}\overline{\varphi}$ for some $\overline{v}{:}\overline{\varphi}$ in $\overline{\Psi}_k$, which contradicts the induction hypothesis):

- $c_u\varphi_1 \ldots \varphi_n$, then, for any $\overline{v}{:}\overline{\varphi}$ in $\overline{\Psi}_{k+1}$ which is $\overline{\Psi}_k \cup \{v_{k+1}{:}\varphi_{k+1}\} \cup \{\nu_n{:}\varphi_n\}$, to show $\Psi_{k^+} \nvdash \overline{v}{:}\overline{\varphi}$ is equivalent to show $\Psi_k, r_u v_{k+1}\nu_1 \ldots \nu_n, \nu_1{:}\varphi_1, \ldots, \nu_{n-1}{:}\varphi_{n-1} \nvdash_E \overline{v}{:}\overline{\varphi}$. We now divide the proof in three cases depending on $\overline{v}{:}\overline{\varphi}$. If $\overline{v}{:}\overline{\varphi}$ is:

– $v_{k+1}{:}\varphi_{k+1}$, then, since (i) we are in the context of a clfob deduction system with locality, see Definition 4.4.2, (ii) $\Psi_k \nvdash_E v_{k+1}{:}\varphi_{k+1}$, and (iii) $\nu_1, \ldots, \nu_n$ are in E but not in $\text{lsv}(\Psi_k) \cup \text{lsv}(v)$, we can use either Lemma 4.4.6 or Proposition 4.4.5, depending if c_u is unary and its relation reflexive, to conclude that $\Psi_k, r_u v_{k+1}\nu_1 \ldots \nu_n, \nu_1{:}\varphi_1, \ldots, \nu_{n-1}{:}\varphi_{n-1} \nvdash_E v_{k+1}{:}\varphi_{k+1}$.

– $\nu_n{:}\varphi_n$, then, suppose $\Psi_k, r_u v\nu_1 \ldots \nu_n, \nu_1{:}\varphi_1, \ldots, \nu_{n-1}{:}\varphi_{n-1} \vdash_E \nu_n{:}\varphi_n$. Thus we could use Proposition 3.5.10 and rule c_{uI} to conclude $\Psi_k \vdash_E v_{k+1}{:}\varphi_{k+1}$ which contradicts the fact above proven that $\Psi_k \nvdash_E v_{k+1}{:}\varphi_{k+1}$.

– in $\overline{\Psi}_k$ then, since (i) we are in the context of a clfob deduction system with locality, see Definition 4.4.2, (ii) $\Psi_k \nvdash_E \overline{v}{:}\overline{\varphi}$, and (iii) $\nu_1, \ldots, \nu_n$ are in E but not in $\text{lsv}(\Psi_k) \cup \text{lsv}(\overline{\Psi}_k)$, we can use either Lemma 4.4.6 or Proposition 4.4.5, depending if c_u is unary and its relation reflexive, to conclude that $\Psi_k, r_u v\nu_1 \ldots \nu_n$, $\nu_1{:}\varphi_1, \ldots, \nu_{n-1}{:}\varphi_{n-1} \nvdash_E \overline{v}{:}\overline{\varphi}$.

- $c_{u^+}\varphi_1 \ldots \varphi_n$, then, the proof is similar to the case that φ_{k+1} is $c_u\varphi_1 \ldots \varphi_n$ so we omit it.

- otherwise for any $\overline{v}{:}\overline{\varphi}$ in $\overline{\Psi}_{k+1}$ which is $\overline{\Psi}_k \cup \{v_{k+1}{:}\varphi_{k+1}\}$, to show that $\Psi_{k^+} \vdash_E \overline{v}{:}\overline{\varphi}$ is equivalent to show that $\Psi_k \nvdash_E \overline{v}{:}\overline{\varphi}$. We now divide the proof in two cases depending on $\overline{v}{:}\overline{\varphi}$. If $\overline{v}{:}\overline{\varphi}$ is:

- $v_{k+1}{:}\varphi_{k+1}$, then the result follows taking into account idempotence, as was shown above;
- in $\overline{\Psi}_k$ then the result follows by induction hypothesis.

Hence $\Psi_{k^+} \nvdash_E \overline{v}{:}\overline{\varphi}$ for any $\overline{v}{:}\overline{\varphi}$ in $\overline{\Psi}_{k+1}$. In order to show that $\Psi_{k+1} \nvdash_E \overline{v}{:}\overline{\varphi}$ for any $\overline{v}{:}\overline{\varphi}$ in $\overline{\Psi}_{k+1}$, according to the definition of Ψ_{k+1}, we have two cases to consider:

• *exh* is not in $\mathcal{D}$. Then Ψ_{k+1} is Ψ_{k^+}, and so $\Psi_{k+1} \nvdash_E \overline{v}{:}\overline{\varphi}$ for any $\overline{v}{:}\overline{\varphi}$ in $\overline{\Psi}_{k+1}$, as we wanted to show.

• *exh* is in $\mathcal{D}$. Then Ψ_{k+1} is Ψ_{k^+,m_k} and $\Psi_{k+1} \nvdash_E \overline{v}{:}\overline{\varphi}$ for any $\overline{v}{:}\overline{\varphi}$ in $\overline{\Psi}_{k+1}$ because $\Psi_{k^+,i} \nvdash_E \overline{v}{:}\overline{\varphi}$ for any $\overline{v}{:}\overline{\varphi}$ in $\overline{\Psi}_{k+1}$ for all $i = 1, \ldots, m_k$, as we show now by induction:

Base: $\Psi_{k^+,0} \nvdash_E \overline{v}{:}\overline{\varphi}$ for any $\overline{v}{:}\overline{\varphi}$ in $\overline{\Psi}_{k+1}$, since $\Psi_{k^+,0}$ is Ψ_{k^+} and $\Psi_{k^+} \nvdash_E \overline{v}{:}\overline{\varphi}$ for any $\overline{v}{:}\overline{\varphi}$ in $\overline{\Psi}_{k+1}$ as we just proved.

Step: assuming $\Psi_{k^+,i} \nvdash_E \overline{v}{:}\overline{\varphi}$ for any $\overline{v}{:}\overline{\varphi}$ in $\overline{\Psi}_{k+1}$, we want to show that $\Psi_{k^+,i+1} \nvdash_E \overline{v}{:}\overline{\varphi}$ for any $\overline{v}{:}\overline{\varphi}$ in $\overline{\Psi}_{k+1}$. So, following the definition of $\Psi_{k^+,i+1}$, we divide the proof in four cases:

- if $\Psi_{k^+,i} \vdash_E v_{i+1} \equiv_\omega v, v \equiv_\alpha v^{\cdot}_{i+1}, v_{i+1} \equiv_\alpha v^{\cdot}, v^{\cdot} \equiv_\omega v^{\cdot}_{i+1}$ for some v and $v^{\cdot}$ in $T_{\mathrm{lab},E}$ then $\Psi_{k^+,i+1}$ is $\Psi_{k^+,i}$ and so the result follows by the induction hypothesis.

- if $\Psi_{k^+,i} \vdash_E v_{i+1} \equiv_\omega v, v \equiv_\alpha v^{\cdot}_{i+1}$ for some v in $T_{\mathrm{lab},E}$ and $\Psi_{k^+,i} \nvdash_E v_{i+1} \equiv_\alpha v^{\cdot}, v^{\cdot} \equiv_\omega v^{\cdot}_{i+1}$ for any $v^{\cdot}$ in $T_{\mathrm{lab},E}$ then $\Psi_{k^+,i+1}$ is $\Psi_{k^+,i} \cup \{v_{i+1} \equiv_\alpha \nu, \nu \equiv_\omega v^{\cdot}_{i+1}\}$. So, if there were $\overline{v}{:}\overline{\varphi}$ in $\overline{\Psi}_{k+1}$ with $\Psi_{k^+,i+1} \vdash_E \overline{v}{:}\overline{\varphi}$ then, by idempotence, Proposition 2.3.15 since $v^{\cdot}_{i+1} \equiv_\omega \nu \vdash_{v^{\cdot}_{i+1},\nu} \nu \equiv_\omega v^{\cdot}_{i+1}$ and $\nu \equiv_\alpha v_{i+1} \vdash_{v_{i+1},\nu} v_{i+1} \equiv_\alpha \nu$ we have that $\Psi_{k^+,i}, v^{\cdot}_{i+1} \equiv_\omega \nu, \nu \equiv_\alpha v_{i+1} \vdash_E \overline{v}{:}\overline{\varphi}$, and thus by Proposition 3.5.10 and rule *exh*, we would have $\Psi_{k^+,i} \vdash_E \overline{v}{:}\overline{\varphi}$, which would contradict the initial assumption. Thus $\Psi_{k^+,i+1} \nvdash_E \overline{v}{:}\overline{\varphi}$ for any $\overline{v}{:}\overline{\varphi}$ in $\overline{\Psi}_{k+1}$, as we wanted to show.

- if $\Psi_{k^+,i} \nvdash_E v_{i+1} \equiv_\omega v, v \equiv_\alpha v^{\cdot}_{i+1}$ for any v in $T_{\mathrm{lab},E}$ and $\Psi_{k^+,i} \vdash_E v_{i+1} \equiv_\alpha v^{\cdot}, v^{\cdot} \equiv_\omega v^{\cdot}_{i+1}$ for some $v^{\cdot}$ in $T_{\mathrm{lab},E}$ then the proof is similar to the preceding one so we omit it.

- if $\Psi_{k^+,i} \nvdash_E v_{i+1} \equiv_\omega v, v \equiv_\alpha v^{\cdot}_{i+1}$ and $\Psi_{k^+,i} \nvdash_E v_{i+1} \equiv_\alpha v^{\cdot}, v^{\cdot} \equiv_\omega v^{\cdot}_{i+1}$, for any v and $v^{\cdot}$ in $T_{\mathrm{lab},E}$ then $\Psi_{k^+,i+1}$ is $\Psi_{k^+,i} \cup \{v_{i+1} \equiv_\omega \nu, \nu \equiv_\alpha v^{\cdot}_{i+1}, v_{i+1} \equiv_\alpha \nu_1, \nu_1 \equiv_\omega v^{\cdot}_{i+1}\}$. So, if there is $\overline{v}{:}\overline{\varphi}$ in $\overline{\Psi}_{k+1}$ with $\Psi_{k^+,i+1} \vdash_E \overline{v}{:}\overline{\varphi}$ then, by Proposition 3.5.10 and rule *exh* we would have $\Psi_{k^+,i}, v_{i+1} \equiv_\alpha \nu_1, \nu_1 \equiv_\omega v^{\cdot}_{i+1} \vdash_E \overline{v}{:}\overline{\varphi}$, and thus, by idempotence, Proposition 2.3.15 we would have that $\Psi_{k^+,i}, v^{\cdot}_{i+1} \equiv_\omega \nu_1, \nu_1 \equiv_\alpha v_{i+1} \vdash_E \overline{v}{:}\overline{\varphi}$. Hence, by Proposition 3.5.10 and rule *exh*, since $\overline{v}{:}\overline{\varphi}$ is of the form $v{:}\varphi$, we would have $\Psi_{k^+,i} \vdash_E \overline{v}{:}\overline{\varphi}$, which would contradict the initial assumption. Thus $\Psi_{k^+,i+1} \nvdash_E \overline{v}{:}\overline{\varphi}$ for any $\overline{v}{:}\overline{\varphi}$ in $\overline{\Psi}_{k+1}$, as we wanted to show.

Henceforth, $\Psi_{k+1} \nvdash_E \overline{v}{:}\overline{\varphi}$ for any $\overline{v}{:}\overline{\varphi}$ in $\overline{\Psi}_{k+1}$ as we wanted to show, since $\Psi_{k^+,m_k} \nvdash_E \overline{v}{:}\overline{\varphi}$ for any $\overline{v}{:}\overline{\varphi}$ in $\overline{\Psi}_{k+1}$.
End of proof of auxiliary fact

Finally, define Ψ as follows:

$$\Psi = \bigcup_{i \geq 0} (\Psi_i)^{\vdash_E}$$

and $\overline{\Psi}$ as follows:

$$\overline{\Psi} = \bigcup_{i \geq 0} \overline{\Psi}_i.$$

Before showing the desired properties mentioned in the proposition, we show a last auxiliary fact:

0. $v{:}\varphi \in \Psi$ iff $\Psi_k, v{:}\varphi \not\vdash \overline{v}{:}\overline{\varphi}$ for any $\overline{v}{:}\overline{\varphi}$ in $\overline{\Psi}_k$, where $k+1$ is the position of $v{:}\varphi$ in the enumeration above mentioned:

Only if Suppose $\Psi_k, v{:}\varphi \vdash_E \overline{v}'{:}\overline{\varphi}'$ for some $\overline{v}'{:}\overline{\varphi}'$ in $\overline{\Psi}_k$. Then, by construction of $\overline{\Psi}_{k+1}$, $v{:}\varphi$ is in $\overline{\Psi}_{k+1}$. Suppose $v{:}\varphi$ is in Ψ and let l be the less natural such that $v{:}\varphi$ is in $(\Psi_l)^{\vdash_E}$, i.e., $\Psi_l \vdash_E v{:}\varphi$. Consider the case where:

- $l \leq k$. Then $\Psi_k \vdash_E v{:}\varphi$, by monotony Proposition 2.3.13, since Ψ_l is contained in Ψ_k because $\Psi_0 \subseteq \Psi_1 \subseteq \ldots$. So $\Psi_k \vdash_E \overline{v}'{:}\overline{\varphi}'$ by idempotence Proposition 2.3.15. Recall that $\overline{v}'{:}\overline{\varphi}'$ is in $\overline{\Psi}_k$. But this contradicts the fact that for every $i \geq 0$, $\Psi_i \not\vdash_E \overline{v}{:}\overline{\varphi}$ for any $\overline{v}{:}\overline{\varphi}$ in $\overline{\Psi}_i$. Hence $v{:}\varphi$ is not in Ψ.

- $l > k$. Note that $\Psi_l \vdash_E \overline{v}'{:}\overline{\varphi}'$ by idempotence, Proposition 2.3.13, because $\Psi_l \vdash_E v{:}\varphi$, Ψ_k is contained in Ψ_l since $\Psi_0 \subseteq \Psi_1 \subseteq \ldots$, and $\Psi_k, v{:}\varphi \vdash_E \overline{v}'{:}\overline{\varphi}'$. Note also that $\overline{v}'{:}\overline{\varphi}'$ is in $\overline{\Psi}_l$ since $\overline{v}'{:}\overline{\varphi}'$ is in $\overline{\Psi}_k$ and $\overline{\Psi}_k$ is contained in $\overline{\Psi}_l$ because $\overline{\Psi}_0 \subseteq \overline{\Psi}_1 \subseteq \ldots$. But this contradicts the fact that for every $i \geq 0$, $\Psi_i \not\vdash_E \overline{v}{:}\overline{\varphi}$ for any $\overline{v}{:}\overline{\varphi}$ in $\overline{\Psi}_i$. Hence $v{:}\varphi$ is not in Ψ.

So, in both cases $v{:}\varphi$ is not in Ψ, as we wanted to show.

If Suppose $\Psi_k, v{:}\varphi \not\vdash_E \overline{v}{:}\overline{\varphi}$ for any $\overline{v}{:}\overline{\varphi}$ in $\overline{\Psi}_k$. Then, by construction of Ψ_{k+1} and of $\overline{\Psi}_{k+1}$ we have that $v{:}\varphi$ is in Ψ_{k+1} and so in Ψ as we wanted to show.

Finally we now show that Ψ satisfies the properties mentioned in the proposition:

1. and 2. The result follows by definition of Ψ and $\overline{\Psi}$ since Ψ_0 is Ψ°, and $\overline{\Psi}_0$ is $\{v'{:}\varphi'\}$.

3. Suppose $\Psi \vdash_E \overline{v}'{:}\overline{\varphi}'$ for some $\overline{v}'{:}\overline{\varphi}'$ in $\overline{\Psi}$. Then, there is a finite deduction sequence ending in $\langle \overline{v}'{:}\overline{\varphi}', \Psi', 1\rangle$ for $\Psi \vdash_E \overline{v}'{:}\overline{\varphi}'$ where Ψ' is contained in Ψ. Let $\bar{\imath}$ be the less natural such that $\overline{v}'{:}\overline{\varphi}'$ is in $\overline{\Psi}_{\bar{\imath}}$, and l the less natural such that Ψ' is contained in $(\Psi_l)^{\vdash_E}$. Observe that $(\Psi_1)^{\vdash_E} \subseteq (\Psi_2)^{\vdash_E} \subseteq \ldots$ by monotony, Proposition 2.3.13, since $\Psi_1 \subseteq \Psi_2 \subseteq \ldots$. Note that $\Psi_l \vdash_E \overline{v}'{:}\overline{\varphi}'$, by idempotence, Proposition 2.3.15, since $\Psi' \vdash_E \overline{v}'{:}\overline{\varphi}'$ and $\Psi_l \vdash_E \eta'$ for any η' in Ψ'. Consider the case where:

- $l \leq \bar{\imath}$. Then $\Psi_{\bar{\imath}} \vdash_E \overline{v}'{:}\overline{\varphi}'$ by monotony, Proposition 2.3.13, since Ψ_l is contained in $\Psi_{\bar{\imath}}$ because $\Psi_0 \subseteq \Psi_1 \subseteq \ldots$. Recall that $\overline{v}'{:}\overline{\varphi}'$ is in $\overline{\Psi}_{\bar{\imath}}$. But this contradicts the fact that for every $i \geq 0$, $\Psi_i \not\vdash_E \overline{v}{:}\overline{\varphi}$ for any $\overline{v}{:}\overline{\varphi}$ in $\overline{\Psi}_i$. So, $\Psi \not\vdash_E \overline{v}{:}\overline{\varphi}$ for any $\overline{v}{:}\overline{\varphi}$ in $\overline{\Psi}$, as we wanted to show.

- $l > \bar{\imath}$. Note that $\overline{v}'{:}\overline{\varphi}'$ is in $\overline{\Psi}_l$ since $\overline{\Psi}_{\bar{\imath}}$ is contained in $\overline{\Psi}_l$ because $\overline{\Psi}_0 \subseteq \overline{\Psi}_1 \subseteq \ldots$. But this contradicts the fact that for every $i \geq 0$, $\Psi_i \not\vdash_E \overline{v}{:}\overline{\varphi}$ for any $\overline{v}{:}\overline{\varphi}$ in $\overline{\Psi}_i$. So, $\Psi \not\vdash_E \overline{v}{:}\overline{\varphi}$ for any $\overline{v}{:}\overline{\varphi}$ in $\overline{\Psi}$, as we wanted to show.

4. *Only if* Suppose $\Psi, v{:}\varphi \vdash_E \overline{v}'{:}\overline{\varphi}'$ for some $\overline{v}'{:}\overline{\varphi}'$ in $\overline{\Psi}$. Then, there is a sober deduction sequence ending in $\langle \overline{v}'{:}\overline{\varphi}', \Psi', 1\rangle$ for $\Psi, v{:}\varphi \vdash_E \overline{v}'{:}\overline{\varphi}'$, where Ψ' is contained in $\Psi \cup \{v{:}\varphi\}$. Let

$\bar{\imath}$ be the less natural such that $\overline{v}':\overline{\varphi}'$ is in $\overline{\Psi}_{\bar{\imath}}$, and l the less natural such that $\Psi' \backslash \{v{:}\varphi\}$ is contained in $(\Psi_l)^{\vdash_E}$. Observe that $(\Psi_1)^{\vdash_E} \subseteq (\Psi_2)^{\vdash_E} \subseteq \ldots$ by monotony, Proposition 2.3.13, since $\Psi_1 \subseteq \Psi_2 \subseteq \ldots$. Note that $\Psi_l, v{:}\varphi \vdash_E \overline{v}':\overline{\varphi}'$, by idempotence, Proposition 2.3.15, since $\Psi' \vdash_E \overline{v}':\overline{\varphi}'$ and $\Psi_l \vdash_E \eta'$ for any η' in $\Psi' \setminus \{v{:}\varphi\}$. Consider the case where:

- $l \leq \bar{\imath}$. (i) Suppose $v{:}\varphi$ is in $(\Psi_i)^{\vdash_E}$, i.e., $\Psi_i \vdash_E v{:}\varphi$, for some $i \leq \bar{\imath}$. Then $\Psi_{\bar{i}} \vdash_E v{:}\varphi$ by monotony, Proposition 2.3.13, since $\Psi_0 \subseteq \Psi_1 \subseteq \ldots$. Moreover Ψ_l is contained in $\Psi_{\bar{\imath}}$ because $\Psi_0 \subseteq \Psi_1 \subseteq \ldots$. So $\Psi_{\bar{\imath}} \vdash_E \overline{v}':\overline{\varphi}'$, by idempotence, Proposition 2.3.15, because $\Psi_l, v{:}\varphi \vdash_E \overline{v}':\overline{\varphi}'$, which contradicts the fact that for every $j \geq 0$, $\Psi_j \not\vdash_E \overline{v}:\overline{\varphi}$ for any $\overline{v}:\overline{\varphi}$ in $\overline{\Psi}_j$. Hence $v{:}\varphi$ is not in $(\Psi_i)^{\vdash_E}$ for any $i \leq \bar{\imath}$. (ii) Suppose $v{:}\varphi$ is in $(\Psi_i)^{\vdash_E}$, i.e., $\Psi_i \vdash_E v{:}\varphi$, for some $i > \bar{\imath}$. Then, $\Psi_i \vdash_E \overline{v}':\overline{\varphi}'$, by idempotence, since $\Psi_l, v{:}\varphi \vdash_E \overline{v}':\overline{\varphi}'$, $\Psi_i \vdash_E v{:}\varphi$, and Ψ_l is contained in Ψ_i because $\Psi_0 \subseteq \Psi_1 \subseteq \ldots$. Note that $\overline{v}':\overline{\varphi}'$ is in $\overline{\Psi}_i$ since $\overline{\Psi}_{\bar{\imath}}$ is contained in $\overline{\Psi}_i$ because $\overline{\Psi}_0 \subseteq \overline{\Psi}_1 \subseteq \ldots$. But this contradicts the fact that for every $j \geq 0$, $\Psi_j \not\vdash_E \overline{v}:\overline{\varphi}$ for any $\overline{v}:\overline{\varphi}$ in $\overline{\Psi}_j$. So $v{:}\varphi$ is not in $(\Psi_i)^{\vdash_E}$ for any $i > \bar{\imath}$. Therefore $v{:}\varphi$ is not in $(\Psi_i)^{\vdash_E}$ for every $i \geq 0$. Henceforth it is not in Ψ, as we wanted to show.

- $l > \bar{\imath}$. (i) Suppose $v{:}\varphi$ is in $(\Psi_i)^{\vdash_E}$, i.e., $\Psi_i \vdash_E v{:}\varphi$, for some $i \leq l$. Then $\Psi_l \vdash_E v{:}\varphi$ by monotony, Proposition 2.3.13, because $\Psi_0 \subseteq \Psi_1 \subseteq \ldots$. So $\Psi_l \vdash_E \overline{v}':\overline{\varphi}'$, by idempotence, Proposition 2.3.15. Note that $\overline{v}':\overline{\varphi}'$ is in $\overline{\Psi}_l$ because $\overline{\Psi}_0 \subseteq \overline{\Psi}_1 \subseteq \ldots$. But this contradicts the fact that for every $j \geq 0$, $\Psi_j \not\vdash_E \overline{v}:\overline{\varphi}$ for any $\overline{v}:\overline{\varphi}$ in $\overline{\Psi}_j$. Hence $v{:}\varphi$ is not in $(\Psi_i)^{\vdash_E}$ for any $i \leq l$. (ii) Suppose $v{:}\varphi$ is in $(\Psi_i)^{\vdash_E}$, i.e., $\Psi_i \vdash_E v{:}\varphi$, for some $i > l$. Then, $\Psi_i \vdash_E \overline{v}':\overline{\varphi}'$, by idempotence, since $\Psi_i \vdash_E v{:}\varphi$, and Ψ_l is contained in Ψ_i because $\Psi_0 \subseteq \Psi_1 \subseteq \ldots$. Note that $\overline{v}':\overline{\varphi}'$ is in $\overline{\Psi}_i$ since $\overline{\Psi}_{\bar{\imath}}$ is contained in $\overline{\Psi}_i$ because $\overline{\Psi}_0 \subseteq \overline{\Psi}_1 \subseteq \ldots$. But this contradicts the fact that for every $j \geq 0$, $\Psi_j \not\vdash_E \overline{v}:\overline{\varphi}$ for any $\overline{v}:\overline{\varphi}$ in $\overline{\Psi}_j$. So $v{:}\varphi$ is not in $(\Psi_i)^{\vdash_E}$ for any $i > l$ and so $v{:}\varphi$ is not in $(\Psi_i)^{\vdash_E}$ for every $i \geq 0$. Henceforth it is not in Ψ, as we wanted to show.

So, in both cases $v{:}\varphi$ is not in Ψ, as we wanted to show.

If Suppose $\Psi, v{:}\varphi \not\vdash_E \overline{v}:\overline{\varphi}$ for any $\overline{v}:\overline{\varphi}$ in $\overline{\Psi}$. Let k be the position of $v{:}\varphi$ in the enumeration of labelled schema formulae considered in the construction of Ψ. Then, by monotony, Proposition 2.3.13, we have that, $\Psi_{k-1}, v{:}\varphi \not\vdash_E \overline{v}:\overline{\varphi}$ for any $\overline{v}:\overline{\varphi}$ in $\overline{\Psi}_{k-1}$. So, by construction of Ψ_k and of $\overline{\Psi}_k$ we have that $v{:}\varphi$ is in Ψ_k and so in Ψ as we wanted to show.

5. Suppose $\Psi \vdash_E \eta$. Then, there is a finite deduction sequence ending in $\langle \eta, \Psi', 1 \rangle$ for $\Psi \vdash_E \eta$. Note that $(\Psi_1)^{\vdash_E} \subseteq (\Psi_2)^{\vdash_E} \subseteq \ldots$ by monotony, Proposition 2.3.13, because $\Psi_1 \subseteq \Psi_2 \subseteq \ldots$. Since Ψ' is contained in Ψ and is finite then there is i with Ψ' contained in $(\Psi_i)^{\vdash_E}$. So, by idempotence, Proposition 2.3.15, ${\Psi_i}^{\vdash_E} \vdash_E \eta$, since $\Psi' \vdash_E \eta$, and, also by the same proposition, $\Psi_i \vdash_E \eta$, since $\Psi_i \vdash_E {\Psi_i}^{\vdash_E}$. So η is in $(\Psi_i)^{\vdash_E}$, and thus, η is in Ψ, as we wanted to show.

6. Suppose *exh* is in $\mathcal{D}$. Let v and $v^{\cdot}$ be label schema terms in Ψ. Suppose $v{:}\varphi$ and $v^{\cdot}{:}\varphi^{\cdot}$ occur in the enumeration of labelled schema formulae at positions i and $i^{\cdot}$, respectively, and suppose without loss of generality that $i \leq i^{\cdot}$. Then, by construction of $\Psi_{i^{\cdot}+1}$, we have that $\Psi_{i^{\cdot}+1} \vdash_E v \equiv_\omega v_1, v_1 \equiv_\alpha v^{\cdot}$, for some v_1 in $T_{\mathrm{lab},E}$. So, taking into account the definition of Ψ, there is v_1 in $T_{\mathrm{lab},E}$ with $v \equiv_\omega v_1$ and $v_1 \equiv_\alpha v^{\cdot}$ in Ψ.

7. *Only if.* Let c_u be a universal connective in $\mathcal{D}$, and $v_1, \ldots, v_n$ label schema terms in

$T_{\mathrm{lab},E}$. Suppose $v{:}c_u\varphi_1\ldots\varphi_n$, $r_u v v_1 \ldots v_n$, and $v_1{:}\varphi_1, \ldots, v_{n-1}{:}\varphi_{n-1}$ are in Ψ. Then

1	$v{:}c_u\varphi_1\ldots\varphi_n$	$\{v{:}c_u\varphi_1\ldots\varphi_n\}$	1	asp	
2	$r_u v v_1 \ldots v_n$	$\{r_u v v_1 \ldots v_n\}$	1	asp	
3	$v_1{:}\varphi_1$	$\{v_1{:}\varphi_1\}$	1	asp	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$n+1$	$v_{n-1}{:}\varphi_{n-1}$	$\{v_{n-1}{:}\varphi_{n-1}\}$	1	asp	
$n+2$	$v_n{:}\varphi_n$	Ψ'	1	c_{uE}	$1, 2, 3, \ldots, n+1$

where Ψ' is $\{v{:}c_u\varphi_1\ldots\varphi_n, r_u v v_1 \ldots v_n, v_1{:}\varphi_1, \ldots, v_{n-1}{:}\varphi_{n-1}\}$, is a sober deduction sequence for $\Psi \vdash_E v_n{:}\varphi_n$. So, $v_n{:}\varphi_n$ is in Ψ since by 5., Ψ is deductively closed.

If. Suppose $v{:}c_u\varphi_1\ldots\varphi_n$ is not in Ψ. Then, by 0., $\Psi_k, v{:}c_u\varphi_1\ldots\varphi_n \vdash_E \overline{v}'{:}\overline{\varphi}'$ for some $\overline{v}'{:}\overline{\varphi}'$ in $\overline{\Psi}_k$, where $k+1$ is the position of $v{:}c_u\varphi_1\ldots\varphi_n$ in the enumeration of labelled schema formulae above mentioned. So by construction of Ψ there are label schema variables $\nu_1, \ldots, \nu_n$ in E such that $r_u v v_1 \ldots v_n, \nu_1{:}\varphi_1, \ldots, \nu_{n-1}{:}\varphi_{n-1}$ are in Ψ_{k+1} and so in Ψ, and $\nu_n{:}\varphi_n$ is in $\overline{\Psi}_{k+1}$ and so in $\overline{\Psi}$, thus, by 4., not in Ψ, as we wanted to show.

8. The proof is similar to 7. so we will omit it.

9. *Only if.* Suppose $v{:}c_e\varphi_1\ldots\varphi_n$ is in Ψ. The result follows because, by construction of Ψ, there are label schema variables $\nu_1, \ldots, \nu_n$, such that $r_e v \nu_1 \ldots \nu_n$ and $\nu_1{:}\varphi_1, \ldots, \nu_n{:}\varphi_n$ are in Ψ.

If. Suppose there are $v_1, \ldots, v_n$ in $T_{\mathrm{lab},E}$ such that $r_e v v_1 \ldots v_n, v_1{:}\varphi_1, \ldots, v_n{:}\varphi_n$ are in Ψ. Then

1	$r_e v v_1 \ldots v_n$	$\{r_e v v_1 \ldots v_n\}$	1	asp	
2	$v_1{:}\varphi_1$	$\{v_1{:}\varphi_1\}$	1	asp	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$n+1$	$v_n{:}\varphi_n$	$\{v_n{:}\varphi_n\}$	1	asp	
$n+2$	$v{:}c_e\varphi_1\ldots\varphi_n$	Ψ'	1	c_{eI}	$1, 2, \ldots, n+1$

where Ψ' is the set $\{r_e v v_1 \ldots v_n, v_1{:}\varphi_1, \ldots, v_n{:}\varphi_n\}$, is a sober deduction sequence for $\Psi \vdash_E v{:}c_e\varphi_1\ldots\varphi_n$. So, $v{:}c_e\varphi_1\ldots\varphi_n$ is in Ψ since by 5., Ψ is deductively closed.

10. The proof is similar to 9. so we will omit it.

11. *Only if.* Suppose $v{:}-\varphi$ and $v^*{:}\varphi$ are in Ψ. Let v' be a label schema term in $T_{\mathrm{lab},E}$. Then

1	$v{:}-\varphi$	$\{v{:}-\varphi\}$	1	asp	
2	$v^*{:}\varphi$	$\{v^*{:}\varphi\}$	1	asp	
3	$v'{:}\bot$	$\{v{:}-\varphi, v^*{:}\varphi\}$	1	$-_E$	$1, 2$

is a sober deduction sequence for $\Psi \vdash_E v'{:}\bot$. So, $v'{:}\bot$ is in Ψ since by 5., Ψ is deductively closed.

If. Suppose $v{:}-\varphi$ is not in Ψ and $v^*{:}\varphi$ is in Ψ. Then, for some v', if $v'{:}\bot$ were in Ψ then $\Psi \vdash_E v{:}-\varphi$ and since Ψ is deductively closed by 5., $v{:}-\varphi$ would be in Ψ which contradicts our initial assumption. So $v'{:}\bot$ is not in Ψ for all label schema terms v'.

12. The proof is similar to 11. so we will omit it.

13. The proof follows straightforwardly.

14. The proof follows straightforwardly so it is omitted.

15. *Only if* Suppose $v{:}\varphi_1 \rightarrow \varphi_2$ and $v{:}\varphi_1$ are in Ψ. Then, it is straightforward to see that $\Psi \vdash_E v{:}\varphi_2$, and so by 5., that $v{:}\varphi_2$ is in Ψ.

16. The proof proceeds by case analysis on the possible rules in $\mathcal{D}$. Suppose the rule is:

- c_{u^+I} or c_{u^+E} where c_{u^+} is a constrained universal connective in $\mathcal{D}$. The result follows by 7.

- c_{e^+I} or c_{e^+E} where c_{e^+} is a constrained existential connective in $\mathcal{D}$. The result follows by 9.

- c_{uI} or c_{uE} where c_u is a universal connective in $\mathcal{D}$. The result follows by 7.

- c_{eI} or c_{eE} where c_e is an existential connective in $\mathcal{D}$. The result follows by 9.

- $-_I$ or $-_E$ where $-$ is a non-local negation in $\mathcal{D}$. The result follows by 11.

- $\neg_I$ or $\neg_E$ where $\neg$ is a local negation in $\mathcal{D}$. The result follows by 12.

- $\wedge_I$ or $\wedge_E^1$ or $\wedge_E^2$ where $\wedge$ is a conjunction in $\mathcal{D}$. The result follows by 13.

- $\vee_I^1$ or $\vee_I^2$ where $\vee$ is a disjunction in $\mathcal{D}$. The result follows by 14.

- $\rightarrow_E$ where $\rightarrow$ is a implication in $\mathcal{D}$. The result follows by 15.

- $\top_I$. Let σ be a schema substitution within L_E. We have that $\nu\sigma{:}\top$ is in Ψ since Ψ is deductively closed, by 5., and $\Psi \vdash_E \nu\sigma{:}\top$.

- *exh.* Let σ be a schema substitution within L_E such that $(\pi_\Sigma\sigma_{\mathrm{nl}}) = 1$ for each π in P_f and $(\pi_{\Sigma;E}\sigma) = 1$ for each π in P_d, and suppose that for any v' in $T_{\mathrm{lab},E}$ if $\nu_1\sigma \equiv_\omega v'$ and $v' \equiv_\alpha \nu_2\sigma$ are in Ψ then $\nu''\sigma{:}\xi''\sigma$ is in Ψ. Let $v^{\cdot}$ be a label schema term such that $\nu_1\sigma \equiv_\omega v^{\cdot}$ and $v^{\cdot} \equiv_\alpha \nu_2\sigma$ are in Ψ, which exists by 6.. Then $\nu''\sigma{:}\xi''\sigma$ is in Ψ, as we wanted to show.

- $-_c$. Let σ be a schema substitution within L_E. Suppose $\nu\sigma^*{:}\xi\sigma$ is not in Ψ, and $\nu\sigma{:}-\xi\sigma$ is in Ψ. Then, by 4., $\Psi, \nu\sigma^*{:}\xi\sigma \vdash_E \overline{v}'{:}\overline{\varphi}'$ for some $\overline{v}'{:}\overline{\varphi}'$ in $\overline{\Psi}$. So, if $\nu'\sigma{:}\bot$ is in Ψ then, since Ψ is deductively closed by item 5., we have that $\Psi \vdash_E \nu\sigma^*{:}\xi\sigma$, and so, by idempotence, Proposition 2.3.15, we would have $\Psi \vdash_E \overline{v}'{:}\overline{\varphi}'$ which contradicts 3.. Hence $\nu'\sigma{:}\bot$ is not in Ψ, as we wanted to show.

- $\neg_c$ where $\neg$ is a local negation in $\mathcal{D}$. The proof is similar to $\neg_c$ so we will omit it.

- c_i. The proof is similar to $-_c$ so we will omit it.

- ${c_u}^{\mathrm{spc}}$, where c_u is a universal connective in $\mathcal{D}$. Let σ be a schema substitution within L_E such that $(\pi_\Sigma\sigma_{\mathrm{nl}}) = 1$ for each π in P_f. Suppose $\nu\sigma{:}c_u\,\xi\sigma$ is in Ψ. Then, $\Psi \vdash_E \nu\sigma{:}\xi'\sigma$ by rule ${c_u}^{\mathrm{spc}}$ since σ satisfy the proviso in P_f, and the result follows since Ψ is deductively closed, see 5.

- ${c_{u^+}}^{\mathrm{spc}}$, where c_{u^+} is a constrained universal connective in $\mathcal{D}$. The proof is similar to ${c_u}^{\mathrm{spc}}$ so we will omit it.

- *a term inheritance rule.* Let σ be a schema substitution within L_E such that $(\pi_\Sigma\sigma_{\mathrm{nl}}) = 1$ for each π in P_f. Suppose $s\,\nu_1\sigma \ldots \nu_n\sigma$ is in Ψ. Then, $\Psi \vdash_E \nu\sigma{:}\theta\sigma =_g \nu'\sigma{:}\theta\sigma$. So, $\nu\sigma{:}\theta\sigma =_g \nu'\sigma{:}\theta\sigma$ is in Ψ since by 5. Ψ is deductively closed.

- *a formula inheritance rule.* Let σ be a schema substitution within L_E such that $(\pi_\Sigma \sigma_{\mathrm{nl}}) = 1$ for each π in P_f. Suppose $s\,\nu_1\sigma \ldots \nu_n\sigma$ and $\nu_i\sigma{:}\xi\sigma$ are in Ψ. Then, $\Psi \vdash_E \nu_{i'}\sigma{:}\xi\sigma$. So, $\nu_{i'}\sigma{:}\xi\sigma$ is in Ψ since by 5. Ψ is deductively closed.

- *a general formula inheritance rule over* $\equiv_\omega$. Straightforward.

- *a docked rule* and *label docked rule.* Straightforward.

- *a relational rule.* Straightforward.

- *a generalization rule.* Straightforward.

- *a laxed generalization rule.* Straightforward.

- *a laxed generalization rule for a label function symbol.* Straightforward.

- *the 1 assignment rule.* Straightforward.

- *the 1 individual rule.* Straightforward.

- *a rule common to all lfob according to Definition 2.3.10.* Straightforward. *QED*

Remark 4.4.8 Observe that in the previous proposition we were not able to show that if $v{:}\varphi_1 \vee \varphi_2$ is in Ψ then $v{:}\varphi_1$ or $v{:}\varphi_2$ are in Ψ, and if $v{:}\varphi_1$ is in Ψ implies $v{:}\varphi_2$ is in Ψ then $v{:}\varphi_1 \rightarrow \varphi_2$ is in Ψ. Note that we are not assuming any kind of negation in the system. The study we made for clfob logic systems with a classical negation, demonstrated that, in their context, it was possible the construction of an appropriate set extending a consistent set, satisfying the appropriateness conditions for all the possible rules.

To see why that property of disjunction does not hold, suppose that $v{:}\varphi_1 \vee \varphi_2$ is in Ψ and both $v{:}\varphi_1$ and $v{:}\varphi_2$ are not in Ψ. Then, by item 4. in Proposition 4.4.7, there are labelled schema formulae $\overline{v}_1{:}\overline{\varphi}_1$ and $\overline{v}_2{:}\overline{\varphi}_2$ in $\overline{\Psi}$ with $\Psi, v{:}\varphi_1 \vdash \overline{v}_1{:}\overline{\varphi}_1$ and $\Psi, v{:}\varphi_2 \vdash \overline{v}_2{:}\overline{\varphi}_2$. The problem is that $\overline{v}_1{:}\overline{\varphi}_1$ and $\overline{v}_2{:}\overline{\varphi}_2$ may be different. One way to solve the problem is to enrich our labelled deduction systems with a disjunction connective $\ddot{\vee}$ and rules, for labelled schema formulae, such that $v{:}\varphi_1 \vee \varphi_2 \vdash_E v{:}\varphi_1 \ddot{\vee} v{:}\varphi_2$ and vice-versa, and when inductively constructing the sets Ψ_{k+1}, we add $v_{k+1}{:}\varphi_{k+1}$ to Ψ_k if $\Psi_k, v_{k+1}{:}\varphi_{k+1} \nvdash_E \overline{v}_1{:}\overline{\varphi}_1 \ddot{\vee} \ldots \ddot{\vee} \overline{v}_n{:}\overline{\varphi}_n$ for any $\overline{v}_1{:}\overline{\varphi}_1, \ldots, \overline{v}_n{:}\overline{\varphi}_n$ in $\overline{\Psi}_k$. Then the result would follow straightforwardly since if both $v{:}\varphi_1$ and $v{:}\varphi_2$ are not in Ψ then $v{:}\varphi_1 \ddot{\vee} v{:}\varphi_2$ is also not in Ψ and so $v{:}\varphi_1 \vee \varphi_2$ is not in Ψ as we wanted to show. We decided not to include that connective because of two reasons: i. the logic system that would result from that addition would be closer to hybrid logic than to labelled deduction, which is out of the scope of this thesis, ii. we think it is normal and acceptable that some not positive results appear since simultaneously there are several positive ones and since these results appear in the context of lfob logic systems with very special characteristics. Nevertheless, the inclusion of a connective for disjunction of labelled schema formulae is an interesting issue that we intend to pursue in the future, in close relation with hybrid logic.

In Proposition 4.4.7 we were also not able to show that if $v{:}\varphi_1$ is in Ψ implies $v{:}\varphi_2$ is in Ψ then $v{:}\varphi_1 \rightarrow \varphi_2$ is in Ψ. To see this does not happen suppose that $v{:}\varphi_1 \rightarrow \varphi_2$ is not in Ψ. Then we can prove straightforwardly that $v{:}\varphi_2$ is not in Ψ. The problem is to show that $v{:}\varphi_1$ is in Ψ. One way to circumvent this would be, during the construction of Ψ, to add explicitly $v{:}\varphi_1$ to Ψ_{k+1} whenever $\Psi_k, v{:}\varphi_1 \rightarrow v{:}\varphi_2 \vdash_E \overline{v}{:}\overline{\varphi}$ for some $\overline{v}{:}\overline{\varphi}$ in $\overline{\Psi}_k$. But this has the problem that, in this situation, we can not prove $\Psi_k, v{:}\varphi_1 \nvdash_E \overline{v}{:}\overline{\varphi}$ for any $\overline{v}{:}\overline{\varphi}$ in $\overline{\Psi}_k$.

After having shown Proposition 4.4.7, we can now establish that, in the context of a clfob deduction system with locality and without disjunction and implication, every $v'{:}\varphi'$ consistent set can be extended to an appropriate set also $v'{:}\varphi'$ consistent.

Proposition 4.4.9 Let $\mathcal{D}$ be a clfob deduction system with locality and without disjunction and implication. Then, for each $v'{:}\varphi'$ consistent set Ψ°, there is a set E of label schema variables, and an appropriate set Ψ within L_E, extending Ψ°, which is strong if the rule *exh* is in $\mathcal{D}$.

The proof of the preceding proposition is omitted since it is completely analogous to the proof of Proposition 4.3.9. We are now able to state the important fact that any clfob logic system with locality and without disjunction and implication is appropriate. Note that its proof is omitted since it follows straightforwardly from Proposition 4.4.9.

Theorem 4.4.10 Any clfob logic system with locality and without disjunction and implication is appropriate. Moreover if it contains rule *exh* it is strong appropriate.

Chapter 5

Examples

In this chapter we develop original labelled presentations for relevance logic R, for the $\wedge$-$\Rightarrow$ fragment of basic relevance logic B, for first-order modal logic with decreasing domains, and for the $\wedge$ fragment of first-order modal logic T. We devote a section for each logic. Each section starts by describing the logic in question: its non-labelled signatures, its usual models and non-labelled logic systems. After that, we present a lfob logic system that we claim to be a labelled presentation for the logic, and we show that it is sound and complete. Finally we end each section by proving that, effectively, the lfob logic system proposed is a labelled presentation for the logic in question. After the sections dedicated to the examples we have a section with the common definitions and results needed along the chapter, and ending the chapter we draw some remarks and conclusions.

In each of the lfob logic systems proposed, soundness is showed relying on Theorem 2.4.6, by proving that all the structures of the lfob logic system in question satisfy its rules. Note that we capitalize in Proposition 2.4.7 in order to avoid to show that the structures satisfy the common rules. Completeness is accomplished making use of Theorem 3.3.3, after proving that the lfob logic system is rich. In order to show this we have to prove that, for any consistent set, there is a canonical structure in the lfob logic system, over an appropriate set extending that set and consistent with respect to the same formula. So, when defining the lfob logic system, we establish that it has all the structures induced by the usual models for the logic, plus, the structures equivalent in terms of satisfaction of a certain subset of formulae to those structures. Henceforth, we guarantee that the canonical structure is in the lfob logic system by checking that it is equivalent in terms of satisfaction to a structure induced by a usual model for the logic. In the examples considered we proved that the canonical structure is equivalent in terms of satisfaction with the structure induced by the canonical model for the logic. We hope these examples help to clarify and illustrate our results, and show that the results apply to a broad class of logics.

Note that our labelled presentations are different from those existing in the literature. For instance, the labelled presentation we propose in Example 5.1.1 for first-order modal logic with decreasing domains is different from those proposed in [10, 70, 79] for that logic, as well as our labelled presentation for relevance logic R in Example 5.3.1 with respect to the ones proposed in [11, 79] for that logic. These distinctions are justified not only by the differences in the deduction framework, but also because our completeness and equivalence results imply the presence or the avoidance of some rules. Moreover these distinctions are justified also by a different treatment of some connectives.

5.1 First-order modal logic with decreasing domains

In this section we present the lfob logic system $\mathcal{L}_{\text{FOMdD}}$ and show that it is a labelled presentation for first-order modal logic with decreasing domains and flexible symbols [48, 49, 12, 79, 39, 51, 78] with respect to formulae without schema variables. See Definitions 2.4.9 and 2.4.10, to recall when a lfob logic system is a labelled presentation for a non-labelled logic with respect to a set of non-labelled schema formulae. Note that, in order to lighten the presentation, we will not mention that the logic is with flexible symbols. We now briefly describe what is a non-labelled fob signature, a relational model and a non-labelled fob logic system, for first-order modal logic with decreasing domains. Recall the definitions of non-labelled fob signature and non-labelled fob logic system in Definition 2.4.8. We consider that $X^{\cdot}$, $\Xi^{\cdot}_t$ and $\Xi^{\cdot}_f$ are equal to X, Ξ_t and Ξ_f, respectively. In order to recall the importance of $X^{\cdot}$, $\Xi^{\cdot}_t$ and $\Xi^{\cdot}_f$ see Definition 2.4.8.

A *non-labelled fob signature* $\Sigma^{\cdot}$ *for first-order modal logic with decreasing domains* is a non-labelled fob signature such that $P^{\cdot}_0$ is a non-empty set, ε is in $P^{\cdot}_1$, $C^{\cdot}_0 = \{\bot\}$, $C^{\cdot}_1 = \{\forall, \Box, \neg\}$, $C^{\cdot}_2 = \{\wedge\}$, and $C^{\cdot}_k = \emptyset$ for $k \geq 3$.

A *relational model* m *over* $\Sigma^{\cdot}$ *for first-order modal logic with decreasing domains* is a tuple $\langle W, R^{\circ}_{\Box}, D, \_^F, \_^P \rangle$ where D and W are non-empty sets, $R^{\circ}_{\Box}$ is contained in $W \times W$, $\_^F$ is a family $\{\_^F{}_{k,w}\}_{k \in I\!N, w \in W}$ where $\_^F{}_{k,w} : F_k \to D^k \to D$, and $\_^P$ is a family $\{\_^P{}_{k,w}\}_{k \in I\!N, w \in W}$ where $\_^P{}_{k,w} : P_k \to D^k \to 2$, such that $\varepsilon^P_{1,w_2}(d) = 1$ implies $\varepsilon^P_{1,w_1}(d) = 1$ if $w_1 R^{\circ}_{\Box} w_2$, for any w_1, w_2 in W and d in D. A *schema assignment* α over m is a pair $\langle \alpha_t, \alpha_f \rangle$ such that α_t is a map that given an element of $\Xi^{\cdot}_t$ returns a map from $W \times D^X$ to D, and α_f is a map that associates to an element of $\Xi^{\cdot}_f$ a subset of $W \times D^X$. Note that $m, \alpha, w \Vdash^a \gamma$ is inductively defined as expected. We just recall the satisfaction clauses for formulae whose main connective is $\forall_x$ or $\Box$:

$$\begin{array}{lll} m, \alpha, w \Vdash^a \forall_x \gamma & \text{iff} & \text{for any } a_1, \\ & & \text{if } a_1 \text{ is x co-equivalent to } a \text{ and } m, \alpha, w \Vdash^{a_1} \varepsilon(x) \\ & & \text{then } m, \alpha, w \Vdash^{a_1} \gamma \\ m, \alpha, w \Vdash^a \Box \gamma & \text{iff} & \text{for any } w_1, \text{ if } w R^{\circ}_{\Box} w_1 \text{ then } m, \alpha, w_1 \Vdash^a \gamma. \end{array}$$

A *non-labelled fob logic system for first-order modal logic with decreasing domains* is a pair $\langle \Sigma^{\cdot}, \propto \rangle$ where $\Sigma^{\cdot}$ is a non-labelled fob signature for this logic and $\propto$ is such that

$$\Phi \propto \gamma \quad \text{iff} \quad \text{if } m, \alpha, w \Vdash^a \Phi \text{ then } m, \alpha, w \Vdash^a \gamma$$

for any relational model m for first-order modal logic with decreasing domains over $\Sigma^{\cdot}$, any schema assignment α over m, and world w and assignment a over m.

5.1.1 Logic system

Example 5.1.1 *First-order modal logic with decreasing domains lfob logic system.* Consider the lfob logic system $\mathcal{L}_{\text{FOMdD}}$, where Σ_{FOMdD} is

- $F^l_k = \emptyset$ for $k \geq 0$,
- $S_2 = \{R_{\Box}\} \cup \{\equiv_x \mid x \in X\}$, and $S_k = \emptyset$ for $k \geq 3$ and $k = 1$,

- $\langle F, P\rangle$ is a first-order alphabet, and ε is in P_1;
- $C_0 = \{\bot\}$, $C_1 = \{\neg\}$, $C_2 = \{\wedge\}$, and $C_k = \emptyset$ for $k \geq 3$ and $k = 1$,
- $Q_1 = \{\forall\}$, $O_1 = \{\Box\}$, and $Q_k = O_k = \emptyset$ for $k \geq 2$,

and R, besides the rules specified in Definition 2.3.10 common to all lfob deduction systems, contains:

$$\frac{\nu{:}\xi_1 \qquad \nu{:}\xi_2}{\nu{:}\xi_1 \wedge \xi_2}\ \wedge_I \qquad\qquad \frac{\nu{:}\xi_1 \wedge \xi_2}{\nu{:}\xi_1}\ \wedge_E^1 \qquad\qquad \frac{\nu{:}\xi_1 \wedge \xi_2}{\nu{:}\xi_2}\ \wedge_E^2$$

$$\frac{\nu{:}\xi \,/\, \nu'{:}\bot}{\nu{:}\neg\xi}\ \neg_I \qquad\qquad \frac{\nu{:}\neg\xi \qquad \nu{:}\xi}{\nu'{:}\bot}\ \neg_E \qquad\qquad \frac{\nu{:}\neg\xi \,/\, \nu'{:}\bot}{\nu{:}\xi}\ \neg_c$$

$$\frac{\nu \equiv_x \nu', \nu'{:}\varepsilon(x) \,/\, \nu'{:}\xi}{\nu{:}\forall_x \xi}\ \forall_{xI};\ \text{fresh}(\nu', \langle \vartheta_1, \{\nu \equiv_x \nu', \nu'{:}\varepsilon(x)\}, \nu'{:}\xi\rangle)$$

$$\frac{\nu{:}\forall_x \xi \quad \nu \equiv_x \nu' \quad \nu'{:}\varepsilon(x)}{\nu'{:}\xi}\ \forall_{xE} \qquad\qquad \frac{\nu{:}\forall_x \xi \quad \nu{:}\varepsilon(\theta)}{\nu{:}\xi'}\ {\forall_x}^{\text{spc}};\ \xi' \lhd \xi_\theta^x$$

$$\frac{\nu R_\Box \nu' \quad \nu'{:}\varepsilon(x)}{\nu{:}\varepsilon(x)}\ dD$$

$$\frac{\nu R_\Box \nu' \,/\, \nu'{:}\xi}{\nu{:}\Box\xi}\ \Box_I;\ \text{fresh}(\nu', \langle \vartheta_1, \{\nu R_\Box \nu'\}, \nu'{:}\xi\rangle) \qquad\qquad \frac{\nu{:}\Box\xi \quad \nu R_\Box \nu'}{\nu'{:}\xi}\ \Box_E$$

$$\frac{\nu R_\Box \nu'}{\nu \equiv_\alpha \nu'}\ R_{\Box\alpha^{1,2}} \qquad \frac{\nu R_\Box \nu_1 \quad \nu \equiv_\omega \nu' \quad \nu_1 \equiv_\omega \nu_1' \quad \nu' \equiv_\alpha \nu_1'}{\nu' R_\Box \nu_1'}\ R_{\Box\alpha}^{\text{gen}}$$

$$\frac{\nu \equiv_x \nu'}{\nu \equiv_\omega \nu'}\ \equiv_{x\omega^{1,2}} \qquad \frac{\nu \equiv_x \nu_1 \quad \nu \equiv_\alpha \nu' \quad \nu_1 \equiv_\alpha \nu_1' \quad \nu' \equiv_\omega \nu_1'}{\nu' \equiv_x \nu_1'}\ \equiv_{x\omega}^{\text{gen}}$$

$$\frac{\nu \equiv_x \nu'}{\nu{:}\theta =_g \nu'{:}\theta}\ \equiv_{xg^{1,2}};\ \text{vardif}(x, \theta)$$

$$\frac{\nu_1 \equiv_\omega \nu', \nu' \equiv_\alpha \nu_2 \,/\, \nu''{:}\xi''}{\nu''{:}\xi''}\ \mathit{exh};\ \text{fresh}(\nu', \langle \vartheta_1, \{\nu_1 \equiv_\omega \nu', \nu' \equiv_\alpha \nu_2\}, \nu''{:}\xi''\rangle)$$

$$\frac{\nu \equiv_x \nu' \quad \nu' \equiv_x \nu''}{\nu \equiv_x \nu''}\ \equiv_{xt} \qquad \frac{}{\nu \equiv_x \nu}\ \equiv_{xr} \qquad \frac{\nu' \equiv_x \nu}{\nu \equiv_x \nu'}\ \equiv_{xs}$$

$$\frac{\nu \equiv_\omega \nu' \qquad \nu{:}\theta_1 =_g \nu'{:}\theta_1 \quad \dots \quad \nu{:}\theta_k =_g \nu'{:}\theta_k \qquad \nu{:}\xi}{\nu'{:}\xi}\ \mathit{gmon}^k_{\equiv_\omega};\ \text{p-gmon}_{\equiv_\omega}(\xi, \theta_1, \dots, \theta_k)$$

for every x in X, where the provisos $\xi' \lhd \xi^x_\theta$, $\text{vardif}(x, \theta)$, and $\text{p-gmon}_{\equiv_\omega}(\xi, \theta_1, \dots, \theta_k)$ are such that

- $\text{vardif}(x, \theta)_{\Sigma_{\text{FOMdD}}}(\rho_{\text{nl}}) = 1$ iff $\theta\rho_t$ is a variable different of x,
- $\text{p-gmon}_{\equiv_\omega}(\xi, \theta_1, \dots, \theta_k)_{\Sigma_{\text{FOMdD}}}(\rho_{\text{nl}}) = 1$ iff $\xi\rho_f$ is a formula whose free variables are $\theta_1\rho_t, \dots, \theta_k\rho_t$,
- $\xi' \lhd \xi^x_{\theta\,\Sigma_{\text{FOMdD}}}(\rho_{\text{nl}}) = 1$ iff $\xi'\rho_f$ is obtained by replacing the free occurrences of x in $\xi\rho_f$ by a term $\theta\rho_t$ equal to a variable whenever some free x in $\xi\rho_f$ is in the scope of a modality, such that no variable in $\theta\rho_t$ is captured by a quantifier,

M is composed of all the relational models for first-order modal logic with decreasing domains over the non-labelled fob signature $\langle F, P, C \cup Q \cup O \rangle$,

and $\breve{\cdot}$ associates to each relational model $\langle W, R^\circ_\Box, D, \_^F, \_^P \rangle$ denoted by m, a class, such that s is in $\breve{m}$ iff either s is the structure induced by m, denoted by s_m, i.e., the structure $\langle U, A, W, \alpha, \omega, D, \mathcal{E}, \mathcal{B}, [.] \rangle$ such that

- $A = D^X$, $\quad U = W \times A$, $\quad \alpha(\langle w, a \rangle) = a$, $\quad \omega(\langle w, a \rangle) = w$,
- $\mathcal{E}$ is D^U and $\quad \mathcal{B}$ is 2^U,

and

- $[x]_a = a(x)$,
- $[f]_w = f^F_{k,w}$ for f in F_k,
- $[p]_w = p^P_{k,w}$ for p in P_k,
- $[\bot]_{\omega(u)\alpha(u)}(u) = 0$,
- $[\neg]_{\omega(u)\alpha(u)}(b)(u) = 1$ iff $b(u) = 0$,
- $[\wedge]_{\omega(u)\alpha(u)}(b_1, b_2)(u) = 1$ iff $b_1(u) = 1$ and $b_2(u) = 1$,
- $[\equiv_x] = \{\langle u, u_1 \rangle \mid \omega(u) = \omega(u_1),$ and $\alpha(u)$ and $\alpha(u_1)$ are x co-equivalent$\}$,
- $[\forall_x]_{\omega(u)}(b)(u) = 1$ iff for all u_1, if $\langle u, u_1 \rangle$ is in $[\equiv_x]$ and $[\varepsilon]_{\omega(u_1)}(\alpha(u_1)(x)) = 1$ then u_1 is in b,
- $[R_\Box] = \{\langle u, u_1 \rangle \mid \alpha(u) = \alpha(u_1)$ and $\omega(u) R^\circ_\Box \omega(u_1)\}$,
- $[\Box](b)(u) = 1$ iff for all u_1, if $\langle u, u_1 \rangle$ is in $[R_\Box]$ then u_1 is in b.

or s satisfies all the rules, and, for any schema assignment α over s there exists schema assignment $\alpha^{\cdot}$ over s_m such that, for any formula ϕ in $\mathcal{L}^{\cdot}_{\text{FOMdD}}$, $s, \alpha \Vdash \nu{:}\phi$ iff $s_m, \alpha^{\cdot} \Vdash \nu{:}\phi$.

In order for the structures considered in $\mathcal{L}_{\text{FOMdD}}$ induced by a natural model to be indeed structures it is necessary that the algebraic map $\widehat{\ }$ associated to each structure be well defined, see Definition 2.2.1. It is straightforward to see this is the case since any such structure accepts all possible truth values and all possible denotations of terms. To conclude the subsection we present a proposition showing that the underlying deduction system of $\mathcal{L}_{\text{FOMdD}}$ is uniform and with a universal quantifier; see Definition 5.5.8 to recall when a lfob deduction system is uniform and with a universal quantifier. We omit the proof since it is straightforward.

Proposition 5.1.2 The lfob deduction system in $\mathcal{L}_{\text{FOMdD}}$ is uniform and with a universal quantifier.

5.1.2 Soundness

Theorem 5.1.3 The lfob logic system $\mathcal{L}_{\text{FOMdD}}$ is sound.

Proof The proof follows by Theorem 2.4.6 since by Proposition 5.1.4 all structures of $\mathcal{L}_{\text{FOMdD}}$ satisfy its rules. *QED*

Note that, according to Proposition 2.4.7, in order to show that all structures of $\mathcal{L}_{\text{FOMdD}}$ satisfy its rules, it is sufficient to show that the structures satisfy the specific rules since every structure satisfies the rules common to all lfob deduction systems. In this subsection, we first present the proof that all structures of $\mathcal{L}_{\text{FOMdD}}$ satisfy its rules and only after we present the three lemmas needed along this proof.

Proposition 5.1.4 All structures of $\mathcal{L}_{\text{FOMdD}}$ satisfy its rules.

Proof Let $m = \langle W, R^{\circ}_{\Box}, D, \_^{F}, \_^{P}\rangle$ be a model in $\mathcal{L}_{\text{FOMdD}}$ and $s = \langle U, A, W, \alpha, \omega, D, \mathcal{E}, \mathcal{B}, [.]\rangle$ a structure in $\breve{m}$. Then, according to the definition of $\breve{m}$, either (i) s is a structure induced by m, or (ii) s is equivalent in terms of satisfaction to a structure induced by the model. Suppose (ii) holds. Then, by definition, s satisfies all rules and we are done. Suppose (i) holds. Then, we have to check that s, in this case denoted by s_m, satisfies all the rules. Note that, by Proposition 2.4.7, s_m satisfies the rules common to all lfob logic systems, hence, it is only needed to check that it satisfies the non-common rules. So, let $r = \langle\{\langle\vartheta_1, \Psi_1, \eta_1\rangle, \ldots, \langle\vartheta_k, \Psi_k, \eta_k\rangle\}, \eta, P_f, P_d\rangle$ be a rule in $\mathcal{L}_{\text{FOMdD}}$ that is not a rule common to all lfob logic systems, E a set of label schema variables, σ a schema substitution within L_E such that $(\pi_{\Sigma}\sigma_{\text{nl}})$ is 1 for each π in P_f, and $\pi_{\Sigma;E}(\sigma)$ is 1 for each π in P_d, and α a schema assignment over s_m and E. Suppose r is

$gmon^{k}_{\equiv_\omega}$, and suppose that $s_m, \alpha \Vdash \nu\sigma \equiv_\omega \nu'\sigma$, $s_m, \alpha \Vdash \nu\sigma{:}\theta_i\sigma =_g \nu'\sigma{:}\theta_i\sigma$ for each $i = 1, \ldots, k$, and $s_m, \alpha \Vdash \nu\sigma{:}\xi\sigma$, and that $\xi\sigma$ is a formula whose free variables are precisely $\theta_1\sigma, \ldots, \theta_k\sigma$. Note that $\omega(\llbracket\nu\sigma\rrbracket^{s_m}_{\alpha}) = \omega(\llbracket\nu'\sigma\rrbracket^{s_m}_{\alpha})$, $\alpha(\llbracket\nu\sigma\rrbracket^{s_m}_{\alpha})(\theta_i\sigma) = \alpha(\llbracket\nu'\sigma\rrbracket^{s_m}_{\alpha})(\theta'_i\sigma)$ for each $i = 1, \ldots, k$, and $\llbracket\xi\sigma\rrbracket^{s_m}_{\alpha}(\llbracket\nu\sigma\rrbracket^{s_m}_{\alpha}) = 1$. So, $\llbracket\xi\sigma\rrbracket^{s_m}_{\alpha}(\llbracket\nu'\sigma\rrbracket^{s_m}_{\alpha}) = 1$, by item 2. of Lemma 5.1.6, as we wanted to show.

${\forall_x}^{\text{spc}}$, and suppose $s_m, \alpha \Vdash \nu\sigma{:}\forall_x\xi\sigma$, $s_m, \alpha \Vdash \nu\sigma{:}\varepsilon(\theta\sigma)$ and that $\xi\sigma$ is a schema formula where all schema variables appear under the scope of a $\forall_x$ or a $\exists_x$, and $\xi'\sigma$ is $\xi\sigma$ if there are no free x's in $\xi\sigma$, otherwise $\xi'\sigma$ is obtained by replacing the free occurrences of x in

$\xi\sigma$ by a schema term $\theta\sigma$ either equal to a variable whenever some free x in $\xi\sigma$ is in the scope of a modality, or not having schema variables whenever some free x in $\xi\sigma$ is in the scope of a quantifier, such that no variable in $\theta\sigma$ is captured by a quantifier. We want to show that $s_m, \alpha \Vdash \nu\sigma{:}\xi'\sigma$. Consider the point u such that $\omega(u) = \omega(\llbracket \nu\sigma \rrbracket_\alpha^{s_m})$, $\alpha(u)$ and $\alpha(\llbracket \nu\sigma \rrbracket_\alpha^{s_m})$ are x co-equivalent, and $\alpha(u)(x) = \llbracket \theta\sigma \rrbracket_\alpha^{s_m}(\llbracket \nu\sigma \rrbracket_\alpha^{s_m})$. Then, $[\varepsilon]_{\omega(u)}(\alpha(u)(x)) = [\varepsilon]_{\omega(\llbracket \nu\sigma \rrbracket_\alpha^{s_m})}(\llbracket \theta\sigma \rrbracket_\alpha^{s_m}(\llbracket \nu\sigma \rrbracket_\alpha^{s_m})) = 1$. Note that, by item 2. of Lemma 5.1.5, $\llbracket \xi\sigma \rrbracket_\alpha^{s_m}(u) = 1$ iff $\llbracket \xi'\sigma \rrbracket_\alpha^{s_m}(\llbracket \nu\sigma \rrbracket_\alpha^{s_m}) = 1$. So $\llbracket \xi'\sigma \rrbracket_\alpha^{s_m}(\llbracket \nu\sigma \rrbracket_\alpha^{s_m}) = 1$ as we wanted to show since $\llbracket \xi\sigma \rrbracket_\alpha^{s_m}(u) = 1$ because $\llbracket \forall_x \xi\sigma \rrbracket_\alpha^{s_m}(\llbracket \nu\sigma \rrbracket_\alpha^{s_m}) = 1$.

$\forall_{xI}$, and suppose that, for every α' $\{\nu'\sigma\}$ co-equivalent to α, if $s_m, \alpha' \Vdash \nu\sigma \equiv_x \nu'\sigma$ and $s_m, \alpha' \Vdash \nu'\sigma{:}\varepsilon(x)$ then $s_m, \alpha' \Vdash \nu'\sigma{:}\xi\sigma$. Let u be a point such that $\omega(\llbracket \nu\sigma \rrbracket_\alpha^{s_m}) = \omega(u)$, $\alpha(\llbracket \nu\sigma \rrbracket_\alpha^{s_m})$ and $\alpha(u)$ are x co-equivalent, and $[\varepsilon]_{\omega(u)}(\alpha(u)(x)) = 1$. Consider the assignment α' $\{\nu'\sigma\}$ co-equivalent to α with $\nu'\alpha' = u$. Note that $\llbracket \nu\sigma \rrbracket_{\alpha'}^{s_m} = \llbracket \nu\sigma \rrbracket_\alpha^{s_m}$ and $\llbracket \xi\sigma \rrbracket_{\alpha'}^{s_m} = \llbracket \xi\sigma \rrbracket_\alpha^{s_m}$, by Proposition 2.4.4, since $\nu_1\alpha' = \nu_1\alpha$ for each ν_1 in $\mathrm{lsv}(\nu\sigma)$, since $\nu'\sigma$ is not in $\mathrm{lsv}(\nu\sigma)$, because $\mathrm{fresh}(\nu', \langle \vartheta, \{\nu \equiv_x \nu', \nu'{:}\varepsilon(x)\}, \nu'{:}\xi' \rangle)(\sigma) = 1$. Then, by the initial assumption, $\llbracket \xi\sigma \rrbracket_{\alpha'}^{s_m}(\llbracket \nu'\sigma \rrbracket_{\alpha'}^{s_m}) = 1$, i.e., $\llbracket \xi\sigma \rrbracket_\alpha^{s_m}(u) = 1$, by Proposition 2.4.4, as we wanted to show.

$\Box_I$. The proof that s_m satisfies $\Box_I$ is similar to the proof for $\forall_{xI}$, so it is omitted.

$\Box_E$, and suppose that $s_m, \alpha \Vdash \nu\sigma{:}\Box\xi\sigma$ and $s_m, \alpha \Vdash \nu\sigma R_\Box \nu'\sigma$. Then, for all u, if $\langle \llbracket \nu\sigma \rrbracket_\alpha^{s_m}, u \rangle$ is in $[R_\Box]$ then $\llbracket \xi\sigma \rrbracket_\alpha^{s_m}(u) = 1$. Note that $\langle \llbracket \nu\sigma \rrbracket_\alpha^{s_m}, \llbracket \nu'\sigma \rrbracket_\alpha^{s_m} \rangle$ is in $[R_\Box]$. So $\llbracket \xi\sigma \rrbracket_\alpha^{s_m}(\llbracket \nu'\sigma \rrbracket_\alpha^{s_m}) = 1$. Therefore $s_m, \alpha \Vdash \nu'\sigma{:}\xi\sigma$.

$\forall_{xE}$. The proof is similar to the one for $\Box_E$, so it is omitted.

$\wedge_I$ or $\wedge_E^1$ or $\wedge_E^2$ or $\neg_I$ or $\neg_E$ or $\neg_c$. Then, the proof follows straightforwardly.

${R_\Box}_\alpha^{\text{gen}}$, and suppose that $s_m, \alpha \Vdash \nu\sigma R_\Box \nu_1\sigma$, $s_m, \alpha \Vdash \nu\sigma \equiv_\omega \nu'\sigma$, $s_m, \alpha \Vdash \nu_1\sigma \equiv_\omega \nu_1'\sigma$, $s_m, \alpha \Vdash \nu'\sigma \equiv_\alpha \nu_1'\sigma$. Therefore $\alpha(\llbracket \nu\sigma \rrbracket_\alpha^{s_m}) = \alpha(\llbracket \nu_1\sigma \rrbracket_\alpha^{s_m})$, $\omega(\llbracket \nu\sigma \rrbracket_\alpha^{s_m}) R_\Box^\circ \omega(\llbracket \nu_1\sigma \rrbracket_\alpha^{s_m})$, $\omega(\llbracket \nu\sigma \rrbracket_\alpha^{s_m}) = \omega(\llbracket \nu'\sigma \rrbracket_\alpha^{s_m})$, $\omega(\llbracket \nu_1\sigma \rrbracket_\alpha^{s_m}) = \omega(\llbracket \nu_1'\sigma \rrbracket_\alpha^{s_m})$, and $\alpha(\llbracket \nu'\sigma \rrbracket_\alpha^{s_m}) = \alpha(\llbracket \nu_1'\sigma \rrbracket_\alpha^{s_m})$. So $\omega(\llbracket \nu'\sigma \rrbracket_\alpha^{s_m}) R_\Box^\circ \omega(\llbracket \nu_1'\sigma \rrbracket_\alpha^{s_m})$. Thus the pair $\langle \llbracket \nu'\sigma \rrbracket_\alpha^{s_m}, \llbracket \nu_1'\sigma \rrbracket_\alpha^{s_m} \rangle$ is in $[R_\Box]$. Hence $s_m, \alpha \Vdash \nu'\sigma R_\Box \nu_1'\sigma$, as we wanted to show.

$\equiv_{x\omega}^{\text{gen}}$, and suppose that $s_m, \alpha \Vdash \nu\sigma \equiv_x \nu_1\sigma$, $s_m, \alpha \Vdash \nu\sigma \equiv_\alpha \nu'\sigma$, $s_m, \alpha \Vdash \nu_1\sigma \equiv_\alpha \nu_1'\sigma$, $s_m, \alpha \Vdash \nu'\sigma \equiv_\omega \nu_1'\sigma$. Therefore we have $\omega(\llbracket \nu\sigma \rrbracket_\alpha^{s_m}) = \omega(\llbracket \nu_1\sigma \rrbracket_\alpha^{s_m})$, $\alpha(\llbracket \nu\sigma \rrbracket_\alpha^{s_m})$ and $\alpha(\llbracket \nu_1\sigma \rrbracket_\alpha^{s_m})$ are x co-equivalent, $\alpha(\llbracket \nu\sigma \rrbracket_\alpha^{s_m}) = \alpha(\llbracket \nu'\sigma \rrbracket_\alpha^{s_m})$, $\alpha(\llbracket \nu_1\sigma \rrbracket_\alpha^{s_m}) = \alpha(\llbracket \nu_1'\sigma \rrbracket_\alpha^{s_m})$, and $\omega(\llbracket \nu'\sigma \rrbracket_\alpha^{s_m}) = \omega(\llbracket \nu_1'\sigma \rrbracket_\alpha^{s_m})$. So $\alpha(\llbracket \nu'\sigma \rrbracket_\alpha^{s_m})$ and $\alpha(\llbracket \nu_1'\sigma \rrbracket_\alpha^{s_m})$ are also x co-equivalent, and thus $s_m, \alpha \Vdash \nu'\sigma \equiv_x \nu_1'\sigma$, as we wanted to show.

$R_{\Box\alpha^{1,2}}$, and suppose that $s_m, \alpha \Vdash \nu\sigma R_\Box \nu'\sigma$. Then $\langle \llbracket \nu\sigma \rrbracket_\alpha^{s_m}, \llbracket \nu'\sigma \rrbracket_\alpha^{s_m} \rangle$ is in $[R_\Box]$. So, according to the definition of $[R_\Box]$, $\alpha(\llbracket \nu\sigma \rrbracket_\alpha^{s_m}) = \alpha(\llbracket \nu'\sigma \rrbracket_\alpha^{s_m})$. Therefore $s_m, \alpha \Vdash \nu\sigma \equiv_\alpha \nu'\sigma$.

$\equiv_{x\omega^{1,2}}$. The proof is similar to the one for $R_{\Box\alpha^{1,2}}$, so it is omitted.

$\equiv_{xg^{1,2}}$, and suppose that $s_m, \alpha \Vdash \nu\sigma \equiv_x \nu'\sigma$ and that $\theta\rho_t$ is a variable distinct of x. Then $\langle \llbracket \nu\sigma \rrbracket_\alpha^{s_m}, \llbracket \nu'\sigma \rrbracket_\alpha^{s_m} \rangle$ is in $[\equiv_x]$. So, according to the definition of $[\equiv_x]$, $\alpha(\llbracket \nu\sigma \rrbracket_\alpha^{s_m})$ and

$\alpha(\llbracket\nu'\sigma\rrbracket_\alpha^{s_m})$ are x co-equivalent. Thus $\llbracket\theta\rho_t\rrbracket_\alpha^{s_m}(\llbracket\nu\sigma\rrbracket_\alpha^{s_m}) = [\theta\rho_t]_{\alpha(\llbracket\nu\sigma\rrbracket_\alpha^{s_m})} = \alpha(\llbracket\nu\sigma\rrbracket_\alpha^{s_m})(\theta\rho_t) = \alpha(\llbracket\nu'\sigma\rrbracket_\alpha^{s_m})(\theta\rho_t) = [\theta\rho_t]_{\alpha(\llbracket\nu'\sigma\rrbracket_\alpha^{s_m})} = \llbracket\theta\sigma\rrbracket_\alpha^{s_m}(\llbracket\nu'\sigma\rrbracket_\alpha^{s_m})$. Therefore $s_m, \alpha \Vdash \nu\sigma{:}\theta\sigma =_g \nu'\sigma{:}\theta\sigma$.

dD, and suppose that $s_m, \alpha \Vdash \nu\sigma R_\Box \nu'\sigma$ and $s_m, \alpha \Vdash \nu'\sigma{:}\varepsilon(x)$. Note that $\alpha(\llbracket\nu\sigma\rrbracket_\alpha^{s_m}) = \alpha(\llbracket\nu'\sigma\rrbracket_\alpha^{s_m})$, $\omega(\llbracket\nu\sigma\rrbracket_\alpha^{s_m}) R^{\circ}_{\Box} \omega(\llbracket\nu'\sigma\rrbracket_\alpha^{s_m})$ and $\alpha(\llbracket\nu'\sigma\rrbracket_\alpha^{s_m})(x)$ is in $[\varepsilon]_{\omega(\llbracket\nu'\sigma\rrbracket_\alpha^{s_m})}$. So $\alpha(\llbracket\nu\sigma\rrbracket_\alpha^{s_m})(x)$ is in $[\varepsilon]_{\omega(\llbracket\nu'\sigma\rrbracket_\alpha^{s_m})}$, and since in m the domains decrease, we have that $\alpha(\llbracket\nu\sigma\rrbracket_\alpha^{s_m})(x)$ is in $[\varepsilon]_{\omega(\llbracket\nu\sigma\rrbracket_\alpha^{s_m})}$. So, $s_m, \alpha \Vdash \nu\sigma{:}\varepsilon(x)$, as we wanted to show.

exh, and suppose that for any α' $\{\nu'\sigma\}$ co-equivalent to α if $s_m, \alpha' \Vdash \nu_1\sigma \equiv_\omega \nu'\sigma$ and $s_m, \alpha' \Vdash \nu'\sigma \equiv_\alpha \nu_2\sigma$ then $s_m, \alpha' \Vdash \nu''\sigma{:}\xi''\sigma$. Consider the point $\langle w, a\rangle$ in s_m with $w = \omega(\llbracket\nu_1\sigma\rrbracket_\alpha^{s_m})$ and $a = \alpha(\llbracket\nu_2\sigma\rrbracket_\alpha^{s_m})$. Consider also the schema assignment $\alpha^{\cdot}$ $\{\nu'\sigma\}$ co-equivalent to α with $\nu'\sigma\alpha^{\cdot} = \langle w, a\rangle$. Note that $\nu^{\cdot}\alpha^{\cdot} = \nu^{\cdot}\alpha$ for any $\nu^{\cdot}$ in $\mathrm{lsv}(\nu_1\sigma) \cup \mathrm{lsv}(\nu_2\sigma)$, since $\nu'\sigma$ is not in $\mathrm{lsv}(\nu_1\sigma) \cup \mathrm{lsv}(\nu_2\sigma)$ because $\mathrm{fresh}(\nu', \langle\vartheta_1, \{\nu_1 \equiv_\omega \nu', \nu' \equiv_\alpha \nu_2\}, \nu''{:}\xi''\rangle)(\sigma) = 1$. So, by Proposition 2.4.4, we have that $\llbracket\nu_1\sigma\rrbracket_\alpha^{s_m} = \llbracket\nu_1\sigma\rrbracket_{\alpha^{\cdot}}^{s_m}$ and $\llbracket\nu_2\sigma\rrbracket_\alpha^{s_m} = \llbracket\nu_2\sigma\rrbracket_{\alpha^{\cdot}}^{s_m}$. Therefore $\omega(\llbracket\nu'\sigma\rrbracket_{\alpha^{\cdot}}^{s_m}) = \omega(\llbracket\nu_1\sigma\rrbracket_{\alpha^{\cdot}}^{s_m})$ and $\alpha(\llbracket\nu'\sigma\rrbracket_{\alpha^{\cdot}}^{s_m}) = \alpha(\llbracket\nu_2\sigma\rrbracket_{\alpha^{\cdot}}^{s_m})$. Then $s_m, \alpha^{\cdot} \Vdash \nu_1\sigma \equiv_\omega \nu'\sigma$ and $s_m, \alpha^{\cdot} \Vdash \nu'\sigma \equiv_\alpha \nu_2\sigma$. So, by the initial assumption, $s_m, \alpha^{\cdot} \Vdash \nu''\sigma{:}\xi''\sigma$. Then, again by Proposition 2.4.4, we have, as we wanted to show $s_m, \alpha \Vdash \nu''\sigma{:}\xi''\sigma$ since $\nu^{\cdot}\alpha^{\cdot} = \nu^{\cdot}\alpha$ for any $\nu^{\cdot}$ in $\mathrm{lsv}(\nu''\sigma)$.

$\equiv_{xt}$, $\equiv_{xr}$, $\equiv_{xs}$. Straightforward. *QED*

Now, we present the three lemmas needed along the proof of Proposition 5.1.4.

Lemma 5.1.5 For any model m in $\mathcal{L}_{\mathrm{FOMdD}}$, structure s_m induced by m, and variable x

1. $\llbracket t'\rrbracket_\alpha^{s_m}(u) = \llbracket t'[t/x]\rrbracket_\alpha^{s_m}(u')$, whenever t' is a term and $\alpha(u)(x) = \llbracket t\rrbracket_\alpha^{s_m}(u')$,
2. $\llbracket\varphi\rrbracket_\alpha^{s_m}(u) = 1$ iff $\llbracket\varphi'\rrbracket_\alpha^{s_m}(u') = 1$, whenever φ is a schema formula where all schema variables appear under the scope of a $\forall_x$ or a $\exists_x$, and φ' is φ if there are no free x's in φ, otherwise φ' is obtained by replacing the free occurrences of x in φ by a schema term t either equal to a variable whenever some free x in φ is in the scope of a modality, or not having schema variables if some free x in φ is in the scope of a quantifier, such that no variable in t is captured by a quantifier, and $\alpha(u)(x) = \llbracket t\rrbracket_\alpha^{s_m}(u')$,

for any points u and u' such that $\omega(u) = \omega(u')$, and $\alpha(u)$ and $\alpha(u')$ are x co-equivalent.

Proof 1. The proof follows straightforwardly by induction on the structure of t', so we omit it.

2. The proof follows by induction on the structure of φ:

φ is $p(t_1, \ldots, t_k)$. Observe that $t_1, \ldots, t_k$ do not have schema variables. Therefore $\llbracket p(t_1, \ldots, t_k)\rrbracket_\alpha^{s_m}(u) = 1$ iff $[p]_{\omega(u)}(\llbracket t_1\rrbracket_\alpha^{s_m}(u), \ldots, \llbracket t_k\rrbracket_\alpha^{s_m}(u)) = 1$ equivalent, by fact 1., to $[p]_{\omega(u')}(\llbracket t_1[t/x]\rrbracket_\alpha^{s_m}(u'), \ldots, \llbracket t_k[t/x]\rrbracket_\alpha^{s_m}(u')) = 1$, and so $\llbracket p(t_1[t/x], \ldots, t_k[t/x])\rrbracket_\alpha^{s_m}(u') = 1$.

φ is $t_1 = t_2$. Note that t_1 and t_2 do not have schema variables. Then $\llbracket\varphi\rrbracket_\alpha^{s_m}(u) = 1$ iff $\llbracket t_1\rrbracket_\alpha^{s_m}(u) = \llbracket t_2\rrbracket_\alpha^{s_m}(u)$ iff (by item 1.) $\llbracket t_1[t/x]\rrbracket_\alpha^{s_m}(u') = \llbracket t_2[t/x]\rrbracket_\alpha^{s_m}(u')$ iff $\llbracket\varphi[t/x]\rrbracket_\alpha^{s_m}(u') = 1$ as we wanted to show.

φ is $\perp$. Then $\llbracket\perp\rrbracket_{\alpha}^{s_m}(u) = 0$ and $\llbracket\perp\rrbracket_{\alpha}^{s_m}(u') = 0$, as we wanted to show.

φ *is* $\neg\varphi_1$. Then $\llbracket\varphi\rrbracket_{\alpha}^{s_m}(u) = 1$ iff $\llbracket\varphi_1\rrbracket_{\alpha}^{s_m}(u) = 0$ iff (by induction hypothesis) $\llbracket\varphi_1'\rrbracket_{\alpha}^{s_m}(u') = 0$ iff $\llbracket\varphi'\rrbracket_{\alpha}^{s_m}(u') = 1$ as we wanted to show.

φ *is* $\varphi_1 \wedge \varphi_2$. Then $\llbracket\varphi\rrbracket_{\alpha}^{s_m}(u) = 1$ iff $\llbracket\varphi_1\rrbracket_{\alpha}^{s_m}(u) = 1$ and $\llbracket\varphi_2\rrbracket_{\alpha}^{s_m}(u) = 1$ iff (by induction hypothesis) $\llbracket\varphi_1'\rrbracket_{\alpha}^{s_m}(u') = 1$ and $\llbracket\varphi_2'\rrbracket_{\alpha}^{s_m}(u') = 1$ iff $\llbracket\varphi'\rrbracket_{\alpha}^{s_m}(u') = 1$ as we wanted to show.

φ is $\forall_x\varphi_1$. Then φ is a schema formula equal to φ' because φ do not have free x's. So we want to show that $\llbracket\forall_x\varphi_1\rrbracket_{\alpha}^{s_m}(u) = 1$ iff $\llbracket\forall_x\varphi_1\rrbracket_{\alpha}^{s_m}(u') = 1$. But this follows by item 2. of Lemma 5.1.7.

φ is $\forall_y\varphi_1$, and y is distinct of x. Then, φ' is equal to $\forall_y\varphi_1'$, φ_1 is a schema formula where all schema variables appear under the scope of a $\forall_x$ or a $\exists_x$, and φ_1' is φ_1 whenever there are no free x's in φ_1, otherwise φ_1' is obtained by replacing the free occurrences of x in φ_1 by the term t which is equal to a variable whenever some free x in φ_1 is in the scope of a modality, such that no variable in t is captured by a quantifier. Consider two cases:

- φ_1 has no free occurrences of x. Note that we want to show that $\llbracket\forall_y\varphi_1\rrbracket_{\alpha}^{s_m}(u) = 1$ iff $\llbracket\forall_y\varphi_1\rrbracket_{\alpha}^{s_m}(u') = 1$. But this result follows by item 2. of Lemma 5.1.7.

- φ_1 has free occurrences of x. Then t has no occurrences of y and has no schema variables. Assume we have $\llbracket\forall_y\varphi_1\rrbracket_{\alpha}^{s_m}(u) = 1$. Let u_1' be a point such that $\omega(u_1') = \omega(u')$, $\alpha(u_1')$ and $\alpha(u')$ are y co-equivalent, and $[\varepsilon]_{\omega(u_1')}(\alpha(u_1')(y)) = 1$. Then $\llbracket t\rrbracket_{\alpha}^{s_m}(u_1') = \llbracket t\rrbracket_{\alpha}^{s_m}(u')$ by item 1. of Lemma 5.1.6. Consider the point u_1 with $\omega(u_1) = \omega(u_1')$, $\alpha(u_1)$ and $\alpha(u_1')$ x co-equivalent, and $\alpha(u_1)(x) = \alpha(u)(x)$. Then $\alpha(u_1)(x) = \llbracket t\rrbracket_{\alpha}^{s_m}(u_1')$. Note that $\llbracket\varphi_1\rrbracket_{\alpha}^{s_m}(u_1) = 1$ since $\omega(u_1) = \omega(u)$, $\alpha(u_1)$ and $\alpha(u)$ are y co-equivalent, and $[\varepsilon]_{\omega(u_1)}(\alpha(u_1)(y)) = [\varepsilon]_{\omega(u_1')}(\alpha(u_1')(y)) = 1$. Hence, by induction hypothesis, $\llbracket\varphi_1'\rrbracket_{\alpha}^{s_m}(u_1') = 1$, as we wanted to show. The proof of the other direction is similar so it is omitted.

φ is $\Box\varphi_1$ and φ' is equal to $\Box\varphi_1'$. Then φ_1 is a schema formula where all schema variables appear under the scope of a $\forall_x$ or a $\exists_x$, and φ_1' is φ_1 whenever there are no free x's in φ_1, otherwise φ_1' is obtained by replacing the free occurrences of x in φ_1 by a variable t such that t is not captured by a quantifier and $\alpha(u)(x) = \alpha(u')(t)$. Consider two cases:

- φ_1 has no free occurrences of x. Note that we want to show that $\llbracket\Box\varphi_1\rrbracket_{\alpha}^{s_m}(u) = 1$ iff $\llbracket\Box\varphi_1\rrbracket_{\alpha}^{s_m}(u') = 1$. But this result follows by item 2. of Lemma 5.1.7.

- φ_1 has free occurrences of x. Then t is a variable and $\alpha(u)(x) = \alpha(u')(t)$. Assume $\llbracket\Box\varphi_1'\rrbracket_{\alpha}^{s_m}(u') = 1$. Let u_1 be a point with $\alpha(u) = \alpha(u_1)$ and $\omega(u)R_{\Box}^{\circ}\omega(u_1)$. Consider u_1' with $\alpha(u') = \alpha(u_1')$ and $\omega(u_1) = \omega(u_1')$. Then $\omega(u')R_{\Box}^{\circ}\omega(u_1')$, $\alpha(u_1)$ and $\alpha(u_1')$ are x co-equivalent, and $\alpha(u_1)(x) = \alpha(u_1')(t)$. Thus $\llbracket\varphi_1'\rrbracket_{\alpha}^{s_m}(u_1') = 1$, and so, by induction hypothesis, $\llbracket\varphi_1\rrbracket_{\alpha}^{s_m}(u_1) = 1$, as we wanted to show. The proof of the other direction is similar. *QED*

Lemma 5.1.6 For any model m in $\mathcal{L}_{\mathrm{FOMdD}}$ and structure s_m induced by m

1. $\llbracket t\rrbracket_{\alpha}^{s_m}(u) = \llbracket t\rrbracket_{\alpha}^{s_m}(u')$, whenever t is a term and $\alpha(u)(x) = \alpha(u')(x)$ for any variable x in t,

2. $\llbracket\varphi\rrbracket_{\alpha}^{s_m}(u) = 1$ iff $\llbracket\varphi\rrbracket_{\alpha}^{s_m}(u') = 1$, whenever φ is a formula where $\alpha(u)(x) = \alpha(u')(x)$ for any free variable x in φ,

for any points u and u' with $\omega(u) = \omega(u')$.

Proof 1. The proof follows straightforwardly by induction on the structure of t, so we omit it.

2. We show this by induction on the structure of φ.

If φ *is* $p(t_1, \ldots, t_k)$ then $\llbracket p(t_1, \ldots, t_k)\rrbracket^{s_m}_\alpha(u) = 1$ iff $[p]_{\omega(u)}(\llbracket t_1\rrbracket^{s_m}_\alpha(u), \ldots, \llbracket t_k\rrbracket^{s_m}_\alpha(u)) = 1$ iff $[p]_{\omega(u')}(\llbracket t_1\rrbracket^{s_m}_\alpha(u'), \ldots, \llbracket t_k\rrbracket^{s_m}_\alpha(u')) = 1$ iff $\llbracket p(t_1, \ldots, t_k)\rrbracket^{s_m}_\alpha(u') = 1$.

If φ *is* $t_1 = t_2$ then $\llbracket\varphi\rrbracket^{s_m}_\alpha(u) = 1$ iff $\llbracket t_1\rrbracket^{s_m}_\alpha(u) = \llbracket t_2\rrbracket^{s_m}_\alpha(u)$ iff (by item 1.) $\llbracket t_1\rrbracket^{s_m}_\alpha(u') = \llbracket t_2\rrbracket^{s_m}_\alpha(u')$ iff $\llbracket\varphi\rrbracket^{s_m}_\alpha(u') = 1$ as we wanted to show.

If φ is $\bot$ then $\llbracket\bot\rrbracket^{s_m}_\alpha(u) = 0$ and $\llbracket\bot\rrbracket^{s_m}_\alpha(u') = 0$, as we wanted to show.

If φ is $\neg\varphi_1$ then $\llbracket\neg\varphi_1\rrbracket^{s_m}_\alpha(u) = 1$ iff $\llbracket\varphi_1\rrbracket^{s_m}_\alpha(u) = 0$ iff (by induction hypothesis) $\llbracket\varphi_1\rrbracket^{s_m}_\alpha(u') = 0$ iff $\llbracket\neg\varphi_1\rrbracket^{s_m}_\alpha(u') = 1$.

If φ is $\varphi_1 \wedge \varphi_2$ then $\llbracket\varphi_1 \wedge \varphi_2\rrbracket^{s_m}_\alpha(u) = 1$ iff $\llbracket\varphi_1\rrbracket^{s_m}_\alpha(u) = 1$ and $\llbracket\varphi_2\rrbracket^{s_m}_\alpha(u) = 1$ iff (by induction hypothesis) $\llbracket\varphi_1\rrbracket^{s_m}_\alpha(u') = 1$ and $\llbracket\varphi_2\rrbracket^{s_m}_\alpha(u') = 1$ iff $\llbracket\varphi_1 \wedge \varphi_2\rrbracket^{s_m}_\alpha(u') = 1$.

If φ is $\forall_x\varphi_1$, where x may be free in φ_1 then assume $\llbracket\forall_x\varphi_1\rrbracket^{s_m}_\alpha(u) = 1$. Let u'_1 be a point such that $\omega(u'_1) = \omega(u')$, $\alpha(u'_1)$ and $\alpha(u')$ are x co-equivalent, and $[\varepsilon]_{\omega(u'_1)}(\alpha(u'_1)(x)) = 1$. Consider a point u_1 such that $\omega(u_1) = \omega(u'_1)$, $\alpha(u_1)$ and $\alpha(u)$ are x co-equivalent, and $\alpha(u_1)(x) = \alpha(u'_1)(x)$. Note that $\omega(u_1) = \omega(u'_1) = \omega(u)$ and $[\varepsilon]_{\omega(u_1)}(\alpha(u_1)(x)) = [\varepsilon]_{\omega(u'_1)}(\alpha(u'_1)(x)) = 1$. So $\llbracket\varphi_1\rrbracket^{s_m}_\alpha(u_1) = 1$. We now show that $\alpha(u_1)(y) = \alpha(u'_1)(y)$ for any variable y free in φ_1. Let y be a variable free in φ_1. Consider two cases (i) y is distinct of x. Then y is free in $\forall_x\varphi_1$. So $\alpha(u_1)(y) = \alpha(u)(y) = \alpha(u')(y) = \alpha(u'_1)(y)$, as we want, and (ii) y is x. Then $\alpha(u_1)(x) = \alpha(u'_1)(x)$ by definition of u_1. So, by induction hypothesis $\llbracket\varphi_1\rrbracket^{s_m}_\alpha(u'_1) = 1$, as we wanted to show. The proof of the other direction follows in a similar way.

If φ is $\Box\varphi_1$ then assume $\llbracket\Box\varphi_1\rrbracket^{s_m}_\alpha(u) = 1$. Let u'_1 be a point with $\alpha(u') = \alpha(u'_1)$ and $\omega(u')R^{\circ}_{\Box}\omega(u'_1)$. Consider the point u_1 with $\omega(u_1) = \omega(u'_1)$ and $\alpha(u_1) = \alpha(u)$. Then $\omega(u)R^{\circ}_{\Box}\omega(u_1)$ and so $\llbracket\varphi_1\rrbracket^{s_m}_\alpha(u_1) = 1$. Note that $\alpha(u_1)(x) = \alpha(u'_1)(x)$ for any free variable x in φ_1. To see this let x be a free variable in φ_1. Then $\alpha(u_1)(x) = \alpha(u)(x) = \alpha(u')(x) = \alpha(u'_1)(x)$ as we wanted to show. Hence, by induction hypothesis, we can conclude that $\llbracket\varphi_1\rrbracket^{s_m}_\alpha(u'_1) = 1$, as we wanted to show. The proof for the other direction is similar. *QED*

Lemma 5.1.7 For any model m in $\mathcal{L}_{\text{FOMdD}}$ and structure s_m induced by m

1. $\llbracket t\rrbracket^{s_m}_\alpha(u) = \llbracket t\rrbracket^{s_m}_\alpha(u')$, whenever t is a term without schema variables where x does not appear,
2. $\llbracket\varphi\rrbracket^{s_m}_\alpha(u) = 1$ iff $\llbracket\varphi\rrbracket^{s_m}_\alpha(u') = 1$, whenever φ is a schema formula where x does not appear free and all schema variables appear under the scope of a $\forall_x$ or a $\exists_x$,

for any points u and u' with $\omega(u) = \omega(u')$, and $\alpha(u)$ and $\alpha(u')$ x co-equivalent.

Proof 1. The proof follows straightforwardly by induction on the structure of t, so we omit it.

2. The proof follows by induction on the structure of φ.

Assume φ *is* $p(t_1, \ldots, t_k)$ and $t_1, \ldots, t_k$ do not contain any x and any term schema variable. Then $\llbracket p(t_1, \ldots, t_k)\rrbracket^{s_m}_\alpha(u) = 1$ iff $[p]_{\omega(u)}(\llbracket t_1\rrbracket^{s_m}_\alpha(u), \ldots, \llbracket t_k\rrbracket^{s_m}_\alpha(u)) = 1$, which, by fact

1., is equivalent to $[p]_{\omega(u')}(\llbracket t_1\rrbracket^{s_m}_\alpha(u'),\ldots,\llbracket t_k\rrbracket^{s_m}_\alpha(u'))=1$ iff $\llbracket p(t_1,\ldots,t_k)\rrbracket^{s_m}_\alpha(u')=1$, as desired.

Assume φ *is* $t_1=t_2$ and t_1 and t_2 do not contain any x and any term schema variable. Then $\llbracket\varphi\rrbracket^{s_m}_\alpha(u)=1$ iff $\llbracket t_1\rrbracket^{s_m}_\alpha(u)=\llbracket t_2\rrbracket^{s_m}_\alpha(u)$ iff (by item 1.) $\llbracket t_1\rrbracket^{s_m}_\alpha(u')=\llbracket t_2\rrbracket^{s_m}_\alpha(u')$ iff $\llbracket\varphi\rrbracket^{s_m}_\alpha(u')=1$ as we wanted to show.

Assume φ is $\perp$. Then $\llbracket\perp\rrbracket^{s_m}_\alpha(u)=0$ and $\llbracket\perp\rrbracket^{s_m}_\alpha(u')=0$, as we wanted to show.

Assume φ *is* $\neg\varphi_1$. So $\llbracket\neg\varphi_1\rrbracket^{s_m}_\alpha(u)=1$ iff $[\neg]_{\omega(u)\alpha(u)}(\llbracket\varphi_1\rrbracket^{s_m}_\alpha\cap U_{\omega(u)\alpha(u)})(u)=1$, iff $\llbracket\varphi_1\rrbracket^{s_m}_\alpha(u)=0$ iff (using the induction hypothesis), $\llbracket\varphi_1\rrbracket^{s_m}_\alpha(u')=0$ iff $[\neg]_{\omega(u')\alpha(u')}(\llbracket\varphi_1\rrbracket^{s_m}_\alpha\cap U_{\omega(u')\alpha(u')})(u')=1$ iff $\llbracket\neg\varphi_1\rrbracket^{s_m}_\alpha(u')=1$.

Assume φ *is* $\varphi_1\wedge\varphi_2$. So $\llbracket\varphi_1\wedge\varphi_2\rrbracket^{s_m}_\alpha(u)=1$ iff $[\wedge]_{\omega(u)\alpha(u)}(\llbracket\varphi_1\rrbracket^{s_m}_\alpha\cap U_{\omega(u)\alpha(u)},\llbracket\varphi_2\rrbracket^{s_m}_\alpha\cap U_{\omega(u)\alpha(u)})(u)=1$, iff $\llbracket\varphi_1\rrbracket^{s_m}_\alpha(u)=1$ and $\llbracket\varphi_2\rrbracket^{s_m}_\alpha(u)=1$, iff (using the induction hypothesis), $\llbracket\varphi_1\rrbracket^{s_m}_\alpha(u')=1$ and $\llbracket\varphi_2\rrbracket^{s_m}_\alpha(u')=1$, iff $[\wedge]_{\omega(u')\alpha(u')}(\llbracket\varphi_1\rrbracket^{s_m}_\alpha\cap U_{\omega(u')\alpha(u')},\llbracket\varphi_2\rrbracket^{s_m}_\alpha\cap U_{\omega(u')\alpha(u')})(u')=1$ iff $\llbracket\varphi_1\wedge\varphi_2\rrbracket^{s_m}_\alpha(u')=1$.

Assume φ *is* $\forall_x\varphi_1$. Assume $\llbracket\forall_x\varphi_1\rrbracket^{s_m}_\alpha(u)=1$. Let u'_1 be a point such that $\omega(u'_1)=\omega(u')$, $\alpha(u'_1)$ and $\alpha(u')$ are x co-equivalent, and $[\varepsilon]_{\omega(u'_1)}(\alpha(u'_1)(x))=1$. Then $\omega(u'_1)=\omega(u)$ and $\alpha(u'_1)$ and $\alpha(u)$ are x co-equivalent, and so, by the assumption, $\llbracket\forall_x\varphi_1\rrbracket^{s_m}_\alpha(u'_1)=1$, as we wanted to show. The proof of the other direction is similar.

Assume φ *is* $\forall_y\varphi_1$, where y is distinct of x. Assume $\llbracket\forall_y\varphi_1\rrbracket^{s_m}_\alpha(u)=1$. Let u'_1 be a point such that $\omega(u'_1)=\omega(u')$, $\alpha(u'_1)$ and $\alpha(u')$ are y co-equivalent, and $[\varepsilon]_{\omega(u'_1)}(\alpha(u'_1)(y))=1$. Consider a point u_1 such that $\omega(u'_1)=\omega(u_1)$, $\alpha(u'_1)$ and $\alpha(u_1)$ are x co-equivalent and $\alpha(u_1)(x)=\alpha(u)(x)$. So $\alpha(u_1)$ and $\alpha(u)$ are y co-equivalent. Note also that $\omega(u_1)=\omega(u)$ and $[\varepsilon]_{\omega(u_1)}(\alpha(u_1)(y))=[\varepsilon]_{\omega(u'_1)}(\alpha(u'_1)(y))=1$. So $\llbracket\varphi_1\rrbracket^{s_m}_\alpha(u_1)=1$. Hence, by induction hypothesis, $\llbracket\varphi_1\rrbracket^{s_m}_\alpha(u'_1)=1$ as we wanted to show, because φ_1 is also a schema formula where x does not appear free and all schema variables appear under the scope of a $\forall_x$ or a $\exists_x$. The proof of the other direction follows in a similar way.

Assume φ *is* $\Box\varphi_1$. Assume $\llbracket\Box\varphi_1\rrbracket^{s_m}_\alpha(u)=1$. Let u'_1 be a point such that $\alpha(u')=\alpha(u'_1)$ and $\omega(u')R^\circ_\Box\omega(u'_1)$. Consider a point u_1 with $\omega(u_1)=\omega(u'_1)$ and $\alpha(u_1)=\alpha(u)$. Then $\omega(u)R^\circ_\Box\omega(u_1)$. So, $\llbracket\varphi_1\rrbracket^{s_m}_\alpha(u_1)=1$. Hence, by induction hypothesis, $\llbracket\varphi_1\rrbracket^{s_m}_\alpha(u'_1)=1$, since $\omega(u_1)=\omega(u'_1)$ and $\alpha(u_1)$ and $\alpha(u'_1)$ are x co-equivalent, and φ_1 is also a schema formula where x does not appear free and all schema variables appear under the scope of a $\forall_x$ or a $\exists_x$. The proof of the other direction follows in a similar way. *QED*

5.1.3 Completeness

The goal of this subsection is to show the following theorem, which we prove at the end of the section.

Theorem 5.1.8 The lfob logic system $\mathcal{L}_{\text{FOMdD}}$ is complete.

To prove this theorem we show that $\mathcal{L}_{\text{FOMdD}}$ is rich and then we invoke Theorem 3.3.3 which says that a rich lfob logic system is complete. But before we need to show some auxiliary propositions. So, we start to prove that $\mathcal{L}_{\text{FOMdD}}$ is a connected lfob logic system with a classical negation.

Proposition 5.1.9 The lfob logic system $\mathcal{L}_{\text{FOMdD}}$ is connected and with a classical negation.

Proof It is straightforward to see that $\mathcal{L}_{\mathrm{FOMdD}}$ is connected and that it has a local classical negation. Recall Definition 4.1.3 of a connected lfob logic system and Definition 4.3.2 of a clfob deduction system with a local classical negation. *QED*

We need now to introduce the notion of a relational model for first-order modal logic with decreasing domains, induced by an appropriate set, in the context of $\mathcal{L}_{\mathrm{FOMdD}}$ of Example 5.1.1.

Definition 5.1.10 The model m_Ψ *induced by* an E appropriate set Ψ in the context of $\mathcal{L}_{\mathrm{FOMdD}}$ is $\langle W, R^{\circ}_{\Box}, D, \_^{F}, \_^{P}\rangle$, defined as follows:

- W is $\{[v]^{\Psi,E}_{\omega} \mid v \text{ in } T_{\mathrm{lab},E}\}$,
- $R^{\circ}_{\Box}$ is $\{\langle [v_1]^{\Psi,E}_{\omega}, [v_2]^{\Psi,E}_{\omega}\rangle \mid R_{\Box}v_1v_2 \text{ is in } \Psi\}$,
- D is $\{[v{:}t]^{\Psi,E}_{g} \mid v \text{ in } T_{\mathrm{lab},E} \text{ and } t \text{ in } T\}$,
- $\_^{F}$ is a family of function denotations induced by Ψ, (see Definition 5.5.3),
- $\_^{P}$ is a family of predicate denotations induced by Ψ, (see Definition 5.5.3).

We now show the equivalence between satisfaction in the canonical structure and in the structure induced by the canonical model, for labelled schema formulae $\nu{:}\phi$ where ϕ is without schema variables.

Lemma 5.1.11 In the context of $\mathcal{L}_{\mathrm{FOMdD}}$, given an E strong appropriate set Ψ, a formula ϕ in $\mathcal{L}^{\cdot}_{\mathrm{FOMdD}}$, and a label schema variable ν,

$$s_\Psi, \alpha \Vdash \nu{:}\phi \quad \text{iff} \quad s_{m_\Psi}, \alpha^{\cdot} \Vdash \nu{:}\phi$$

for any schema assignments α over s_Ψ and $\alpha^{\cdot}$ over s_{m_Ψ} with $\nu\alpha^{\cdot} = [\nu\alpha]^{\Psi,E}_{\omega\times\iota}$.

Proof Recall first the definition of ι^{Ψ}_{DX} and $[v]^{\Psi,E}_{\omega\times\iota}$ in Remark 5.5.1, and the definition of a Ψ strongly uniform map in Definition 5.5.10, and let f be a Ψ strongly uniform map. Note that, in the context of $\mathcal{L}_{\mathrm{FOMdD}}$, the class of Ψ strong uniform maps is not empty. For example, the map f defined as $f(|\phi|^{s_\Psi}) = [|\phi|^{s_\Psi}]^{\Psi,E}_{\omega\times\iota} \cup \{\langle w, a\rangle \mid w \text{ is } [v]^{\Psi,E}_{\omega} \text{ for some } v \text{ and } x \text{ with } [v]^{\Psi,E}_{x} \subseteq |\phi|^{s_\Psi} \text{ and } a \ x \text{ co-equivalent to } \iota^{\Psi}_{DX}(v)\}$ for any formula ϕ, and $f(|\phi|^{s_\Psi}) = [|\phi|^{s_\Psi}]^{\Psi,E}_{\omega\times\iota}$ for any ϕ having schema variables, is a Ψ strongly uniform map. Note that, by Proposition 5.1.2, we are in the context of a uniform lfob deduction system with a universal quantifier. So, we now show some auxiliary results and after that we prove the lemma.

1. $[\![t]\!]^{s_\Psi}_{\alpha}(v) = [\![t]\!]^{s_{m_\Psi}}_{\alpha^{\cdot}}([v]^{\Psi,E}_{\omega\times\iota})$, where t is a term in $\mathcal{L}^{\cdot}_{\mathrm{FOMdD}}$, and v is a label schema term. The proof follows by induction on the possible cases for the term t in $\mathcal{L}^{\cdot}_{\mathrm{FOMdD}}$:

t is in X. Then $[\![t]\!]^{s_\Psi}_{\alpha}(v) = [t]^{s_\Psi}_{\alpha_{s_\Psi}(v)} =$ (see the definition of canonical structure in Section 3.1) $= [v{:}t]^{\Psi,E}_{g} =$ (see the definition of ι^{Ψ}_{DX}, Remark 5.5.1) $= \iota^{\Psi}_{DX}(v)(t) = [t]^{s_{m_\Psi}}_{\iota^{\Psi}_{DX}(v)} = [t]^{s_{m_\Psi}}_{\alpha_{s_{m_\Psi}}([v]^{\Psi,E}_{\omega\times\iota})} = [\![t]\!]^{s_{m_\Psi}}_{\alpha^{\cdot}}([v]^{\Psi,E}_{\omega\times\iota})$.

t is in F_0. Then, $\llbracket t\rrbracket_{\alpha}^{s_\Psi}(v) = [t]^{s_\Psi}_{\omega(v)} = |t|(v) = [v{:}t]_g^{\Psi,E} = t^F{}_{[v]_\omega^{\Psi,E}} = [t]^{sm_\Psi}_{\omega([v]^{\Psi,E}_{\omega\times\iota})} = \llbracket t\rrbracket_{\alpha^{\cdot}}^{sm_\Psi}([v]^{\Psi,E}_{\omega\times\iota})$.

t is $f(t_1,\ldots,t_k)$. Therefore, $\llbracket t\rrbracket_{\alpha}^{s_\Psi}(v) = [f]^{s_\Psi}_{\omega(v)}(\llbracket t_1\rrbracket_{\alpha}^{s_\Psi}(v),\ldots,\llbracket t_k\rrbracket_{\alpha}^{s_\Psi}(v)) =$ (by Proposition 3.1.3) $= [f]^{s_\Psi}_{\omega(v)}(|t_1|(v),\ldots,|t_k|(v)) = |f(t_1,\ldots,t_k)|(v) = [v{:}f(t_1,\ldots,t_k)]_g^{\Psi,E} = f^F{}_{[v]_\omega^{\Psi,E}}([v{:}t_1]_g^{\Psi,E},\ldots,[v{:}t_k]_g^{\Psi,E}) = [f]^{sm_\Psi}_{\omega([v]^{\Psi,E}_{\omega\times\iota})}(\llbracket t_1\rrbracket_{\alpha}^{s_\Psi}(v),\ldots,\llbracket t_k\rrbracket_{\alpha}^{s_\Psi}(v)) =$ (by induction hypothesis) $= [f]^{sm_\Psi}_{\omega([v]^{\Psi,E}_{\omega\times\iota})}(\llbracket t_1\rrbracket_{\alpha^{\cdot}}^{sm_\Psi}([v]^{\Psi,E}_{\omega\times\iota}),\ldots,\llbracket t_k\rrbracket_{\alpha^{\cdot}}^{sm_\Psi}([v]^{\Psi,E}_{\omega\times\iota})) = \llbracket t\rrbracket_{\alpha^{\cdot}}^{sm_\Psi}([v]^{\Psi,E}_{\omega\times\iota})$.

2. $[v]^{\Psi,E}_{\omega\times\iota}$ is in $f(\llbracket\phi\rrbracket_{\alpha}^{s_\Psi})$ iff $[v]^{\Psi,E}_{\omega\times\iota}$ is in $\llbracket\phi\rrbracket_{\alpha^{\cdot}}^{sm_\Psi}$. The proof follows by induction on the structure of a formula ϕ in $\mathcal{L}^{\cdot}_{\mathrm{FOMdD}}$:

Suppose ϕ is $t_1 = t_2$. Then $[v]^{\Psi,E}_{\omega\times\iota}$ is in $f(\llbracket\phi\rrbracket_{\alpha}^{s_\Psi})$ iff (by Proposition 5.5.11) v is in $\llbracket\phi\rrbracket_{\alpha}^{s_\Psi}$ iff $\llbracket t_1\rrbracket_{\alpha}^{s_\Psi}(v) = \llbracket t_2\rrbracket_{\alpha}^{s_\Psi}(v)$ iff (by item 1. above proven) $\llbracket t_1\rrbracket_{\alpha^{\cdot}}^{sm_\Psi}([v]^{\Psi,E}_{\omega\times\iota}) = \llbracket t_2\rrbracket_{\alpha^{\cdot}}^{sm_\Psi}([v]^{\Psi,E}_{\omega\times\iota})$ iff $[v]^{\Psi,E}_{\omega\times\iota}$ is in $\llbracket\phi\rrbracket_{\alpha^{\cdot}}^{sm_\Psi}$, as we wanted to show.

Suppose ϕ is $p(t_1,\ldots,t_k)$. Then $[v]^{\Psi,E}_{\omega\times\iota}$ is in $f(\llbracket\phi\rrbracket_{\alpha}^{s_\Psi})$ iff (by Proposition 5.5.11) $v \in \llbracket\phi\rrbracket_{\alpha}^{s_\Psi}$ iff (by Proposition 3.1.3) $v \in |\phi|^{s_\Psi}$ iff $v{:}p(t_1,\ldots,t_k) \in \Psi$ iff $p^P{}_{[v]_\omega^{\Psi,E}}([v{:}t_1]_g^{\Psi,E},\ldots,[v{:}t_k]_g^{\Psi,E}) = 1$ iff $[p]^{sm_\Psi}_{[v]_\omega^{\Psi,E}}([v{:}t_1]_g^{\Psi,E},\ldots,[v{:}t_k]_g^{\Psi,E}) = 1$ iff (by Proposition 3.1.3) $[p]^{sm_\Psi}_{[v]_\omega^{\Psi,E}}(\llbracket t_1\rrbracket_{\alpha}^{s_\Psi}(v),\ldots,\llbracket t_k\rrbracket_{\alpha}^{s_\Psi}(v)) = 1$ iff (by item 1.) $[p]^{sm_\Psi}_{\omega([v]^{\Psi,E}_{\omega\times\iota})}(\llbracket t_1\rrbracket_{\alpha^{\cdot}}^{sm_\Psi}([v]^{\Psi,E}_{\omega\times\iota}),\ldots,\llbracket t_k\rrbracket_{\alpha^{\cdot}}^{sm_\Psi}([v]^{\Psi,E}_{\omega\times\iota})) = 1$ iff $\llbracket\phi\rrbracket_{\alpha^{\cdot}}^{sm_\Psi}([v]^{\Psi,E}_{\omega\times\iota}) = 1$ as we wanted to show.

Suppose γ is $\perp$. Then $[v]^{\Psi,E}_{\omega\times\iota}$ is not in $f(\llbracket\perp\rrbracket_{\alpha}^{s_\Psi})$ and $[v]^{\Psi,E}_{\omega\times\iota}$ is not in $\llbracket\perp\rrbracket_{\alpha^{\cdot}}^{sm_\Psi}$. We show first that $v{:}\perp$ is not in Ψ for any label schema term v. Denote by $v'{:}\varphi'$ the labelled schema formula with respect to which Ψ is consistent and suppose that $v{:}\perp$ is in Ψ. Then, by rule $\neg_c$, we have that $v'{:}\varphi'$ is in Ψ since Ψ is deductively closed, which is a contradiction. So, $v{:}\perp$ is not in Ψ, as we wanted to show. Hence, v is not in $|\perp|^{s_\Psi}$ and by Proposition 3.1.3 it is not in $\llbracket\perp\rrbracket_{\alpha}^{s_\Psi}$. Thus, by Proposition 5.5.11. $[v]^{\Psi,E}_{\omega\times\iota}$ is not in $f(\llbracket\perp\rrbracket_{\alpha}^{s_\Psi})$. The fact that $[v]^{\Psi,E}_{\omega\times\iota}$ is not in $\llbracket\perp\rrbracket_{\alpha^{\cdot}}^{sm_\Psi}$ follows by definition of $[\perp]^{sm_\Psi}$.

Suppose ϕ is $\neg\phi_1$. Then $[v]^{\Psi,E}_{\omega\times\iota}$ is in $f(\llbracket\phi\rrbracket_{\alpha}^{s_\Psi})$ iff (by Proposition 5.5.11) v is in $\llbracket\phi\rrbracket_{\alpha}^{s_\Psi}$ iff (by Proposition 3.1.3) v is in $|\phi|^{s_\Psi}$ iff $v{:}\neg\phi_1$ is in Ψ iff (since Ψ is an appropriate set) $v{:}\phi_1$ is not in Ψ iff (by Proposition 3.1.3) v is not in $\llbracket\phi_1\rrbracket_{\alpha}^{s_\Psi}$ iff (by Proposition 5.5.11) $[v]^{\Psi,E}_{\omega\times\iota}$ is not in $f(\llbracket\phi_1\rrbracket_{\alpha}^{s_\Psi})$ iff (by induction hypothesis) $\llbracket\phi_1\rrbracket_{\alpha^{\cdot}}^{sm_\Psi}([v]^{\Psi,E}_{\omega\times\iota}) = 0$ iff $[\neg]_{\omega([v]^{\Psi,E}_{\omega\times\iota})\alpha([v]^{\Psi,E}_{\omega\times\iota})}(\llbracket\phi_1\rrbracket_{\alpha^{\cdot}}^{sm_\Psi})([v]^{\Psi,E}_{\omega\times\iota}) = 1$ iff $[v]^{\Psi,E}_{\omega\times\iota}$ is in $\llbracket\neg\phi_1\rrbracket_{\alpha^{\cdot}}^{sm_\Psi}$, as we wanted to show.

Suppose ϕ is $\phi_1 \wedge \phi_2$. Then $[v]^{\Psi,E}_{\omega\times\iota}$ is in $f(\llbracket\phi\rrbracket_{\alpha}^{s_\Psi})$ iff (by Proposition 5.5.11) v is in $\llbracket\phi\rrbracket_{\alpha}^{s_\Psi}$ iff (by Proposition 3.1.3) v is in $|\phi|^{s_\Psi}$ iff $v{:}\phi_1 \wedge \phi_2$ is in Ψ iff (since Ψ is an appropriate set) $v{:}\phi_1$ is in Ψ and $v{:}\phi_2$ is in Ψ iff (by Proposition 3.1.3) v is in $\llbracket\phi_1\rrbracket_{\alpha}^{s_\Psi}$ and v is in $\llbracket\phi_2\rrbracket_{\alpha}^{s_\Psi}$ iff (by Proposition 5.5.11) $[v]^{\Psi,E}_{\omega\times\iota}\,\mathrm{inf}(\llbracket\phi_1\rrbracket_{\alpha}^{s_\Psi})$ and $[v]^{\Psi,E}_{\omega\times\iota}\,\mathrm{inf}(\llbracket\phi_2\rrbracket_{\alpha}^{s_\Psi})$ iff (by induction hypothesis) $[v]^{\Psi,E}_{\omega\times\iota}$ is in $\llbracket\phi_1\rrbracket_{\alpha^{\cdot}}^{sm_\Psi}$ and in $\llbracket\phi_2\rrbracket_{\alpha^{\cdot}}^{sm_\Psi}$ iff $[\wedge]_{\omega([v]^{\Psi,E}_{\omega\times\iota})\alpha([v]^{\Psi,E}_{\omega\times\iota})}(\llbracket\phi_1\rrbracket_{\alpha^{\cdot}}^{sm_\Psi},\llbracket\phi_2\rrbracket_{\alpha^{\cdot}}^{sm_\Psi})([v]^{\Psi,E}_{\omega\times\iota}) = 1$ iff $[v]^{\Psi,E}_{\omega\times\iota}$ is in $\llbracket\phi_1\wedge\phi_2\rrbracket_{\alpha^{\cdot}}^{sm_\Psi}$, as we wanted to show.

Suppose ϕ is $\Box\phi_1$. Then $[v]^{\Psi,E}_{\omega\times\iota}$ is in $f(\llbracket\phi\rrbracket_{\alpha}^{s_\Psi})$ iff (by Proposition 5.5.11) v is in $\llbracket\phi\rrbracket_{\alpha}^{s_\Psi}$ iff (by Proposition 3.1.3) v is in $|\phi|^{s_\Psi}$ iff $v{:}\Box\phi_1$ is in Ψ iff (since Ψ is an appropriate set) for all v_1 if $vR_\Box v_1$ is in Ψ then $v_1{:}\phi_1$ is in Ψ iff (by Proposition 5.5.14) for any assignment a_1

in ${D_\Psi}^X$ and label schema term $v^{\cdot}_1$ if exists a label schema term $v^{\cdot}$ with $v \equiv_\omega v^{\cdot}$ and $v^{\cdot} R_\Box v^{\cdot}_1$ in Ψ, and $\iota^\Psi_{D^X}(v) = a_1$ then $\langle [v^{\cdot}_1]^{\Psi,E}_\omega, a_1\rangle$ is in $f([\![\phi_1]\!]^{s_\Psi}_\alpha)$ iff (by induction hypothesis) for all $\langle [v^{\cdot}_1]^{\Psi,E}_\omega, a_1\rangle)$ in $U^{s_{m_\Psi}}$ if $\omega([v]^{\Psi,E}_{\omega\times\iota}) R^\circ_\Box \omega(\langle [v^{\cdot}_1]^{\Psi,E}_\omega, a_1\rangle)$ and $\alpha([v]^{\Psi,E}_{\omega\times\iota}) = \alpha(\langle [v^{\cdot}_1]^{\Psi,E}_\omega, a_1\rangle)$ then $\langle [v^{\cdot}_1]^{\Psi,E}_\omega, a_1\rangle$ is in $[\![\phi_1]\!]^{s_{m_\Psi}}_{\alpha^{\cdot}}$ iff $[\Box]^{s_{m_\Psi}}([\![\phi_1]\!]^{s_{m_\Psi}}_{\alpha^{\cdot}})([v]^{\Psi,E}_{\omega\times\iota}) = 1$ iff $[v]^{\Psi,E}_{\omega\times\iota}$ is in $[\![\Box\phi_1]\!]^{s_{m_\Psi}}_{\alpha^{\cdot}}$ as we wanted to show.

Suppose ϕ is $\forall_x \phi_1$. Then $[v]^{\Psi,E}_{\omega\times\iota}$ is in $f([\![\phi]\!]^{s_\Psi}_\alpha)$ iff (by Proposition 5.5.11) v is in $[\![\phi]\!]^{s_\Psi}_\alpha$ iff (by Proposition 3.1.3) v is in $|\phi|^{s_\Psi}$ iff $v{:}\forall_x\phi_1$ is in Ψ iff (by Proposition 5.5.13) for all a_1 if a_1 is x co-equivalent to $\iota^\Psi_{D^X}(v)$ and for some v_1 and t, $a_1(x)$ is equal to $[v_1{:}t]^{\Psi,E}_g$, and $v_1{:}\varepsilon(t)$ and $v_1 \equiv_\omega v$ in Ψ then $\langle [v]^{\Psi,E}_\omega, a_1\rangle$ is in $f([\![\phi_1]\!]^{s_\Psi}_\alpha)$ iff (by induction hypothesis) for all $\langle [v^{\cdot}_1]^{\Psi,E}_\omega, a_1\rangle)$ in $U^{s_{m_\Psi}}$ if $\omega([v]^{\Psi,E}_{\omega\times\iota}) = \omega(\langle [v^{\cdot}_1]^{\Psi,E}_\omega, a_1\rangle)$ and $\alpha([v]^{\Psi,E}_{\omega\times\iota})$ is x co-equivalent to $\alpha(\langle [v^{\cdot}_1]^{\Psi,E}_\omega, a_1\rangle)$ and $a_1(x)$ is in $[\varepsilon]_{\omega(\langle [v^{\cdot}_1]^{\Psi,E}_\omega, a_1\rangle)}$ then $\langle [v^{\cdot}_1]^{\Psi,E}_\omega, a_1\rangle$ is in $[\![\phi_1]\!]^{s_{m_\Psi}}_{\alpha^{\cdot}}$ iff $[\forall_x]^{s_{m_\Psi}}_{\alpha([v]^{\Psi,E}_{\omega\times\iota})}([\![\phi_1]\!]^{s_{m_\Psi}}_{\alpha^{\cdot}})([v]^{\Psi,E}_{\omega\times\iota}) = 1$ iff $[v]^{\Psi,E}_{\omega\times\iota}$ is in $[\![\forall_x\phi_1]\!]^{s_{m_\Psi}}_{\alpha^{\cdot}}$ as we wanted to show.

Finally the main proof. $s_\Psi, \alpha \Vdash \nu{:}\phi$ iff $[\![\nu]\!]^{s_\Psi}_\alpha$ is in $[\![\phi]\!]^{s_\Psi}_\alpha$ iff (by Proposition 5.5.11) $[\nu\alpha]^{\Psi,E}_{\omega\times\iota}$ is in $f([\![\phi]\!]^{s_\Psi}_\alpha)$ iff (by item 2. above proven) $\nu\alpha^{\cdot}$ is in $[\![\phi]\!]^{s_{m_\Psi}}_{\alpha^{\cdot}}$ iff $s_{m_\Psi}, \alpha^{\cdot} \Vdash \nu{:}\phi$. *QED*

Proposition 5.1.12 The lfob logic system $\mathcal{L}_{\text{FOMdD}}$ is rich.

Proof In the context of $\mathcal{L}_{\text{FOMdD}}$, let Ψ_0 be a $v{:}\varphi$ consistent set. Then, by Proposition 4.3.9, there exists a strong appropriate set Ψ extending Ψ_0 and also $v{:}\varphi$ consistent, since $\mathcal{L}_{\text{FOMdD}}$ contains the rule *exh*, and, by Proposition 5.1.9, is a clfob deduction system with a classical negation. So, it is possible to consider the canonical structure s_Ψ. Let m_Ψ be the model induced by Ψ, as defined in Definition 5.1.10. Recall that s_{m_Ψ} is the structure induced by m_Ψ and so s_{m_Ψ} is in $\breve{m}_\Psi$. Then, taking into account Lemma 5.1.11 and the definition of $\breve{m}_\Psi$ in Example 5.1.1, we have that s_Ψ is in $\breve{m}_\Psi$, as we wanted to show. *QED*

Proof (Completeness Theorem 5.1.8) The proof follows immediately using Theorem 3.3.3 since, by Proposition 5.1.12, $\mathcal{L}_{\text{FOMdD}}$ is rich. *QED*

5.1.4 Equivalence to non-labelled logic system

In this subsection we show that the lfob logic system $\mathcal{L}_{\text{FOMdD}}$ is indeed a labelled presentation for first-order modal logic with decreasing domains with respect to formulae without schema variables, as is stated in the next theorem. This theorem is proved at the end of the section.

Theorem 5.1.13 The lfob logic system $\mathcal{L}_{\text{FOMdD}}$ is a labelled presentation for first-order modal logic with decreasing domains with respect to formulae without schema variables.

The proof of the theorem relies on finding a non-labelled logic system for first-order modal logic with decreasing domains such that $\mathcal{L}_{\text{FOMdD}}$ is a labelled presentation for that system with respect to formulae without schema variables. Recall Definition 2.4.10 where it is defined what is a lfob logic system be a labelled presentation for a logic with respect to some set of schema formulae. So, consider the following system.

Example 5.1.14 *First-order modal logic with decreasing domains non-labelled logic system.* Let $\mathcal{L}^{\cdot}_{\mathrm{FOMdD}}$ be the non-labelled logic system $\langle \Sigma^{\cdot}, \propto\rangle$ for first-order modal logic with decreasing domains having as its family of propositions the family of propositions of $\mathcal{L}_{\mathrm{FOMdD}}$, and having as its family of function symbols the family of function symbols of $\mathcal{L}_{\mathrm{FOMdD}}$.

Observe that the only components that can change between non-labelled logic systems for first-order modal logic with decreasing domains are the family of propositions and the family of function symbols, and, motivated by that change, the relation $\propto$ between schema formulae associated to the system.

Before presenting the proof that $\mathcal{L}_{\mathrm{FOMdD}}$ is indeed a labelled presentation for first-order modal logic with decreasing domains with respect to formulae without schema variables, we show two auxiliary lemmas proving the equivalence for formulae without schema variables, between the usual semantics for the logic and the labelled semantics. Lemma 5.1.15 proves the equivalence for formulae and Lemma 5.1.16 proves the equivalence for entailment.

Lemma 5.1.15 In the context of $\mathcal{L}_{\mathrm{FOMdD}}$ and $\mathcal{L}^{\cdot}_{\mathrm{FOMdD}}$,

$$m, \alpha^{\cdot}, w \Vdash^{a} \gamma \quad \text{iff} \quad s_m, \alpha \Vdash \nu{:}\gamma$$

where γ is a schema formula in $\mathcal{L}^{\cdot}_{\mathrm{FOMdD}}$ and α and $\alpha^{\cdot}$ are such that $\alpha_t = \alpha^{\cdot}_t$, $\alpha_f = \alpha^{\cdot}_f$ and $\nu\alpha_l = \langle w, a\rangle$.

Proof First we state an auxiliary result and only after we show the proposition.

1. $[\![t]\!]^{s_m}_{\alpha}(\langle w, a\rangle) = t^{m,\alpha^{\cdot}}_{w,a}$, where t is schema term. We omit the proof since it follows straightforwardly by induction on the structure of the schema term t in $\mathcal{L}^{\cdot}_{\mathrm{FOMdD}}$.

Finally the main proof, which follows by induction on γ:

if γ is $t_1 = t_2$ then $m, \alpha^{\cdot}, w \Vdash^{a} t_1 = t_2$ iff ${t_1}^{m,\alpha^{\cdot}}_{w,a} = {t_2}^{m,\alpha^{\cdot}}_{w,a}$ iff (by item 1. proven above and the definition of s_m) $[\![t_1]\!]^{s_m}_{\alpha}(\nu\alpha) = [\![t_2]\!]^{s_m}_{\alpha}(\nu\alpha)$ iff $s_m, \alpha \Vdash \nu{:}t_1 = t_2$.

if γ is $p(t_1, \ldots, t_k)$ then $m, \alpha^{\cdot}, w \Vdash^{a} \gamma$ iff $p^{P}_{w}({t_1}^{m,\alpha^{\cdot}}_{w,a}, \ldots, {t_k}^{m,\alpha^{\cdot}}_{w,a}) = 1$ iff (by item 1. and the definition of s_m) $[p]_w([\![t_1]\!]^{s_m}_{\alpha}(w, a), \ldots, [\![t_k]\!]^{s_m}_{\alpha}(w, a)) = 1$ iff $[\![p(t_1, \ldots, t_k)]\!](\nu\alpha) = 1$ iff $s_m, \alpha \Vdash \nu{:}\gamma$.

if γ is in $\Xi^{\cdot}_f (= \Xi_f)$ then $m, \alpha^{\cdot}, w \Vdash^{a} \gamma$ iff $\langle w, a\rangle$ is in $\gamma\alpha^{\cdot}_f$ iff $\nu\alpha$ is in $\gamma\alpha_f$ iff $s_m, \alpha \Vdash \nu{:}\gamma$, as we wanted to show.

if γ is $\perp$ then the result follows because $m, \alpha^{\cdot}, w \nVdash^{a} \perp$ and $s_m, \alpha \nVdash \nu{:}\perp$.

if γ is $\neg\gamma_1$ then it is straightforward to see that $m, \alpha^{\cdot}, w \Vdash^{a} \gamma$ iff $s_m, \alpha \Vdash \nu{:}\gamma$.

if γ is $\gamma_1 \wedge \gamma_2$ then it is straightforward to see that $m, \alpha^{\cdot}, w \Vdash^{a} \gamma$ iff $s_m, \alpha \Vdash \nu{:}\gamma$.

if γ is $\forall_x \gamma_1$ then the key point of this proof is to show that the assertion, for any a_1 if a_1 is x co-equivalent to a and $m, \alpha^{\cdot}, w \Vdash^{a_1} \varepsilon(x)$ then $m, \alpha^{\cdot}, w \Vdash^{a_1} \gamma_1$, is equivalent to, for any $\langle w^{\cdot}, a^{\cdot}\rangle$ if $\langle\langle w, a\rangle, \langle w^{\cdot}, a^{\cdot}\rangle\rangle$ is in $[\equiv_x]^{s_m}$ and $[\varepsilon]^{s_m}_{\omega(u)}(\alpha(u)(x)) = 1$ then $\langle w^{\cdot}, a^{\cdot}\rangle$ is in $[\![\gamma_1]\!]^{s_m}_{\alpha}$. So, suppose that for any a_1 if a_1 is x co-equivalent to a and $m, \alpha^{\cdot}, w \Vdash^{a_1} \varepsilon(x)$ then $m, \alpha^{\cdot}, w \Vdash^{a_1} \gamma_1$ and let $\langle w^{\cdot}, a^{\cdot}\rangle$ be such that $\langle\langle w, a\rangle, \langle w^{\cdot}, a^{\cdot}\rangle\rangle$ is in $[\equiv_x]^{s_m}$ and $[\varepsilon]^{s_m}_{w^{\cdot}}(a^{\cdot}(x)) = 1$. Then $w = w^{\cdot}$, a and $a^{\cdot}$ are x co-equivalent, and $m, \alpha^{\cdot}, w^{\cdot} \Vdash^{a^{\cdot}} \varepsilon(x)$. Thus

$m, \alpha^{\cdot}, w^{\cdot} \Vdash^{a^{\cdot}} \gamma_1$. Hence, by induction hypothesis, $s_m, \alpha_1 \Vdash \nu_1{:}\gamma_1$ where $\alpha_{1t} = \alpha^{\cdot}_t$, $\alpha_{1f} = \alpha^{\cdot}_f$ and $\nu_1\alpha_1 = \langle w^{\cdot}, a^{\cdot}\rangle$. So, $\nu_1\alpha_1$ is in $\llbracket\gamma_1\rrbracket^{s_m}_{\alpha_1}$, i.e., by Proposition 2.4.4, $\langle w^{\cdot}, a^{\cdot}\rangle$ is in $\llbracket\gamma_1\rrbracket^{s_m}_{\alpha}$, as we wanted to show. For the other direction the proof is similar so we omit it.

if γ is $\Box\gamma_1$ then the proof is similar to the case where γ is $\forall_x\gamma_1$, so we omit it. *QED*

Lemma 5.1.16 In the context of $\mathcal{L}_{\text{FOMdD}}$ and $\mathcal{L}^{\cdot}_{\text{FOMdD}}$,

$$\Phi \propto \gamma \quad \text{iff} \quad \{\nu{:}\phi \mid \phi \text{ in } \Phi\} \vDash \nu{:}\gamma$$

where ν is a label schema variable, Φ is a set of formulae in $\mathcal{L}^{\cdot}_{\text{FOMdD}}$ and γ is a formula in $\mathcal{L}^{\cdot}_{\text{FOMdD}}$.

Proof Assume $\Phi \propto \gamma$, and let s be a structure in $\mathcal{L}_{\text{FOMdD}}$ and α an assignment over s such that $s, \alpha \Vdash \{\nu{:}\phi \mid \phi \text{ in } \Phi\}$. Then, we have to consider two cases, (i) s is a structure induced by a relational model m for first-order modal logic with decreasing domains, or (ii) s is a structure equivalent in terms of satisfaction of formulae with a structure $s^{\cdot}_m$ induced by a relational model m for first-order modal logic with decreasing domains. If (i) holds then, denoting by $\langle w, a\rangle$ the point $\nu\alpha$, we can use Lemma 5.1.15 to conclude that $m, \alpha, w \Vdash^a \Phi$. Thus, using the fact that $\Phi \propto \gamma$ we have that $m, \alpha, w \Vdash^a \gamma$. Hence, using again Lemma 5.1.15 we can conclude that $s, \alpha \Vdash \nu{:}\gamma$ as we wanted to show. If (ii) holds then there is a assignment $\alpha^{\cdot}$ over $s^{\cdot}_m$ such that $s^{\cdot}_m, \alpha^{\cdot} \Vdash \{\nu{:}\phi \mid \phi \text{ in } \Phi\}$. Therefore, denoting by $\langle w, a\rangle$ the point $\nu\alpha^{\cdot}$, we can use Lemma 5.1.15 to conclude that $m, \alpha^{\cdot}, w \Vdash^a \Phi$. Thus, using the fact that $\Phi \propto \gamma$ we have that $m, \alpha^{\cdot}, w \Vdash^a \gamma$. Hence, using again Lemma 5.1.15 we can conclude that $s^{\cdot}_m, \alpha^{\cdot} \Vdash \nu{:}\gamma$. So, since s is a structure equivalent in terms of satisfaction of formulae with $s^{\cdot}_m$ we have that $s, \alpha \Vdash \nu{:}\gamma$ as we wanted to show.

Assume $\{\nu{:}\phi \mid \phi \text{ in } \Phi\} \vDash \nu{:}\gamma$ and let m be a relational model for the first-order modal logic with decreasing domains over $\Sigma^{\cdot}$, $\alpha^{\cdot}$ a assignment over m, w a world of m and a a quantification assignment with $m, \alpha^{\cdot}, w \Vdash^a \Phi$. Then, by Lemma 5.1.15, we have that $s_m, \alpha \Vdash \{\nu{:}\phi \mid \phi \in \Phi\}$ where α is such that α_t is equal to $\alpha^{\cdot}_t$, α_f is equal to $\alpha^{\cdot}_f$, and $\nu\alpha_l$ is $\langle w, a\rangle$. So, $s_m, \alpha \Vdash \nu{:}\gamma$ since $\{\nu{:}\phi \mid \phi \text{ in } \Phi\} \vDash \nu{:}\gamma$ and s_m is in $\breve{m}$ and m is in M. Therefore using again Lemma 5.1.15, we have that $m, \alpha^{\cdot}, w \Vdash^a \gamma$, as we wanted to show. *QED*

Proof (Labelled presentation Theorem 5.1.13) According to Definition 2.4.10 it is sufficient to show that $\mathcal{L}_{\text{FOMdD}}$ is a labelled presentation for $\mathcal{L}^{\cdot}_{\text{FOMdD}}$ with respect to formulae without schema variables. Recall Definition 2.4.9 where it is said what is a lfob logic system be a labelled presentation for a non-labelled logic system with respect to a set of non-labelled schema formulae. Note that the set of formulae of $\mathcal{L}_{\text{FOMdD}}$ coincide with the set of formulae of $\mathcal{L}^{\cdot}_{\text{FOMdD}}$. So, the lfob logic system $\mathcal{L}_{\text{FOMdD}}$ is a labelled presentation for $\mathcal{L}^{\cdot}_{\text{FOMdD}}$ with respect to formulae without schema variables because:

- $\mathcal{L}_{\text{FOMdD}}$ is sound and complete. See Theorem 5.1.8 for completeness and Theorem 5.1.3 for soundness.

- $X^{\cdot}$, $\Xi^{\cdot}_t$ and $\Xi^{\cdot}_f$ coincide with X, Ξ_t and Ξ_f.

- $F^{\cdot}$ is equal to F, $P^{\cdot}$ is equal to P, and $C^{\cdot}$ is equal to $C \cup O \cup Q$, by definition of $\mathcal{L}_{\text{FOMdD}}$ (see Example 5.1.1), and of $\mathcal{L}^{\cdot}_{\text{FOMdD}}$ (see Example 5.1.14).

- $\Phi \propto \gamma$ iff $\{\nu{:}\phi \mid \phi \in \Phi\} \vDash \nu{:}\gamma$, where ν is a label schema variable, Φ is a set of formulae in $\mathcal{L}^{\cdot}_{\text{FOMdD}}$ and γ is a formula in $\mathcal{L}^{\cdot}_{\text{FOMdD}}$. This result follows by Lemma 5.1.16. *QED*

5.2 $\wedge$ fragment of first-order modal logic T

In this section we present the lfob logic system $\mathcal{L}_{\text{FOT}_\wedge}$ and show that it is a labelled presentation for the $\wedge$ fragment of first-order modal logic T with constant domains and flexible symbols, with respect to formulae without schema variables [48, 39, 51, 79]. See Definitions 2.4.9 and 2.4.10 to recall when a lfob logic system is a labelled presentation for a non-labelled logic with respect to a set of non-labelled schema formulae. Note that, in order to lighten the presentation, we will not mention that the logic is with flexible symbols. We now briefly describe what is a non-labelled fob signature, a relational model and a non-labelled fob logic system for the $\wedge$ fragment of first-order modal logic T with constant domains. Recall the definitions of non-labelled fob signature and non-labelled fob logic system in Definition 2.4.8. We consider that $X^{\cdot}$, $\Xi^{\cdot}_t$ and $\Xi^{\cdot}_f$ are equal to X, Ξ_t and Ξ_f, respectively. In order to recall the importance of $X^{\cdot}$, $\Xi^{\cdot}_t$ and $\Xi^{\cdot}_f$ see Definition 2.4.8.

A *non-labelled fob signature* $\Sigma^{\cdot}$ *for the* $\wedge$ *fragment of first-order modal logic* T *with constant domains* is a non-labelled fob signature such that $P^{\cdot}_0$ is a non-empty set, $C^{\cdot}_1 = \{\forall, \exists, \Box, \Diamond\}$, $C^{\cdot}_2 = \{\wedge\}$, and $C^{\cdot}_k = \emptyset$ for $k \geq 3$ and $k = 0$.

A *relational model* m *over* $\Sigma^{\cdot}$ *for the* $\wedge$ *fragment of first-order modal logic* T *with constant domains* is a tuple $\langle W, R^{\circ}_{\Box}, D, \_^{F}, \_^{P}\rangle$ where D and W are non-empty sets, $R^{\circ}_{\Box}$ is contained in $W \times W$ and is reflexive, $\_^{F}$ is a family $\{\_^{F}{}_{k,w}\}_{k \in \mathbb{N}, w \in W}$ where $\_^{F}{}_{k,w} : F_k \to D^k \to D$ and $\_^{P}$ is a family $\{\_^{P}{}_{k,w}\}_{k \in \mathbb{N}, w \in W}$ where $\_^{P}{}_{k,w} : P_k \to D^k \to 2$. A *schema assignment* α over m is a pair $\langle \alpha_t, \alpha_f\rangle$ where α_t is a map that given an element of $\Xi^{\cdot}_t$ returns a map from $W \times D^X$ to D, and α_f is a map that associates to an element of $\Xi^{\cdot}_f$ a subset of $W \times D^X$. Note that $m, \alpha, w \Vdash^a \gamma$ is inductively defined as expected.

A *non-labelled fob logic system for the* $\wedge$ *fragment of first-order modal logic* T *with constant domains* is a pair $\langle \Sigma^{\cdot}, \propto\rangle$ where $\Sigma^{\cdot}$ is a non-labelled fob signature for this logic and $\propto$ is such that

$$\Phi \propto \gamma \quad \text{iff} \quad \text{if } m, \alpha, w \Vdash^a \Phi \text{ then } m, \alpha, w \Vdash^a \gamma$$

for any relational model m for the $\wedge$ fragment of first-order modal logic T with constant domains over $\Sigma^{\cdot}$, any schema assignment α over m, and world w and assignment a over m.

5.2.1 Logic system

Example 5.2.1 *$\wedge$ fragment of first-order modal logic* T *with constant domains lfob logic system.* Consider the lfob logic system $\langle \Sigma, R, M, \breve{:}\rangle$, in the sequel denoted by $\mathcal{L}_{\text{FOT}_\wedge}$, where Σ is

- $F^l_k = \emptyset$ for $k \geq 0$,
- $S_2 = \{R_{\Box}\} \cup \{\equiv_x \mid x \in X\}$, and $S_k = \emptyset$ for $k \geq 3$, and $k = 1$,
- $\langle F, P\rangle$ is a first-order alphabet,

- $C_2 = \{\wedge\}$, and $C_k = \emptyset$ for $k \geq 3$, $k = 1$, and $k = 0$,
- $Q_1 = \{\forall, \exists\}$, $O_1 = \{\Box, \Diamond\}$, and $Q_k = O_k = \emptyset$ for $k \geq 2$,

and R, besides the rules specified in Definition 2.3.10 common to all lfob deduction systems, contains:

$$\frac{\nu{:}\xi_1 \qquad \nu{:}\xi_2}{\nu{:}\xi_1 \wedge \xi_2}\ \wedge_I \qquad\qquad \frac{\nu{:}\xi_1 \wedge \xi_2}{\nu{:}\xi_1}\ \wedge_E^1 \qquad\qquad \frac{\nu{:}\xi_1 \wedge \xi_2}{\nu{:}\xi_2}\ \wedge_E^2$$

$$\frac{\nu \equiv_x \nu' \,/\, \nu'{:}\xi}{\nu{:}\forall_x \xi}\ \forall_{xI};\ \text{fresh}(\nu', \langle \vartheta_1, \{\nu \equiv_x \nu'\}, \nu'{:}\xi\rangle) \qquad\qquad \frac{\nu{:}\forall_x \xi \quad \nu \equiv_x \nu'}{\nu'{:}\xi}\ \forall_{xE}$$

$$\frac{\nu{:}\forall_x \xi}{\nu{:}\xi'}\ {\forall_x}^{\text{spc}};\ \ \xi' \triangleleft \xi^x_\theta \qquad\qquad \frac{\nu \equiv_x \nu' \quad \nu'{:}\xi}{\nu{:}\exists_x \xi}\ \exists_{xI}$$

$$\frac{\nu{:}\exists_x \xi \quad \nu \equiv_x \nu', \nu'{:}\xi \,/\, \nu''{:}\xi''}{\nu''{:}\xi''}\ \exists_{xE};\ \ \text{fresh}(\nu', \langle \vartheta_2, \{\nu \equiv_x \nu', \nu'{:}\xi\}, \nu''{:}\xi''\rangle)$$

$$\frac{\nu R_\Box \nu' \,/\, \nu'{:}\xi}{\nu{:}\Box\xi}\ \Box_I;\ \text{fresh}(\nu', \langle \vartheta_1, \{\nu R_\Box \nu'\}, \nu'{:}\xi\rangle) \qquad\qquad \frac{\nu{:}\Box\xi \quad \nu R_\Box \nu'}{\nu'{:}\xi}\ \Box_E$$

$$\frac{\nu R_\Box \nu' \quad \nu'{:}\xi}{\nu{:}\Diamond\xi}\ \Diamond_I$$

$$\frac{\nu{:}\Diamond\xi \quad \nu R_\Box \nu', \nu'{:}\xi \,/\, \nu''{:}\xi''}{\nu''{:}\xi''}\ \Diamond_E;\ \ \text{fresh}(\nu', \langle \vartheta_2, \{\nu R_\Box \nu', \nu'{:}\xi\}, \nu''{:}\xi''\rangle)$$

$$\frac{\nu R_\Box \nu'}{\nu \equiv_\alpha \nu'}\ R_{\Box\alpha^{1,2}} \qquad\qquad \frac{\nu R_\Box \nu_1 \quad \nu \equiv_\omega \nu' \quad \nu_1 \equiv_\omega \nu_1' \quad \nu' \equiv_\alpha \nu_1'}{\nu' R_\Box \nu_1'}\ R_{\Box}{}^{\text{gen}}_{\alpha} \qquad\qquad \frac{}{\nu R_\Box \nu}\ R_{\Box r}$$

$$\frac{\nu_1 \equiv_\omega \nu', \nu' \equiv_\alpha \nu_2 \,/\, \nu''{:}\xi''}{\nu''{:}\xi''}\ \mathit{exh};\ \text{fresh}(\nu', \langle \vartheta_1, \{\nu_1 \equiv_\omega \nu', \nu' \equiv_\alpha \nu_2\}, \nu''{:}\xi''\rangle)$$

$$\frac{\nu \equiv_x \nu'}{\nu{:}\theta =_g \nu'{:}\theta}\ \equiv_{xg^{1,2}};\ \text{vardif}(x, \theta) \qquad\qquad \frac{\nu \equiv_x \nu_1 \quad \nu \equiv_\alpha \nu' \quad \nu_1 \equiv_\alpha \nu_1' \quad \nu' \equiv_\omega \nu_1'}{\nu' \equiv_x \nu_1'}\ \equiv_{x\omega}^{\text{gen}}$$

$$\frac{\nu \equiv_x \nu'}{\nu \equiv_\omega \nu'}\ \equiv_{x\omega^{1,2}}$$

$$\frac{\nu \equiv_x \nu' \quad \nu' \equiv_x \nu''}{\nu \equiv_x \nu''}\ \equiv_{xt} \qquad\qquad \frac{}{\nu \equiv_x \nu}\ \equiv_{xr} \qquad\qquad \frac{\nu' \equiv_x \nu}{\nu \equiv_x \nu'}\ \equiv_{xs}$$

$$\frac{\nu \equiv_\omega \nu' \qquad \nu{:}\theta_1 =_g \nu'{:}\theta_1 \quad \dots \quad \nu{:}\theta_k =_g \nu'{:}\theta_k \qquad \nu{:}\xi}{\nu'{:}\xi}\ \mathit{gmon}^k_{\equiv_\omega};\ \text{p-gmon}_{\equiv_\omega}(\xi, \theta_1, \dots, \theta_k)$$

for every x in X, where the provisos $\xi' \lhd \xi^x_\theta$, $\text{vardif}(x, \theta)$, and $\text{p-gmon}_{\equiv_\omega}(\xi, \theta_1, \dots, \theta_k)$ are such that

- $\text{vardif}(x, \theta)_{\Sigma_{\text{FOT}_\wedge}}(\rho_{\text{nl}}) = 1$ iff $\theta\rho_t$ is a variable different of x,
- $\text{p-gmon}_{\equiv_\omega}(\xi, \theta_1, \dots, \theta_k)_{\Sigma_{\text{FOT}_\wedge}}(\rho_{\text{nl}}) = 1$ iff $\xi\rho_f$ is a formula whose free variables are $\theta_1\rho_t, \dots, \theta_k\rho_t$,
- $\xi' \lhd \xi^x_\theta{}_{\Sigma_{\text{FOT}_\wedge}}(\rho_{\text{nl}}) = 1$ iff $\xi'\rho_f$ is obtained by replacing the free occurrences of x in $\xi\rho_f$ by a term $\theta\rho_t$ equal to a variable whenever some free x in $\xi\rho_f$ is in the scope of a modality, such that no variable in $\theta\rho_t$ is captured by a quantifier,

M is composed of all the relational models for this logic over the non-labelled fob signature $\langle F, P, C \cup Q \cup O\rangle$,

and $\breve{\cdot}$ associates to each relational model $\langle W, R^\circ_\Box, D, \_^F, \_^P\rangle$ denoted by m, a class, such that s is in $\breve{m}$ iff either s is the structure induced by m, denoted by s_m, i.e., the structure $\langle U, A, W, \alpha, \omega, D, \mathcal{E}, \mathcal{B}, [.]\rangle$ such that

- $A = D^X, \quad U = W \times A, \quad \alpha(\langle w, a\rangle) = a, \quad \omega(\langle w, a\rangle) = w,$
- $\mathcal{E}$ is D^U and $\mathcal{B}$ is 2^U,

and

- $[x]_a = a(x)$,
- $[f]_w = f^F_{k,w}$ for f in F_k,
- $[p]_w = p^P_{k,w}$ for p in P_k,
- $[\wedge]_{\omega(u)\alpha(u)}(b_1, b_2)(u) = 1$ iff $b_1(u) = 1$ and $b_2(u) = 1$,
- $[\equiv_x] = \{\langle u, u_1\rangle \mid \omega(u) = \omega(u_1),$ and $\alpha(u)$ and $\alpha(u_1)$ are x co-equivalent$\}$,
- $[\forall_x]_{\omega(u)}(b)(u) = 1$ iff for all u_1, if $\langle u, u_1\rangle$ is in $[\equiv_x]$ then u_1 is in b,
- $[\exists_x]_{\omega(u)}(b)(u) = 1$ iff exists u_1 with $\langle u, u_1\rangle$ in $[\equiv_x]$ and $b(u_1) = 1$,
- $[R_\Box] = \{\langle u, u_1\rangle \mid \alpha(u) = \alpha(u_1)$ and $\omega(u) R^\circ_\Box \omega(u_1)\}$,
- $[\Box](b)(u) = 1$ iff for all u_1, if $\langle u, u_1\rangle$ is in $[R_\Box]$ then u_1 is in b,
- $[\Diamond](b)(u) = 1$ iff exists u_1 with $\langle u, u_1\rangle$ in $[R_\Box]$ and u_1 in b,

or s satisfies all the rules, and, for any schema assignment α over s there exists schema assignment $\alpha^\cdot$ over s_m such that, for any formula ϕ in $\mathcal{L}^\cdot_{\text{FOT}_\wedge}$, $s, \alpha \Vdash \nu{:}\phi$ iff $s_m, \alpha^\cdot \Vdash \nu{:}\phi$.

In order for the structures considered in $\mathcal{L}_{\mathrm{FOT}_\wedge}$ induced by a model to be indeed structures it is necessary that the algebraic map $\widehat{\ }$ associated to each structure be well defined, see Definition 2.2.1. It is straightforward to see this is the case since any such structure accepts all possible truth values and all possible denotations of terms. To conclude the subsection we present a proposition showing that the underlying deduction system of $\mathcal{L}_{\mathrm{FOT}_\wedge}$ is uniform with a universal and an existential quantifier; see Definition 5.5.16 to recall when a lfob deduction system is uniform with a universal and an existential quantifier. We omit the proof since it is straightforward.

Proposition 5.2.2 The lfob deduction system in $\mathcal{L}_{\mathrm{FOT}_\wedge}$ is uniform with a universal and an existential quantifier.

5.2.2 Soundness

Theorem 5.2.3 The lfob logic system $\mathcal{L}_{\mathrm{FOT}_\wedge}$ is sound.

Proof The proof follows by Theorem 2.4.6 since by Proposition 5.2.4 all structures of $\mathcal{L}_{\mathrm{FOT}_\wedge}$ satisfy its rules. *QED*

Note that, according to Proposition 2.4.7, in order to show that all structures of $\mathcal{L}_{\mathrm{FOT}_\wedge}$ satisfy its rules, it is sufficient to show that the structures satisfy the specific rules since every structure satisfies the rules common to all lfob deduction systems. In this subsection, we first present the proof that all structures of $\mathcal{L}_{\mathrm{FOT}_\wedge}$ satisfy its rules and only after we present the three lemmas needed along this proof.

Proposition 5.2.4 All structures of $\mathcal{L}_{\mathrm{FOT}_\wedge}$ satisfy its rules.

Proof Let $m = \langle W, R^{\circ}_{\Box}, D, \_^F, \_^P\rangle$ be a model in $\mathcal{L}_{\mathrm{FOT}_\wedge}$ and $s = \langle U, A, W, \alpha, \ \omega, D, \mathcal{E}, \mathcal{B}, [.]\rangle$ a structure in $\breve{m}$. Then, according to the definition of $\breve{m}$, either (i) s is a structure induced by m, or (ii) s is equivalent in terms of satisfaction to a structure induced by the model. Suppose (ii) holds. Then, by definition, s satisfies all rules and we are done. Suppose (i) holds. Then, we have to check that s, in this case denoted by s_m, satisfies all the rules. Note that, by Proposition 2.4.7, s_m satisfies the rules common to all lfob logic systems, hence, it is only needed to check that it satisfies the non-common rules. So, let $r = \langle\{\langle\vartheta_1, \Psi_1, \eta_1\rangle, \dots, \langle\vartheta_k, \Psi_k, \eta_k\rangle\}, \eta, P_f, P_d\rangle$ be a rule in $\mathcal{L}_{\mathrm{FOT}_\wedge}$ that is not a rule common to all lfob logic systems, E a set of label schema variables, σ a schema substitution within L_E such that $(\pi_\Sigma \sigma_{\mathrm{nl}})$ is 1 for each π in P_f, and $\pi_{\Sigma;E}(\sigma)$ is 1 for each π in P_d, and α a schema assignment over s_m and E. Thus,

suppose r is $gmon^k_{\equiv_\omega}$, and that $s_m, \alpha \Vdash \nu\sigma \equiv_\omega \nu'\sigma$, $s_m, \alpha \Vdash \nu{:}\theta_i =_g \nu'{:}\theta_i$ for each $i = 1, \dots, k$, and $s_m, \alpha \Vdash \nu\sigma{:}\xi\sigma$, and that $\xi\sigma$ is a formula whose free variables are precisely $\theta_1\sigma, \dots, \theta_k\sigma$. Note that $\omega([\![\nu\sigma]\!]^{s_m}_\alpha) = \omega([\![\nu'\sigma]\!]^{s_m}_\alpha)$, $\alpha([\![\nu\sigma]\!]^{s_m}_\alpha)(\theta_i\sigma) = \alpha([\![\nu'\sigma]\!]^{s_m}_\alpha)(\theta'_i\sigma)$ for each $i = 1, \dots, k$, and $[\![\xi\sigma]\!]^{s_m}_\alpha([\![\nu\sigma]\!]^{s_m}_\alpha) = 1$. Therefore, by item 2. of Lemma 5.2.6, $[\![\xi\sigma]\!]^{s_m}_\alpha([\![\nu'\sigma]\!]^{s_m}_\alpha) = 1$ as we wanted to show.

suppose r is ${\forall_x}^{\mathrm{spc}}$, and that $s_m, \alpha \Vdash \nu\sigma{:}\forall_x\xi\sigma$ and that $\xi\sigma$ is a schema formula where all schema variables appear under the scope of a $\forall_x$ or a $\exists_x$, and $\xi'\sigma$ is $\xi\sigma$ if there are no free x's in $\xi\sigma$, otherwise $\xi'\sigma$ is obtained by replacing the free occurrences of x in $\xi\sigma$ by a schema term $\theta\sigma$ either equal to a variable whenever some free x in $\xi\sigma$ is in the scope of a modality, or not having schema variables whenever some free x in $\xi\sigma$ is in the scope of a quantifier, such that no variable in $\theta\sigma$ is captured by a quantifier. We want to show that $s_m, \alpha \Vdash \nu\sigma{:}\xi'\sigma$. Consider the point u such that $\omega(u) = \omega(\llbracket \nu\sigma \rrbracket_{\alpha}^{s_m})$, $\alpha(u)$ and $\alpha(\llbracket \nu\sigma \rrbracket_{\alpha}^{s_m})$ are x co-equivalent, and $\alpha(u)(x) = \llbracket \theta\sigma \rrbracket_{\alpha}^{s_m}(\llbracket \nu\sigma \rrbracket_{\alpha}^{s_m})$. Then, by item 2. of Lemma 5.2.5, $\llbracket \xi\sigma \rrbracket_{\alpha}^{s_m}(u) = 1$ iff $\llbracket \xi'\sigma \rrbracket_{\alpha}^{s_m}(\llbracket \nu\sigma \rrbracket_{\alpha}^{s_m}) = 1$. So $\llbracket \xi'\sigma \rrbracket_{\alpha}^{s_m}(\llbracket \nu\sigma \rrbracket_{\alpha}^{s_m}) = 1$ as we wanted to show since $\llbracket \xi\sigma \rrbracket_{\alpha}^{s_m}(u) = 1$ because $s_m, \alpha \Vdash \nu\sigma{:}\forall_x\xi\sigma$.

suppose r is $\forall_{xI}$, and that, for every α' $\{\nu'\sigma\}$ co-equivalent to α, if $s_m, \alpha' \Vdash \nu\sigma \equiv_x \nu'\sigma$ then $s_m, \alpha' \Vdash \nu'\sigma{:}\xi\sigma$. Let u be a point such that $\omega(\llbracket \nu\sigma \rrbracket_{\alpha}^{s_m}) = \omega(u)$ and $\alpha(\llbracket \nu\sigma \rrbracket_{\alpha}^{s_m})$ and $\alpha(u)$ are x co-equivalent. Consider the assignment α' $\{\nu'\sigma\}$ co-equivalent to α with $\nu'\alpha' = u$. Note that $\llbracket \nu\sigma \rrbracket_{\alpha'}^{s_m} = \llbracket \nu\sigma \rrbracket_{\alpha}^{s_m}$ and $\llbracket \xi\sigma \rrbracket_{\alpha'}^{s_m} = \llbracket \xi\sigma \rrbracket_{\alpha}^{s_m}$, by Proposition 2.4.4, since $\nu_1\alpha' = \nu_1\alpha$ for each ν_1 in $\mathrm{lsv}(\nu\sigma)$, since $\nu'\sigma$ is not in $\mathrm{lsv}(\nu\sigma)$, because $\mathrm{fresh}(\nu', \langle \vartheta, \{\nu \equiv_x \nu'\}, \nu'{:}\xi' \rangle)(\sigma) = 1$. Then, by the initial assumption, $\llbracket \xi\sigma \rrbracket_{\alpha'}^{s_m}(\llbracket \nu'\sigma \rrbracket_{\alpha'}^{s_m}) = 1$, i.e., $\llbracket \xi\sigma \rrbracket_{\alpha}^{s_m}(u) = 1$, by Proposition 2.4.4, as we wanted to show.

suppose r is $\Box_I$. The proof that s satisfies $\Box_I$ is similar to the proof for $\forall_{xI}$, so it is omitted.

suppose r is $\Box_E$, and that $s_m, \alpha \Vdash \nu\sigma{:}\Box\xi\sigma$ and $s_m, \alpha \Vdash \nu\sigma R_\Box \nu'\sigma$. Then, for all u, if $\langle \llbracket \nu\sigma \rrbracket_{\alpha}^{s_m}, u \rangle$ is in $[R_\Box]^{s_m}$ then $\llbracket \xi\sigma \rrbracket_{\alpha}^{s_m}(u) = 1$. Note that $\langle \llbracket \nu\sigma \rrbracket_{\alpha}^{s_m}, \llbracket \nu'\sigma \rrbracket_{\alpha}^{s_m} \rangle$ is in $[R_\Box]^{s_m}$. So $\llbracket \xi\sigma \rrbracket_{\alpha}^{s_m}(\llbracket \nu'\sigma \rrbracket_{\alpha}^{s_m}) = 1$. Therefore $s_m, \alpha \Vdash \nu'\sigma{:}\xi\sigma$.

suppose r is $\forall_{xE}$. The proof that s satisfies $\forall_{xE}$ is similar to the one for $\Box_E$, so it is omitted.

suppose r is $\exists_{xI}$, and that $s_m, \alpha \Vdash \nu\sigma \equiv_x \nu'\sigma$ and $s_m, \alpha \Vdash \nu'\sigma{:}\xi\sigma$. Then, $\llbracket \nu'\sigma \rrbracket_{\alpha}^{s_m}$ is such that $\langle \llbracket \nu\sigma \rrbracket_{\alpha}^{s_m}, \llbracket \nu'\sigma \rrbracket_{\alpha}^{s_m} \rangle$ is in $[\equiv_x]^{s_m}$ and $\llbracket \xi\sigma \rrbracket_{\alpha}^{s_m}(\llbracket \nu'\sigma \rrbracket_{\alpha}^{s_m}) = 1$. Thus, $\llbracket \Diamond\xi\sigma \rrbracket_{\alpha}^{s_m}(\llbracket \nu\sigma \rrbracket_{\alpha}^{s_m}) = 1$, as we wanted to show.

suppose r is $\Diamond_I$. The proof that s satisfies $\Diamond_I$ is similar to the proof for $\exists_{xI}$, so it is omitted.

suppose r is $\Diamond_E$, and that $s_m, \alpha \vdash \nu\sigma{:}\Diamond\xi\sigma$ and that for every α' $\{\nu'\sigma\}$ co-equivalent to α, if $s_m, \alpha' \Vdash \nu\sigma R_\Box \nu'\sigma$ and $s_m, \alpha' \Vdash \nu'\sigma{:}\xi\sigma$ then $s_m, \alpha' \Vdash \nu''\sigma{:}\xi''\sigma$. Note that there is a point u with $\langle \llbracket \nu\sigma \rrbracket_{\alpha}^{s_m}, u \rangle$ in $[R_\Box]^{s_m}$ and $\llbracket \xi\sigma \rrbracket_{\alpha}^{s_m}(u) = 1$ since $s_m, \alpha \vdash \nu\sigma{:}\Diamond\xi\sigma$. Consider the assignment $\alpha^{\cdot}$ $\{\nu'\sigma\}$ co-equivalent to α such that $\alpha^{\cdot}(\nu'\sigma) = u$. Then $\llbracket \xi''\sigma \rrbracket_{\alpha^{\cdot}}^{s_m}(\llbracket \nu''\sigma \rrbracket_{\alpha^{\cdot}}^{s_m}) = 1$. So, we can conclude as we want that $\llbracket \xi''\sigma \rrbracket_{\alpha}^{s_m}(\llbracket \nu''\sigma \rrbracket_{\alpha}^{s_m}) = 1$ by Proposition 2.4.4, because $\nu_1\alpha^{\cdot} = \nu_1\alpha$ for each ν_1 in $\mathrm{lsv}(\nu''\sigma)$, since $\nu'\sigma$ is not in $\mathrm{lsv}(\nu''\sigma)$ due to the fact that $\mathrm{fresh}(\nu', \langle \vartheta, \{\nu \equiv_x \nu', \nu'{:}\xi\}, \nu''{:}\xi'' \rangle)_{\Sigma;E}(\sigma) = 1$.

suppose r is $\exists_{xE}$. The proof that s satisfies $\exists_{xE}$ is similar to the one for $\Diamond_E$, so it is omitted.

suppose r is $R_\Box{}_{\alpha}^{\mathrm{gen}}$, and that $s_m, \alpha \Vdash \nu\sigma R_\Box \nu_1\sigma$, $s_m, \alpha \Vdash \nu\sigma \equiv_\omega \nu'\sigma$, $s_m, \alpha \Vdash \nu_1\sigma \equiv_\omega \nu_1'\sigma$, $s_m, \alpha \Vdash \nu'\sigma \equiv_\alpha \nu_1'\sigma$. So $\alpha(\llbracket \nu\sigma \rrbracket_{\alpha}^{s_m}) = \alpha(\llbracket \nu_1\sigma \rrbracket_{\alpha}^{s_m})$, $\omega(\llbracket \nu\sigma \rrbracket_{\alpha}^{s_m}) R_\Box^\circ \omega(\llbracket \nu_1\sigma \rrbracket_{\alpha}^{s_m})$, $\omega(\llbracket \nu\sigma \rrbracket_{\alpha}^{s_m}) = \omega(\llbracket \nu'\sigma \rrbracket_{\alpha}^{s_m})$, $\omega(\llbracket \nu_1\sigma \rrbracket_{\alpha}^{s_m}) = \omega(\llbracket \nu_1'\sigma \rrbracket_{\alpha}^{s_m})$, and $\alpha(\llbracket \nu'\sigma \rrbracket_{\alpha}^{s_m}) = \alpha(\llbracket \nu_1'\sigma \rrbracket_{\alpha}^{s_m})$. Then we have that $\omega(\llbracket \nu'\sigma \rrbracket_{\alpha}^{s_m}) R_\Box^\circ \omega(\llbracket \nu_1'\sigma \rrbracket_{\alpha}^{s_m})$. Thus the pair $\langle \llbracket \nu'\sigma \rrbracket_{\alpha}^{s_m}, \llbracket \nu_1'\sigma \rrbracket_{\alpha}^{s_m} \rangle$ is in $[R_\Box]^{s_m}$. Hence $s_m, \alpha \Vdash \nu'\sigma R_\Box \nu_1'\sigma$, as we wanted to show.

suppose r is $\equiv_{x\omega}^{\mathrm{gen}}$, and suppose that $s_m, \alpha \Vdash \nu\sigma \equiv_x \nu_1\sigma$, $s_m, \alpha \Vdash \nu\sigma \equiv_\alpha \nu'\sigma$, $s_m, \alpha \Vdash$

$\nu_1\sigma \equiv_\alpha \nu_1'\sigma$, $s_m, \alpha \Vdash \nu'\sigma \equiv_\omega \nu_1'\sigma$. Therefore we have $\omega(\llbracket\nu\sigma\rrbracket_\alpha^{s_m}) = \omega(\llbracket\nu_1\sigma\rrbracket_\alpha^{s_m})$, $\alpha(\llbracket\nu\sigma\rrbracket_\alpha^{s_m})$ and $\alpha(\llbracket\nu_1\sigma\rrbracket_\alpha^{s_m})$ are x co-equivalent, $\alpha(\llbracket\nu\sigma\rrbracket_\alpha^{s_m}) = \alpha(\llbracket\nu'\sigma\rrbracket_\alpha^{s_m})$, $\alpha(\llbracket\nu_1\sigma\rrbracket_\alpha^{s_m}) = \alpha(\llbracket\nu_1'\sigma\rrbracket_\alpha^{s_m})$, and $\omega(\llbracket\nu'\sigma\rrbracket_\alpha^{s_m}) = \omega(\llbracket\nu_1'\sigma\rrbracket_\alpha^{s_m})$. So $\alpha(\llbracket\nu'\sigma\rrbracket_\alpha^{s_m})$ and $\alpha(\llbracket\nu_1'\sigma\rrbracket_\alpha^{s_m})$ are also x co-equivalent, and thus $s_m, \alpha \Vdash \nu'\sigma \equiv_x \nu_1'\sigma$, as we wanted to show.

suppose r is $R_{\Box_\alpha^{1,2}}$, and that $s_m, \alpha \Vdash \nu\sigma R_\Box \nu'\sigma$. Then $\langle\llbracket\nu\sigma\rrbracket_\alpha^{s_m}, \llbracket\nu'\sigma\rrbracket_\alpha^{s_m}\rangle$ is in $[R_\Box]^{s_m}$. So, according to the definition of $[R_\Box]^{s_m}$, $\alpha(\llbracket\nu\sigma\rrbracket_\alpha^{s_m}) = \alpha(\llbracket\nu'\sigma\rrbracket_\alpha^{s_m})$. Therefore $s_m, \alpha \Vdash \nu\sigma \equiv_\alpha \nu'\sigma$.

suppose r is $\equiv_{x_\omega^{1,2}}$. The proof is similar to the one for $R_{\Box_\alpha^{1,2}}$, so it is omitted.

suppose r is $\equiv_{x_g^{1,2}}$, and that $s_m, \alpha \Vdash \nu\sigma \equiv_x \nu'\sigma$ and that $\theta\rho_t$ is a variable distinct of x. Then $\langle\llbracket\nu\sigma\rrbracket_\alpha^{s_m}, \llbracket\nu'\sigma\rrbracket_\alpha^{s_m}\rangle$ is in $[\equiv_x]^{s_m}$. So, according to the definition of $[\equiv_x]^{s_m}$, $\alpha(\llbracket\nu\sigma\rrbracket_\alpha^{s_m})$ and $\alpha(\llbracket\nu'\sigma\rrbracket_\alpha^{s_m})$ are x co-equivalent. Thus $\llbracket\theta\rho_t\rrbracket_\alpha^{s_m}(\llbracket\nu\sigma\rrbracket_\alpha^{s_m}) = [\theta\rho_t]_{\alpha(\llbracket\nu\sigma\rrbracket_\alpha^{s_m})} = \alpha(\llbracket\nu\sigma\rrbracket_\alpha^{s_m})(\theta\rho_t) = \alpha(\llbracket\nu'\sigma\rrbracket_\alpha^{s_m})(\theta\rho_t) = [\theta\rho_t]_{\alpha(\llbracket\nu'\sigma\rrbracket_\alpha^{s_m})} = \llbracket\theta\sigma\rrbracket_\alpha^{s_m}(\llbracket\nu'\sigma\rrbracket_\alpha^{s_m})$. Therefore $s_m, \alpha \Vdash \nu\sigma{:}\theta\sigma =_g \nu'\sigma{:}\theta\sigma$.

suppose r is $R_{\Box_r}$. Note that $\langle u, u\rangle \in [R_\Box]^{s_m}$ for any point u. So, $s_m, \alpha \Vdash \nu\sigma R_\Box \nu\sigma$.

suppose r is *exh* and that, for any α' $\{\nu'\sigma\}$ co-equivalent to α if $s_m, \alpha' \Vdash \nu_1\sigma \equiv_\omega \nu'\sigma$ and $s_m, \alpha' \Vdash \nu'\sigma \equiv_\alpha \nu_2\sigma$ then $s_m, \alpha' \Vdash \nu''\sigma{:}\xi''\sigma$. Consider the point $\langle w, a\rangle$ in s_m with $w = \omega(\llbracket\nu_1\sigma\rrbracket_\alpha^{s_m})$ and $a = \alpha(\llbracket\nu_2\sigma\rrbracket_\alpha^{s_m})$. Consider also the schema assignment $\alpha^{\cdot}$ $\{\nu'\sigma\}$ co-equivalent to α with $\nu'\sigma\alpha^{\cdot} = \langle w, a\rangle$. Note that $\nu^{\cdot}\alpha^{\cdot} = \nu^{\cdot}\alpha$ for any $\nu^{\cdot}$ in $\mathrm{lsv}(\nu_1\sigma) \cup \mathrm{lsv}(\nu_2\sigma)$, since $\nu'\sigma$ is not in $\mathrm{lsv}(\nu_1\sigma) \cup \mathrm{lsv}(\nu_2\sigma)$ because $\mathrm{fresh}(\nu', \langle\vartheta_1, \{\nu_1 \equiv_\omega \nu', \nu' \equiv_\alpha \nu_2\}, \nu''{:}\xi''\rangle)(\sigma) = 1$. So, by Proposition 2.4.4, we have that $\llbracket\nu_1\sigma\rrbracket_\alpha^{s_m} = \llbracket\nu_1\sigma\rrbracket_{\alpha^{\cdot}}^{s_m}$ and $\llbracket\nu_2\sigma\rrbracket_\alpha^{s_m} = \llbracket\nu_2\sigma\rrbracket_{\alpha^{\cdot}}^{s_m}$. Therefore $\omega(\llbracket\nu'\sigma\rrbracket_{\alpha^{\cdot}}^{s_m}) = \omega(\llbracket\nu_1\sigma\rrbracket_{\alpha^{\cdot}}^{s_m})$ and $\alpha(\llbracket\nu'\sigma\rrbracket_{\alpha^{\cdot}}^{s_m}) = \alpha(\llbracket\nu_2\sigma\rrbracket_{\alpha^{\cdot}}^{s_m})$. Then $s_m, \alpha^{\cdot} \Vdash \nu_1\sigma \equiv_\omega \nu'\sigma$ and $s_m, \alpha^{\cdot} \Vdash \nu'\sigma \equiv_\alpha \nu_2\sigma$. So, by the initial assumption, $s_m, \alpha^{\cdot} \Vdash \nu''\sigma{:}\xi''\sigma$. Then, again by Proposition 2.4.4, we have, as we wanted to show $s_m, \alpha \Vdash \nu''\sigma{:}\xi''\sigma$ since $\nu^{\cdot}\alpha^{\cdot} = \nu^{\cdot}\alpha$ for any $\nu^{\cdot}$ in $\mathrm{lsv}(\nu''\sigma)$.

suppose r is $\equiv_{xt}$, $\equiv_{xr}$, $\equiv_{xs}$, $\wedge_I$ or $\wedge_E$. Then the proof follows straightforwardly. *QED*

Now we show the three lemmas needed along the proof of Proposition 5.2.4.

Lemma 5.2.5 For any model m in $\mathcal{L}_{\mathrm{FOT}_\wedge}$, structure s_m induced by m, and variable x,

1. $\llbracket t'\rrbracket_\alpha^{s_m}(u) = \llbracket t'[t/x]\rrbracket_\alpha^{s_m}(u')$, whenever t' is a term without schema variables, and $\alpha(u)(x) = \llbracket t\rrbracket_\alpha^{s_m}(u')$,
2. $\llbracket\varphi\rrbracket_\alpha^{s_m}(u) = 1$ iff $\llbracket\varphi'\rrbracket_\alpha^{s_m}(u') = 1$, whenever φ is a schema formula where all schema variables are under the scope of a $\forall_x$ or a $\exists_x$, and φ' is φ if there are no free x's in φ, otherwise φ' is obtained by replacing the free occurrences of x in φ by a schema term t either equal to a variable whenever some free x in φ is in the scope of a modality, or not having schema variables whenever some free x in φ is in the scope of a quantifier, such that no variable in t is captured by a quantifier, and $\alpha(u)(x) = \llbracket t\rrbracket_\alpha^{s_m}(u')$,

for any points u and u' with $\omega(u) = \omega(u')$, and $\alpha(u)$ and $\alpha(u')$ x co-equivalent.

Proof 1. The proof is omitted since it follows straightforwardly by induction on the structure of t'.

2. The proof follows by induction on the structure of φ:

φ is $p(t_1, \ldots, t_k)$. Note that the terms $t_1, \ldots, t_k$ do not have schema variables. So $[\![p(t_1, \ldots, t_k)]\!]^{s_m}_{\alpha}(u) = 1$ iff $[p]_{\omega(u)}([\![t_1]\!]^{s_m}_{\alpha}(u), \ldots, [\![t_k]\!]^{s_m}_{\alpha}(u)) = 1$ equivalent, by fact 1., to $[p]_{\omega(u)}([\![t_1[t/x]]\!]^{s_m}_{\alpha}(u'), \ldots, [\![t_k[t/x]]\!]^{s_m}_{\alpha}(u')) = 1$, so $[\![p(t_1[t/x], \ldots, t_k[t/x])]\!]^{s_m}_{\alpha}(u') = 1$.

φ is $t_1 = t_2$. Note that t_1 and t_2 do not have schema variables. Then $[\![\varphi]\!]^{s_m}_{\alpha}(u) = 1$ iff $[\![t_1]\!]^{s_m}_{\alpha}(u) = [\![t_2]\!]^{s_m}_{\alpha}(u)$ iff (by item 1.) $[\![t_1[t/x]]\!]^{s_m}_{\alpha}(u') = [\![t_2[t/x]]\!]^{s_m}_{\alpha}(u')$ iff $[\![\varphi[t/x]]\!]^{s_m}_{\alpha}(u') = 1$ as we wanted to show.

φ is $\varphi_1 \wedge \varphi_2$. Note that φ' is $\varphi'_1 \wedge \varphi'_2$ where φ_1 and φ'_1, and φ_2 and φ'_2 satisfy the sufficient conditions to apply the induction hypothesis. Then $[\![\varphi]\!]^{s_m}_{\alpha}(u) = 1$ iff $[\![\varphi_1]\!]^{s_m}_{\alpha}(u) = 1$ and $[\![\varphi_2]\!]^{s_m}_{\alpha}(u) = 1$ iff (by induction hypothesis) $[\![\varphi'_1]\!]^{s_m}_{\alpha}(u') = 1$ and $[\![\varphi'_2]\!]^{s_m}_{\alpha}(u') = 1$ iff $[\![\varphi']\!]^{s_m}_{\alpha}(u') = 1$ as we wanted to show.

φ is $\forall_x \varphi_1$. Then φ is a schema formula equal to φ' because φ do not have free x's. So we want to show that $[\![\forall_x \varphi_1]\!]^{s_m}_{\alpha}(u) = 1$ iff $[\![\forall_x \varphi_1]\!]^{s_m}_{\alpha}(u') = 1$. But this follows by item 2. of Lemma 5.2.7.

φ is $\exists_x \varphi_1$. The proof is similar to that for $\forall_x \varphi_1$.

φ is $\forall_y \varphi_1$, and y is distinct of x. Then, φ' is equal to $\forall_y \varphi'_1$, φ_1 is a schema formula where all schema variables appear under the scope of a $\forall_x$ or a $\exists_x$, and φ'_1 is φ_1 whenever there are no free x's in φ_1, otherwise φ'_1 is obtained by replacing the free occurrences of x in φ_1 by the term t which is equal to a variable whenever some free x in φ_1 is in the scope of a modality, such that no variable in t is captured by a quantifier. Consider two cases:

- φ_1 has no free occurrences of x. So we want to show $[\![\forall_y \varphi_1]\!]^{s_m}_{\alpha}(u) = 1$ iff $[\![\forall_y \varphi_1]\!]^{s_m}_{\alpha}(u') = 1$. But this result follows by item 2. of Lemma 5.2.7.

- φ_1 has free occurrences of x. Then t has no occurrences of y and has no schema variables. Assume we have $[\![\forall_y \varphi_1]\!]^{s_m}_{\alpha}(u) = 1$. Let u'_1 be a point such that $\omega(u'_1) = \omega(u')$ and $\alpha(u'_1)$ and $\alpha(u')$ are y co-equivalent. Then $[\![t]\!]^{s_m}_{\alpha}(u'_1) = [\![t]\!]^{s_m}_{\alpha}(u')$ by item 1. of Lemma 5.2.6. Consider the point u_1 with $\omega(u_1) = \omega(u'_1)$, $\alpha(u_1)$ and $\alpha(u'_1)$ x co-equivalent, and $\alpha(u_1)(x) = \alpha(u)(x)$. Then $\alpha(u_1)(x) = [\![t]\!]^{s_m}_{\alpha}(u'_1)$. Note that $[\![\varphi_1]\!]^{s_m}_{\alpha}(u_1) = 1$ since $\omega(u_1) = \omega(u)$ and $\alpha(u_1)$ and $\alpha(u)$ are y co-equivalent. So, by induction hypothesis, $[\![\varphi'_1]\!]^{s_m}_{\alpha}(u'_1) = 1$, as we wanted to show. The proof of the other direction is similar so it is omitted.

φ is $\exists_y \varphi_1$. The proof is similar to that for $\forall_y \varphi_1$.

φ is $\Diamond \varphi_1$ and φ' is equal to $\Diamond \varphi'_1$. Then φ_1 is a schema formula where all schema variables appear under the scope of a $\forall_x$ or a $\exists_x$, and φ'_1 is φ_1 whenever there are no free x's in φ_1, otherwise φ'_1 is obtained by replacing the free occurrences of x in φ_1 by a variable t such that t is not captured by a quantifier and $\alpha(u)(x) = \alpha(u')(t)$. Consider two cases:

- φ_1 has no free occurrences of x. Hence we want to show that $[\![\Diamond \varphi_1]\!]^{s_m}_{\alpha}(u) = 1$ iff $[\![\Diamond \varphi_1]\!]^{s_m}_{\alpha}(u') = 1$. But this result follows by item 2. of Lemma 5.2.7.

- φ_1 has free occurrences of x. Then t is a variable and $\alpha(u)(x) = \alpha(u')(t)$. Assume we have $[\![\Diamond \varphi'_1]\!]^{s_m}_{\alpha}(u') = 1$. Then there is a point u'_1 with $\alpha(u'_1) = \alpha(u')$, $\omega(u')R^{\circ}_{\Box}\omega(u'_1)$ and $[\![\varphi'_1]\!]^{s_m}_{\alpha}(u'_1)$. Consider the point u_1 with $\alpha(u_1) = \alpha(u)$ and $\omega(u_1) = \omega(u'_1)$. Then $\omega(u)R^{\circ}_{\Box}\omega(u_1)$ and $\alpha(u_1)(x) = \alpha(u'_1)(t)$. So we can apply the induction hypothesis and conclude $[\![\varphi_1]\!]_\alpha^{s_m}(u_1)$ as we want. The proof of the other direction is similar.

φ is $\Box\varphi_1$. The proof is similar to that for $\Diamond\varphi_1$ so it is omitted. *QED*

Lemma 5.2.6 For any model m in $\mathcal{L}_{\mathrm{FOT}_\wedge}$ and structure s_m induced by m

1. $[\![t]\!]_\alpha^{s_m}(u) = [\![t]\!]_\alpha^{s_m}(u')$, whenever t is a term and $\alpha(u)(x) = \alpha(u')(x)$ for any variable x in t,
2. $[\![\varphi]\!]_\alpha^{s_m}(u) = 1$ iff $[\![\varphi]\!]_\alpha^{s_m}(u') = 1$, whenever φ is a formula where $\alpha(u)(x) = \alpha(u')(x)$ for any free variable x in φ,

for any points u and u' with $\omega(u) = \omega(u')$.

Proof 1. The proof is omitted since it follows straightforwardly by induction on the structure of t.

2. We show this by induction on the structure of φ:

Let φ *be* $p(t_1,\dots,t_k)$. Then $[\![p(t_1,\dots,t_k)]\!]_\alpha^{s_m}(u) = 1$ iff $[p]_{\omega(u)}([\![t_1]\!]_\alpha^{s_m}(u),\dots,[\![t_k]\!]_\alpha^{s_m}(u)) = 1$ iff $[p]_{\omega(u')}([\![t_1]\!]_\alpha^{s_m}(u'),\dots,[\![t_k]\!]_\alpha^{s_m}(u')) = 1$ iff $[\![p(t_1,\dots,t_k)]\!]_\alpha^{s_m}(u') = 1$.

Letφ *be* $t_1 = t_2$. Then $[\![\varphi]\!]_\alpha^{s_m}(u) = 1$ iff $[\![t_1]\!]_\alpha^{s_m}(u) = [\![t_2]\!]_\alpha^{s_m}(u)$ iff (by item 1.) $[\![t_1]\!]_\alpha^{s_m}(u') = [\![t_2]\!]_\alpha^{s_m}(u')$ iff $[\![\varphi]\!]_\alpha^{s_m}(u') = 1$ as we wanted to show.

Let φ *be* $\forall_x\varphi_1$, where x may be free in φ_1. Assume $[\![\forall_x\varphi_1]\!]_\alpha^{s_m}(u) = 1$. Let u'' be a point such that $\omega(u'') = \omega(u')$, and $\alpha(u'')$ and $\alpha(u')$ are x co-equivalent. Consider a point $u''^{\cdot}$ such that $\omega(u''^{\cdot}) = \omega(u'')$, $\alpha(u''^{\cdot})$ and $\alpha(u)$ are x co-equivalent, and $\alpha(u''^{\cdot})(x) = \alpha(u'')(x)$. Note that $\omega(u'') = \omega(u)$ and so $[\![\varphi_1]\!]_\alpha^{s_m}(u''^{\cdot}) = 1$. We now show that $\alpha(u''^{\cdot})(y) = \alpha(u'')(y)$ for any variable y free in φ_1. Let y be a variable free in φ_1. Consider two cases (i) y is distinct of x. Then y is free in $\forall_x\varphi_1$. So $\alpha(u''^{\cdot})(y) = \alpha(u)(y) = \alpha(u')(y) = \alpha(u'')(y)$, as we want, and (ii) y is x. Then $\alpha(u''^{\cdot})(x) = \alpha(u'')(x)$ by definition of $u''^{\cdot}$. So, by induction hypothesis $[\![\varphi_1]\!]_\alpha^{s_m}(u'') = 1$. The proof of the other direction follows in a similar way.

Let φ *be* $\exists_x\varphi_1$. Then the proof is similar to that for $\forall_x\varphi_1$.

Let φ *be* $\Diamond\varphi_1$. Assume $[\![\Diamond\varphi_1]\!]_\alpha^{s_m}(u) = 1$. Then exists u_1 with $\alpha(u) = \alpha(u_1)$, $\omega(u)R_\Box^\circ\omega(u_1)$, and $[\![\varphi_1]\!]_\alpha^{s_m}(u_1) = 1$. Consider the point u_1' with $\alpha(u_1') = \alpha(u')$ and $\omega(u_1') = \omega(u_1)$. Then $\omega(u')R_\Box^\circ\omega(u_1')$. It is straightforward to see that $\alpha(u_1')(x) = \alpha(u_1)(x)$ for any variable x free in φ_1. Then by induction hypothesis $[\![\varphi_1]\!]_\alpha^{s_m}(u_1') = 1$. The proof of the other direction follows in a similar way.

Let φ *be* $\Box\varphi_1$. Then the proof is similar to that for $\Diamond\varphi_1$.

Let φ *be* $\varphi_1 \wedge \varphi_2$. Then the proof is straightforward so it is omitted. *QED*

Lemma 5.2.7 For any model m in $\mathcal{L}_{\mathrm{FOT}_\wedge}$ and structure s_m induced by m

1. $[\![t]\!]_\alpha^{s_m}(u) = [\![t]\!]_\alpha^{s_m}(u')$, whenever t is a term without schema variables where x does not appear,
2. $[\![\varphi]\!]_\alpha^{s_m}(u) = 1$ iff $[\![\varphi]\!]_\alpha^{s_m}(u') = 1$, whenever φ is a schema formula where x does not appear free and all schema variables appear under the scope of a $\forall_x$ or a $\exists_x$,

for any points u and u' with $\omega(u) = \omega(u')$, and $\alpha(u)$ and $\alpha(u')$ x co-equivalent.

Proof 1. The proof is omitted since it follows straightforwardly by induction on the structure of t.

2. The proof follows by induction on the structure of φ.

Assume φ *is* $p(t_1, \ldots, t_k)$ and $t_1, \ldots, t_k$ do not contain any x and any term schema variable. Then $\llbracket p(t_1, \ldots, t_k)\rrbracket_\alpha^{s_m}(u) = 1$ iff $[p]_{\omega(u)}(\llbracket t_1\rrbracket_\alpha^{s_m}(u), \ldots, \llbracket t_k\rrbracket_\alpha^{s_m}(u)) = 1$, which is, by fact 1., equivalent to $[p]_{\omega(u')}(\llbracket t_1\rrbracket_\alpha^{s_m}(u'), \ldots, \llbracket t_k\rrbracket_\alpha^{s_m}(u')) = 1$ iff $\llbracket p(t_1, \ldots, t_k)\rrbracket_\alpha^{s_m}(u') = 1$, as desired.

Assume φ *is* $t_1 = t_2$ and t_1 and t_2 does not contain any x and any term schema variable. Then $\llbracket \varphi\rrbracket_\alpha^{s_m}(u) = 1$ iff $\llbracket t_1\rrbracket_\alpha^{s_m}(u) = \llbracket t_2\rrbracket_\alpha^{s_m}(u)$ iff (by item 1.) $\llbracket t_1\rrbracket_\alpha^{s_m}(u') = \llbracket t_2\rrbracket_\alpha^{s_m}(u')$ iff $\llbracket \varphi\rrbracket_\alpha^{s_m}(u') = 1$ as we wanted to show.

Assume φ *is* $\forall_x \varphi_1$ and $\llbracket \forall_x \varphi_1\rrbracket_\alpha^{s}(u) = 1$. Let u'_1 be a point such that $\omega(u'_1) = \omega(u')$ and $\alpha(u'_1)$ and $\alpha(u')$ are x co-equivalent. Then $\omega(u'_1) = \omega(u)$ and $\alpha(u'_1)$ and $\alpha(u)$ are x co-equivalent, and so, by the assumption, $\llbracket \forall_x \varphi_1\rrbracket_\alpha^{s}(u'_1) = 1$, as we wanted to show. The proof of the other direction is similar.

Assume φ *is* $\exists_x \varphi_1$. Then the proof is similar to the case where φ is $\forall_x \varphi_1$.

Assume φ *is* $\forall_y \varphi_1$, where y is distinct of x. Assume $\llbracket \forall_y \varphi_1\rrbracket_\alpha^{s_m}(u) = 1$. Let u'_1 be a point such that $\omega(u'_1) = \omega(u')$ and $\alpha(u'_1)$ and $\alpha(u')$ are y co-equivalent. Consider a point u_1 such that $\omega(u'_1) = \omega(u_1)$, $\alpha(u'_1)$ and $\alpha(u_1)$ are x co-equivalent and $\alpha(u_1)(x) = \alpha(u)(x)$. So $\alpha(u_1)$ and $\alpha(u)$ are y co-equivalent. Note also that $\omega(u_1) = \omega(u)$. So $\llbracket \varphi_1\rrbracket_\alpha^{s_m}(u_1) = 1$. Hence, by induction hypothesis, $\llbracket \varphi_1\rrbracket_\alpha^{s_m}(u'_1) = 1$ as we wanted to show, because φ_1 is also a schema formula where x does not appear free and all schema variables appear under the scope of a $\forall_x$ or a $\exists_x$. The proof of the other direction follows in a similar way;

φ *is* $\exists_y \varphi_1$. Then the proof is similar to the case where φ is $\forall_y \varphi_1$.

Assume φ *is* $\Box \varphi_1$ and $\llbracket \Box \varphi_1\rrbracket_\alpha^{s_m}(u) = 1$. Let u'_1 be a point with $\alpha(u') = \alpha(u'_1)$ and $\omega(u') R_\Box^\circ \omega(u'_1)$. Consider a point u_1 with $\omega(u_1) = \omega(u'_1)$ and $\alpha(u_1) = \alpha(u)$. Then $\omega(u) R_\Box^\circ \omega(u_1)$. So, $\llbracket \varphi_1\rrbracket_\alpha^{s_m}(u_1) = 1$. Hence, by induction hypothesis, $\llbracket \varphi_1\rrbracket_\alpha^{s_m}(u'_1) = 1$, since $\omega(u_1) = \omega(u'_1)$ and $\alpha(u_1)$ and $\alpha(u'_1)$ are x co-equivalent, and φ_1 is also a schema formula where x does not appear free and all schema variables appear under the scope of a $\forall_x$ or a $\exists_x$. The proof of the other direction follows in a similar way;

Assume φ *is* $\Diamond \varphi_1$. Then the proof is similar to the case where φ is $\Box \varphi_1$;

Assume φ *is* $\varphi_1 \wedge \varphi_2$. So $\llbracket \varphi_1 \wedge \varphi_2\rrbracket_\alpha^{s_m}(u) = 1$ iff $[\wedge]_{\omega(u)\alpha(u)}(\llbracket \varphi_1\rrbracket_\alpha^{s_m} \cap U_{\omega(u)\alpha(u)}, \llbracket \varphi_2\rrbracket_\alpha^{s_m} \cap U_{\omega(u)\alpha(u)})(u) = 1$, iff $\llbracket \varphi_1\rrbracket_\alpha^{s_m}(u) = 1$ and $\llbracket \varphi_2\rrbracket_\alpha^{s_m}(u) = 1$, iff (using the induction hypothesis), $\llbracket \varphi_1\rrbracket_\alpha^{s_m}(u') = 1$ and $\llbracket \varphi_2\rrbracket_\alpha^{s_m}(u') = 1$, iff $[\wedge]_{\omega(u')\alpha(u')}(\llbracket \varphi_1\rrbracket_\alpha^{s_m} \cap U_{\omega(u')\alpha(u')}, \llbracket \varphi_2\rrbracket_\alpha^{s_m} \cap U_{\omega(u')\alpha(u')})(u') = 1$ iff $\llbracket \varphi_1 \wedge \varphi_2\rrbracket_\alpha^{s_m}(u') = 1$. *QED*

5.2.3 Completeness

The goal of this subsection is to show the following theorem, which we prove at the end of the section.

Theorem 5.2.8 The lfob logic system $\mathcal{L}_{\mathrm{FOT}_\wedge}$ is complete.

To prove this theorem we show that $\mathcal{L}_{\mathrm{FOT}_\wedge}$ is rich and then we invoke Theorem 3.3.3 which says that a rich lfob logic system is complete. But before we need to show some auxiliary propositions. So, we start to prove that $\mathcal{L}_{\mathrm{FOT}_\wedge}$ is a connected lfob logic system without a disjunction and with locality.

Proposition 5.2.9 The lfob logic system $\mathcal{L}_{\mathrm{FOT}_\wedge}$ is connected with locality and without a disjunction and implication.

Proof It is straightforward to see that $\mathcal{L}_{\mathrm{FOT}_\wedge}$ is connected and do not have a disjunction neither a implication. Recall Definition 4.1.3 of a connected lfob logic system, and Definition 4.1.1 of disjunction and implication connectives. Note that all universal connectives in $\mathcal{L}_{\mathrm{FOT}_\wedge}$ are unary and are associated to reflexive relations. So, it is straightforward to see that $\mathcal{L}_{\mathrm{FOT}_\wedge}$ is with locality. Recall the notion of a lfob deduction system with locality in Definition 4.4.2. *QED*

We need now to introduce the notion of a relational model for the $\wedge$ fragment of first-order modal logic T with constant domains, induced by an appropriate set, in the context of $\mathcal{L}_{\mathrm{FOT}_\wedge}$.

Definition 5.2.10 The model m_Ψ *induced by* an E appropriate set Ψ in the context of $\mathcal{L}_{\mathrm{FOT}_\wedge}$ is $\langle W, R^\circ_\Box, D, \_^F, \_^P\rangle$, defined as follows:

- W is $\{[v]^{\Psi,E}_\omega \mid v \text{ in } T_{\mathrm{lab},E}\}$,
- $R^\circ_\Box$ is $\{\langle [v_1]^{\Psi,E}_\omega, [v_2]^{\Psi,E}_\omega\rangle \mid R_\Box v_1 v_2 \text{ is in } \Psi\}$,
- D is $\{[v{:}t]^{\Psi,E}_g \mid v \text{ in } T_{\mathrm{lab},E} \text{ and } t \text{ in } T\}$,
- $\_^F$ is a family of function denotations induced by Ψ (see Definition 5.5.3),
- $\_^P$ is a family of predicate denotations induced by Ψ (see Definition 5.5.3).

In the next proposition we show that the structure induced by the canonical model induced by an appropriate set is equivalent in terms of satisfaction of formulae in $\mathcal{L}^{\cdot}_{\mathrm{FOT}_\wedge}$ to the canonical structure over that set.

Lemma 5.2.11 Given an E strong appropriate set Ψ, a formula ϕ in $\mathcal{L}^{\cdot}_{\mathrm{FOT}_\wedge}$, and a label schema variable ν,

$$s_\Psi, \alpha \Vdash \nu{:}\phi \quad \text{iff} \quad s_{m_\Psi}, \alpha^{\cdot} \Vdash \nu{:}\phi$$

for any schema assignments α over s_Ψ and $\alpha^{\cdot}$ over s_{m_Ψ} with $\nu\alpha^{\cdot} = [\nu\alpha]^{\Psi,E}_{\omega\times\iota}$.

Proof Recall first the definition of ι^Ψ_{DX} and $[v]^{\Psi,E}_{\omega\times\iota}$ in Remark 5.5.1, and, the definition of a Ψ strongly uniform map in Definition 5.5.10, and let f be a Ψ strongly uniform map. Note that, in the context of $\mathcal{L}_{\mathrm{FOT}_\wedge}$, the class of Ψ strong uniform maps is not empty. For example, the map f defined as $f(|\phi|^{s_\Psi}) = [|\phi|^{s_\Psi}]^{\Psi,E}_{\omega\times\iota} \cup \{\langle w, a\rangle \mid w \text{ is } [v]^{\Psi,E}_\omega \text{ for some } v \text{ and } x \text{ with } [v]^{\Psi,E}_x \subseteq |\phi|^{s_\Psi} \text{ and } a\ x \text{ co-equivalent to } \iota^\Psi_{DX}(v)\}$ for any formula ϕ, and $f(|\phi|^{s_\Psi}) = [|\phi|^{s_\Psi}]^{\Psi,E}_{\omega\times\iota}$ for any ϕ having schema variables, is a Ψ strongly uniform map. Recall, by Proposition 5.2.2, that we are in the context of a uniform lfob deduction system with a universal and an existential quantifier. So, we now show some auxiliary results and after that we prove the lemma.

1. $\llbracket t\rrbracket_{\alpha}^{s_\Psi}(v) = \llbracket t\rrbracket_{\alpha^{\cdot}}^{s_{m_\Psi}}([v]_{\omega\times\iota}^{\Psi,E})$, where t is a term in $\mathcal{L}^{\cdot}_{\mathrm{FOT}_\wedge}$, and v is a label schema term. The proof follows by induction on the possible cases for the term t in $\mathcal{L}^{\cdot}_{\mathrm{FOT}_\wedge}$.

Suppose t is in X. Then $\llbracket t\rrbracket_{\alpha}^{s_\Psi}(v) = [t]^{s_\Psi}_{\alpha_{s_\Psi}(v)}$ = (see the definition of canonical structure in Section 3.1) $= [v{:}t]_g^{\Psi,E}$ = (see the definition of $\iota^{\Psi}_{D^X}$, Remark 5.5.1) $= \iota^{\Psi}_{D^X}(v)(t) = [t]^{s_{m_\Psi}}_{\iota^{\Psi}_{D^X}(v)} = [t]^{s_{m_\Psi}}_{\alpha_{s_{m_\Psi}}([v]_{\omega\times\iota}^{\Psi,E})} = \llbracket t\rrbracket_{\alpha^{\cdot}}^{s_{m_\Psi}}([v]_{\omega\times\iota}^{\Psi,E})$.

Suppose t is in F_0. Then, $\llbracket t\rrbracket_{\alpha}^{s_\Psi}(v) = [t]^{s_\Psi}_{\omega(v)} = |t|(v) = [v{:}t]_g^{\Psi,E} = {t^F}_{[v]_{\omega}^{\Psi,E}} = [t]^{s_{m_\Psi}}_{\omega([v]_{\omega\times\iota}^{\Psi,E})} = \llbracket t\rrbracket_{\alpha^{\cdot}}^{s_{m_\Psi}}([v]_{\omega\times\iota}^{\Psi,E})$.

Suppose t is in $f(t_1,\ldots,t_k)$. Therefore, $\llbracket t\rrbracket_{\alpha}^{s_\Psi}(v) = [f]^{s_\Psi}_{\omega(v)}(\llbracket t_1\rrbracket_{\alpha}^{s_\Psi}(v),\ldots,\llbracket t_k\rrbracket_{\alpha}^{s_\Psi}(v))$ = (by Proposition 3.1.3) $= [f]^{s_\Psi}_{\omega(v)}(|t_1|(v),\ldots,|t_k|(v)) = |f(t_1,\ldots,t_k)|(v) = [v{:}f(t_1,\ldots,t_k)]_g^{\Psi,E} = {f^F}_{[v]_{\omega}^{\Psi,E}}([v{:}t_1]_g^{\Psi,E},\ldots,[v{:}t_k]_g^{\Psi,E}) = [f]^{s_{m_\Psi}}_{\omega([v]_{\omega\times\iota}^{\Psi,E})}(\llbracket t_1\rrbracket_{\alpha}^{s_\Psi}(v),\ldots,\llbracket t_k\rrbracket_{\alpha}^{s_\Psi}(v))$ = (by induction hypothesis) $= [f]^{s_{m_\Psi}}_{\omega([v]_{\omega\times\iota}^{\Psi,E})}(\llbracket t_1\rrbracket_{\alpha^{\cdot}}^{s_{m_\Psi}}([v]_{\omega\times\iota}^{\Psi,E}),\ldots,\llbracket t_k\rrbracket_{\alpha^{\cdot}}^{s_{m_\Psi}}([v]_{\omega\times\iota}^{\Psi,E})) = \llbracket t\rrbracket_{\alpha^{\cdot}}^{s_{m_\Psi}}([v]_{\omega\times\iota}^{\Psi,E})$.

2. $[v]_{\omega\times\iota}^{\Psi,E}$ is in $f(\llbracket\phi\rrbracket_{\alpha}^{s_\Psi})$ iff $[v]_{\omega\times\iota}^{\Psi,E}$ is in $\llbracket\phi\rrbracket_{\alpha^{\cdot}}^{s_{m_\Psi}}$. The proof follows by induction on the structure of a formula ϕ in $\mathcal{L}^{\cdot}_{\mathrm{FOT}_\wedge}$.

Suppose ϕ is $t_1 = t_2$. Then $[v]_{\omega\times\iota}^{\Psi,E}$ is in $f(\llbracket\phi\rrbracket_{\alpha}^{s_\Psi})$ iff (by Proposition 5.5.11) v is in $\llbracket\phi\rrbracket_{\alpha}^{s_\Psi}$ iff $\llbracket t_1\rrbracket_{\alpha}^{s_\Psi}(v) = \llbracket t_2\rrbracket_{\alpha}^{s_\Psi}(v)$ iff (by item 1. above proven) $\llbracket t_1\rrbracket_{\alpha^{\cdot}}^{s_{m_\Psi}}([v]_{\omega\times\iota}^{\Psi,E}) = \llbracket t_2\rrbracket_{\alpha^{\cdot}}^{s_{m_\Psi}}([v]_{\omega\times\iota}^{\Psi,E})$ iff $[v]_{\omega\times\iota}^{\Psi,E}$ is in $\llbracket\phi\rrbracket_{\alpha^{\cdot}}^{s_{m_\Psi}}$, as we wanted to show.

Suppose ϕ is $p(t_1,\ldots,t_k)$. Then $[v]_{\omega\times\iota}^{\Psi,E}$ is in $f(\llbracket\phi\rrbracket_{\alpha}^{s_\Psi})$ iff (by Proposition 5.5.11) v is in $\llbracket\phi\rrbracket_{\alpha}^{s_\Psi}$ iff (by Proposition 3.1.3) $v \in |\phi|^{s_\Psi}$ iff $v{:}p(t_1,\ldots,t_k) \in \Psi$ iff ${p^P}_{[v]_{\omega}^{\Psi,E}}([v{:}t_1]_g^{\Psi,E},\ldots,[v{:}t_k]_g^{\Psi,E})$ $= 1$ iff $[p]^{s_{m_\Psi}}_{[v]_{\omega}^{\Psi,E}}([v{:}t_1]_g^{\Psi,E},\ldots,[v{:}t_k]_g^{\Psi,E}) = 1$ iff (by Proposition 3.1.3) $[p]^{s_{m_\Psi}}_{[v]_{\omega}^{\Psi,E}}(\llbracket t_1\rrbracket_{\alpha}^{s_\Psi}(v),\ldots,$ $\llbracket t_k\rrbracket_{\alpha}^{s_\Psi}(v)) = 1$ iff (by item 1.) $[p]^{s_{m_\Psi}}_{\omega([v]_{\omega\times\iota}^{\Psi,E})}(\llbracket t_1\rrbracket_{\alpha^{\cdot}}^{s_{m_\Psi}}([v]_{\omega\times\iota}^{\Psi,E}),\ldots,\llbracket t_k\rrbracket_{\alpha^{\cdot}}^{s_{m_\Psi}}([v]_{\omega\times\iota}^{\Psi,E})) = 1$ iff $\llbracket\phi\rrbracket_{\alpha^{\cdot}}^{s_{m_\Psi}}([v]_{\omega\times\iota}^{\Psi,E}) = 1$ as we wanted to show.

Suppose ϕ is $\Diamond\phi_1$. Then $[v]_{\omega\times\iota}^{\Psi,E}$ is in $f(\llbracket\phi\rrbracket_{\alpha}^{s_\Psi})$ iff (by Proposition 5.5.11) v is in $\llbracket\phi\rrbracket_{\alpha}^{s_\Psi}$ iff (by Proposition 3.1.3) v is in $|\phi|^{s_\Psi}$ iff $v{:}\Diamond\phi_1$ is in Ψ iff (since Ψ is an appropriate set) exists v_1 with $vR_\Box v_1$ and $v_1{:}\phi_1$ in Ψ iff (by Proposition 5.5.15) exists a_1 in ${D_\Psi}^X$ and label schema terms $v^{\cdot}$ and $v^{\cdot}_1$ with $v \equiv_\omega v^{\cdot}$ and $v^{\cdot}R_\Box v^{\cdot}_1$ in Ψ, $\iota^{\Psi}_{D^X}(v) = a_1$ and $\langle [v^{\cdot}_1]_{\omega}^{\Psi,E}, a_1\rangle$ in $f(\llbracket\phi_1\rrbracket_{\alpha}^{s_\Psi})$ iff (by induction hypothesis) $\omega([v]_{\omega\times\iota}^{\Psi,E})R^{\circ}_{\Box}\omega(\langle [v^{\cdot}_1]_{\omega}^{\Psi,E}, a_1\rangle)$, $\alpha([v]_{\omega\times\iota}^{\Psi,E}) = \alpha(\langle [v^{\cdot}_1]_{\omega}^{\Psi,E}, a_1\rangle)$, and $\langle [v^{\cdot}_1]_{\omega}^{\Psi,E}, a_1\rangle$ is in $\llbracket\phi_1\rrbracket_{\alpha^{\cdot}}^{s_{m_\Psi}}$ iff $[\Diamond]^{s_{m_\Psi}}(\llbracket\phi_1\rrbracket_{\alpha^{\cdot}}^{s_{m_\Psi}})([v]_{\omega\times\iota}^{\Psi,E}) = 1$ iff $[v]_{\omega\times\iota}^{\Psi,E}$ is in $\llbracket\Diamond\phi_1\rrbracket_{\alpha^{\cdot}}^{s_{m_\Psi}}$ as we wanted to show.

Suppose ϕ is $\Box\phi_1$. Then $[v]_{\omega\times\iota}^{\Psi,E}$ is in $f(\llbracket\phi\rrbracket_{\alpha}^{s_\Psi})$ iff (by Proposition 5.5.11) v is in $\llbracket\phi\rrbracket_{\alpha}^{s_\Psi}$ iff (by Proposition 3.1.3) v is in $|\phi|^{s_\Psi}$ iff $v{:}\Box\phi_1$ is in Ψ iff (since Ψ is an appropriate set) for all v_1 if $vR_\Box v_1$ is in Ψ then $v_1{:}\phi_1$ is in Ψ iff (by Proposition 5.5.14) for any assignment a_1 in ${D_\Psi}^X$ and label schema term $v^{\cdot}_1$ if exists a label schema term $v^{\cdot}$ with $v \equiv_\omega v^{\cdot}$ and $v^{\cdot}R_\Box v^{\cdot}_1$ in Ψ, and $\iota^{\Psi}_{D^X}(v) = a_1$ then $\langle [v^{\cdot}_1]_{\omega}^{\Psi,E}, a_1\rangle$ is in $f(\llbracket\phi_1\rrbracket_{\alpha}^{s_\Psi})$ iff (by induction hypothesis) for all $\langle [v^{\cdot}_1]_{\omega}^{\Psi,E}, a_1\rangle)$ in $U^{s_{m_\Psi}}$ if $\omega([v]_{\omega\times\iota}^{\Psi,E})R^{\circ}_{\Box}\omega(\langle [v^{\cdot}_1]_{\omega}^{\Psi,E}, a_1\rangle)$ and $\alpha([v]_{\omega\times\iota}^{\Psi,E}) = \alpha(\langle [v^{\cdot}_1]_{\omega}^{\Psi,E}, a_1\rangle)$ then $\langle [v^{\cdot}_1]_{\omega}^{\Psi,E}, a_1\rangle$ is in $\llbracket\phi_1\rrbracket_{\alpha^{\cdot}}^{s_{m_\Psi}}$ iff $[\Box]^{s_{m_\Psi}}(\llbracket\phi_1\rrbracket_{\alpha^{\cdot}}^{s_{m_\Psi}})([v]_{\omega\times\iota}^{\Psi,E}) = 1$ iff $[v]_{\omega\times\iota}^{\Psi,E}$ is in $\llbracket\Box\phi_1\rrbracket_{\alpha^{\cdot}}^{s_{m_\Psi}}$ as we wanted to show.

Suppose ϕ is $\exists_x\phi_1$. Then $[v]^{\Psi,E}_{\omega\times\iota}$ is in $f(\llbracket\phi\rrbracket^{s_\Psi}_\alpha)$ iff (by Proposition 5.5.11) v is in $\llbracket\phi\rrbracket^{s_\Psi}_\alpha$ iff (by Proposition 3.1.3) v is in $|\phi|^{s_\Psi}$ iff $v{:}\exists_x\phi_1$ is in Ψ iff (by Proposition 5.5.17) exists a_1 with a_1 x co-equivalent to $\iota^{\Psi}_{DX}(v)$, and $\langle[v]^{\Psi,E}_{\omega}, a_1\rangle$ in $f(\llbracket\phi_1\rrbracket^{s_\Psi}_\alpha)$ iff (by induction hypothesis) exists $\langle[\dot{v}_1]^{\Psi,E}_{\omega}, a_1\rangle$ with $\omega([v]^{\Psi,E}_{\omega\times\iota}) = \omega(\langle[\dot{v}_1]^{\Psi,E}_{\omega}, a_1\rangle)$, $\alpha([v]^{\Psi,E}_{\omega\times\iota})$ is x co-equivalent to $\alpha(\langle[\dot{v}_1]^{\Psi,E}_{\omega}, a_1\rangle)$, and $\langle[\dot{v}_1]^{\Psi,E}_{\omega}, a_1\rangle$ is in $\llbracket\phi_1\rrbracket^{s_{m_\Psi}}_{\alpha^{\cdot}}$ iff $[\exists_x]^{s_{m_\Psi}}_{\omega([v]^{\Psi,E}_{\omega\times\iota})}(\llbracket\phi_1\rrbracket^{s_{m_\Psi}}_{\alpha^{\cdot}})([v]^{\Psi,E}_{\omega\times\iota}) = 1$ iff $[v]^{\Psi,E}_{\omega\times\iota}$ is in $\llbracket\exists_x\phi_1\rrbracket^{s_{m_\Psi}}_{\alpha^{\cdot}}$ as we wanted to show.

Suppose ϕ is $\forall_x\phi_1$. Then $[v]^{\Psi,E}_{\omega\times\iota}$ is in $f(\llbracket\phi\rrbracket^{s_\Psi}_\alpha)$ iff (by Proposition 5.5.11) v is in $\llbracket\phi\rrbracket^{s_\Psi}_\alpha$ iff (by Proposition 3.1.3) v is in $|\phi|^{s_\Psi}$ iff $v{:}\forall_x\phi_1$ is in Ψ iff (by Proposition 5.5.12) for all a_1 if a_1 is x co-equivalent to $\iota^{\Psi}_{DX}(v)$ then $\langle[v]^{\Psi,E}_{\omega}, a_1\rangle$ is in $[\llbracket\phi_1\rrbracket^{s_\Psi}_\alpha]^{\Psi,E}_{\omega\times\iota^+}$ iff (by induction hypothesis) for all $\langle[\dot{v}_1]^{\Psi,E}_{\omega}, a_1\rangle)$ in $U^{s_{m_\Psi}}$ if $\omega([v]^{\Psi,E}_{\omega\times\iota}) = \omega(\langle[\dot{v}_1]^{\Psi,E}_{\omega}, a_1\rangle)$ and $\alpha([v]^{\Psi,E}_{\omega\times\iota})$ is x co-equivalent to $\alpha(\langle[\dot{v}_1]^{\Psi,E}_{\omega}, a_1\rangle)$ then $\langle[\dot{v}_1]^{\Psi,E}_{\omega}, a_1\rangle$ is in $\llbracket\phi_1\rrbracket^{s_{m_\Psi}}_{\alpha^{\cdot}}$ iff $[\forall_x]^{s_{m_\Psi}}_{\alpha([v]^{\Psi,E}_{\omega\times\iota})}(\llbracket\phi_1\rrbracket^{s_{m_\Psi}}_{\alpha^{\cdot}})([v]^{\Psi,E}_{\omega\times\iota}) = 1$ iff $[v]^{\Psi,E}_{\omega\times\iota}$ is in $\llbracket\forall_x\phi_1\rrbracket^{s_{m_\Psi}}_{\alpha^{\cdot}}$ as we wanted to show.

Suppose ϕ is $\phi_1 \wedge \phi_2$. Then $[v]^{\Psi,E}_{\omega\times\iota}$ is in $f(\llbracket\phi\rrbracket^{s_\Psi}_\alpha)$ iff (by Proposition 5.5.11) v is in $\llbracket\phi\rrbracket^{s_\Psi}_\alpha$ iff (by Proposition 3.1.3) v is in $|\phi|^{s_\Psi}$ iff $v{:}\phi_1 \wedge \phi_2$ is in Ψ iff (since Ψ is an appropriate set) $v{:}\phi_1$ is in Ψ and $v{:}\phi_2$ is in Ψ iff (by Proposition 3.1.3) v is in $\llbracket\phi_1\rrbracket^{s_\Psi}_\alpha$ and v is in $\llbracket\phi_2\rrbracket^{s_\Psi}_\alpha$ iff (by Proposition 5.5.11) $[v]^{\Psi,E}_{\omega\times\iota}$ is in $f(\llbracket\phi_1\rrbracket^{s_\Psi}_\alpha)$ and $[v]^{\Psi,E}_{\omega\times\iota}$ is in $f(\llbracket\phi_2\rrbracket^{s_\Psi}_\alpha)$ iff (induction hypothesis) $[v]^{\Psi,E}_{\omega\times\iota}$ is in $\llbracket\phi_1\rrbracket^{s_{m_\Psi}}_{\alpha^{\cdot}}$ and $[v]^{\Psi,E}_{\omega\times\iota}$ is in $\llbracket\phi_2\rrbracket^{s_{m_\Psi}}_{\alpha^{\cdot}}$ iff $[\wedge]_{\omega([v]^{\Psi,E}_{\omega\times\iota})\alpha([v]^{\Psi,E}_{\omega\times\iota})}(\llbracket\phi_1\rrbracket^{s_{m_\Psi}}_{\alpha^{\cdot}}, \llbracket\phi_2\rrbracket^{s_{m_\Psi}}_{\alpha^{\cdot}})([v]^{\Psi,E}_{\omega\times\iota}) = 1$ iff $[v]^{\Psi,E}_{\omega\times\iota}$ is in $\llbracket\phi_1 \wedge \phi_2\rrbracket^{s_{m_\Psi}}_{\alpha^{\cdot}}$, as we wanted to show.

Finally the main proof: $s_\Psi, \alpha \Vdash \nu{:}\phi$ iff $\llbracket\nu\rrbracket^{s_\Psi}_\alpha$ is in $\llbracket\phi\rrbracket^{s_\Psi}_\alpha$ iff (by Proposition 5.5.11) $[\nu\alpha]^{\Psi,E}_{\omega\times\iota}$ is in $f(\llbracket\phi\rrbracket^{s_\Psi}_\alpha)$ iff (by item 2. above proven) $\nu\alpha^{\cdot}$ is in $\llbracket\phi\rrbracket^{s_{m_\Psi}}_{\alpha^{\cdot}}$ iff $s_{m_\Psi}, \alpha^{\cdot} \Vdash \nu{:}\phi$. *QED*

Proposition 5.2.12 The lfob logic system $\mathcal{L}_{\mathrm{FOT}_\wedge}$ is rich.

Proof In the context of $\mathcal{L}_{\mathrm{FOT}_\wedge}$, let Ψ_0 be a $v{:}\varphi$ consistent set. Then, by Proposition 4.4.9, there exists a strong appropriate set Ψ extending Ψ_0 and also $v{:}\varphi$ consistent, since $\mathcal{L}_{\mathrm{FOT}_\wedge}$ contains the rule *exh*, and, by Proposition 5.2.9, is a clfob deduction system without a disjunction and a implication and with locality. So, it is possible to consider the canonical structure s_Ψ. Let m_Ψ be the model induced by Ψ, as defined in Definition 5.2.10. Recall that s_{m_Ψ} is the structure induced by m_Ψ and so s_{m_Ψ} is in $\breve{m}_\Psi$. Then, taking into account Lemma 5.2.11 and the definition of $\breve{m}_\Psi$ in Example 5.2.1, we have that s_Ψ is in $\breve{m}_\Psi$, as we wanted to show. *QED*

Proof (Completeness Theorem 5.2.8) The proof follows immediately using Theorem 3.3.3 since, by Proposition 5.2.12, $\mathcal{L}_{\mathrm{FOT}_\wedge}$ is rich. *QED*

5.2.4 Equivalence to non-labelled logic system

In this subsection we show that the lfob logic system $\mathcal{L}_{\mathrm{FOT}_\wedge}$ is indeed a labelled presentation for the $\wedge$ fragment of first-order modal logic T with constant domains with respect to formulae without schema variables, as is stated in the next theorem. This theorem is proved at the end of the section.

Theorem 5.2.13 The lfob logic system $\mathcal{L}_{\mathrm{FOT}_\wedge}$ is a labelled presentation for the $\wedge$ fragment of first-order modal logic T with constant domains with respect to formulae without schema variables.

The proof of the theorem relies on finding a non-labelled logic system for the $\wedge$ fragment of first-order modal logic T with constant domains such that $\mathcal{L}_{\mathrm{FOT}_\wedge}$ is a labelled presentation for that system with respect to formulae without schema variables. Recall Definition 2.4.10 where it is defined what is a lfob logic system be a labelled presentation for a logic with respect to some set of schema formulae. So, consider the following system.

Example 5.2.14 $\wedge$ *fragment of first-order modal logic* T *with constant domains non-labelled logic system.* Let $\mathcal{L}^{\cdot}_{\mathrm{FOT}_\wedge}$ be the non-labelled logic system $\langle \Sigma^{\cdot}, \propto \rangle$ for the $\wedge$ fragment of first-order modal logic T with constant domains having as its family of propositions, the family of propositions of $\mathcal{L}_{\mathrm{FOT}_\wedge}$, and having as its family of function symbols, the family of function symbols of $\mathcal{L}_{\mathrm{FOT}_\wedge}$.

Observe that the only components that can change between non-labelled logic systems for the $\wedge$ fragment of first-order modal logic T with constant domains are the family of propositions and the family of function symbols, and, motivated by that change, the relation $\propto$ between schema formulae associated to the system.

Before presenting the proof that $\mathcal{L}_{\mathrm{FOT}_\wedge}$ is indeed a labelled presentation for the $\wedge$ fragment of first-order modal logic T with constant domains with respect to formulae without schema variables, we show two auxiliary lemmas proving the equivalence, for formulae without schema variables, between the usual semantics for the logic and the labelled semantics. Lemma 5.2.15 proves the equivalence for formulae and Lemma 5.2.16 proves the equivalence for entailment.

Lemma 5.2.15 In the context of $\mathcal{L}_{\mathrm{FOT}_\wedge}$ and $\mathcal{L}^{\cdot}_{\mathrm{FOT}_\wedge}$,

$$m, \alpha^{\cdot}, w \Vdash^{a} \gamma \quad \text{iff} \quad s_m, \alpha \Vdash \nu{:}\gamma$$

where γ is a schema formula in $\mathcal{L}^{\cdot}_{\mathrm{FOT}_\wedge}$ and α and $\alpha^{\cdot}$ are such that $\alpha_t = \alpha^{\cdot}_t$, $\alpha_f = \alpha^{\cdot}_f$ and $\nu\alpha_l = \langle w, a \rangle$.

Proof First we show an auxiliary result and only after we show the proposition.

1. $\llbracket t \rrbracket^{s_m}_{\alpha}(\langle w, a \rangle) = t^{m,\alpha^{\cdot}}_{w,a}$, where t is schema term. We omit the proof since it follows straightforwardly by induction on the schema term t in $\mathcal{L}^{\cdot}_{\mathrm{FOT}_\wedge}$.

Finally the main proof, which follows by induction on γ:

if γ is $t_1 = t_2$ then $m, \alpha^{\cdot}, w \Vdash^{a} t_1 = t_2$ iff ${t_1}^{m,\alpha^{\cdot}}_{w,a} = {t_2}^{m,\alpha^{\cdot}}_{w,a}$ iff (by item 1. proven above and the definition of s_m) $\llbracket t_1 \rrbracket^{s_m}_{\alpha}(\nu\alpha) = \llbracket t_2 \rrbracket^{s_m}_{\alpha}(\nu\alpha)$ iff $s_m, \alpha \Vdash \nu{:}t_1 = t_2$.

if γ is $p(t_1, \ldots, t_k)$ then $m, \alpha^{\cdot}, w \Vdash^{a} \gamma$ iff $p^{P}_{w}({t_1}^{m,\alpha^{\cdot}}_{w,a}, \ldots, {t_k}^{m,\alpha^{\cdot}}_{w,a}) = 1$ iff (by item 1. above and the definition of s_m) $[p]_w(\llbracket t_1 \rrbracket^{s_m}_{\alpha}(w, a), \ldots, \llbracket t_k \rrbracket^{s_m}_{\alpha}(w, a)) = 1$ iff $\llbracket p(t_1, \ldots, t_k) \rrbracket(\nu\alpha) = 1$ iff $s_m, \alpha \Vdash \nu{:}\gamma$.

if γ is in $\Xi^{\cdot}_f (= \Xi_f)$ then $m, \alpha^{\cdot}, w \Vdash^{a} \gamma$ iff $\langle w, a \rangle$ is in $\gamma\alpha^{\cdot}_f$ iff $\nu\alpha$ is in $\gamma\alpha_f$ iff $s_m, \alpha \Vdash \nu{:}\gamma$, as we wanted to show.

if γ is $\gamma_1 \wedge \gamma_2$ then $m, \alpha^{\cdot}, w \Vdash \gamma$ iff $m, \alpha^{\cdot}, w \Vdash^{a} \gamma_1$ and $m, \alpha^{\cdot}, w \Vdash^{a} \gamma_2$ iff (by induction hypothesis) $s_m, \alpha \Vdash \nu{:}\gamma_1$ and $s_m, \alpha \Vdash \nu{:}\gamma_2$ iff $[\![\gamma_1]\!]^{s_m}_{\alpha}([\![\nu]\!]^{s_m}_{\alpha}) = 1$ and $[\![\gamma_2]\!]^{s_m}_{\alpha}([\![\nu]\!]^{s_m}_{\alpha}) = 1$ iff $\widehat{\wedge}([\![\gamma_1]\!]^{s_m}_{\alpha}, [\![\gamma_2]\!]^{s_m}_{\alpha})([\![\nu]\!]^{s_m}_{\alpha}) = 1$ iff $s_m, \alpha \Vdash \nu{:}\gamma$.

if γ is $\forall_x \gamma_1$ then the key point of this proof is to show that the assertion, for any a_1 x co-equivalent to a we have $m, \alpha^{\cdot}, w \Vdash^{a_1} \gamma_1$, is equivalent to, for any $\langle w^{\cdot}, a^{\cdot}\rangle$ if $\langle\langle w, a\rangle, \langle w^{\cdot}, a^{\cdot}\rangle\rangle$ is in $[\equiv_x]^{s_m}$ then $\langle w^{\cdot}, a^{\cdot}\rangle$ is in $[\![\gamma_1]\!]^{s_m}_{\alpha}$. So, suppose that for any a_1 x co-equivalent to a we have $m, \alpha^{\cdot}, w \Vdash^{a_1} \gamma_1$ and let $\langle w^{\cdot}, a^{\cdot}\rangle$ be such that $\langle\langle w, a\rangle, \langle w^{\cdot}, a^{\cdot}\rangle\rangle$ is in $[\equiv_x]^{s_m}$. Then $w = w^{\cdot}$ and a and $a^{\cdot}$ are x co-equivalent. Thus $m, \alpha^{\cdot}, w^{\cdot} \Vdash^{a^{\cdot}} \gamma_1$. Hence, by induction hypothesis, $s_m, \alpha_1 \Vdash \nu_1{:}\gamma_1$ where $\alpha_{1t} = \alpha^{\cdot}_t$, $\alpha_{1f} = \alpha^{\cdot}_f$ and $\nu_1\alpha_1 = \langle w^{\cdot}, a^{\cdot}\rangle$. So, $\nu_1\alpha_1$ is in $[\![\gamma_1]\!]^{s_m}_{\alpha_1}$, i.e., by Proposition 2.4.4, $\langle w^{\cdot}, a^{\cdot}\rangle$ is in $[\![\gamma_1]\!]^{s_m}_{\alpha}$, as we wanted to show. For the other direction the proof is similar so we omit it.

if γ is $\Box\gamma_1$ then the proof is similar to the case where γ is $\forall_x\gamma_1$, so we omit it.

if γ is $\Diamond\gamma_1$ then the key point of this proof is to show that the assertion, exists w_1 with $wR^{\circ}_{\Box}w_1$ and $m, \alpha^{\cdot}, w_1 \Vdash^{a} \gamma_1$, is equivalent to, exists $\langle w_1, a_1\rangle$ with $\langle\langle w, a\rangle, \langle w_1, a_1\rangle\rangle$ in $[R_{\Box}]^{s_m}$ and $\langle w_1, a_1\rangle$ is in $[\![\gamma_1]\!]^{s_m}_{\alpha}$. So, suppose that exists $\langle w_1, a_1\rangle$ with $\langle\langle w, a\rangle, \langle w_1, a_1\rangle\rangle$ in $[R_{\Box}]^{s_m}$ and $\langle w_1, a_1\rangle$ in $[\![\gamma_1]\!]^{s_m}_{\alpha}$. Then, by definition of $[R_{\Box}]^{s_m}$ we have that $a = a_1$ and $wR^{\circ}_{\Box}w_1$. Consider a assignment α_1 $\{\nu_1\}$ co-equivalent to α with $\nu_1\alpha_1$ equal to $\langle w_1, a_1\rangle$. Then, by Proposition 2.4.4 we have that $\nu_1\alpha_1$ is in $[\![\gamma_1]\!]^{s_m}_{\alpha_1}$, and so $s_m, \alpha_1 \Vdash \nu_1{:}\gamma_1$, thus, by induction hypothesis, we have that $m, \alpha^{\cdot}, w_1 \Vdash^{a_1} \gamma_1$ as we wanted to show. The proof of the other direction the proof is similar so we omit it.

if γ is $\exists_x\gamma_1$ then the proof is similar to the case where γ is $\Diamond\gamma_1$, so we omit it. *QED*

Lemma 5.2.16 In the context of $\mathcal{L}_{\mathrm{FOT}_\wedge}$ and $\mathcal{L}^{\cdot}_{\mathrm{FOT}_\wedge}$,

$$\Phi \propto \gamma \quad \text{iff} \quad \{\nu{:}\phi \mid \phi \text{ in } \Phi\} \vDash \nu{:}\gamma$$

where ν is a label schema variable, Φ is a set of formulae and γ is a formula in $\mathcal{L}^{\cdot}_{\mathrm{FOT}_\wedge}$.

Proof Assume $\Phi \propto \gamma$, and let s be a structure in $\mathcal{L}_{\mathrm{FOT}_\wedge}$ and α an assignment over s such that $s, \alpha \Vdash \{\nu{:}\phi \mid \phi \text{ in } \Phi\}$. Then, we have to consider two cases, (i) s is a structure induced by a relational model m for the $\wedge$ fragment of first-order modal logic T with constant domains, or (ii) s is a structure equivalent in terms of satisfaction with a structure $s^{\cdot}_m$ induced by a relational model m for the $\wedge$ fragment of first-order modal logic T. If (i) holds then, denoting by $\langle w, a\rangle$ the point $\nu\alpha$, we can use Lemma 5.2.15 to conclude that $m, w \Vdash^{a} \Phi$. Thus, using the fact that $\Phi \propto \gamma$ we have that $m, w \Vdash^{a} \gamma$. Hence, using again Lemma 5.2.15 we can conclude that $s, \alpha \Vdash \nu{:}\gamma$ as we wanted to show. If (ii) holds then there is a assignment $\alpha^{\cdot}$ over $s^{\cdot}_m$ such that $s^{\cdot}_m, \alpha^{\cdot} \Vdash \{\nu{:}\phi \mid \phi \text{ in } \Phi\}$. Therefore, denoting by $\langle w, a\rangle$ the point $\nu\alpha^{\cdot}$, we can use Lemma 5.2.15 to conclude that $m, w \Vdash^{a} \Phi$. Thus, using the fact that $\Phi \propto \gamma$ we have that $m, w \Vdash^{a} \gamma$. Hence, using again Lemma 5.2.15 we can conclude that $s^{\cdot}_m, \alpha^{\cdot} \Vdash \nu{:}\gamma$. So, since s is a structure equivalent in terms of satisfaction with $s^{\cdot}_m$ we have that $s, \alpha \Vdash \nu{:}\gamma$ as we wanted to show.

Assume $\{\nu{:}\phi \mid \phi \text{ in } \Phi\} \vDash \nu{:}\gamma$ and let m be a relational model for the $\wedge$ fragment of first-order modal logic T with constant domains over $\Sigma^{\cdot}$, $\alpha^{\cdot}$ a assignment over m, w a world of m and a a quantification assignment with $m, \alpha^{\cdot}, w \Vdash^{a} \Phi$. Then, by Lemma 5.2.15, we have that $s_m, \alpha \Vdash \{\nu{:}\phi \mid \phi \in \Phi\}$ where α is such that α_t is equal to $\alpha^{\cdot}_t$, α_f is equal to

$\alpha^{\cdot}_{f}$, and $\nu\alpha_l$ is $\langle w, a\rangle$. So, $s_m, \alpha \Vdash \nu{:}\gamma$ since $\{\nu{:}\phi \mid \phi \text{ in } \Phi\} \vDash \nu{:}\gamma$ and s_m is in $\breve{m}$ and m is in M. Therefore using again Lemma 5.2.15, we have that $m, \alpha^{\cdot}, w \Vdash^{a} \gamma$, as we wanted to show. *QED*

Proof (Labelled presentation Theorem 5.2.13) According to Definition 2.4.10 it is sufficient to show that $\mathcal{L}_{\mathrm{FOT}_\wedge}$ is a labelled presentation for $\mathcal{L}^{\cdot}_{\mathrm{FOT}_\wedge}$ with respect to formulae without schema variables. Recall Definition 2.4.9 where it is said what is a lfob logic system be a labelled presentation for a non-labelled logic system with respect to a set of non-labelled schema formulae. Note that the set of formulae of $\mathcal{L}_{\mathrm{FOT}_\wedge}$ coincide with the set of formulae of $\mathcal{L}^{\cdot}_{\mathrm{FOT}_\wedge}$. So, the lfob logic system $\mathcal{L}_{\mathrm{FOT}_\wedge}$ is a labelled presentation for $\mathcal{L}^{\cdot}_{\mathrm{FOT}_\wedge}$ with respect to formulae without schema variables because:

- $\mathcal{L}_{\mathrm{FOT}_\wedge}$ is sound and complete. See Theorem 5.2.8 for completeness and Theorem 5.2.3 for soundness.

- $X^{\cdot}$, $\Xi^{\cdot}_{t}$ and $\Xi^{\cdot}_{f}$ coincide with X, Ξ_t and Ξ_f.

- $F^{\cdot}$ is equal to F, $P^{\cdot}$ is equal to P, and $C^{\cdot}$ is equal to $C \cup O \cup Q$, by definition of $\mathcal{L}_{\mathrm{FOT}_\wedge}$ (see Example 5.2.1), and of $\mathcal{L}^{\cdot}_{\mathrm{FOT}_\wedge}$ (see Example 5.2.14).

- $\Phi \propto \gamma$ iff $\{\nu{:}\phi \mid \phi \in \Phi\} \vDash \nu{:}\gamma$, where ν is a label schema variable, Φ is a set of formulae in $\mathcal{L}^{\cdot}_{\mathrm{FOT}_\wedge}$ and γ is a formula in $\mathcal{L}^{\cdot}_{\mathrm{FOT}_\wedge}$. This result follows by Lemma 5.2.16. *QED*

5.3 Relevance logic R

In this section we present a lfob logic system $\mathcal{L}_{\mathrm{R}}$ for relevance logic R [63, 59, 79, 1, 2, 32, 58, 68, 66, 67, 77]. See Definitions 2.4.9 and 2.4.10 to recall when a lfob logic system is a labelled presentation for a non-labelled logic system or is a labelled presentation for a non-labelled logic. We now briefly describe what is a non-labelled fob signature, a relational model and a non-labelled fob logic system for relevance logic R. Recall the definitions of non-labelled fob signature and non-labelled fob logic system in Definition 2.4.8. We consider that $X^{\cdot}$ and $\Xi^{\cdot}_{t}$ are empty sets, and $\Xi^{\cdot}_{f}$ is equal to Ξ_f. In order to recall the importance of $X^{\cdot}$, $\Xi^{\cdot}_{t}$ and $\Xi^{\cdot}_{f}$ see Definition 2.4.8.

A *non-labelled fob signature* $\Sigma^{\cdot}$ *for relevance logic* R is a non-labelled fob signature such that $F^{\cdot}_{k} = \emptyset$ for $k \geq 0$, $P^{\cdot}_{0}$ is a non-empty set, $P^{\cdot}_{k} = \emptyset$ for $k \geq 1$, $C^{\cdot}_{0} = \{\bot\}$, $C^{\cdot}_{1} = \{-\}$, $C^{\cdot}_{2} = \{\wedge, \vee, \Rightarrow\}$, and $C^{\cdot}_{k} = \emptyset$ for $k \geq 3$.

A *relational model* m *over* $\Sigma^{\cdot}$ *for relevance logic* R is a tuple $\langle W, G, R^{\circ}_{\Rightarrow}, *_{\circ}, f_c{}^{\circ}, f_s{}^{\circ}, V\rangle$ where W is a non-empty set, G is in W, $*_{\circ}$ is a map from W to W, $f_c{}^{\circ}$ is a map from W^3 to W, $f_s{}^{\circ}$ is a map from W^5 to W, $R^{\circ}_{\Rightarrow}$ is contained in $W \times W \times W$, such that

- $R^{\circ}_{\Rightarrow} w w_2 w_3$ whenever $R^{\circ}_{\Rightarrow} G w w_1$ and $R^{\circ}_{\Rightarrow} w_1 w_2 w_3$,
- $R^{\circ}_{\Rightarrow} w_1 w w_3$ whenever $R^{\circ}_{\Rightarrow} G w w_2$ and $R^{\circ}_{\Rightarrow} w_1 w_2 w_3$,
- $R^{\circ}_{\Rightarrow} w_1 w_2 w$ whenever $R^{\circ}_{\Rightarrow} G w_3 w$ and $R^{\circ}_{\Rightarrow} w_1 w_2 w_3$,
- $R^{\circ}_{\Rightarrow} G w w$,
- $R^{\circ}_{\Rightarrow} G w^{*_{\circ}*_{\circ}} w$,
- $R^{\circ}_{\Rightarrow} G w w^{*_{\circ}*_{\circ}}$,

- $R^{\circ}_{\Rightarrow} w_2 w_1 w_3$ whenever $R^{\circ}_{\Rightarrow} w_1 w_2 w_3$,
- $R^{\circ}_{\Rightarrow} w_1 w_3^{*\circ} w_2^{*\circ}$ whenever $R^{\circ}_{\Rightarrow} w_1 w_2 w_3$,
- $R^{\circ}_{\Rightarrow} w_1 w_2 f_c{}^{\circ}(w_1, w_2, w_3)$ whenever $R^{\circ}_{\Rightarrow} w_1 w_2 w_3$,
- $R^{\circ}_{\Rightarrow} f_c{}^{\circ}(w_1, w_2, w_3) w_2 w_3$ whenever $R^{\circ}_{\Rightarrow} w_1 w_2 w_3$,
- $R^{\circ}_{\Rightarrow} w_1 w_3 f_s{}^{\circ}(w_1, w_2, w_3, w_4, w)$ whenever $R^{\circ}_{\Rightarrow} w_1 w_2 w$ and $R^{\circ}_{\Rightarrow} w w_3 w_4$,
- $R^{\circ}_{\Rightarrow} w_2 f_s{}^{\circ}(w_1, w_2, w_3, w_4, w) w_4$ whenever $R^{\circ}_{\Rightarrow} w_1 w_2 w$ and $R^{\circ}_{\Rightarrow} w w_3 w_4$,

and V is a map $V : P^{\cdot}_0 \times W \to 2$ satisfying the atomic hereditary condition: if $V(p, w) = 1$ and $R^{\circ}_{\Rightarrow} G w w'$ then $V(p, w') = 1$. $m, \alpha, w \Vdash \varphi$, is defined inductively as follows,

$$\begin{array}{lll}
m, \alpha, w \Vdash \xi & \text{iff} & w \text{ is in } \xi\alpha \\
m, \alpha, w \nVdash \bot & & \\
m, \alpha, w \Vdash p & \text{iff} & V(p, w) = 1 \\
m, \alpha, w \Vdash -\varphi & \text{iff} & m, \alpha, w^{*\circ} \nVdash \varphi \\
m, \alpha, w \Vdash \varphi_1 \wedge \varphi_2 & \text{iff} & m, \alpha, w \Vdash \varphi_1 \text{ and } m, \alpha, w \Vdash \varphi_2 \\
m, \alpha, w \Vdash \varphi_1 \vee \varphi_2 & \text{iff} & m, \alpha, w \Vdash \varphi_1 \text{ or } m, \alpha, w \Vdash \varphi_2 \\
m, \alpha, w \Vdash \varphi_1 \Rightarrow \varphi_2 & \text{iff} & \text{for any } w_1 \text{ and } w_2, \\
 & & \text{if } R^{\circ}_{\Rightarrow} w w_1 w_2 \text{ and } m, \alpha, w_1 \Vdash \varphi_1 \text{ then } m, \alpha, w_2 \Vdash \varphi_2
\end{array}$$

where α is a *schema assignment* over m, i.e., a map that associates to an element of Ξ_f a *hereditary set* in m, i.e., a set of worlds of m such that if $R^{\circ}_{\Rightarrow} G w w^{\cdot}$ holds and w is in that set then $w^{\cdot}$ is also in that set.

In order to understand better $*$ and $-$, and their role in the representation of the relevant negation in the relational semantics for R we recommend the reading of [65, 33, 34, 64].

A *non-labelled fob logic system for relevance logic R* is a pair $\langle \Sigma^{\cdot}, \propto \rangle$ where $\Sigma^{\cdot}$ is a non-labelled signature for this logic, and $\propto$ is such that

$$\Phi \propto \gamma \quad \text{iff} \quad \text{if } m, \alpha, w \Vdash \Phi \text{ then } m, \alpha, w \Vdash \gamma$$

for any relational model m for relevance logic R over $\Sigma^{\cdot}$, assignment α over m, and world w in W_m.

Observe that the only distinct aspects between two different non-labelled logic systems for relevance logic R are, the sets of zero-ary propositions and, caused by that, their relations.

5.3.1 Logic system

Example 5.3.1 *Relevance logic* R *lfob logic system.* Consider the lfob logic system $\langle \Sigma, R, M, \breve{\cdot} \rangle$, in the sequel denoted by $\mathcal{L}_{\mathrm{R}}$, where Σ is such that

- $F^l_0 = \{0\}$, $F^l_1 = \{^*\}$, $F^l_3 = \{f_c\}$, $F^l_5 = \{f_s\}$, and $F^l_k = \emptyset$ for $k \geq 6$ and $k = 2$ or 4,
- $S_3 = \{R_{\Rightarrow}\}$, and $S_k = \emptyset$ for $k \geq 4$ and $k = 2$ and $k = 1$,
- $F_k = \emptyset$ for $k \geq 0$,

- P_0 is a non-empty set, and $P_k = \emptyset$ for $k \geq 1$,
- $C_0 = \{\bot\}$, $C_2 = \{\wedge, \vee\}$, and $C_k = \emptyset$ for $k \geq 3$ and $k = 1$ and $k = 0$,
- $Q_k = \emptyset$ for $k \geq 1$,
- $O_1 = \{-\}$, $O_2 = \{\Rightarrow\}$, and $O_k = \emptyset$ for $k \geq 3$ and $k = 1$,

and R, besides the rules specified in Definition 2.3.10 common to all lfob deduction systems, contains:

$$\frac{\nu{:}\xi_1 \vee \xi_2 \quad \nu{:}\xi_1 \,/\, \nu''{:}\xi'' \quad \nu{:}\xi_2 \,/\, \nu''{:}\xi''}{\nu''{:}\xi''}\ \vee_E \qquad \frac{\nu{:}\xi_1}{\nu{:}\xi_1 \vee \xi_2}\ \vee_I^1 \qquad \frac{\nu{:}\xi_2}{\nu{:}\xi_1 \vee \xi_2}\ \vee_I^2$$

$$\frac{\nu{:}\xi_1 \qquad \nu{:}\xi_2}{\nu{:}\xi_1 \wedge \xi_2}\ \wedge_I \qquad \frac{\nu{:}\xi_1 \wedge \xi_2}{\nu{:}\xi_1}\ \wedge_E^1 \qquad \frac{\nu{:}\xi_1 \wedge \xi_2}{\nu{:}\xi_2}\ \wedge_E^2$$

$$\frac{\nu^*{:}\xi \,/\, \nu'{:}\bot}{\nu{:}-\xi}\ -_I \qquad \frac{\nu{:}-\xi \qquad \nu^*{:}\xi}{\nu'{:}\bot}\ -_E \qquad \frac{\nu{:}-\xi \,/\, \nu'{:}\bot}{\nu^*{:}\xi}\ -_c$$

$$\frac{R_{\Rightarrow}\nu\nu_1\nu_2, \nu_1{:}\xi_1 \,/\, \nu_2{:}\xi_2}{\nu{:}\xi_1 \Rightarrow \xi_2}\ \Rightarrow_I;\ \text{fresh}(\{\nu_1, \nu_2\}, \langle \vartheta_1, \{R_{\Rightarrow}\nu\nu_1\nu_2, \nu_1{:}\xi_1\}, \nu_2{:}\xi_2\rangle)$$

$$\frac{\nu{:}\xi_1 \Rightarrow \xi_2 \qquad \nu_1{:}\xi_1 \qquad R_{\Rightarrow}\nu\nu_1\nu_2}{\nu_2{:}\xi_2}\ \Rightarrow_E$$

$$\frac{R_{\Rightarrow}0\nu\nu_1 \quad R_{\Rightarrow}\nu_1\nu_2\nu_3}{R_{\Rightarrow}\nu\nu_2\nu_3}\ \mathit{rmon1} \qquad \frac{R_{\Rightarrow}0\nu\nu_2 \quad R_{\Rightarrow}\nu_1\nu_2\nu_3}{R_{\Rightarrow}\nu_1\nu\nu_3}\ \mathit{rmon2}$$

$$\frac{R_{\Rightarrow}0\nu_3\nu \quad R_{\Rightarrow}\nu_1\nu_2\nu_3}{R_{\Rightarrow}\nu_1\nu_2\nu}\ \mathit{rmon3}$$

$$\frac{R_{\Rightarrow}0\nu_1\nu_2 \qquad \nu_1{:}\xi}{\nu_2{:}\xi}\ \mathit{mon}_{R_{\Rightarrow}0} \qquad \frac{}{R_{\Rightarrow}0\nu\nu}\ \mathit{iden} \qquad \frac{}{\nu_1{:}\theta_1 =_g \nu_2{:}\theta_2}\ \mathit{1\ ind}$$

$$\frac{R_{\Rightarrow}\nu\nu_1\nu_2 \quad \nu \equiv_\omega \nu' \quad \nu_1 \equiv_\omega \nu_1' \quad \nu_2 \equiv_\omega \nu_2'}{R_{\Rightarrow}\nu'\nu_1'\nu_2'}\ R_{\Rightarrow}{}_\alpha^{\text{lgen}} \qquad \frac{}{\nu_1 \equiv_\alpha \nu_2}\ \mathit{1\ assg}$$

$$\frac{\nu_1 \equiv_\omega \nu_1' \quad \nu_2 \equiv_\omega \nu_2' \quad \nu_3 \equiv_\omega \nu_3' \quad \nu_4 \equiv_\omega \nu_4' \quad \nu_5 \equiv_\omega \nu_5'}{f_s(\nu_1, \nu_2, \nu_3, \nu_4, \nu_5) \equiv_\omega f_s(\nu_1', \nu_2', \nu_3', \nu_4', \nu_5')}\ f_s{}_\alpha^{\text{lgen}} \qquad \frac{\nu \equiv_\omega \nu'}{\nu^* \equiv_\omega \nu'^*}\ {}^*{}_\alpha^{\text{lgen}}$$

$$\frac{\nu_1 \equiv_\omega \nu_1' \quad \nu_2 \equiv_\omega \nu_2' \quad \nu_3 \equiv_\omega \nu_3'}{f_c(\nu_1, \nu_2, \nu_3) \equiv_\omega f_c(\nu_1', \nu_2', \nu_3')}\ f_c{}_\alpha^{\text{lgen}}$$

$$\frac{}{R_{\Rightarrow}0\nu\nu^{**}}\ {}^{**}i \qquad \frac{}{R_{\Rightarrow}0\nu^{**}\nu}\ {}^{**}c$$

$$\frac{R_{\Rightarrow}\nu_1\nu_2\nu_3}{R_{\Rightarrow}\nu_1\nu_2 f_c(\nu_1,\nu_2,\nu_3)}\ \mathit{cont1} \qquad \frac{R_{\Rightarrow}\nu_1\nu_2\nu_3}{R_{\Rightarrow}f_c(\nu_1,\nu_2,\nu_3)\nu_2\nu_3}\ \mathit{cont2} \qquad \frac{R_{\Rightarrow}\nu_1\nu_2\nu_3}{R_{\Rightarrow}\nu_2\nu_1\nu_3}\ \mathit{comm}$$

$$\frac{R_{\Rightarrow}\nu_1\nu_2\nu \quad R_{\Rightarrow}\nu\nu_3\nu_4}{R_{\Rightarrow}\nu_1\nu_3 f_s(\nu_1,\nu_2,\nu_3,\nu_4,\nu)}\ \mathit{suff1} \qquad \frac{R_{\Rightarrow}\nu_1\nu_2\nu \quad R_{\Rightarrow}\nu\nu_3\nu_4}{R_{\Rightarrow}\nu_2\nu_4 f_s(\nu_1,\nu_2,\nu_3,\nu_4,\nu)}\ \mathit{suff2} \qquad \frac{R_{\Rightarrow}\nu_1\nu_2\nu_3}{R_{\Rightarrow}\nu_1\nu_3^*\nu_2^*}\ \mathit{inv}$$

M is composed of all the relational models over the non-labelled fob signature $\langle F, P, C \cup Q \cup O\rangle$ for relevance logic R,

and $\breve{\cdot}$ associates to each relational model $\langle W, G, R^{\circ}_{\Rightarrow}, {}^{*\circ}, f_c{}^{\circ}, f_s{}^{\circ}, V\rangle$ denoted by m, a class, such that s is in $\breve{m}$ iff either s is a structure induced by m, denoted by s_m, i.e., a structure $\langle U, A, W, \alpha, \omega, D, \mathcal{E}, \mathcal{B}, [.]\rangle$ such that

- D is a singleton,
- $A = D^X$ and $U = W \times A$,
- $\alpha(\langle w, a\rangle) = a$ and $\omega(\langle w, a\rangle) = w$,
- $\mathcal{E}$ is D^U,
- b is in $\mathcal{B}$ iff
    - b is contained in U,
    - if $R^{\circ}_{\Rightarrow}G\omega(u_1)\omega(u_2)$, and $b(u_1) = 1$ then $b(u_2) = 1$,

and, denoting by a the unique assignment in D^X,

- $[0] = \langle G, a\rangle$,
- $[^*](u) = \langle \omega(u)^{*\circ}, a\rangle$,
- $[f_c](u_1, u_2, u_3) = \langle f_c{}^{\circ}(\omega(u_1), \omega(u_2), \omega(u_3)), a\rangle$,
- $[f_s](u_1, u_2, u_3, u_4, u_5) = \langle f_s{}^{\circ}(\omega(u_1), \omega(u_2), \omega(u_3), \omega(u_4), \omega(u_5)), a\rangle$,
- $[x]_a = a(x)$,
- $[p]_w = V(p, w)$ for p in P_0,
- $[\bot]_{\omega(u)\alpha(u)} = 0$,
- $[\wedge]_{\omega(u)\alpha(u)}(b_1, b_2)(u) = 1$ iff $b_1(u) = 1$ and $b_2(u) = 1$,
- $[\vee]_{\omega(u)\alpha(u)}(b_1, b_2)(u) = 1$ iff $b_1(u) = 1$ or $b_2(u) = 1$,
- $[R_{\Rightarrow}] = \{\langle u, u_1, u_2\rangle \mid R^{\circ}_{\Rightarrow}\omega(u)\omega(u_1)\omega(u_2)\}$,
- $[-](b)(u) = 1$ iff $\langle \omega(u)^{*\circ}, a\rangle$ is not in b,

- $[\Rightarrow](b_1, b_2)(u) = 1$ iff for all u_1 and u_2, if $\langle u, u_1, u_2\rangle$ is in $[R_{\Rightarrow}]$ and u_1 is in b_1 then u_2 is in b_2,

or s satisfies all the rules, and, for any assignment α over s there exists a induced structure $s^{\cdot}_m$ by m and a schema assignment $\alpha^{\cdot}$ over $s^{\cdot}_m$ such that for any labelled schema formulae η, $s, \alpha \Vdash \eta$ iff $s^{\cdot}_m, \alpha^{\cdot} \Vdash \eta$.

In order for the structures considered in $\mathcal{L}_{\mathrm{R}}$ induced by a model to be indeed structures it is necessary that the algebraic map $\widehat{\ }$ associated to each structure be well defined, see Definition 2.2.1. We now show that this is effectively the case and so that we did not made a mistake when we assumed that those tuples were structures.

Proposition 5.3.2 The algebraic map $\widehat{\ }$ associated to any structure in $\mathcal{L}_{\mathrm{R}}$ induced by a model is well defined.

Proof Let s be a structure induced by a relational model m for positive basic relevance logic R, and $\widehat{\ }$ the algebraic map associated to s. Then,

- $\widehat{x}$ is in $\mathcal{E}^s$ whenever x is a quantification variable. Straightforward since $\mathcal{E}^s$ is D^U.

- $\widehat{f}$ is a map from $\mathcal{E}^{sk}$ to $\mathcal{E}^s$ for any f in F_k and $k \geq 0$. Straightforward.

- $\widehat{p}$ is a map from $\mathcal{E}^{sk}$ to $\mathcal{B}^s$ for any p in P_k and $k \geq 0$. We only have to check this for $k = 0$ since $P_k = \emptyset$ for $k \geq 1$. This happens because $\widehat{p}(u) = [p]_{\omega(u)} = V(p, \omega(u)) \in 2$, and because if $R^{\circ}_{\Rightarrow}G\omega(u_1)\omega(u_2)$, and $\widehat{p}(u_1) = 1$ then $\widehat{p}(u_2) = 1$. To show this last fact, let u_1 and u_2 be points in s and suppose $R^{\circ}_{\Rightarrow}G\omega(u_1)\omega(u_2)$ and $\widehat{p}(u_1) = 1$. Then, $[p]_{\omega(u_1)} = 1$ and so $V(p, \omega(u_1)) = 1$. Then, since $R^{\circ}_{\Rightarrow}G\omega(u_1)\omega(u_2)$ and m satisfies the atomic hereditary condition, we have that $V(p, \omega(u_2)) = 1$. Hence $[p]_{\omega(u_2)} = 1$ and so $\widehat{p}(u_2) = 1$ as we wanted to show.

- $\widehat{=}$ is a map from $\mathcal{E}^{s2}$ to $\mathcal{B}^s$. This happens because $\widehat{=}(e_1, e_2)(u) \in 2$ by definition of $\widehat{=}$, and because, given e_1 and e_2 in $\mathcal{E}^s$, if $R^{\circ}_{\Rightarrow}G\omega(u_1)\omega(u_2)$, and $\widehat{=}(e_1, e_2)(u_1) = 1$ then $\widehat{=}(e_1, e_2)(u_2) = 1$. To show this note that $e_1(u_2) =$ (since D is a singleton, see the definition of s) $= e_1(u_1) = e_2(u_1) = e_2(u_2)$, as we wanted to show.

- $\widehat{\wedge}$ is a map from $\mathcal{B}^{s2}$ to $\mathcal{B}^s$. This happens because $\widehat{\wedge}(b_1, b_2)(u) \in 2$ by definition of $\widehat{\wedge}$, and because, given b_1 and b_2 in $\mathcal{B}^s$, if $R^{\circ}_{\Rightarrow}G\omega(u)\omega(u^{\cdot})$, and $\widehat{\wedge}(b_1, b_2)(u) = 1$ then $\widehat{\wedge}(b_1, b_2)(u^{\cdot}) = 1$. To show this let u and $u^{\cdot}$ be points in U^s and suppose $R^{\circ}_{\Rightarrow}G\omega(u)\omega(u^{\cdot})$, and $\widehat{\wedge}(b_1, b_2)(u) = 1$. Then $b_1(u) = 1$ and $b_2(u) = 1$, and so, due to the condition imposed to truth values in Example 5.3.1, $b_1(u^{\cdot}) = 1$ and $b_2(u^{\cdot}) = 1$. Therefore $\widehat{\wedge}(b_1, b_2)(u^{\cdot}) = 1$, as we wanted to show.

- $\widehat{\vee}$ is a map from $\mathcal{B}^{s2}$ to $\mathcal{B}^s$. We omit this proof since it is identical to the proof for $\wedge$.

- $\widehat{-}$ is a map from $\mathcal{B}^s$ to $\mathcal{B}^s$. This happens because $\widehat{-}(b)(u) \in 2$ by definition of $\widehat{-}$, and because, given b in $\mathcal{B}^s$, if $R^{\circ}_{\Rightarrow}G\omega(u)\omega(u^{\cdot})$, and $\widehat{-}(b)(u) = 1$ then $\widehat{-}(b)(u^{\cdot}) = 1$. To show this let a be the only assignment in D^X, and u and $u^{\cdot}$ points in U^s and suppose

$R^{\circ}_{\Rightarrow}G\omega(u)\omega(u^{\cdot})$, and $\widehat{-}(b)(u) = 1$. Then $b(\langle \omega(u)^{*\circ}, a\rangle) = 0$. Note that $R^{\circ}_{\Rightarrow}G\omega(u^{\cdot})^{*\circ}\omega(u)^{*\circ}$. Hence $b(\langle \omega(u^{\cdot})^{*\circ}, a\rangle) = 0$. and so $\widehat{-}(b)(u^{\cdot}) = 1$, as we wanted to show.

- $\widehat{\Rightarrow}$ is a map from $\mathcal{B}^{s2}$ to $\mathcal{B}^{s}$. This happens because $\widehat{\Rightarrow}(b_1, b_2)(u) \in 2$ by definition of $\widehat{\Rightarrow}$, and because, given b_1 and b_2 in $\mathcal{B}^s$, if $R^{\circ}_{\Rightarrow}G\omega(u)\omega(u^{\cdot})$, and $\widehat{\Rightarrow}(b_1, b_2)(u) = 1$ then $\widehat{\Rightarrow}(b_1, b_2)(u^{\cdot}) = 1$. To show this let u and $u^{\cdot}$ be points in U^s and suppose $R^{\circ}_{\Rightarrow}G\omega(u)\omega(u^{\cdot})$, and $\widehat{\Rightarrow}(b_1, b_2)(u) = 1$. Let $u^{\cdot}_1$ and $u^{\cdot}_2$ be points in U^s such that $\langle u^{\cdot}, u^{\cdot}_1, u^{\cdot}_2\rangle$ is in $[R_{\Rightarrow}]$ and $b_1(u^{\cdot}_1) = 1$. So $R^{\circ}_{\Rightarrow}\omega(u^{\cdot})\omega(u^{\cdot}_1)\omega(u^{\cdot}_2)$. Then, using one of the properties of $R^{\circ}_{\Rightarrow}$ we have that $R^{\circ}_{\Rightarrow}\omega(u)\omega(u^{\cdot}_1)\omega(u^{\cdot}_2)$, i.e., $\langle u, u^{\cdot}_1, u^{\cdot}_2\rangle$ is in $[R_{\Rightarrow}]$. Hence, because $\widehat{\Rightarrow}(b_1, b_2)(u) = 1$ and using also the fact that $b_1(u^{\cdot}_1) = 1$ we have that $b_2(u^{\cdot}_2) = 1$. Therefore $\widehat{\Rightarrow}(b_1, b_2)(u^{\cdot}) = 1$ as we wanted to show. *QED*

Ending this subsection we present a proposition that shows that, in the context of a structure induced by a model in $\mathcal{L}_{\mathrm{R}}$, there exists a strong relationship between truth values and hereditary sets. This relationship is expected, and gives us a feeling, proved in next subsections, that indeed our structures represent all and only the information of the usual models for the logic R. We omit the proof since it follows straightforwardly.

Proposition 5.3.3 Given a structure s in $\mathcal{L}_{\mathrm{R}}$ induced by a model m there is a bijection between the truth values in s and the hereditary sets in m.

5.3.2 Soundness

Theorem 5.3.4 The lfob logic system $\mathcal{L}_{\mathrm{R}}$ is sound.

Proof The proof follows by Theorem 2.4.6 since by Proposition 5.3.5 all structures of $\mathcal{L}_{\mathrm{R}}$ satisfy its rules. *QED*

Note that, according to Proposition 2.4.7, in order to show that all structures of $\mathcal{L}_{\mathrm{R}}$ satisfy its rules, it is sufficient to show that the structures satisfy the specific rules since every structure satisfies the rules common to all lfob deduction systems. In this subsection, we first present the proof that all structures of $\mathcal{L}_{\mathrm{R}}$ satisfy its rules and only after we show the auxiliary lemma needed along this proof.

Proposition 5.3.5 All structures of $\mathcal{L}_{\mathrm{R}}$ satisfy its rules.

Proof Let $m = \langle W, G, R^{\circ}_{\Rightarrow}, *_{\circ}, {f_c}^{\circ}, {f_s}^{\circ}, V\rangle$ be a model in $\mathcal{L}_{\mathrm{R}}$ and $s = \langle U, A, W, \alpha, \omega, D, \mathcal{E}, \mathcal{B}, [.]\rangle$ a structure in $\breve{m}$. Then, according to the definition of $\breve{m}$, either (i) s is a structure induced by m, or (ii) s is equivalent in terms of satisfaction to a structure induced by the model. Suppose (ii) holds. Then, by definition, s satisfies all rules and we are done. Suppose (i) holds. Then, we have to check that s, in this case denoted by s_m, satisfies all the rules. Note that, by Proposition 2.4.7, s_m satisfies the rules common to all lfob logic systems, hence, it is only needed to check that it satisfies the non-common rules. So, let a be the unique assignment in D^X, $r = \langle\{\langle\vartheta_1, \Psi_1, \eta_1\rangle, \ldots, \langle\vartheta_k, \Psi_k, \eta_k\rangle\}, \eta, P_f, P_d\rangle$ be a rule in $\mathcal{L}_{\mathrm{R}}$ that is not a rule common to all lfob logic systems, E a set of label schema variables, σ a schema substitution within L_E such that $(\pi_{\Sigma}\sigma_{\mathrm{nl}})$ is 1 for each π in P_f, and

$\pi_{\Sigma;E}(\sigma)$ is 1 for each π in P_d, and α a schema assignment over s_m and E. Thus,

suppose r is $mon_{R_{\Rightarrow}0}$, and that $s_m, \alpha \Vdash R_{\Rightarrow}0\nu_1\sigma\nu_2\sigma$ and $s_m, \alpha \Vdash \nu_1\sigma{:}\xi\sigma$. Observe that $R^{\circ}_{\Rightarrow}G\omega(\llbracket \nu_1\sigma \rrbracket^{s_m}_{\alpha})\omega(\llbracket \nu_2\sigma \rrbracket^{s_m}_{\alpha})$ and $\llbracket \xi\sigma \rrbracket^{s_m}_{\alpha}(\llbracket \nu_1\sigma \rrbracket^{s_m}_{\alpha}) = 1$. Hence $\llbracket \xi\sigma \rrbracket^{s_m}_{\alpha}(\llbracket \nu_2\sigma \rrbracket^{s_m}_{\alpha}) = 1$ by Lemma 5.3.6 and thus $s_m, \alpha \Vdash \nu_2\sigma{:}\xi\sigma$ as we wanted to show.

suppose r is *1 ind.* Note that $\llbracket \theta_1\sigma \rrbracket^{s_m}_{\alpha}(\llbracket \nu_1\sigma \rrbracket^{s_m}_{\alpha}) = \llbracket \theta_2\sigma \rrbracket^{s_m}_{\alpha}(\llbracket \nu_2\sigma \rrbracket^{s_m}_{\alpha})$, since D is a singleton. So, $s_m, \alpha \Vdash \nu_1\sigma{:}\theta_1\sigma =_g \nu_2\sigma{:}\theta_2\sigma$ as we wanted to show.

suppose r is *1 assg.* Note that $\alpha(\llbracket \nu_1\sigma \rrbracket^{s_m}_{\alpha}) = \alpha(\llbracket \nu_2\sigma \rrbracket^{s_m}_{\alpha})$, since there is only one assignment because D is a singleton. So, $s_m, \alpha \Vdash \nu_1\sigma \equiv_{\alpha} \nu_2\sigma$ as we wanted to show.

suppose r is $\Rightarrow_I$, and, for any $\alpha^{\cdot}$ $\{\nu_1\sigma, \nu_2\sigma\}$ co-equivalent to α, if $s_m, \alpha^{\cdot} \Vdash R_{\Rightarrow}\nu\sigma\nu_1\sigma\nu_2\sigma$ and $s_m, \alpha^{\cdot} \Vdash \nu_1\sigma{:}\xi_1\sigma$ then $s_m, \alpha^{\cdot} \Vdash \nu_2\sigma{:}\xi_2\sigma$. Let u_1' and u_2' be points with $\langle \llbracket \nu\sigma \rrbracket^{s_m}_{\alpha}, u_1', u_2' \rangle$ in $[R_{\Rightarrow}]^{s_m}$ and such that $\llbracket \xi_1\sigma \rrbracket^{s_m}_{\alpha}(u_1') = 1$. Consider the assignment $\alpha^{\cdot}$ $\{\nu_1\sigma, \nu_2\sigma\}$ co-equivalent to α, with $\nu_1\sigma\alpha^{\cdot} = u_1'$ and $\nu_2\sigma\alpha^{\cdot} = u_2'$. Note that $\llbracket \xi_1\sigma \rrbracket^{s_m}_{\alpha}$ is equal to $\llbracket \xi_1\sigma \rrbracket^{s_m}_{\alpha^{\cdot}}$, by Proposition 2.4.4, and $\llbracket \nu\sigma \rrbracket^{s_m}_{\alpha}$ is equal to $\llbracket \nu\sigma \rrbracket^{s_m}_{\alpha^{\cdot}}$, also by Proposition 2.4.4, since $\nu_1\sigma$ and $\nu_2\sigma$ are not in $\mathrm{lsv}(\nu\sigma)$. Then, $\llbracket \xi_1\sigma \rrbracket^{s_m}_{\alpha^{\cdot}}(\llbracket \nu_1\sigma \rrbracket^{s_m}_{\alpha^{\cdot}}) = 1$ and $\langle \llbracket \nu\sigma \rrbracket^{s}_{\alpha^{\cdot}}, \llbracket \nu_1\sigma \rrbracket^{s}_{\alpha^{\cdot}}, \llbracket \nu_2\sigma \rrbracket^{s}_{\alpha^{\cdot}} \rangle$ is in $[R_{\Rightarrow}]^{s_m}$. So, by the initial assumption, $\llbracket \xi_2\sigma \rrbracket^{s_m}_{\alpha^{\cdot}}(\llbracket \nu_2\sigma \rrbracket^{s_m}_{\alpha^{\cdot}}) = 1$, and thus $\llbracket \xi_2\sigma \rrbracket^{s_m}_{\alpha}(u_2') = 1$ by Proposition 2.4.4 and definition of $\alpha^{\cdot}$. Hence, $[\Rightarrow]^{s_m}_{\alpha(\llbracket \nu\sigma \rrbracket^{s_m}_{\alpha})}(\llbracket \xi_1\sigma \rrbracket^{s_m}_{\alpha}, \llbracket \xi_2\sigma \rrbracket^{s_m}_{\alpha})(\llbracket \nu\sigma \rrbracket^{s_m}_{\alpha}) = 1$ and therefore $s_m, \alpha \Vdash \nu\sigma{:}\xi_1\sigma \Rightarrow \xi_2\sigma$ as we wanted to show.

suppose r is $\Rightarrow_E$, and that $s_m, \alpha \Vdash R_{\Rightarrow}\nu\sigma\nu_1\sigma\nu_2\sigma$, $s_m, \alpha \Vdash \nu\sigma{:}\xi_1\sigma \Rightarrow \xi_2\sigma$, and $s_m, \alpha \Vdash \nu_1\sigma{:}\xi_1\sigma$. Observe that $\llbracket \xi_1\sigma \Rightarrow \xi_2\sigma \rrbracket^{s_m}_{\alpha}(\llbracket \nu\sigma \rrbracket^{s_m}_{\alpha}) = 1$, $\langle \llbracket \nu\sigma \rrbracket^{s_m}_{\alpha}, \llbracket \nu_1\sigma \rrbracket^{s_m}_{\alpha}, \llbracket \nu_2\sigma \rrbracket^{s_m}_{\alpha} \rangle$ is in $[R_{\Rightarrow}]$ and $\llbracket \xi_1\sigma \rrbracket^{s_m}_{\alpha}(\llbracket \nu_1\sigma \rrbracket^{s_m}_{\alpha}) = 1$. Then $\llbracket \xi_2\sigma \rrbracket^{s_m}_{\alpha}(\llbracket \nu_2\sigma \rrbracket^{s_m}_{\alpha}) = 1$ and so $s_m, \alpha \Vdash \nu_2\sigma{:}\xi_2\sigma$.

suppose r is $-_I$, and that if $s_m, \alpha \Vdash \nu\sigma^*{:}\xi\sigma$ then $s_m, \alpha \Vdash \nu'\sigma{:}\bot$. Note that $s_m, \alpha \nVdash \nu'\sigma{:}\bot$. So $s_m, \alpha \nVdash \nu\sigma^*{:}\xi\sigma$ and thus $\llbracket \xi\sigma \rrbracket^{s_m}_{\alpha}(\llbracket \nu\sigma^* \rrbracket^{s_m}_{\alpha}) = 0$. Note that $\llbracket \nu\sigma^* \rrbracket^{s_m}_{\alpha} = [*](\llbracket \nu\sigma \rrbracket^{s_m}_{\alpha}) = \langle \omega(\llbracket \nu\sigma \rrbracket^{s_m}_{\alpha})^{*\circ}, a \rangle$. So $\llbracket \xi\sigma \rrbracket^{s_m}_{\alpha}(\langle \omega(\llbracket \nu\sigma \rrbracket^{s_m}_{\alpha})^{*\circ}, a \rangle) = 0$. Hence $[-](\llbracket \xi\sigma \rrbracket^{s_m}_{\alpha})(\langle \omega(\llbracket \nu\sigma \rrbracket^{s_m}_{\alpha}), a \rangle) = 1$ and so $s_m, \alpha \Vdash \nu\sigma{:}-\xi\sigma$ as we wanted to show.

suppose r is $-_E$, and that $s_m, \alpha \Vdash \nu\sigma{:}-\xi\sigma$ and $s_m, \alpha \Vdash \nu\sigma^*{:}\xi\sigma$. It is straightforward to see that from this assumption we obtain a contradiction and so we can deduce everything. In particular $s_m, \alpha \Vdash \nu'\sigma{:}\bot$ as we wanted.

suppose r is $-_c$, and that if $s_m, \alpha \Vdash \nu\sigma{:}-\xi\sigma$ then $s_m, \alpha \Vdash \nu'\sigma{:}\bot$. Observe that $s_m, \alpha \nVdash \nu'\sigma{:}\bot$. So $s_m, \alpha \nVdash \nu\sigma{:}-\xi\sigma$ and thus $[-](\llbracket \xi\sigma \rrbracket^{s_m}_{\alpha})(\langle \omega(\llbracket \nu\sigma \rrbracket^{s_m}_{\alpha}), a \rangle) = 0$. Therefore we can conclude $\llbracket \xi\sigma \rrbracket^{s_m}_{\alpha}(\langle \omega(\llbracket \nu\sigma \rrbracket^{s_m}_{\alpha})^{*\circ}, a \rangle) = 1$. Note that $\langle \omega(\llbracket \nu\sigma \rrbracket^{s_m}_{\alpha})^{*\circ}, a \rangle = [*](\llbracket \nu\sigma \rrbracket^{s_m}_{\alpha}) = \llbracket \nu\sigma^* \rrbracket^{s_m}_{\alpha}$. Hence $\llbracket \xi\sigma \rrbracket^{s_m}_{\alpha}(\llbracket \nu\sigma^* \rrbracket^{s_m}_{\alpha}) = 1$. Thus $s_m, \alpha \Vdash \nu\sigma^*{:}\xi\sigma$ as we wanted to show.

suppose r is $\wedge_I$, $\wedge^1_E$, $\wedge^2_E$, $\vee_E$, $\vee^1_I$ or $\vee^2_I$. The proofs are similar to that for $\Rightarrow_E$, so they are omitted.

suppose r is $R_{\Rightarrow}{}^{\mathrm{lgen}}_{\alpha}$, and that $s_m, \alpha \Vdash R_{\Rightarrow}\nu_1\sigma\nu_2\sigma\nu_3\sigma$, $s_m, \alpha \Vdash \nu_1\sigma \equiv_{\omega} \nu_1'\sigma$, $s_m, \alpha \Vdash \nu_2\sigma \equiv_{\omega} \nu_2'\sigma$, and $s_m, \alpha \Vdash \nu_3\sigma \equiv_{\omega} \nu_3'\sigma$. Hence $R^{\circ}_{\Rightarrow}\omega(\llbracket \nu_1\sigma \rrbracket^{s_m}_{\alpha})\omega(\llbracket \nu_2\sigma \rrbracket^{s_m}_{\alpha})\omega(\llbracket \nu_3\sigma \rrbracket^{s_m}_{\alpha})$, $\omega(\llbracket \nu_1\sigma \rrbracket^{s_m}_{\alpha}) = \omega(\llbracket \nu_1'\sigma \rrbracket^{s_m}_{\alpha})$, $\omega(\llbracket \nu_2\sigma \rrbracket^{s_m}_{\alpha}) = \omega(\llbracket \nu_2'\sigma \rrbracket^{s_m}_{\alpha})$ and $\omega(\llbracket \nu_3\sigma \rrbracket^{s_m}_{\alpha}) = \omega(\llbracket \nu_3'\sigma \rrbracket^{s_m}_{\alpha})$. Therefore we have that $R^{\circ}_{\Rightarrow}\omega([\![\nu'_1\sigma]\!]^{s_m}_{\alpha})\omega([\![\nu'_2\sigma]\!]^{s_m}_{\alpha})\omega([\![\nu'_3\sigma]\!]^{s_m}_{\alpha})$ and so $s_m, \alpha \Vdash R_{\Rightarrow}\nu'_1\sigma\nu'_2\sigma\nu'_3\sigma$.

suppose r is *rmon1*, and that $s_m, \alpha \Vdash R_{\Rightarrow}0\nu\sigma\nu_1\sigma$ and $s_m, \alpha \Vdash R_{\Rightarrow}\nu_1\sigma\nu_2\sigma\nu_3\sigma$. Therefore $R^{\circ}_{\Rightarrow}G\omega([\![\nu\sigma]\!]^{s_m}_{\alpha})\omega([\![\nu_1\sigma]\!]^{s_m}_{\alpha})$ and $R^{\circ}_{\Rightarrow}\omega([\![\nu_1\sigma]\!]^{s_m}_{\alpha})\omega([\![\nu_2\sigma]\!]^{s_m}_{\alpha})\omega([\![\nu_3\sigma]\!]^{s_m}_{\alpha})$. Henceforth, taking into account the definition of a relational model for relevance logic R, we have that $R^{\circ}_{\Rightarrow}\omega([\![\nu\sigma]\!]^{s_m}_{\alpha})\omega([\![\nu_2\sigma]\!]^{s_m}_{\alpha})\omega([\![\nu_3\sigma]\!]^{s_m}_{\alpha})$ and so $s_m, \alpha \Vdash R_{\Rightarrow}\nu\sigma\nu_2\sigma\nu_3\sigma$ as we wanted to show.

suppose r is *rmon2*, *rmon3*, *suff1*, *suff2*, *cont1*, *cont2*, *comm*, *inv*, $^{**}{}_i$, $^{**}{}_c$ or *iden*. The proofs are similar to the proof for *rmon1*, so they are omitted. *QED*

We now show the auxiliary lemma needed in the proof of Proposition 5.3.5.

Lemma 5.3.6 We have, for any structure s_m induced by a model m in $\mathcal{L}_{\mathrm{R}}$,

$$[\![\varphi]\!]^{s_m}_{\alpha}(u_2) = 1 \quad \text{whenever} \quad [\![\varphi]\!]^{s_m}_{\alpha}(u_1) = 1 \text{ and } R^{\circ}_{\Rightarrow}G\omega(u_1)\omega(u_2)$$

for any schema formula φ.

Proof The proof follows by induction on the structure of the schema formula φ. Let a be the unique assignment in D^X and assume $[\![\varphi]\!]^{s_m}_{\alpha}(u_1) = 1$ and $R^{\circ}_{\Rightarrow}G\omega(u_1)\omega(u_2)$. So,

suppose φ *is in* P_0. Note that $V(\varphi, \omega(u_1)) = 1$. So, by the atomic hereditary condition in the definition of a relational model for the relevance logic R, $V(\varphi, \omega(u_2)) = 1$, and thus, $[\![\varphi]\!]^{s_m}_{\alpha}(u_2) = 1$, as we wanted to show.

suppose φ *is in* Ξ_f. Suppose $[\![\varphi]\!]^{s_m}_{\alpha}(u_1) = 1$ and $R^{\circ}_{\Rightarrow}G\omega(u_1)\omega(u_2)$. Then u_1 is in $\varphi\alpha_f$. So, u_2 is in $\varphi\alpha_f$ because of the condition imposed to the truth values in $\mathcal{L}_{\mathrm{R}}$. Hence $[\![\varphi]\!]^{s_m}_{\alpha}(u_2) = 1$ as we wanted to show.

suppose φ *is* $x = y$. Then, $[\![\varphi]\!]^{s_m}_{\alpha}(u_2) = 1$ because $\alpha(u_2)(x) = \alpha(u_2)(y)$ since D is a singleton.

suppose φ *is* $-\varphi_1$. Then we have that $[\![\varphi_1]\!]^{s_m}_{\alpha}(\langle\omega(u_1)^{*\circ}, a\rangle) = 0$ and $R^{\circ}_{\Rightarrow}G\omega(u_2)^{*\circ}\omega(u_1)^{*\circ}$. Therefore $[\![\varphi_1]\!]^{s_m}_{\alpha}(\langle\omega(u_2)^{*\circ}, a\rangle) = 0$. and so $[\![\varphi]\!]^{s_m}_{\alpha}(u_2) = 1$, as we wanted to show.

suppose φ *is* $\bot$. Note that the assumption $[\![\bot]\!]^{s_m}_{\alpha}(u_1) = 1$ contradicts the definition of s_m. So we can conclude $[\![\bot]\!]^{s_m}_{\alpha}(u_2) = 1$ since from a contradiction we can assert anything.

suppose φ *is* $\varphi_1 \Rightarrow \varphi_2$. Let u'_2 and u''_2 be points such that $R^{\circ}_{\Rightarrow}\omega(u_2)\omega(u'_2)\omega(u''_2)$ and $[\![\varphi_1]\!]^{s_m}_{\alpha}(u'_2) = 1$. Then $R^{\circ}_{\Rightarrow}\omega(u_1)\omega(u'_2)\omega(u''_2)$, and so $[\![\varphi_2]\!]^{s_m}_{\alpha}(u''_2) = 1$.

suppose φ *is* $\varphi_1 \wedge \varphi_2$. The result follows because $[\![\varphi_1]\!]^{s_m}_{\alpha}(u_1) = 1$ and $[\![\varphi_2]\!]^{s_m}_{\alpha}(u_1) = 1$, and so, by the induction hypothesis, $[\![\varphi_1]\!]^{s_m}_{\alpha}(u_2) = 1$ and $[\![\varphi_2]\!]^{s_m}_{\alpha}(u_2) = 1$, as we wanted to show.

suppose φ *is* $\varphi_1 \vee \varphi_2$. Then the proof is similar to the case where φ is $\varphi_1 \wedge \varphi_2$. *QED*

5.3.3 Completeness

The goal of this subsection is to show the following theorem, which we prove at the end of the section.

Theorem 5.3.7 The lfob logic system $\mathcal{L}_{\mathrm{R}}$ is complete.

To prove this theorem we show that $\mathcal{L}_{\mathrm{R}}$ is rich and then we invoke Theorem 3.3.3 which says that a rich lfob logic system is complete. But before we need to show some auxiliary propositions.

Proposition 5.3.8 The lfob logic system $\mathcal{L}_{\mathrm{R}}$ is connected and with a classical negation.

Proof It is straightforward to see that $\mathcal{L}_{\mathrm{R}}$ is connected and that it has a non-local classical negation. Recall Definition 4.1.3 of a connected lfob logic system and Definition 4.3.1 of a clfob deduction system with a non-local classical negation. *QED*

We need now to introduce the notion of a relational model for relevance logic R, induced by an appropriate set in the context of $\mathcal{L}_{\mathrm{R}}$.

Prop/Definition 5.3.9 In the context of $\mathcal{L}_{\mathrm{R}}$, the model m_Ψ *induced by* an E appropriate set Ψ is defined as follows:

- W is $\{[v]_\omega^{\Psi,E} \mid v \text{ in } T_{\mathrm{lab},E}\}$,
- G is $[0]_\omega^{\Psi,E}$,
- $R^\circ_\Rightarrow$ is $\{\langle [v_1]_\omega^{\Psi,E}, [v_2]_\omega^{\Psi,E}, [v_3]_\omega^{\Psi,E}\rangle \mid R_\Rightarrow v_1 v_2 v_3 \text{ is in } \Psi\}$,
- ${*}_\circ([v]_\omega^{\Psi,E}) = [v^*]_\omega^{\Psi,E}$,
- ${f_c}^\circ([v_1]_\omega^{\Psi,E}, [v_2]_\omega^{\Psi,E}, [v_3]_\omega^{\Psi,E}) = [f_c(v_1, v_2, v_3)]_\omega^{\Psi,E}$,
- ${f_s}^\circ([v_1]_\omega^{\Psi,E}, [v_2]_\omega^{\Psi,E}, [v_3]_\omega^{\Psi,E}, [v_4]_\omega^{\Psi,E}, [v_5]_\omega^{\Psi,E}) = [f_s(v_1, v_2, v_3, v_4, v_5)]_\omega^{\Psi,E}$,
- $V(p, [v]_\omega^{\Psi,E}) = 1$ iff $v{:}p$ is in Ψ (note that if $v{:}p, v \equiv_\omega v' \in \Psi$ then $v'{:}p \in \Psi$).

We denote by s_{m_Ψ} the structure induced by m_Ψ where

$$D^{s_{m_\Psi}} = \{[v{:}t]_g^{\Psi,E} \mid v \text{ is in } T_{\mathrm{lab},E} \text{ and } t \text{ is in } T\},$$

which is a singleton by rule *1 ind*.

Proof Given an E appropriate set Ψ the model m_Ψ is indeed a relational model for relevance logic R because:

- ${*}_\circ$, ${f_c}^\circ$ and ${f_s}^\circ$ are well defined. The proof follows straightforwardly by rules ${*}_\alpha^{\mathrm{lgen}}$, ${f_c}_\alpha^{\mathrm{lgen}}$, and ${f_s}_\alpha^{\mathrm{lgen}}$.

- $R^\circ_\Rightarrow [v]_\omega^{\Psi,E} [v_2]_\omega^{\Psi,E} [v_3]_\omega^{\Psi,E}$ whenever $R^\circ_\Rightarrow [0]_\omega^{\Psi,E} [v]_\omega^{\Psi,E} [v1]_\omega^{\Psi,E}$ and $R^\circ_\Rightarrow [v_1]_\omega^{\Psi,E} [v_2]_\omega^{\Psi,E} [v_3]_\omega^{\Psi,E}$. To see this suppose $R^\circ_\Rightarrow [0]_\omega^{\Psi,E} [v]_\omega^{\Psi,E} [v1]_\omega^{\Psi,E}$ and $R^\circ_\Rightarrow [v_1]_\omega^{\Psi,E} [v_2]_\omega^{\Psi,E} [v_3]_\omega^{\Psi,E}$. Then, by definition of m_Ψ and by rule ${R_\Rightarrow}_\alpha^{\mathrm{lgen}}$, we have that $R_\Rightarrow 0vv1$ and $R_\Rightarrow v_1 v_2 v_3$ are in Ψ. So $R_\Rightarrow v v_2 v_3$ is in Ψ and thus $R^\circ_\Rightarrow [v]_\omega^{\Psi,E} [v_2]_\omega^{\Psi,E} [v_3]_\omega^{\Psi,E}$ as we wanted to show.

- for the other conditions on $R^\circ_\Rightarrow$ the proof follows similarly. *QED*

Remark 5.3.10 In the following, in the context of $\mathcal{L}_{\mathrm{R}}$, given an E appropriate set Ψ and a label schema term v, we denote by 1 the unique assignment in $D^{s_{m_\Psi}X}$, and by $[v]_{\omega\times 1}^{\Psi,E}$ the pair $\langle [v]_\omega^{\Psi,E}, 1\rangle$ in $U^{s_{m_\Psi}}$. Moreover, given a set V of label schema terms, we denote by $[V]_\omega^{\Psi,E}$ the set $\{[v]_\omega^{\Psi,E} \mid v \in V\}$, and by $[V]_{\omega\times 1}^{\Psi,E}$ the set $\{[v]_{\omega\times 1}^{\Psi,E} \mid v \in V\}$. Finally, we extend in an obvious way this notation to sets of tuples of label schema terms.

In the next lemma we prove some auxiliary facts that will be needed in the proof of Lemma 5.3.13.

Lemma 5.3.11 In the context of $\mathcal{L}_{\mathrm{R}}$, given an E appropriate set Ψ,

- v is in $|\gamma|^{s_\Psi}$ iff $[v]^{\Psi,E}_{\omega\times 1}$ is in $[|\gamma|^{s_\Psi}]^{\Psi,E}_{\omega\times 1}$,
- if $R_{\Rightarrow}0v_1v_2$ is in Ψ and $[v_1]^{\Psi,E}_{\omega\times 1}$ is in $[|\gamma|^{s_\Psi}]^{\Psi,E}_{\omega\times 1}$ then $[v_2]^{\Psi,E}_{\omega\times 1}$ is in $[|\gamma|^{s_\Psi}]^{\Psi,E}_{\omega\times 1}$,
- $\langle v_1,\ldots,v_n\rangle$ is in $[R_{\Rightarrow}]^{s_\Psi}$ iff $\langle [v_1]^{\Psi,E}_{\omega\times 1},\ldots,[v_n]^{\Psi,E}_{\omega\times 1}\rangle$ is in $[[R_{\Rightarrow}]^{s_\Psi}]^{\Psi,E}_{\omega\times 1}$,
- $[|\gamma|^{s_\Psi}]^{\Psi,E}_{\omega\times 1}$ is in $\mathcal{B}^{s_{m_\Psi}}$,

for any schema formula γ in $\mathcal{L}_{\mathrm{R}}$.

Proof In order to show the first item, suppose $[v]^{\Psi,E}_{\omega\times 1}$ is in $[|\gamma|^{s_\Psi}]^{\Psi,E}_{\omega\times 1}$. Then, there is $v^{\cdot}$ with $v^{\cdot}{:}\gamma$ in Ψ, and $v \equiv_\omega v^{\cdot}$ in Ψ. Note that, by rule *iden*, $R_{\Rightarrow}0v^{\cdot}v^{\cdot}$ is in Ψ, and so, $R_{\Rightarrow}0v^{\cdot}v$ is also in Ψ by rule $R_{\Rightarrow}{}^{\mathrm{lgen}}_{\alpha}$. Then, by rule $mon_{R_{\Rightarrow}0}$, $v{:}\gamma$ is in Ψ. Hence v is in $|\gamma|^{s_\Psi}$ as we would like to show. For the left to right direction the proof is immediate so we omit it.

With respect to the second item, let γ be a schema formula, and v_1 and v_2 be label schema terms. Suppose $R_{\Rightarrow}0v_1v_2$ is in Ψ and $[v_1]^{\Psi,E}_{\omega\times 1}$ is in $[|\gamma|^{s_\Psi}]^{\Psi,E}_{\omega\times 1}$. Then, by the previous item, v_1 is in $|\gamma|^{s_\Psi}$, and so $v_1{:}\gamma$ is in Ψ. Then, by rule $mon_{R_{\Rightarrow}0}$, $v_2{:}\gamma$ is in Ψ. Therefore $[v_2]^{\Psi,E}_{\omega\times 1}$ is in $[|\gamma|]^{\Psi,E}_{\omega\times 1}$, as we wanted to show.

In order to show the third item note that the proof of the left to right direction is immediate. For the opposite direction suppose $\langle [v_1]^{\Psi,E}_{\omega\times 1},\ldots,[v_n]^{\Psi,E}_{\omega\times 1}\rangle$ is in $[[R_{\Rightarrow}]^{s_\Psi}]^{\Psi,E}_{\omega\times 1}$. So, $R_{\Rightarrow}v^{\cdot}_1\ldots v^{\cdot}_n$, and for $i=1,\ldots,n$ $v^{\cdot}_i \equiv_\omega v_i$ are in Ψ, for some label schema terms $v^{\cdot}_1\ldots v^{\cdot}_n$. Therefore, $R_{\Rightarrow}v_1\ldots v_n$ is in Ψ, by rule $R_{\Rightarrow}{}^{\mathrm{lgen}}_{\alpha}$, and so $\langle v_1,\ldots,v_n\rangle$ is in $[R_{\Rightarrow}]^{s_\Psi}$, as we wanted to show.

For the fourth item it is enough, according to the definition of $\mathcal{L}_{\mathrm{R}}$, Example 5.3.1, to prove that:

- $[|\gamma|^{s_\Psi}]^{\Psi,E}_{\omega\times 1}$ is contained in $U^{s_{m_\Psi}}$. This follows due to Definition 5.3.9 of a relational model for the relevance logic R induced by an appropriate set, due to the definition of D in s_{m_Ψ} in Definition 5.3.9, and by definition, in Example 5.3.1, of the set of points U of the structure associated to a model.

- if $R^{\circ}_{\Rightarrow}G^{m_\Psi}\omega_{s_{m_\Psi}}(u_1)\omega_{s_{m_\Psi}}(u_2)$, and $[|\gamma|^{s_\Psi}]^{\Psi,E}_{\omega\times 1}(u_1)=1$ then $[|\gamma|^{s_\Psi}]^{\Psi,E}_{\omega\times 1}(u_2)=1$. Let u_1 equal to $\langle [v_1]^{\Psi,E}_{\omega},1\rangle$ and u_2 equal to $\langle [v_2]^{\Psi,E}_{\omega},1\rangle$ be points in $U^{s_{m_\Psi}}$. Note that u_1 is equal to $[v_1]^{\Psi,E}_{\omega\times 1}$ and u_2 is equal to $[v_2]^{\Psi,E}_{\omega\times 1}$. Suppose that $R^{\circ}_{\Rightarrow}G^{m_\Psi}\omega_{s_{m_\Psi}}(u_1)\omega_{s_{m_\Psi}}(u_2)$ and $[|\gamma|^{s_\Psi}]^{\Psi,E}_{\omega\times 1}(u_1)=1$. So $R^{\circ}_{\Rightarrow}[0]^{\Psi,E}_{\omega}[v_1]^{\Psi,E}_{\omega}[v_2]^{\Psi,E}_{\omega}$, and thus, based on the definition of m_Ψ, Definition 5.3.9, it is straightforward to show that $R_{\Rightarrow}0v_1v_2$ is in Ψ, by rule $R_{\Rightarrow}{}^{\mathrm{lgen}}_{\alpha}$. Then, according to the second item above proven, u_2 i.e., $[v_2]^{\Psi,E}_{\omega\times 1}$ is in $[|\gamma|^{s_\Psi}]^{\Psi,E}_{\omega\times 1}$, as we wanted to show. *QED*

Prop/Definition 5.3.12 In the context of $\mathcal{L}_{\mathrm{R}}$, given an E appropriate set Ψ and a assignment α over s_Ψ, we denote by $\alpha_{\omega\times 1}$ the assignment over s_{m_Ψ} defined as follows

- $\nu\alpha_{\omega\times 1_l} = [\nu\alpha_l]^{\Psi,E}_{\omega\times 1}$,
- $(\theta\alpha_{\omega\times 1_t})([v]^{\Psi,E}_{\omega\times 1}) = (\theta\alpha_t)(v)$
- $\xi\alpha_{\omega\times 1_f} = [\xi\alpha_f]^{\Psi,E}_{\omega\times 1}$

Proof $\alpha_{\omega\times 1_t}$ is well defined. Suppose $\theta\alpha_t$ is $|t|$ for some schema term t. Then, for any label schema terms v and $v^{\cdot}$, $(\theta\alpha_{\omega\times 1_t})([v]^{\Psi,E}_{\omega\times 1}) = (\theta\alpha_t)(v) = [v{:}t]^{\Psi,E}_g$ =(by rule *1 ind* since Ψ is an appropriate set)= $[v^{\cdot}{:}t]^{\Psi,E}_g = (\theta\alpha_t)(v^{\cdot}) = (\theta\alpha_{\omega\times 1_t})([v^{\cdot}]^{\Psi,E}_{\omega\times 1})$ as we wanted to show.

$\theta\alpha_{\omega\times 1_t}$ is in $\mathcal{E}^{s_{m_\Psi}}$. This happens since $\theta\alpha_{\omega\times 1_t}$ is in $D^{s_{m_\Psi}U^{s_{m_\Psi}}}$ as is straightforward to see taking into account the definition of $\theta\alpha_{\omega\times 1_t}$.

$\xi\alpha_{\omega\times 1_f}$ is in $\mathcal{B}^{s_{m_\Psi}}$. Immediate by Lemma 5.3.11. *QED*

We now show that satisfaction in the canonical structure and in the structure induced by the canonical model, for any labelled schema formulae, is equivalent.

Lemma 5.3.13 In the context of $\mathcal{L}_{\mathrm{R}}$, given an E appropriate set Ψ,

$$s_\Psi, \alpha \Vdash \eta \quad \text{iff} \quad s_{m_\Psi}, \alpha_{\omega\times 1} \Vdash \eta.$$

for any assignment α over s_Ψ and labelled schema formula η in $\mathcal{L}_{\mathrm{R}}$.

Proof Note that by Proposition 3.1.4, α is $|\cdot| \circ \sigma$ for some schema substitution σ. First we show some auxiliary results and only after those results we show the proposition.

0. $[\llbracket v\rrbracket^{s_\Psi}_\alpha]^{\Psi,E}_{\omega\times 1} = \llbracket v\rrbracket^{s_{m_\Psi}}_{\alpha_{\omega\times 1}}$, where v is a label schema term. The proof follows by induction on the possible cases for the label schema term v in $\mathcal{L}_{\mathrm{R}}$.

If v is in Ξ_l or in E then $[\llbracket v\rrbracket^{s_\Psi}_\alpha]^{\Psi,E}_{\omega\times 1} = [v\alpha]^{\Psi,E}_{\omega\times 1} = \nu\alpha_{\omega\times 1} = \llbracket v\rrbracket^{s_{m_\Psi}}_{\alpha_{\omega\times 1}}$.

If v is 0 then we have that $[\llbracket 0\rrbracket^{s_\Psi}_\alpha]^{\Psi,E}_{\omega\times 1} = [0]^{\Psi,E}_{\omega\times 1} = \langle [0]^{\Psi,E}_\omega, 1\rangle = \langle G^{m_\Psi}, 1\rangle$ = (by definition of s_{m_Ψ} in Definition 5.3.9) $= [0]^{s_{m_\Psi}} = \llbracket 0\rrbracket^{s_{m_\Psi}}_{\alpha_{\omega\times 1}}$.

If v is $v_1{}^*$ then $[\llbracket v\rrbracket^{s_\Psi}_\alpha]^{\Psi,E}_{\omega\times 1}$ = (by Proposition 3.1.3) $= [v_1\sigma^*]^{\Psi,E}_{\omega\times 1} = \langle [v_1\sigma^*]^{\Psi,E}_\omega, 1\rangle$ = (by definition of m_Ψ, Definition 5.3.9) $= \langle {}^{*\circ}([v_1\sigma]^{\Psi,E}_\omega), 1\rangle = [^*]([v_1\sigma]^{\Psi,E}_{\omega\times 1})$ = (by Proposition 3.1.3) $= [^*]([\llbracket v_1\rrbracket^{s_\Psi}_\alpha]^{\Psi,E}_{\omega\times 1})$ = (by induction hypothesis) $= [^*](\llbracket v_1\rrbracket^{s_{m_\Psi}}_{\alpha_{\omega\times 1}}) = \llbracket v\rrbracket^{s_{m_\Psi}}_{\alpha_{\omega\times 1}}$.

If v is $f_c(v_1, v_2, v_3)$ then $[\llbracket v\rrbracket^{s_\Psi}_\alpha]^{\Psi,E}_{\omega\times 1}$ = (by Proposition 3.1.3) $= [f_c(v_1\sigma, v_2\sigma, v_3\sigma)]^{\Psi,E}_{\omega\times 1} = \langle [f_c(v_1\sigma, v_2\sigma, v_3\sigma)]^{\Psi,E}_\omega, 1\rangle$ = (by definition of m_Ψ) $= \langle f_c{}^\circ([v_1\sigma]^{\Psi,E}_\omega, [v_2\sigma]^{\Psi,E}_\omega, [v_3\sigma]^{\Psi,E}_\omega), 1\rangle = [f_c]([v_1\sigma]^{\Psi,E}_{\omega\times 1}, [v_2\sigma]^{\Psi,E}_{\omega\times 1}, [v_3\sigma]^{\Psi,E}_{\omega\times 1})$ = (by Proposition 3.1.3) $= [f_c]([\llbracket v_1\rrbracket^{s_\Psi}_\alpha]^{\Psi,E}_{\omega\times 1}, [\llbracket v_2\rrbracket^{s_\Psi}_\alpha]^{\Psi,E}_{\omega\times 1}, [\llbracket v_3\rrbracket^{s_\Psi}_\alpha]^{\Psi,E}_{\omega\times 1})$ = (by induction hypothesis) $= [f_c](\llbracket v_1\rrbracket^{s_{m_\Psi}}_{\alpha_{\omega\times 1}}, \llbracket v_2\rrbracket^{s_{m_\Psi}}_{\alpha_{\omega\times 1}}, \llbracket v_3\rrbracket^{s_{m_\Psi}}_{\alpha_{\omega\times 1}}) = \llbracket v\rrbracket^{s_{m_\Psi}}_{\alpha_{\omega\times 1}}$.

If v is $f_s(v_1, v_2, v_3, v_4, v_5)$ then the proof is similar to the case where v is $f_s(v_1, v_2, v_3, v_4, v_5)$.

1. $\llbracket t\rrbracket^{s_\Psi}_\alpha(v) = \llbracket t\rrbracket^{s_{m_\Psi}}_{\alpha_{\omega\times 1}}([v]^{\Psi,E}_{\omega\times 1})$, where t is schema term and v is a label schema term. The proof follows by induction on the possible cases for the schema term t in $\mathcal{L}_{\mathrm{R}}$. Then,

if t is in X then $\llbracket t\rrbracket^{s_\Psi}_\alpha(v) = [t]^{s_\Psi}_{\alpha_{s_\Psi}(v)} = [v{:}t]^{\Psi,E}_g = 1(t) = [t]^{s_{m_\Psi}}_1 = [t]^{s_{m_\Psi}}_{\alpha_{s_{m_\Psi}}([v]^{\Psi,E}_{\omega\times 1})} =$

$\llbracket t\rrbracket^{s_{m_\Psi}}_{\alpha_{\omega\times 1}}([v]^{\Psi,E}_{\omega\times 1})$.

if t is in Ξ_t then $\llbracket t\rrbracket^{s_\Psi}_{\alpha}(v) = (t\alpha_t)(v) =$ (by definition of $\alpha_{\omega\times 1}$, Definition 5.3.12) $= (t\alpha_{\omega\times 1 t})([v]^{\Psi,E}_{\omega\times 1}) = \llbracket t\rrbracket^{s_{m_\Psi}}_{\alpha_{\omega\times 1}}([v]^{\Psi,E}_{\omega\times 1})$.

2. $[\llbracket\gamma\rrbracket^{s_\Psi}_{\alpha}]^{\Psi,E}_{\omega\times 1} = \llbracket\gamma\rrbracket^{s_{m_\Psi}}_{\alpha_{\omega\times 1}}$. The proof follows by induction on the structure of the schema formula γ. So,

suppose γ is in Ξ_f. Then $[\llbracket\gamma\rrbracket^{s_\Psi}_{\alpha}]^{\Psi,E}_{\omega\times 1} = [\gamma\alpha_f]^{\Psi,E}_{\omega\times 1} =$ (by definition of $\alpha_{\omega\times 1}$, Definition 5.3.12) $= \gamma\alpha_{\omega\times 1 f} = \llbracket\gamma\rrbracket^{s_{m_\Psi}}_{\alpha_{\omega\times 1}}$.

suppose γ is $t_1 = t_2$. Then $[v]^{\Psi,E}_{\omega\times 1}$ is in $[\llbracket\gamma\rrbracket^{s_\Psi}_{\alpha}]^{\Psi,E}_{\omega\times 1}$ iff (by Proposition 5.3.11) v is in $\llbracket\gamma\rrbracket^{s_\Psi}_{\alpha}$ iff $\llbracket t_1\rrbracket^{s_\Psi}_{\alpha}(v) = \llbracket t_2\rrbracket^{s_\Psi}_{\alpha}(v)$ iff (by item 1. above proven) $\llbracket t_1\rrbracket^{s_{m_\Psi}}_{\alpha_{\omega\times 1}}([v]^{\Psi,E}_{\omega\times 1}) = \llbracket t_2\rrbracket^{s_{m_\Psi}}_{\alpha_{\omega\times 1}}([v]^{\Psi,E}_{\omega\times 1})$ iff $[v]^{\Psi,E}_{\omega\times 1}$ is in $\llbracket\gamma\rrbracket^{s_{m_\Psi}}_{\alpha_{\omega\times 1}}$, as we wanted to show.

suppose γ is p. Then $[v]^{\Psi,E}_{\omega\times 1}$ is in $[\llbracket\gamma\rrbracket^{s_\Psi}_{\alpha}]^{\Psi,E}_{\omega\times 1}$ iff (by Proposition 5.3.11) v is in $\llbracket\gamma\rrbracket^{s_\Psi}_{\alpha}$ iff (by Proposition 3.1.3) v is in $|\gamma|^{s_\Psi}$ iff $v{:}\gamma$ is in Ψ iff $V^{m_\Psi}(\gamma, [v]^{\Psi,E}_{\omega}) = 1$ iff $[\gamma]^{s_{m_\Psi}}_{[v]^{\Psi,E}_{\omega}} = 1$ iff $[\gamma]^{s_{m_\Psi}}_{\omega([v]^{\Psi,E}_{\omega\times 1})} = 1$ iff $\llbracket\gamma\rrbracket^{s_{m_\Psi}}_{\alpha_{\omega\times 1}}([v]^{\Psi,E}_{\omega\times 1}) = 1$ as we wanted to show.

suppose γ is $\bot$. Note that for any label schema term v we have that $[v]^{\Psi,E}_{\omega\times 1}$ is not in $[\llbracket\bot\rrbracket^{s_\Psi}_{\alpha}]^{\Psi,E}_{\omega\times 1}$ and $[v]^{\Psi,E}_{\omega\times 1}$ is not in $\llbracket\bot\rrbracket^{s_{m_\Psi}}_{\alpha_{\omega\times 1}}$, and so $[\llbracket\bot\rrbracket^{s_\Psi}_{\alpha}]^{\Psi,E}_{\omega\times 1} = \llbracket\bot\rrbracket^{s_{m_\Psi}}_{\alpha_{\omega\times 1}}$. Let v be a label schema term. Denote by $v'{:}\varphi'$ a labelled schema formula with respect to which Ψ is consistent. Suppose $v{:}\bot$ is in Ψ. Then, by rule $-_c$, we have that $v'^{**}{:}\varphi'$ is in Ψ. So, by rules ${}^{**}{}_c$ and $mon_{R_\Rightarrow 0}$, and since Ψ is deductively closed, we can conclude that $v'{:}\varphi'$ is in Ψ which contradicts the fact that Ψ is $v'{:}\varphi'$ consistent. So $v{:}\bot$ is not in Ψ. Hence v is not in $|\bot|^{s_\Psi}$ and thus by Proposition 3.1.3, v is not in $\llbracket\bot\rrbracket^{s_\Psi}_{\alpha}$. Therefore $[v]^{\Psi,E}_{\omega\times 1}$ is not in $[\llbracket\bot\rrbracket^{s_\Psi}_{\alpha}]^{\Psi,E}_{\omega\times 1}$ as we wanted to show, by Proposition 5.3.11. For the other fact it is enough to see that, by definition of induced structure by a relational model, Example 5.3.1, we have that $\llbracket\bot\rrbracket^{s_{m_\Psi}}_{\alpha_{\omega\times 1}}([v]^{\Psi,E}_{\omega\times 1}) = 0$.

suppose γ is $-\gamma_1$. Then $[v]^{\Psi,E}_{\omega\times 1}$ is in $[\llbracket\gamma\rrbracket^{s_\Psi}_{\alpha}]^{\Psi,E}_{\omega\times 1}$ iff (by Proposition 5.3.11) v is in $\llbracket\gamma\rrbracket^{s_\Psi}_{\alpha}$ iff (by Proposition 3.1.3) v is in $|\gamma\sigma|^{s_\Psi}$ iff $v{:}-\gamma_1\sigma$ is in Ψ iff (since Ψ is an appropriate set) if $v^*{:}\gamma_1\sigma$ is in Ψ then $v'{:}\bot$ is in Ψ iff (since $v'{:}\bot$ is not in Ψ) v^* is not in $|\gamma_1\sigma|^{s_\Psi}$ iff (by Proposition 3.1.3) v^* is not in $\llbracket\gamma_1\rrbracket^{s_\Psi}_{\alpha}$ iff $\langle [v^*]^{\Psi,E}_{\omega}, 1\rangle$ is not in $[\llbracket\gamma_1\rrbracket^{s_\Psi}_{\alpha}]^{\Psi,E}_{\omega\times 1}$ iff (by induction hypothesis) $\langle *_\circ([v]^{\Psi,E}_{\omega}), 1\rangle$ is not in $\llbracket\gamma_1\rrbracket^{s_{m_\Psi}}_{\alpha}$ iff $[-](\llbracket\gamma_1\rrbracket^{s_{m_\Psi}}_{\alpha})([v]^{\Psi,E}_{\omega\times 1}) = 1$ iff $\llbracket\gamma\rrbracket^{s_{m_\Psi}}_{\alpha_{\omega\times 1}}([v]^{\Psi,E}_{\omega\times 1}) = 1$.

suppose γ is $\gamma_1 \Rightarrow \gamma_2$. Then $[v]^{\Psi,E}_{\omega\times 1}$ is in $[\llbracket\gamma\rrbracket^{s_\Psi}_{\alpha}]^{\Psi,E}_{\omega\times 1}$ iff (by Lemma 5.3.11) v is in $\llbracket\gamma\rrbracket^{s_\Psi}_{\alpha}$ iff (by Proposition 3.1.3) v is in $|\gamma\sigma|$ iff $v{:}\gamma_1\sigma \Rightarrow \gamma_2\sigma$ is in Ψ iff (since Ψ is an appropriate set) for all v_1 and v_2, if $R_\Rightarrow v v_1 v_2$ and $v_1{:}\gamma_1\sigma$ are in Ψ then $v_2{:}\gamma_2\sigma$ is in Ψ iff for all v_1 and v_2, if $\langle v, v_1, v_2\rangle$ is in $[R_\Rightarrow]^{s_\Psi}$ and v_1 is in $|\gamma_1\sigma|^{s_\Psi}$ then v_2 is in $|\gamma_2\sigma|^{s_\Psi}$ iff (by Lemma 5.3.11) for all u_1 and u_2 in $U^{s_{m_\Psi}}$, if $\langle [v]^{\Psi,E}_{\omega\times 1}, u_1, u_2\rangle$ is in $[[R_\Rightarrow]^{s_\Psi}]^{\Psi,E}_{\omega\times 1}$ and u_1 is in $[|\gamma_1\sigma|^{s_\Psi}]^{\Psi,E}_{\omega\times 1}$ then u_2 is in $[|\gamma_2\sigma|^{s_\Psi}]^{\Psi,E}_{\omega\times 1}$ iff (by Proposition 3.1.3) for all u_1 and u_2 in $U^{s_{m_\Psi}}$, if $\langle [v]^{\Psi,E}_{\omega\times 1}, u_1, u_2\rangle$ is in $[[R_\Rightarrow]^{s_\Psi}]^{\Psi,E}_{\omega\times 1}$ and u_1 is in $[\llbracket\gamma_1\rrbracket^{s_\Psi}_{\alpha}]^{\Psi,E}_{\omega\times 1}$ then u_2 is in $[\llbracket\gamma_2\rrbracket^{s_\Psi}_{\alpha}]^{\Psi,E}_{\omega\times 1}$ iff (by induction hypothesis and fact 3. proven below) for all u_1 and u_2 in $U^{s_{m_\Psi}}$ if $\langle [v]^{\Psi,E}_{\omega\times 1}, u_1, u_2\rangle$ is in $[R_\Rightarrow]^{s_{m_\Psi}}$ and u_1 is in $\llbracket\gamma_1\rrbracket^{s_{m_\Psi}}_{\alpha_{\omega\times 1}}$ then u_2 is in $\llbracket\gamma_2\rrbracket^{s_{m_\Psi}}_{\alpha_{\omega\times 1}}$ iff $[v]^{\Psi,E}_{\omega\times 1}$ is in $\llbracket\gamma\rrbracket^{s_{m_\Psi}}_{\alpha_{\omega\times 1}}$, as we would like to show.

suppose γ is $\gamma_1 \wedge \gamma_2$ or $\gamma_1 \vee \gamma_2$. The proof is similar to the one for $\gamma_1 \Rightarrow \gamma_2$ so it is omitted.

3. $[[R_{\Rightarrow}]^{s_\Psi}]^{\Psi,E}_{\omega\times 1} = [R_{\Rightarrow}]^{s_{m_\Psi}}$. This happens since $\langle [v]^{\Psi,E}_{\omega\times 1}, [v_1]^{\Psi,E}_{\omega\times 1}, [v_2]^{\Psi,E}_{\omega\times 1}\rangle$ is in $[[R_{\Rightarrow}]^{s_\Psi}]^{\Psi,E}_{\omega\times 1}$ iff (by Lemma 5.3.11) $\langle v, v_1, v_2\rangle$ is in $[R_{\Rightarrow}]^{s_\Psi}$ iff $R_{\Rightarrow} v v_1 v_2$ is in Ψ iff (by definition of induced model by Ψ, Definition 5.3.9) $\langle [v]^{\Psi,E}_{\omega}, [v_1]^{\Psi,E}_{\omega}, [v_2]^{\Psi,E}_{\omega}\rangle$ is in $R^{\circ}_{\Rightarrow}{}^{m_\Psi}$ iff (by definition of induced structure from a model) $\langle [v]^{\Psi,E}_{\omega\times 1}, [v_1]^{\Psi,E}_{\omega\times 1}, [v_2]^{\Psi,E}_{\omega\times 1}\rangle$ is in $[R_{\Rightarrow}]^{s_{m_\Psi}}$.

4. $\llbracket v\rrbracket^{s_\Psi}_{\alpha}$ is in $\llbracket\varphi\rrbracket^{s_\Psi}_{\alpha}$ iff $[\llbracket v\rrbracket^{s_\Psi}_{\alpha}]^{\Psi,E}_{\omega\times 1}$ is in $[\llbracket\varphi\rrbracket^{s_\Psi}_{\alpha}]^{\Psi,E}_{\omega\times 1}$. The left to right direction is immediate. For the opposite direction suppose $[\llbracket v\rrbracket^{s_\Psi}_{\alpha}]^{\Psi,E}_{\omega\times 1}$ is in $[\llbracket\varphi\rrbracket^{s_\Psi}_{\alpha}]^{\Psi,E}_{\omega\times 1}$. So, by Proposition 3.1.3, $[v\sigma]^{\Psi,E}_{\omega\times 1}$ is in $[|\varphi\sigma|^{s_\Psi}]^{\Psi,E}_{\omega\times 1}$, which by Lemma 5.3.11 is equivalent to $v\sigma$ is in $|\varphi\sigma|^{s_\Psi}$. Hence, according again to Proposition 3.1.3 we have that $\llbracket v\rrbracket^{s_\Psi}_{\alpha}$ is in $\llbracket\varphi\rrbracket^{s_\Psi}_{\alpha}$, as we wanted to show.

5. $\langle \llbracket v_1\rrbracket^{s_\Psi}_{\alpha}, \llbracket v_2\rrbracket^{s_\Psi}_{\alpha}, \llbracket v_3\rrbracket^{s_\Psi}_{\alpha}\rangle \in [R_{\Rightarrow}]^{s_\Psi}$ iff $\langle [\llbracket v_1\rrbracket^{s_\Psi}_{\alpha}]^{\Psi,E}_{\omega\times 1}, [\llbracket v_2\rrbracket^{s_\Psi}_{\alpha}]^{\Psi,E}_{\omega\times 1}, [\llbracket v_3\rrbracket^{s_\Psi}_{\alpha}]^{\Psi,E}_{\omega\times 1}\rangle \in [[R_{\Rightarrow}]^{s_\Psi}]^{\Psi,E}_{\omega\times 1}$. The left to right direction is immediate. For the opposite direction suppose that the tuple $\langle [\llbracket v_1\rrbracket^{s_\Psi}_{\alpha}]^{\Psi,E}_{\omega\times 1}, [\llbracket v_2\rrbracket^{s_\Psi}_{\alpha}]^{\Psi,E}_{\omega\times 1}, [\llbracket v_3\rrbracket^{s_\Psi}_{\alpha}]^{\Psi,E}_{\omega\times 1}\rangle$ is in $[[R_{\Rightarrow}]^{s_\Psi}]^{\Psi,E}_{\omega\times 1}$. So, $\langle [v_1\sigma]^{\Psi,E}_{\omega\times 1}, [v_2\sigma]^{\Psi,E}_{\omega\times 1}, [v_3\sigma]^{\Psi,E}_{\omega\times 1}\rangle$ is in $[[R_{\Rightarrow}]^{s_\Psi}]^{\Psi,E}_{\omega\times 1}$, by Proposition 3.1.3. So, $\langle v_1\sigma, v_2\sigma, v_3\sigma\rangle$ is in $[R_{\Rightarrow}]^{s_\Psi}$, by Lemma 5.3.11. Thus, using again Proposition 3.1.3, we can conclude $\langle \llbracket v_1\rrbracket^{s_\Psi}_{\alpha}, \llbracket v_2\rrbracket^{s_\Psi}_{\alpha}, \llbracket v_3\rrbracket^{s_\Psi}_{\alpha}\rangle$ is in $[R_{\Rightarrow}]^{s_\Psi}$.

Finally the main proof, which follows by case analysis, on η:

if η is $v{:}\varphi$ then $s_\Psi, \alpha \Vdash v{:}\varphi$ iff $\llbracket v\rrbracket^{s_\Psi}_{\alpha}$ is in $\llbracket\varphi\rrbracket^{s_\Psi}_{\alpha}$ iff (by item 4. above proven) $[\llbracket v\rrbracket^{s_\Psi}_{\alpha}]^{\Psi,E}_{\omega\times 1}$ is in $[\llbracket\varphi\rrbracket^{s_\Psi}_{\alpha}]^{\Psi,E}_{\omega\times 1}$ iff (by items 0. and 2. above proven) $\llbracket v\rrbracket^{s_{m_\Psi}}_{\alpha_{\omega\times 1}}$ is in $\llbracket\varphi\rrbracket^{s_{m_\Psi}}_{\alpha_{\omega\times 1}}$ iff $s_{m_\Psi}, \alpha_{\omega\times 1} \Vdash v{:}\varphi$.

if η is $R_{\Rightarrow} v_1 v_2 v_3$ then, $s_\Psi, \alpha \Vdash R_{\Rightarrow} v_1 v_2 v_3$ iff $\langle \llbracket v_1\rrbracket^{s_\Psi}_{\alpha}, \llbracket v_2\rrbracket^{s_\Psi}_{\alpha}, \llbracket v_3\rrbracket^{s_\Psi}_{\alpha}\rangle$ is in $[R_{\Rightarrow}]^{s_\Psi}$ iff (by item 5.) $\langle [\llbracket v_1\rrbracket^{s_\Psi}_{\alpha}]^{\Psi,E}_{\omega\times 1}, [\llbracket v_2\rrbracket^{s_\Psi}_{\alpha}]^{\Psi,E}_{\omega\times 1}, [\llbracket v_3\rrbracket^{s_\Psi}_{\alpha}]^{\Psi,E}_{\omega\times 1}\rangle$ is in $[[R_{\Rightarrow}]^{s_\Psi}]^{\Psi,E}_{\omega\times 1}$ iff (by items 0. and 3. above proven) $\langle \llbracket v_1\rrbracket^{s_{m_\Psi}}_{\alpha_{\omega\times 1}}, \llbracket v_2\rrbracket^{s_{m_\Psi}}_{\alpha_{\omega\times 1}}, \llbracket v_3\rrbracket^{s_{m_\Psi}}_{\alpha_{\omega\times 1}}\rangle$ is in $[R_{\Rightarrow}]^{s_{m_\Psi}}$ iff $s_{m_\Psi}, \alpha_{\omega\times 1} \Vdash R_{\Rightarrow} v_1 v_2 v_3$.

if η is $v_1 \equiv_\omega v_2$ then $s_\Psi, \alpha \Vdash v_1 \equiv_\omega v_2$ iff $\omega_{s_\Psi}(\llbracket v_1\rrbracket^{s_\Psi}_{\alpha}) = \omega_{s_\Psi}(\llbracket v_2\rrbracket^{s_\Psi}_{\alpha})$ iff $[\llbracket v_1\rrbracket^{s_\Psi}_{\alpha}]^{\Psi,E}_{\omega} = [\llbracket v_2\rrbracket^{s_\Psi}_{\alpha}]^{\Psi,E}_{\omega}$ iff $\omega_{s_{m_\Psi}}([\llbracket v_1\rrbracket^{s_\Psi}_{\alpha}]^{\Psi,E}_{\omega\times 1}) = \omega_{s_{m_\Psi}}([\llbracket v_2\rrbracket^{s_\Psi}_{\alpha}]^{\Psi,E}_{\omega\times 1})$ iff (by item 0.) $\omega_{s_{m_\Psi}}(\llbracket v_1\rrbracket^{s_{m_\Psi}}_{\alpha_{\omega\times 1}}) = \omega_{s_{m_\Psi}}(\llbracket v_2\rrbracket^{s_{m_\Psi}}_{\alpha_{\omega\times 1}})$ iff $s_{m_\Psi}, \alpha_{\omega\times 1} \Vdash v_1 \equiv_\omega v_2$.

if η is $v_1 \equiv_\alpha v_2$ then the result follows because $s_\Psi, \alpha \Vdash v_1 \equiv_\alpha v_2$ and $s_{m_\Psi}, \alpha_{\omega\times 1} \Vdash v_1 \equiv_\alpha v_2$. In order to show that $s_{m_\Psi}, \alpha_{\omega\times 1} \Vdash v_1 \equiv_\alpha v_2$ it is only enough to see that $\alpha_{s_{m_\Psi}}(\llbracket v_1\rrbracket^{s_{m_\Psi}}_{\alpha_{\omega\times 1}}) = 1 = \alpha_{s_{m_\Psi}}(\llbracket v_2\rrbracket^{s_{m_\Psi}}_{\alpha_{\omega\times 1}})$. With respect to $s_\Psi, \alpha \Vdash v_1 \equiv_\alpha v_2$ note that $[v_1\sigma]^{\Psi,E}_{\alpha} = [v_2\sigma]^{\Psi,E}_{\alpha}$ by rule *1 assg* since Ψ is deductively closed.

if η is $v_1{:}t_1 =_g v_2{:}t_2$ then $s_\Psi, \alpha \Vdash v_1{:}t_1 =_g v_2{:}t_2$ and $s_{m_\Psi}, \alpha_{\omega\times 1} \Vdash v_1{:}t_1 =_g v_2{:}t_2$. In order to show that $s_\Psi, \alpha \Vdash v_1{:}t_1 =_g v_2{:}t_2$, it is enough to see that $v_1\sigma{:}t_1\sigma =_g v_2\sigma{:}t_2\sigma$ is in Ψ, due to rule *1 ind*. For $s_{m_\Psi}, \alpha_{\omega\times 1} \Vdash v_1{:}t_1 =_g v_2{:}t_2$ note that $\llbracket t_1\rrbracket^{s_{m_\Psi}}_{\alpha_{\omega\times 1}}(\llbracket v_1\rrbracket^{s_{m_\Psi}}_{\alpha_{\omega\times 1}}) = \llbracket t_2\rrbracket^{s_{m_\Psi}}_{\alpha_{\omega\times 1}}(\llbracket v_2\rrbracket^{s_{m_\Psi}}_{\alpha_{\omega\times 1}})$ since $D^{s_{m_\Psi}}$ is a singleton. *QED*

Proposition 5.3.14 The lfob logic system $\mathcal{L}_{\mathrm{R}}$ is rich.

Proof In the context of $\mathcal{L}_{\mathrm{R}}$, let Ψ_0 be a $v{:}\varphi$ consistent set. Then, by Proposition 4.3.9, there exists an appropriate set Ψ extending Ψ_0 and also $v{:}\varphi$ consistent, since by Proposition 5.3.8, $\mathcal{L}_{\mathrm{R}}$ is a connected lfob logic system with a classical negation. So, it is possible to consider the canonical structure s_Ψ. Consider the model m_Ψ induced by Ψ, as defined in Definition 5.3.9. Recall that, also by Definition 5.3.9, s_{m_Ψ} is a structure induced by m_Ψ and so s_{m_Ψ} is in $\breve{m}_\Psi$. Then, taking into account Lemma 5.3.13 and the definition of $\breve{m}_\Psi$ in Example 5.3.1, we have that s_Ψ is in $\breve{m}_\Psi$, as we wanted to show. *QED*

Proof (Completeness Theorem 5.3.7) The proof follows immediately using Theorem 3.3.3 since by Proposition 5.3.14, $\mathcal{L}_{\mathrm{R}}$ is rich. *QED*

5.3.4 Equivalence to non-labelled logic system

In this subsection we show that the lfob logic system $\mathcal{L}_{\mathrm{R}}$ is indeed a labelled presentation for the relevance logic R, as is stated in the next theorem. This theorem is proved at the end of the section.

Theorem 5.3.15 The lfob logic system $\mathcal{L}_{\mathrm{R}}$ is a labelled presentation for relevance logic R.

The proof of the theorem relies on finding a non-labelled logic system for relevance logic R such that $\mathcal{L}_{\mathrm{R}}$ is a labelled presentation for that system. Recall Definition 2.4.10 where it is defined what is a lfob logic system be a labelled presentation for a logic. So, consider the following system.

Example 5.3.16 *Relevance logic* R *non-labelled logic system.* Let $\mathcal{L}^{\cdot}_{\mathrm{R}}$ be the non-labelled logic system $\langle \Sigma^{\cdot}, \propto \rangle$ for relevance logic R having as its set of zero-ary propositions, the set of zero-ary propositions of $\mathcal{L}_{\mathrm{R}}$.

Observe that the only component that can change between non-labelled logic systems for relevance logic R is the set of zero-ary propositions, and motivated by that change, the relation $\propto$ between schema formulae associated to the system.

Before presenting the proof that $\mathcal{L}_{\mathrm{R}}$ is indeed a labelled presentation for the relevance logic R, we show two auxiliary lemmas proving the equivalence between the usual semantics for the logic system and the labelled semantics. Lemma 5.3.17 proves the equivalence for formulae and Lemma 5.3.18 proves the equivalence for entailment.

Lemma 5.3.17 In the context of $\mathcal{L}_{\mathrm{R}}$ and $\mathcal{L}^{\cdot}_{\mathrm{R}}$,

$$m, \alpha^{\cdot}, w \Vdash \gamma \quad \text{iff} \quad s_m, \alpha \Vdash \nu{:}\gamma$$

where γ is a schema formula in $\mathcal{L}^{\cdot}_{\mathrm{R}}$ and α and $\alpha^{\cdot}$ are such that $(\xi\alpha^{\cdot}) \circ \omega = \xi\alpha$ for any formula schema variable ξ, and $\nu\alpha_l$ is $\langle w, 1 \rangle$.

Proof The proof follows by induction on γ.

Suppose γ is in $P^{\cdot}_0$. Then $m, \alpha^{\cdot}, w \Vdash \gamma$ iff $V(\gamma, w) = 1$ iff $[\gamma]^{s_m}_{w} = 1$ iff $[\gamma]^{s_m}_{\omega(\llbracket\nu\rrbracket^{s_m}_{\alpha})} = 1$ iff $\widehat{\gamma}(\llbracket\nu\rrbracket^{s_m}_{\alpha}) = 1$ iff $\llbracket\gamma\rrbracket^{s_m}_{\alpha}(\llbracket\nu\rrbracket^{s_m}_{\alpha}) = 1$ iff $s_m, \alpha \Vdash \nu{:}\gamma$, as we wanted to show.

Suppose γ is in $\Xi^{\cdot}_f$. Then $m, \alpha^{\cdot}, w \Vdash \gamma$ iff $(\gamma\alpha^{\cdot}_f)(w) = 1$ iff $((\gamma\alpha^{\cdot}_f) \circ \omega)(\langle w, 1 \rangle) = 1$ iff $(\gamma\alpha_f)(\langle w, 1 \rangle) = 1$ iff $\nu\alpha_f$ is in $\gamma\alpha_f$ iff $s_m, \alpha \Vdash \nu{:}\gamma$, as we wanted to show.

Suppose γ is $\bot$. We prove that $m, \alpha^{\cdot}, w \Vdash \gamma$ iff $s_m, \alpha \Vdash \nu{:}\gamma$ by showing that $m, \alpha^{\cdot}, w \nVdash \gamma$ and $s_m, \alpha \nVdash \nu{:}\gamma$. The fact that $m, \alpha^{\cdot}, w \nVdash \gamma$ follows by definition of satisfaction in a relational model. The fact that $s_m, \alpha \nVdash \nu{:}\gamma$ follows similarly, by definition of induced structure by a relational model in Example 5.3.1.

Suppose γ is $-\gamma_1$. Then $m, \alpha^{\cdot}, w \Vdash \gamma$ iff (by definition of the satisfaction relation

$\Vdash$ in the context of a relational model) $m, \alpha^{\cdot}, w^{*\circ} \nVdash \gamma_1$ iff (by induction hypothesis) $s_m, \alpha_1 \nVdash \nu_1{:}\gamma_1$ for ν_1 distinct of ν and α_1 $\{\nu_1\}$ co-equivalent to α with $\nu_1\alpha_1$ equal to $\langle w^{*\circ}, 1\rangle$ iff $\llbracket\gamma_1\rrbracket^{s_m}_{\alpha_1}(\langle w^{*\circ}, 1\rangle) = 0$ iff (by Proposition 2.4.4) $\llbracket\gamma_1\rrbracket^{s_m}_{\alpha}(\langle \omega(\nu\alpha)^{*\circ}, 1\rangle) = 0$ iff $[-](\llbracket\gamma_1\rrbracket^{s_m}_{\alpha})(\llbracket\nu\rrbracket^{\Psi,E}_{\alpha}) = 1$ iff $s_m, \alpha \Vdash \nu{:}\gamma$, as we wanted to show.

Suppose γ is $\gamma_1 \vee \gamma_2$. Then $m, \alpha^{\cdot}, w \Vdash \gamma$ iff $m, \alpha^{\cdot}, w \Vdash \gamma_1$ or $m, \alpha^{\cdot}, w \Vdash \gamma_2$ iff (by induction hypothesis) $s_m, \alpha \Vdash \nu{:}\gamma_1$ or $s_m, \alpha \Vdash \nu{:}\gamma_2$ iff $\llbracket\gamma_1\rrbracket^{s_m}_{\alpha}(\llbracket\nu\rrbracket^{s_m}_{\alpha}) = 1$ or $\llbracket\gamma_2\rrbracket^{s_m}_{\alpha}(\llbracket\nu\rrbracket^{s_m}_{\alpha}) = 1$ iff $\widehat{\vee}(\llbracket\gamma_1\rrbracket^{s_m}_{\alpha}, \llbracket\gamma_2\rrbracket^{s_m}_{\alpha})(\llbracket\nu\rrbracket^{s_m}_{\alpha}) = 1$ iff $s_m, \alpha \Vdash \nu{:}\gamma$, as we wanted to show.

Suppose γ is $\gamma_1 \wedge \gamma_2$. The proof is similar to the case where γ is $\gamma_1 \vee \gamma_2$, so we omit it.

Suppose γ is $\gamma_1 \Rightarrow \gamma_2$. The key point of this proof is to show that, for any w_1 and w_2 if $R^{\circ}_{\Rightarrow} w w_1 w_2$ and $m, \alpha^{\cdot}, w_1 \Vdash \gamma_1$ then $m, \alpha^{\cdot}, w_2 \Vdash \gamma_2$, is equivalent to, for any u_1 and u_2 if $\langle \nu\alpha, u_1, u_2\rangle \in [R_{\Rightarrow}]$ and $\llbracket\gamma_1\rrbracket^{s_m}_{\alpha}(u_1) = 1$ then $\llbracket\gamma_2\rrbracket^{s_m}_{\alpha}(u_2) = 1$. Suppose that, for any w_1 and w_2 if $R^{\circ}_{\Rightarrow} w w_1 w_2$ and $m, \alpha^{\cdot}, w_1 \Vdash \gamma_1$ then $m, \alpha^{\cdot}, w_2 \Vdash \gamma_2$, and let u_1 and u_2 be such that $\langle \nu\alpha, u_1, u_2\rangle \in [R_{\Rightarrow}]$ and $\llbracket\gamma_1\rrbracket^{s_m}_{\alpha}(u_1) = 1$. Then $R^{\circ}_{\Rightarrow}\omega(\nu\alpha)\omega(\nu_1\alpha')\omega(\nu_2\alpha')$ and, by Proposition 2.4.4, $s_m, \alpha' \Vdash \nu_1{:}\gamma_1$ where α' is $\{\nu_1, \nu_2\}$ co-equivalent to α with $\nu_1\alpha' = u_1$ and $\nu_2\alpha' = u_2$. Hence, using the induction hypothesis, we have that $m, \alpha^{\cdot}, \omega(\nu_1\alpha') \Vdash \gamma_1$, and so, $m, \alpha^{\cdot}, \omega(\nu_2\alpha') \Vdash \gamma_2$. Thus, using again the induction hypothesis we have that, $s_m, \alpha' \Vdash \nu_2{:}\gamma_2$, i.e., $\llbracket\gamma_2\rrbracket^{s_m}_{\alpha'}(\nu_2\alpha') = 1$, which, by Proposition 2.4.4, is $\llbracket\gamma_2\rrbracket^{s_m}_{\alpha}(u_2) = 1$ as we wanted to show. For the other direction, suppose for any u_1 and u_2 that if $\langle \nu\alpha, u_1, u_2\rangle \in [R_{\Rightarrow}]$ and $\llbracket\gamma_1\rrbracket^{s_m}_{\alpha}(u_1) = 1$ then $\llbracket\gamma_2\rrbracket^{s_m}_{\alpha}(u_2) = 1$, and let w_1 and w_2 be such that $R^{\circ}_{\Rightarrow} w w_1 w_2$ and $m, \alpha^{\cdot}, w_1 \Vdash \gamma_1$, and α' be $\{\nu_1, \nu_2\}$ co-equivalent to α with $\omega(\nu_1\alpha') = w_1$ and $\omega(\nu_2\alpha') = w_2$. Then, $R^{\circ}_{\Rightarrow}\omega(\nu\alpha)\omega(\nu_1\alpha')\omega(\nu_2\alpha')$ and by induction hypothesis, $s_m, \alpha' \Vdash \nu_1{:}\gamma_1$. Thus, $\langle \nu\alpha, \nu_1\alpha', \nu_2\alpha'\rangle \in [R_{\Rightarrow}]$ and, $\llbracket\gamma_1\rrbracket^{s_m}_{\alpha'}(\nu_1\alpha') = 1$ i.e., $\llbracket\gamma_1\rrbracket^{s_m}_{\alpha}(\nu_1\alpha') = 1$, by Proposition 2.4.4. So $\llbracket\gamma_2\rrbracket^{s_m}_{\alpha}(\nu_2\alpha') = 1$, and by Proposition 2.4.4 $\llbracket\gamma_2\rrbracket^{s_m}_{\alpha'}(\nu_2\alpha') = 1$. Hence $s_m, \alpha' \Vdash \nu_2{:}\gamma_2$ and by induction hypothesis $m, \alpha^{\cdot}, w_2 \Vdash \gamma_2$, as we wanted to show. *QED*

Lemma 5.3.18 In the context of $\mathcal{L}_{\mathrm{R}}$ and $\mathcal{L}^{\cdot}_{\mathrm{R}}$,

$$\Phi \propto \gamma \quad \text{iff} \quad \{\nu{:}\phi \mid \phi \text{ in } \Phi\} \vDash \nu{:}\gamma$$

where ν is a label schema variable, Φ is a set of schema formulae in $\mathcal{L}^{\cdot}_{\mathrm{R}}$ and γ is a schema formula in $\mathcal{L}^{\cdot}_{\mathrm{R}}$.

Proof Assume $\Phi \propto \gamma$, and let s be a structure in $\mathcal{L}_{\mathrm{R}}$ and α an assignment over s such that $s, \alpha \Vdash \{\nu{:}\phi \mid \phi \text{ in } \Phi\}$. Then, we have to consider two cases, (i) s is a structure induced by a relational model m for positive basic relevance logic, or (ii) s is a structure equivalent in terms of satisfaction with a structure $s^{\cdot}_m$ induced by a relational model m for positive basic relevance logic. If (i) holds then, using Lemma 5.3.17, we have that $m, \alpha_{f_\omega}, \omega(\nu\alpha_l) \Vdash \Phi$. Thus, using the fact that $\Phi \propto \gamma$ we have that $m, \alpha_{f_\omega}, \omega(\nu\alpha_l) \Vdash \gamma$. Hence, using again Lemma 5.3.17 we can conclude that $s, \alpha \Vdash \nu{:}\gamma$ as we wanted to show. If (ii) holds then there is a assignment $\alpha^{\cdot}$ over $s^{\cdot}_m$ such that $s^{\cdot}_m, \alpha^{\cdot} \Vdash \{\nu{:}\phi \mid \phi \text{ in } \Phi\}$. Therefore, using Lemma 5.3.17, we have that $m, \alpha^{\cdot}_{f_\omega}, \omega(\nu\alpha^{\cdot}_l) \Vdash \Phi$. Thus, using the fact that $\Phi \propto \gamma$ we have that $m, \alpha^{\cdot}_{f_\omega}, \omega(\nu\alpha^{\cdot}_l) \Vdash \gamma$. Hence, using again Lemma 5.3.17 we can conclude that $s^{\cdot}_m, \alpha^{\cdot} \Vdash \nu{:}\gamma$. So, since s is a structure equivalent in terms of satisfaction with $s^{\cdot}_m$ we have that $s, \alpha \Vdash \nu{:}\gamma$ as we wanted to show.

Assume $\{\nu{:}\phi \mid \phi \text{ in } \Phi\} \vDash \nu{:}\gamma$ and let m be a relational model for the relevance logic R over $\Sigma^{\cdot}$, $\alpha^{\cdot}$ a assignment from $\Xi^{\cdot}_f$ to the class of hereditary sets of m and w a world of m such that $m, \alpha^{\cdot}, w \Vdash \Phi$. Then, by Lemma 5.3.17, we have that $s_m, \alpha \Vdash \{\nu{:}\phi \mid \phi \in \Phi\}$ where α is such that α_{f_ω} is equal to $\alpha^{\cdot}$ and $\omega(\nu\alpha_l) = w$. So, $s_m, \alpha \Vdash \nu{:}\gamma$ since $\{\nu{:}\phi \mid \phi \text{ in } \Phi\} \vDash \nu{:}\gamma$ and s_m is in $\breve{m}$ and m is in M. Therefore using again Lemma 5.3.17, we have that $m, \alpha^{\cdot}, w \Vdash \gamma$, as we wanted to show. *QED*

Proof (Labelled presentation Theorem 5.3.15) According to Definition 2.4.10 it is sufficient to show that $\mathcal{L}_{\mathrm{R}}$ is a labelled presentation for $\mathcal{L}^{\cdot}_{\mathrm{R}}$. Recall Definition 2.4.9 where it is said what is a lfob logic system be a labelled presentation for a non-labelled logic system. So, the lfob logic system $\mathcal{L}_{\mathrm{R}}$ is a labelled presentation for $\mathcal{L}^{\cdot}_{\mathrm{R}}$ because:

- $\mathcal{L}_{\mathrm{R}}$ is sound and complete. See Theorem 5.3.7 for completeness and Theorem 5.3.4 for soundness.

- $X^{\cdot}$ and $\Xi^{\cdot}_t$ are empty sets and $\Xi^{\cdot}_f$ is equal to Ξ_f.

- $F^{\cdot}$ is equal to F, $P^{\cdot}$ is equal to P, and $C^{\cdot}$ is equal to $C \cup O \cup Q$, by definition of $\mathcal{L}_{\mathrm{R}}$, see Example 5.3.1, and of $\mathcal{L}^{\cdot}_{\mathrm{R}}$, see Example 5.3.16.

- $\Phi \propto \gamma$ iff $\{\nu{:}\phi \mid \phi \in \Phi\} \vDash \nu{:}\gamma$, for any set Φ of schema formulae in $\mathcal{L}^{\cdot}_{\mathrm{R}}$ and schema formula γ in $\mathcal{L}^{\cdot}_{\mathrm{R}}$, where ν is a label schema variable. This result follows by Lemma 5.3.18. *QED*

5.4 $\wedge$-$\Rightarrow$ fragment of basic relevance logic B

In this section we present the lfob logic system $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$ and show that it is a labelled presentation for the $\wedge$-$\Rightarrow$ fragment of basic relevance logic B [63, 59, 79, 1, 2, 32, 58, 68, 66, 67, 77] with respect to formulae without schema variables. See Definitions 2.4.9 and 2.4.10, to recall when a lfob logic system is a labelled presentation for a non-labelled logic system or is a labelled presentation for a non-labelled logic, with respect to a set of non-labelled schema formulae. We now briefly describe what is a non-labelled fob signature, a relational model and a non-labelled fob logic system for the $\wedge$-$\Rightarrow$ fragment of basic relevance logic B. Recall the definitions of non-labelled fob signature and non-labelled logic system in Definition 2.4.8. We consider that $X^{\cdot}$ and $\Xi^{\cdot}_t$ are empty sets, and that $\Xi^{\cdot}_f$ is equal to Ξ_f. In order to recall the importance of $X^{\cdot}$, $\Xi^{\cdot}_t$ and $\Xi^{\cdot}_f$ see Definition 2.4.8.

A *non-labelled fob signature* $\Sigma^{\cdot}$ *for the* $\wedge$-$\Rightarrow$ *fragment of basic relevance logic* B is a non-labelled fob signature such that $F^{\cdot}_k = \emptyset$ for $k \geq 0$, $P^{\cdot}_0$ is a non-empty set, $P^{\cdot}_k = \emptyset$ for $k \geq 1$, $C^{\cdot}_2 = \{\wedge, \Rightarrow\}$, and $C^{\cdot}_k = \emptyset$ for $k \geq 3$ and $k = 1$ and $k = 0$.

A *relational model* m *over* $\Sigma^{\cdot}$ *for the* $\wedge$-$\Rightarrow$ *fragment of basic relevance logic* B is a tuple $\langle W, G, R^{\circ}_{\Rightarrow}, V\rangle$ where W is a non-empty set, G is in W, $R^{\circ}_{\Rightarrow}$ is contained in $W \times W \times W$, such that

- $R^{\circ}_{\Rightarrow} w w_2 w_3$ whenever $R^{\circ}_{\Rightarrow} G w w_1$ and $R^{\circ}_{\Rightarrow} w_1 w_2 w_3$,

- $R^{\circ}_{\Rightarrow} w_1 w w_3$ whenever $R^{\circ}_{\Rightarrow} G w w_2$ and $R^{\circ}_{\Rightarrow} w_1 w_2 w_3$,

- $R^{\circ}_{\Rightarrow} w_1 w_2 w$ whenever $R^{\circ}_{\Rightarrow} G w_3 w$ and $R^{\circ}_{\Rightarrow} w_1 w_2 w_3$,

- $R^{\circ}_{\Rightarrow} G w w$,

and V is a map $V : P_0^{\cdot} \times W \rightarrow 2$ satisfying the atomic hereditary condition: if $V(p,w) = 1$ and $R^{\circ}_{\Rightarrow}Gww'$ then $V(p,w') = 1$, such that $m,\alpha,w \Vdash \varphi$ is defined inductively as follows,

$$\begin{array}{lll}
m,\alpha,w \Vdash \xi & \text{iff} & w \text{ is in } \xi\alpha \\
m,\alpha,w \Vdash p & \text{iff} & V(p,w) = 1 \\
m,\alpha,w \Vdash \varphi_1 \wedge \varphi_2 & \text{iff} & m,\alpha,w \Vdash \varphi_1 \text{ and } m,\alpha,w \Vdash \varphi_2 \\
m,\alpha,w \Vdash \varphi_1 \Rightarrow \varphi_2 & \text{iff} & \text{for any } w_1 \text{ and } w_2, \\
 & & \text{if } R^{\circ}_{\Rightarrow}ww_1w_2 \text{ and } m,\alpha,w_1 \Vdash \varphi_1 \text{ then } m,\alpha,w_2 \Vdash \varphi_2,
\end{array}$$

where α is a *schema assignment* over m, i.e., a map that associates to an element of Ξ_f a *hereditary set* in m, i.e., a set of worlds of m such that if $R^{\circ}_{\Rightarrow}Gww^{\cdot}$ holds and w is in that set then $w^{\cdot}$ is also in that set.

A *non-labelled fob logic system for the* $\wedge$-$\Rightarrow$ *fragment of basic relevance logic* B is a pair $\langle \Sigma^{\cdot}, \propto \rangle$ where $\Sigma^{\cdot}$ is a non-labelled signature for this logic, and $\propto$ is such that

$$\Phi \propto \gamma \quad \text{iff} \quad \text{if } m,\alpha,w \Vdash \Phi \text{ then } m,\alpha,w \Vdash \gamma$$

for any relational model m for the $\wedge$-$\Rightarrow$ fragment of basic relevance logic B over $\Sigma^{\cdot}$, assignment α over m, and world w in W_m.

Observe that, the only distinct aspects between two different non-labelled logic systems for the $\wedge$-$\Rightarrow$ fragment of basic relevance logic B are, the sets of zero-ary propositions, and, caused by that, their relations $\propto$. Moreover, we can only associate one non-labelled logic system for the $\wedge$-$\Rightarrow$ fragment of basic relevance logic B to each non-labelled signature for that logic.

5.4.1 Logic system

Example 5.4.1 $\wedge$-$\Rightarrow$ *fragment of basic relevance logic B lfob logic system.* Consider the lfob logic system $\langle \Sigma, R, M, \breve{:} \rangle$, in the sequel denoted by $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$, where Σ, denoted by $\Sigma_{\mathrm{B}_{\wedge,\Rightarrow}}$, is such that

- $F_0^l = \{0\}$, and $F_k^l = \emptyset$ for $k \geq 1$,
- $S_3 = \{R_{\Rightarrow}\}$, and $S_k = \emptyset$ for $k \geq 4$ and $k = 2$ and $k = 1$,
- $F_k = \emptyset$ for $k \geq 0$,
- P_0 is a non-empty set, and $P_k = \emptyset$ for $k \geq 1$,
- $C_2 = \{\wedge\}$, and $C_k = \emptyset$ for $k \geq 3$ and $k = 1$ and $k = 0$,
- $Q_k = \emptyset$ for $k \geq 1$,
- $O_2 = \{\Rightarrow\}$, and $O_k = \emptyset$ for $k \geq 3$ and $k = 1$,

and R, besides the rules specified in Definition 2.3.10 common to all lfob deduction systems, contains:

$$\frac{\nu{:}\xi_1 \qquad \nu{:}\xi_2}{\nu{:}\xi_1 \wedge \xi_2}\,\wedge_I \qquad\qquad \frac{\nu{:}\xi_1 \wedge \xi_2}{\nu{:}\xi_1}\,\wedge_E^1 \qquad\qquad \frac{\nu{:}\xi_1 \wedge \xi_2}{\nu{:}\xi_2}\,\wedge_E^2$$

$$\frac{R_{\Rightarrow}\nu\nu_1\nu_2, \nu_1{:}\xi_1 \,/\, \nu_2{:}\xi_2}{\nu{:}\xi_1 \Rightarrow \xi_2}\ \Rightarrow_I;\ \text{fresh}(\{\nu_1,\nu_2\}, \langle\vartheta_1, \{R_{\Rightarrow}\nu\nu_1\nu_2, \nu_1{:}\xi_1\}, \nu_2{:}\xi_2\rangle)$$

$$\frac{\nu{:}\xi_1 \Rightarrow \xi_2 \qquad \nu_1{:}\xi_1 \qquad R_{\Rightarrow}\nu\nu_1\nu_2}{\nu_2{:}\xi_2}\ \Rightarrow_E$$

$$\frac{R_{\Rightarrow}0\nu\nu_1 \quad R_{\Rightarrow}\nu_1\nu_2\nu_3}{R_{\Rightarrow}\nu\nu_2\nu_3}\ \mathit{rmon1} \qquad\qquad \frac{R_{\Rightarrow}0\nu\nu_2 \quad R_{\Rightarrow}\nu_1\nu_2\nu_3}{R_{\Rightarrow}\nu_1\nu\nu_3}\ \mathit{rmon2}$$

$$\frac{R_{\Rightarrow}0\nu_3\nu \quad R_{\Rightarrow}\nu_1\nu_2\nu_3}{R_{\Rightarrow}\nu_1\nu_2\nu}\ \mathit{rmon3}$$

$$\frac{R_{\Rightarrow}\nu_1'\nu_2'\nu_3' \quad \nu_1 \equiv_\omega \nu_1' \quad \nu_2 \equiv_\omega \nu_2' \quad \nu_3 \equiv_\omega \nu_3'}{R_{\Rightarrow}\nu_1\nu_2\nu_3}\ R_{\Rightarrow}{}^{\text{lgen}}_{\alpha} \qquad\qquad \frac{}{R_{\Rightarrow}0\nu\nu}\ \mathit{iden}$$

$$\frac{R_{\Rightarrow}0\nu_1\nu_2 \qquad \nu_1{:}\xi}{\nu_2{:}\xi}\ \mathit{mon}_{R_{\Rightarrow}0};\ \text{p-mon}_{R_{\Rightarrow}0}(\xi) \qquad\qquad \frac{\nu \equiv_\omega \nu' \qquad \nu{:}\xi}{\nu'{:}\xi}\ \mathit{gmon}_{\equiv_\omega};\ \text{p-gmon}_{\equiv_\omega}(\xi)$$

where the proviso p-gmon$_{\equiv_\omega}(\xi)$ is such that p-gmon$_{\equiv_\omega}(\xi)_{\Sigma_{\mathrm{B}_{\wedge,\Rightarrow}}}(\rho_{\mathrm{nl}}) = 1$ iff $\xi\rho_f$ is a formula without variables, and p-mon$_{R_{\Rightarrow}0}(\xi)$ is such that p-mon$_{R_{\Rightarrow}0}(\xi)_{\Sigma_{\mathrm{B}_{\wedge,\Rightarrow}}}(\rho_{\mathrm{nl}}) = 1$ iff $\xi\rho$ is a formula without variables,

M is composed of all the relational models for this logic over the non-labelled fob signature $\langle F, P, C \cup Q \cup O\rangle$,

and $\check{}$ associates to each relational model $\langle W, G, R^{\circ}_{\Rightarrow}, V\rangle$ denoted by m, a class, such that s is in $\check{m}$ iff either s is a structure induced by m, denoted by s_m, i.e., a structure $\langle U, A, W, \alpha, \omega, D, \mathcal{E}, \mathcal{B}, [.]\rangle$ such that

- D is a non-empty set,
- $A = D^X$ and $U = W \times A$,
- $\alpha(\langle w, a\rangle) = a$ and $\omega(\langle w, a\rangle) = w$,
- $\mathcal{E}$ is D^U,
- $\mathcal{B}$ is 2^U,

and

- $[0] = \langle G, a\rangle$, where a is an assignment,
- $[x]_a = a(x)$,
- $[p]_w = V(p, w)$ for p in P_0,
- $[\wedge]_{\omega(u)\alpha(u)}(b_1, b_2)(u) = 1$ iff $b_1(u) = 1$ and $b_2(u) = 1$,

- $[R_{\Rightarrow}] = \{\langle u_1, u_2, u_3\rangle \mid R^{\circ}_{\Rightarrow}\omega(u_1)\omega(u_2)\omega(u_3)\}$,
- $[\Rightarrow](b_1, b_2)(u) = 1$ iff for all u_1 and u_2, if $\langle u, u_1, u_2\rangle$ is in $[R_{\Rightarrow}]$ and u_1 is in b_1 then u_2 is in b_2,

or s satisfies all rules, and, for any schema assignment α over s exists a induced structure $s^{\cdot}_m$ by m and a schema assignment $\alpha^{\cdot}$ over $s^{\cdot}_m$, such that, for any formula ϕ in $\mathcal{L}^{\cdot}_{\mathrm{B}_{\wedge,\Rightarrow}}$, $s, \alpha \Vdash \nu{:}\phi$ iff $s^{\cdot}_m, \alpha^{\cdot} \Vdash \nu{:}\phi$.

In order for the structures induced by a model in $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$ to be indeed structures it is necessary that the algebraic map $\frown$ associated to each structure be well defined (see Definition 2.2.1). But this is straightforward to see since for each induced structure s the sets $\mathcal{E}^s$ and $\mathcal{B}^s$ coincide respectively with D^{sU^s} and with 2^{U^s}. To conclude the subsection we present a proposition showing that the underlying deduction system of $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$ is uniform; see Definition 5.5.4 to recall when a lfob deduction system is uniform. We omit the proof since it is straightforward.

Proposition 5.4.2 The lfob deduction system in $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$ is uniform.

5.4.2 Soundness

Theorem 5.4.3 The lfob logic system $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$ is sound.

Proof The proof follows by Theorem 2.4.6 since by Proposition 5.4.4 all structures of $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$ satisfy its rules. *QED*

Note that, according to Proposition 2.4.7, in order to show that all structures of $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$ satisfy its rules, it is sufficient to show that the structures satisfy the specific rules since every structure satisfies the rules common to all lfob deduction systems. In this subsection, we first present the proof that all structures of $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$ satisfy its rules and only after we present the two lemmas needed along this proof.

Proposition 5.4.4 All structures of $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$ satisfy its rules.

Proof Let $m = \langle W, G, R^{\circ}_{\Rightarrow}, V\rangle$ be a model in $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$ and $s = \langle U, A, W, \alpha, \omega, D, \mathcal{E}, \mathcal{B}, [.]\rangle$ a structure in $\breve{m}$. Then, according to the definition of $\breve{m}$, either (i) s is a structure induced by m, or (ii) s is equivalent to a structure induced by the model. Suppose (ii) holds. Then, by definition, s satisfies all rules and we are done. Suppose (i) holds. Then, we have to check that s, in this case denoted by s_m, satisfies all the rules. Note that, by Proposition 2.4.7, s_m satisfies the rules common to all lfob logic systems, hence, it is only needed to check that it satisfies the non-common rules. So, let $r = \langle\{\langle\vartheta_1, \Psi_1, \eta_1\rangle, \ldots, \langle\vartheta_k, \Psi_k, \eta_k\rangle\}, \eta, P_f, P_d\rangle$ be a rule in $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$ that is not a rule common to all lfob logic systems, E a set of label schema variables, σ a schema substitution within L_E such that $(\pi_{\Sigma}\sigma_{\mathrm{nl}})$ is 1 for each π in P_f, and $\pi_{\Sigma;E}(\sigma)$ is 1 for each π in P_d, and α a schema assignment over s_m and E. Thus,

suppose r is $mon_{R_{\Rightarrow}0}$, and that $s_m, \alpha \Vdash R_{\Rightarrow}0\nu_1\sigma\nu_2\sigma$, and $s_m, \alpha \Vdash \nu_1\sigma{:}\xi\sigma$, and that $\xi\sigma$ is a formula without variables. Note that $R^{\circ}_{\Rightarrow}G\omega([\![\nu_1\sigma]\!]^{s_m}_{\alpha})\omega([\![\nu_2\sigma]\!]^{s_m}_{\alpha})$ and $[\![\xi\sigma]\!]^{s_m}_{\alpha}([\![\nu_1\sigma]\!]^{s_m}_{\alpha}) = 1$. Then $[\![\xi\sigma]\!]^{s_m}_{\alpha}([\![\nu_2\sigma]\!]^{s_m}_{\alpha}) = 1$ by Lemma 5.4.6 and so $s_m, \alpha \Vdash \nu_2\sigma{:}\xi\sigma$ as we wanted to show.

suppose r is $gmon_{\equiv_\omega}$, and that $s_m, \alpha \Vdash \nu\sigma \equiv_\omega \nu'\sigma$, $s_m, \alpha \Vdash \nu\sigma{:}\xi\sigma$ and that $\xi\sigma$ is a formula without variables. Note that $\omega(\llbracket\nu\sigma\rrbracket_\alpha^{s_m}) = \omega(\llbracket\nu'\sigma\rrbracket_\alpha^{s_m})$ and $\llbracket\xi\sigma\rrbracket_\alpha^{s_m}(\llbracket\nu\sigma\rrbracket_\alpha^{s_m}) = 1$. So, by item 2. of Lemma 5.4.5, $\llbracket\xi\sigma\rrbracket_\alpha^{s_m}(\llbracket\nu'\sigma\rrbracket_\alpha^{s_m}) = 1$ as we wanted to show.

suppose r is $\Rightarrow_I$, and, for any $\alpha^{\cdot}$ $\{\nu_1\sigma, \nu_2\sigma\}$ co-equivalent to α, if $s_m, \alpha^{\cdot} \Vdash R_\Rightarrow\nu\sigma\nu_1\sigma\nu_2\sigma$ and $s_m, \alpha^{\cdot} \Vdash \nu_1\sigma{:}\xi_1\sigma$ then $s_m, \alpha^{\cdot} \Vdash \nu_2\sigma{:}\xi_2\sigma$. Let u_1' and u_2' be points with $\langle\llbracket\nu\sigma\rrbracket_\alpha^{s_m}, u_1', u_2'\rangle$ in $[R_\Rightarrow]^{s_m}$ and such that $\llbracket\xi_1\sigma\rrbracket_\alpha^{s_m}(u_1') = 1$. Consider the assignment $\alpha^{\cdot}$ $\{\nu_1\sigma, \nu_2\sigma\}$ co-equivalent to α, with $\nu_1\sigma\alpha^{\cdot} = u_1'$ and $\nu_2\sigma\alpha^{\cdot} = u_2'$. Note that $\llbracket\xi_1\sigma\rrbracket_\alpha^{s_m}$ is equal to $\llbracket\xi_1\sigma\rrbracket_{\alpha^{\cdot}}^{s_m}$, by Proposition 2.4.4, and $\llbracket\nu\sigma\rrbracket_\alpha^{s_m}$ is equal to $\llbracket\nu\sigma\rrbracket_{\alpha^{\cdot}}^{s_m}$, also by Proposition 2.4.4, since $\nu_1\sigma$ and $\nu_2\sigma$ are not in lsv$(\nu\sigma)$. Then, $\llbracket\xi_1\sigma\rrbracket_{\alpha^{\cdot}}^{s_m}(\llbracket\nu_1\sigma\rrbracket_{\alpha^{\cdot}}^{s_m}) = 1$ and $\langle\llbracket\nu\sigma\rrbracket_{\alpha^{\cdot}}^{s}, \llbracket\nu_1\sigma\rrbracket_{\alpha^{\cdot}}^{s}, \llbracket\nu_2\sigma\rrbracket_{\alpha^{\cdot}}^{s}\rangle$ is in $[R_\Rightarrow]^{s_m}$. So, by the initial assumption, $\llbracket\xi_2\sigma\rrbracket_{\alpha^{\cdot}}^{s_m}(\llbracket\nu_2\sigma\rrbracket_{\alpha^{\cdot}}^{s_m}) = 1$, and thus $\llbracket\xi_2\sigma\rrbracket_\alpha^{s_m}(u_2') = 1$ by Proposition 2.4.4 and definition of $\alpha^{\cdot}$. Hence, $[\Rightarrow]^{s_m}(\llbracket\xi_1\sigma\rrbracket_\alpha^{s_m}, \llbracket\xi_2\sigma\rrbracket_\alpha^{s_m})(\llbracket\nu\sigma\rrbracket_\alpha^{s_m}) = 1$ and therefore $s_m, \alpha \Vdash \nu\sigma{:}\xi_1\sigma \Rightarrow \xi_2\sigma$ as we wanted to show.

suppose r is $\Rightarrow_E$, and that $s_m, \alpha \Vdash R_\Rightarrow\nu\sigma\nu_1\sigma\nu_2\sigma$, $s_m, \alpha \Vdash \nu\sigma{:}\xi_1\sigma \Rightarrow \xi_2\sigma$, and $s_m, \alpha \Vdash \nu_1\sigma{:}\xi_1\sigma$. Observe that $\llbracket\xi_1\sigma \Rightarrow \xi_2\sigma\rrbracket_\alpha^{s_m}(\llbracket\nu\sigma\rrbracket_\alpha^{s_m}) = 1$, $\langle\llbracket\nu\sigma\rrbracket_\alpha^{s_m}, \llbracket\nu_1\sigma\rrbracket_\alpha^{s_m}, \llbracket\nu_2\sigma\rrbracket_\alpha^{s_m}\rangle$ is in $[R_\Rightarrow]$ and $\llbracket\xi_1\sigma\rrbracket_\alpha^{s_m}(\llbracket\nu_1\sigma\rrbracket_\alpha^{s_m}) = 1$. Then $\llbracket\xi_2\sigma\rrbracket_\alpha^{s_m}(\llbracket\nu_2\sigma\rrbracket_\alpha^{s_m}) = 1$ and so $s_m, \alpha \Vdash \nu_2\sigma{:}\xi_2\sigma$ as we wanted to show.

suppose r is $\wedge_I$, $\wedge_E^1$, $\wedge_E^2$. The proofs are similar to the proof for $\Rightarrow_E$, so they are omitted.

suppose r is $R_{\Rightarrow\alpha}^{\text{lgen}}$, and that $s_m, \alpha \Vdash R_\Rightarrow\nu_1\sigma\nu_2\sigma\nu_3\sigma$, $s_m, \alpha \Vdash \nu_1\sigma \equiv_\omega \nu_1'\sigma$, $s_m, \alpha \Vdash \nu_2\sigma \equiv_\omega \nu_2'\sigma$, $s_m, \alpha \Vdash \nu_3\sigma \equiv_\omega \nu_3'\sigma$ and $s_m, \alpha \Vdash \nu_2'\sigma \equiv_\omega \nu_3'\sigma$. Therefore we can conclude that $R_\Rightarrow^\circ\omega(\llbracket\nu_1\sigma\rrbracket_\alpha^{s_m})\omega(\llbracket\nu_2\sigma\rrbracket_\alpha^{s_m})\omega(\llbracket\nu_3\sigma\rrbracket_\alpha^{s_m})$, $\omega(\llbracket\nu_1\sigma\rrbracket_\alpha^{s_m}) = \omega(\llbracket\nu_1'\sigma\rrbracket_\alpha^{s_m})$, $\omega(\llbracket\nu_2\sigma\rrbracket_\alpha^{s_m}) = \omega(\llbracket\nu_2'\sigma\rrbracket_\alpha^{s_m})$, $\omega(\llbracket\nu_3\sigma\rrbracket_\alpha^{s_m}) = \omega(\llbracket\nu_3'\sigma\rrbracket_\alpha^{s_m})$ and $\alpha(\llbracket\nu_2'\sigma\rrbracket_\alpha^{s_m}) = \alpha(\llbracket\nu_3'\sigma\rrbracket_\alpha^{s_m})$. Therefore we can conclude that $R_\Rightarrow^\circ\omega(\llbracket\nu_1'\sigma\rrbracket_\alpha^{s_m})\omega(\llbracket\nu_2'\sigma\rrbracket_\alpha^{s_m})\omega(\llbracket\nu_3'\sigma\rrbracket_\alpha^{s_m})$ and so $s_m, \alpha \Vdash R_\Rightarrow\nu_1'\sigma\nu_2'\sigma\nu_3'\sigma$.

suppose r is *rmon1*, and that $s_m, \alpha \Vdash R_\Rightarrow 0\nu\sigma\nu_1\sigma$ and $s_m, \alpha \Vdash R_\Rightarrow\nu_1\sigma\nu_2\sigma\nu_3\sigma$. Therefore $R_\Rightarrow^\circ G\omega(\llbracket\nu\sigma\rrbracket_\alpha^{s_m})\omega(\llbracket\nu_1\sigma\rrbracket_\alpha^{s_m})$ and $R_\Rightarrow^\circ\omega(\llbracket\nu_1\sigma\rrbracket_\alpha^{s_m})\omega(\llbracket\nu_2\sigma\rrbracket_\alpha^{s_m})\omega(\llbracket\nu_3\sigma\rrbracket_\alpha^{s_m})$. Thus we have that $R_\Rightarrow^\circ\omega(\llbracket\nu\sigma\rrbracket_\alpha^{s_m})\omega(\llbracket\nu_2\sigma\rrbracket_\alpha^{s_m})\omega(\llbracket\nu_3\sigma\rrbracket_\alpha^{s_m})$ and so $s_m, \alpha \Vdash R_\Rightarrow\nu\sigma\nu_2\sigma\nu_3\sigma$ as we wanted to show.

suppose r is *rmon2*, *rmon3* or *iden*. The proofs are similar to the proof for *rmon1*, so they are omitted. *QED*

We now show the two auxiliary lemmas needed in the proof of Proposition 5.4.4.

Lemma 5.4.5 We have, for any model m in $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$,

$$\llbracket\varphi\rrbracket_\alpha^{s_m}(u) = 1 \quad \text{iff} \quad \llbracket\varphi\rrbracket_\alpha^{s_m}(u') = 1,$$

whenever φ is a formula with $\alpha(u)(x) = \alpha(u')(x)$ for any variable x in φ, and u and u' are points with $\omega(u) = \omega(u')$.

Proof The proof follows by induction on the structure of φ:
suppose φ *is in* P_0. Then, $\llbracket\varphi\rrbracket_\alpha^{s_m}(u) = 1$ iff $[\varphi]_{\omega(u)} = 1$ iff $[\varphi]_{\omega(u')} = 1$ iff $\llbracket\varphi\rrbracket_\alpha^{s_m}(u') = 1$.
suppose φ *is* $x = y$. Then, $\llbracket\varphi\rrbracket_\alpha^{s_m}(u) = 1$ iff $\alpha(u)(x) = \alpha(u)(y)$ iff $\alpha(u')(x) = \alpha(u')(y)$ iff $\llbracket\varphi\rrbracket_\alpha^{s_m}(u') = 1$.

suppose φ is $\varphi_1 \Rightarrow \varphi_2$. Assume $[\![\varphi_1 \Rightarrow \varphi_2]\!]_\alpha^{s_m}(u) = 1$. Let u_1' and u_2' be points such that $R_\Rightarrow^\circ \omega(u')\omega(u_1')\omega(u_2')$ and $[\![\varphi_1]\!]_\alpha^{s_m}(u_1') = 1$. Then $R_\Rightarrow^\circ \omega(u)\omega(u_1')\omega(u_2')$. Thus, $[\![\varphi_2]\!]_\alpha^{s_m}(u_2') = 1$, and so $[\![\varphi_1 \Rightarrow \varphi_2]\!]_\alpha^{s_m}(u') = 1$, as we wanted to show.

suppose φ is $\varphi_1 \wedge \varphi_2$. Then the proof is straightforward. *QED*

Lemma 5.4.6 We have, for any structure s_m induced by a model m in $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$,

$$[\![\varphi]\!]_\alpha^{s_m}(u_2) = 1 \quad \text{whenever} \quad [\![\varphi]\!]_\alpha^{s_m}(u_1) = 1 \text{ and } R_\Rightarrow^\circ G\omega(u_1)\omega(u_2)$$

for any formula φ without variables.

Proof The proof follows by induction on the structure of a formula φ without variables. Assume $[\![\varphi]\!]_\alpha^{s_m}(u_1) = 1$ and $R_\Rightarrow^\circ G\omega(u_1)\omega(u_2)$, and

suppose φ *is in* P_0. Note that $V(\varphi, \omega(u_1)) = 1$. So, by the atomic hereditary condition in the definition a relational modal for this logic, $V(\varphi, \omega(u_2)) = 1$, and thus, $[\![\varphi]\!]_\alpha^{s_m}(u_2) = 1$, as we wanted to show.

suppose φ *is* $\varphi_1 \Rightarrow \varphi_2$. Let u_2' and u_2'' be points such that $R_\Rightarrow^\circ \omega(u_2)\omega(u_2')\omega(u_2'')$ and $[\![\varphi_1]\!]_\alpha^{s_m}(u_2') = 1$. Then $R_\Rightarrow^\circ \omega(u_1)\omega(u_2')\omega(u_2'')$, and so $[\![\varphi_2]\!]_\alpha^{s_m}(u_2'') = 1$, as we wanted to show.

suppose φ *is* $\varphi_1 \wedge \varphi_2$. The result follows because $[\![\varphi_1]\!]_\alpha^{s_m}(u_1) = 1$ and $[\![\varphi_2]\!]_\alpha^{s_m}(u_1) = 1$, and so, by the induction hypothesis, $[\![\varphi_1]\!]_\alpha^{s_m}(u_2) = 1$ and $[\![\varphi_2]\!]_\alpha^{s_m}(u_2) = 1$, as we wanted to show. *QED*

5.4.3 Completeness

The goal of this subsection is to show the following theorem, which we prove at the end of the section.

Theorem 5.4.7 The lfob logic system $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$ is complete.

To prove this theorem we show that $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$ is rich and then we invoke Theorem 3.3.3 which says that a rich lfob logic system is complete. But before we need to show some auxiliary propositions. So, we start to prove that $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$ is a connected lfob logic system without a disjunction and implication and with locality.

Proposition 5.4.8 The lfob logic system $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$ is connected, with locality and without a disjunction and implication.

Proof It is straightforward to see that $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$ is connected and do not have a disjunction and a implication. Recall Definition 4.1.3 of a connected lfob logic system, and Definition 4.1.1 of the disjunction and implication connectives. Note that $\Rightarrow$ is a binary universal connective with 3 as its only inheritance position. Moreover, rules *rmon1*, *rmon2* and *rmon3* have at the third position in the conclusion, a label schema variable, which appear in the third position of one of the premises ψ. The other premise have at the third position a label schema variable that appear also in ψ. Observe also that $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$ do not have a negation, and the rule *exh*. So, it is straightforward to see that it is with locality. Recall the notion of a lfob deduction system with locality in Definition 4.4.2. *QED*

We need now to introduce the notion of a relational model for the ∧-⇒ fragment of basic relevance logic B induced by an appropriate set, in the context of $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$.

Definition 5.4.9 In the context of $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$, the model m_Ψ *induced by* an E appropriate set Ψ is defined as follows:

- W is $\{[v]_\omega^{\Psi,E} \mid v \text{ in } T_{\mathrm{lab},E}\}$,
- G is $[0]_\omega^{\Psi,E}$,
- $R_\Rightarrow^\circ$ is $\{\langle [v_1]_\omega^{\Psi,E}, [v_2]_\omega^{\Psi,E}, [v_3]_\omega^{\Psi,E}\rangle \mid R_\Rightarrow v_1 v_2 v_3 \text{ is in } \Psi\}$,
- $V(p, [v]_\omega^{\Psi,E}) = 1$ iff $v{:}p$ is in Ψ (note that if $v{:}p$ and $v \equiv_\omega v'$ are in Ψ then $v'{:}p \in \Psi$).

We denote by s_{m_Ψ} the structure induced by m_Ψ with

$$[0]^{s_{m_\Psi}} = \langle [0]_\omega^{\Psi,E}, \iota_{D^X}^{\Psi}(0)\rangle \qquad \text{and} \qquad D_{s_{m_\Psi}} \text{ equal to } D_\Psi.$$

We now show that satisfaction in the canonical structure and in the structure induced by the canonical model, for labelled schema formulae $\nu{:}\phi$ where ϕ is without schema variables, is equivalent.

Lemma 5.4.10 In the context of $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$, given an E appropriate set Ψ, a formula ϕ in $\mathcal{L}^{\cdot}_{\mathrm{B}_{\wedge,\Rightarrow}}$, and a label schema variable ν,

$$s_\Psi, \alpha \Vdash \nu{:}\phi \quad \text{iff} \quad s_{m_\Psi}, \alpha^{\cdot} \Vdash \nu{:}\phi$$

for any schema assignments α over s_Ψ and $\alpha^{\cdot}$ over s_{m_Ψ} with $\nu\alpha^{\cdot} = [\nu\alpha]_{\omega\times\iota}^{\Psi,E}$.

Proof Recall first the definition of $\iota_{D^X}^{\Psi}$ and $[v]_{\omega\times\iota}^{\Psi,E}$ in Remark 5.5.1, and the definition of a Ψ uniform map in Definition 5.5.5, and let f be a Ψ uniform map. Note that, in the context of the lfob logic system proposed for the ∧-⇒ fragment of basic relevance logic B, the class of Ψ uniform maps is not empty. For example, the map f defined as $f(|\phi|^{s_\Psi}) = \{\langle w, a\rangle \mid w \text{ is } [v]_\omega^{\Psi,E} \text{ for some } v \text{ in } |\phi|^{s_\Psi}, \text{ and } a \in D^X\}$ for any formula ϕ without quantification variables, and $f(|\phi|^{s_\Psi}) = [|\phi|^{s_\Psi}]_{\omega\times\iota}^{\Psi,E}$ for ϕ having schema or quantification variables, is a Ψ uniform map. Recall, by Proposition 5.4.2, that we are in the context of a uniform lfob deduction system. So, we now show a auxiliary result and after we prove the lemma.

1. $[v]_{\omega\times\iota}^{\Psi,E}$ is in $f(\llbracket\phi\rrbracket_\alpha^{s_\Psi})$ iff $[v]_{\omega\times\iota}^{\Psi,E}$ is in $\llbracket\phi\rrbracket_{\alpha^{\cdot}}^{s_{m_\Psi}}$. The proof follows by induction on the structure of a formula ϕ in $\mathcal{L}^{\cdot}_{\mathrm{B}_{\wedge,\Rightarrow}}$:

suppose ϕ is in $P_0^{\cdot}$. Then $[v]_{\omega\times\iota}^{\Psi,E}$ is in $f(\llbracket\phi\rrbracket_\alpha^{s_\Psi})$ iff (by Proposition 5.5.6) v is in $\llbracket\phi\rrbracket_\alpha^{s_\Psi}$ iff (by Proposition 3.1.3) v is in $|\phi|^{s_\Psi}$ iff $v{:}\phi$ is in Ψ iff $V^{m_\Psi}(\phi, [v]_\omega^{\Psi,E}) = 1$ iff $[\phi]_{[v]_\omega^{\Psi,E}}^{s_{m_\Psi}} = 1$ iff $[\phi]_{\omega([v]_{\omega\times\iota}^{\Psi,E})}^{s_{m_\Psi}} = 1$ iff $\llbracket\phi\rrbracket_{\alpha^{\cdot}}^{s_{m_\Psi}}([v]_{\omega\times\iota}^{\Psi,E}) = 1$ as we wanted to show.

suppose ϕ is $\phi_1 \wedge \phi_2$. Then $[v]_{\omega\times\iota}^{\Psi,E}$ is in $f(\llbracket\phi\rrbracket_\alpha^{s_\Psi})$ iff (by Proposition 5.5.6) v is in $\llbracket\phi\rrbracket_\alpha^{s_\Psi}$ iff (by Proposition 3.1.3) v is in $|\phi|^{s_\Psi}$ iff $v{:}\phi_1 \wedge \phi_2$ is in Ψ iff (since Ψ is deductively closed) $v{:}\phi_1$ is in Ψ and $v{:}\phi_2$ is in Ψ iff v is in $|\phi_1|^{s_\Psi}$ and v is in $|\phi_2|^{s_\Psi}$ iff (by Proposition 3.1.3) v

is in $\llbracket\phi_1\rrbracket_\alpha^{s_\Psi}$ and v is in $\llbracket\phi_2\rrbracket_\alpha^{s_\Psi}$ iff (by Proposition 5.5.6) $[v]_{\omega\times\iota}^{\Psi,E}$ is in $f(\llbracket\phi_1\rrbracket_\alpha^{s_\Psi})$ and $[v]_{\omega\times\iota}^{\Psi,E}$ is in $f(\llbracket\phi_2\rrbracket_\alpha^{s_\Psi})$ iff (by induction hypothesis) $[v]_{\omega\times\iota}^{\Psi,E}$ is in $\llbracket\phi_1\rrbracket_{\alpha^\cdot}^{s_{m_\Psi}}$ and $[v]_{\omega\times\iota}^{\Psi,E}$ is in $\llbracket\phi_2\rrbracket_{\alpha^\cdot}^{s_{m_\Psi}}$ iff $[v]_{\omega\times\iota}^{\Psi,E}$ is in $\llbracket\phi_1 \wedge \phi_2\rrbracket_{\alpha^\cdot}^{s_{m_\Psi}}$, as we wanted to show.

suppose ϕ is $\phi_1 \Rightarrow \phi_2$. Then $[v]_{\omega\times\iota}^{\Psi,E}$ is in $f(\llbracket\phi\rrbracket_\alpha^{s_\Psi})$ iff (by Proposition 5.5.6) v is in $\llbracket\phi\rrbracket_\alpha^{s_\Psi}$ iff (by Proposition 3.1.3) v is in $|\phi|^{s_\Psi}$ iff $v{:}\phi_1 \Rightarrow \phi_2$ is in Ψ iff (since Ψ is an appropriate set) for all v_1 and v_2 if $R_\Rightarrow v v_1 v_2$ and $v_1{:}\phi_1$ are in Ψ then $v_2{:}\phi_2$ is in Ψ iff (by Proposition 5.5.7) for any assignments a_1 and a_2 in ${D_\Psi}^X$ and label schema terms $v_1^\cdot$ and $v_2^\cdot$ if exists a label schema term $v^\cdot$ with $v \equiv_\omega v^\cdot$ and $R_\Rightarrow v^\cdot v_1^\cdot v_2^\cdot$ in Ψ, and $\langle [v_1^\cdot]_\omega^{\Psi,E}, a_1\rangle$ in $f(\llbracket\phi_1\rrbracket_\alpha^{s_\Psi})$ then $\langle [v_2^\cdot]_\omega^{\Psi,E}, a_2\rangle$ is in $f(\llbracket\phi_2\rrbracket_\alpha^{s_\Psi})$ iff (by induction hypothesis) for all $\langle [v_1^\cdot]_\omega^{\Psi,E}, a_1\rangle$ and $\langle [v_2^\cdot]_\omega^{\Psi,E}, a_2\rangle$ in $U^{s_{m_\Psi}}$ if $R^\circ_\Rightarrow \omega([v]_{\omega\times\iota}^{\Psi,E})\omega(\langle [v_1^\cdot]_\omega^{\Psi,E}, a_1\rangle)\omega(\langle [v_2^\cdot]_\omega^{\Psi,E}, a_2\rangle)$ and $\langle [v_1^\cdot]_\omega^{\Psi,E}, a_1\rangle$ is in $\llbracket\phi_1\rrbracket_{\alpha^\cdot}^{s_{m_\Psi}}$ then $\langle [v_2^\cdot]_\omega^{\Psi,E}, a_2\rangle$ is in $\llbracket\phi_2\rrbracket_{\alpha^\cdot}^{s_{m_\Psi}}$ iff $[\Rightarrow]^{s_{m_\Psi}}(\llbracket\phi_1\rrbracket_{\alpha^\cdot}^{s_{m_\Psi}}, \llbracket\phi_2\rrbracket_{\alpha^\cdot}^{s_{m_\Psi}})([v]_{\omega\times\iota}^{\Psi,E}) = 1$ iff $[v]_{\omega\times\iota}^{\Psi,E}$ is in $\llbracket\phi_1 \Rightarrow \phi_2\rrbracket_{\alpha^\cdot}^{s_{m_\Psi}}$ as we wanted to show.

Finally the main proof. Then $s_\Psi, \alpha \Vdash \nu{:}\phi$ iff $\llbracket\nu\rrbracket_\alpha^{s_\Psi}$ is in $\llbracket\phi\rrbracket_\alpha^{s_\Psi}$ iff (by Proposition 5.5.6) $[\nu\alpha]_{\omega\times\iota}^{\Psi,E}$ is in $f(\llbracket\phi\rrbracket_\alpha^{s_\Psi})$ iff (by item 1. above proven) $\nu\alpha^\cdot$ is in $\llbracket\phi\rrbracket_{\alpha^\cdot}^{s_{m_\Psi}}$ iff $s_{m_\Psi}, \alpha^\cdot \Vdash \nu{:}\phi$. *QED*

Proposition 5.4.11 The lfob logic system $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$ is rich.

Proof In the context of $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$, let Ψ_0 be a $v{:}\varphi$ consistent set. Then, by Proposition 4.4.9, there exists an appropriate set Ψ extending Ψ_0 and also $v{:}\varphi$ consistent, since by Proposition 5.4.8, $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$ is a clfob deduction system with locality and without a disjunction and a implication. So, it is possible to consider the canonical structure s_Ψ. Let m_Ψ be the model induced by Ψ, as defined in Definition 5.4.9. Recall that, also by Definition 5.4.9, s_{m_Ψ} is a structure induced by m_Ψ and so s_{m_Ψ} is in $\breve{m}_\Psi$. Then, taking into account Lemma 5.4.10 and the definition of $\breve{m}_\Psi$ in Example 5.4.1, we have that s_Ψ is in $\breve{m}_\Psi$, as we wanted to show. *QED*

Proof ($\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$ is complete, Proposition 5.4.7) The proof follows immediately using Theorem 3.3.3 since, by Proposition 5.4.11, $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$ is rich. *QED*

5.4.4 Equivalence to non-labelled logic system

In this subsection we show that the lfob logic system $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$ is indeed a labelled presentation for the $\wedge$-$\Rightarrow$ fragment of basic relevance logic B with respect to formulae without schema variables, as is stated in the next theorem. This theorem is proved at the end of the section.

Theorem 5.4.12 The lfob logic system $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$ is a labelled presentation for the $\wedge$-$\Rightarrow$ fragment of basic relevance logic B with respect to formulae without schema variables.

The proof of the theorem relies on finding a non-labelled logic system for the $\wedge$-$\Rightarrow$ fragment of basic relevance logic B such that $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$ is a labelled presentation for that system with respect to formulae without schema variables. Recall Definition 2.4.10 where it is defined what is a lfob logic system be a labelled presentation for a logic with respect to some set of schema formulae. So, consider the following system.

Example 5.4.13 $\wedge$-$\Rightarrow$ *fragment of basic relevance logic B non-labelled logic system.* Let $\langle \Sigma^{\cdot}, \propto \rangle$ be the non-labelled logic system for the $\wedge$-$\Rightarrow$ fragment of basic relevance logic B which have as its set of zero-ary propositions, the set of zero-ary propositions of $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$. This non-labelled logic system will be denoted by $\mathcal{L}^{\cdot}_{\mathrm{B}_{\wedge,\Rightarrow}}$.

Observe that the only component that can change between non-labelled logic systems for the $\wedge$-$\Rightarrow$ fragment of basic relevance logic B is the set of zero-ary propositions, and, motivated by that change, the relation $\propto$ associated to the system.

Before presenting the proof that $\mathcal{L}_{\mathrm{R}}$ is indeed a labelled presentation for the relevance logic R with respect to formulae without schema variables, we show two auxiliary lemmas proving the equivalence, for formulae without schema variables, between the usual semantics for the logic and the labelled semantics. Lemma 5.4.14 proves the equivalence for formulae and Lemma 5.4.15 proves the equivalence for entailment.

Lemma 5.4.14 In the context of $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$ and $\mathcal{L}^{\cdot}_{\mathrm{B}_{\wedge,\Rightarrow}}$,

$$m, w \Vdash \phi \quad \text{iff} \quad s_m, \alpha \Vdash \nu{:}\phi$$

where ϕ is a formula in $\mathcal{L}^{\cdot}_{\mathrm{B}_{\wedge,\Rightarrow}}$ and α is such that $\omega(\nu\alpha_l) = w$.

Proof The proof follows by induction on ϕ:

suppose ϕ is in $P^{\cdot}_0$. Then $m, w \Vdash \phi$ iff $V(\phi, w) = 1$ iff $[\phi]^{s_m}_{w} = 1$ iff $[\phi]^{s_m}_{\omega(\llbracket \nu \rrbracket^{s_m}_{\alpha})} = 1$ iff $\widehat{\phi}(\llbracket \nu \rrbracket^{s_m}_{\alpha}) = 1$ iff $\llbracket \phi \rrbracket^{s_m}_{\alpha}(\llbracket \nu \rrbracket^{s_m}_{\alpha}) = 1$ iff $s_m, \alpha \Vdash \nu{:}\phi$, as we wanted to show.

suppose ϕ is $\phi_1 \wedge \phi_2$. Then $m, w \Vdash \phi$ iff $m, w \Vdash \phi_1$ and $m, w \Vdash \phi_2$ iff (by induction hypothesis) $s_m, \alpha \Vdash \nu{:}\phi_1$ and $s_m, \alpha \Vdash \nu{:}\phi_2$ iff $\llbracket \phi_1 \rrbracket^{s_m}_{\alpha}(\llbracket \nu \rrbracket^{s_m}_{\alpha}) = 1$ and $\llbracket \phi_2 \rrbracket^{s_m}_{\alpha}(\llbracket \nu \rrbracket^{s_m}_{\alpha}) = 1$ iff $\widehat{\wedge}(\llbracket \phi_1 \rrbracket^{s_m}_{\alpha}, \llbracket \phi_2 \rrbracket^{s_m}_{\alpha})(\llbracket \nu \rrbracket^{s_m}_{\alpha}) = 1$ iff $s_m, \alpha \Vdash \nu{:}\phi$, as we wanted to show.

suppose ϕ is $\phi_1 \Rightarrow \phi_2$. The key point of this proof is to show that, for any w_1 and w_2 if $R^{\circ}_{\Rightarrow} w w_1 w_2$ and $m, w_1 \Vdash \phi_1$ then $m, w_2 \Vdash \phi_2$, is equivalent to, for any u_1 and u_2 if $\langle \nu\alpha, u_1, u_2 \rangle \in [R_{\Rightarrow}]^{s_m}$ and $\llbracket \phi_1 \rrbracket^{s_m}_{\alpha}(u_1) = 1$ then $\llbracket \phi_2 \rrbracket^{s_m}_{\alpha}(u_2) = 1$. Suppose that, for any w_1 and w_2 if $R^{\circ}_{\Rightarrow} w w_1 w_2$ and $m, w_1 \Vdash \phi_1$ then $m, w_2 \Vdash \phi_2$, and let u_1 and u_2 be such that $\langle \nu\alpha, u_1, u_2 \rangle \in [R_{\Rightarrow}]^{s_m}$ and $\llbracket \phi_1 \rrbracket^{s_m}_{\alpha}(u_1) = 1$. Then $R^{\circ}_{\Rightarrow} \omega(\nu\alpha)\omega(\nu_1\alpha^{\cdot})\omega(\nu_2\alpha^{\cdot})$ and $s_m, \alpha^{\cdot} \Vdash \nu_1{:}\phi_1$ where $\alpha^{\cdot}$ is $\{\nu_1, \nu_2\}$ co-equivalent to α with $\nu_1\alpha^{\cdot} = u_1$ and $\nu_2\alpha^{\cdot} = u_2$. Hence, using the induction hypothesis, we have that $m, \omega(\nu_1\alpha^{\cdot}) \Vdash \phi_1$, and so, $m, \omega(\nu_2\alpha^{\cdot}) \Vdash \phi_2$. Thus, using again the induction hypothesis we have that, $s_m, \alpha^{\cdot} \Vdash \nu_2{:}\phi_2$, i.e., $\llbracket \phi_2 \rrbracket^{s_m}_{\alpha^{\cdot}}(\nu_2\alpha^{\cdot}) = 1$, which, by Proposition 2.4.4, is $\llbracket \phi_2 \rrbracket^{s_m}_{\alpha}(u_2) = 1$ as we wanted to show. For the other direction, suppose for any u_1 and u_2 that if $\langle \nu\alpha, u_1, u_2 \rangle \in [R_{\Rightarrow}]^{s_m}$ and $\llbracket \phi_1 \rrbracket^{s_m}_{\alpha}(u_1) = 1$ then $\llbracket \phi_2 \rrbracket^{s_m}_{\alpha}(u_2) = 1$, and let w_1 and w_2 be such that $R^{\circ}_{\Rightarrow} w w_1 w_2$ and $m, w_1 \Vdash \phi_1$, and $\alpha^{\cdot}$ be $\{\nu_1, \nu_2\}$ co-equivalent to α with $\omega(\nu_1\alpha^{\cdot}) = w_1$ and $\omega(\nu_2\alpha^{\cdot}) = w_2$. Then, $R^{\circ}_{\Rightarrow} \omega(\nu\alpha)\omega(\nu_1\alpha^{\cdot})\omega(\nu_2\alpha^{\cdot})$ and by induction hypothesis, $s_m, \alpha^{\cdot} \Vdash \nu_1{:}\phi_1$. Thus, $\langle \nu\alpha, \nu_1\alpha^{\cdot}, \nu_2\alpha^{\cdot} \rangle \in [R_{\Rightarrow}]^{s_m}$ and, $\llbracket \phi_1 \rrbracket^{s_m}_{\alpha^{\cdot}}(\nu_1\alpha^{\cdot}) = 1$ i.e., $\llbracket \phi_1 \rrbracket^{s_m}_{\alpha}(\nu_1\alpha^{\cdot}) = 1$, by Proposition 2.4.4. So $\llbracket \phi_2 \rrbracket^{s_m}_{\alpha}(\nu_2\alpha^{\cdot}) = 1$, and by Proposition 2.4.4 $\llbracket \phi_2 \rrbracket^{s_m}_{\alpha^{\cdot}}(\nu_2\alpha^{\cdot}) = 1$. Hence $s_m, \alpha^{\cdot} \Vdash \nu_2{:}\phi_2$ and by induction hypothesis $m, w_2 \Vdash \phi_2$, as we wanted to show. *QED*

Lemma 5.4.15 In the context of $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$ and $\mathcal{L}^{\cdot}_{\mathrm{B}_{\wedge,\Rightarrow}}$, for a set Φ of formulae and a formula γ,

$$\Phi \propto \gamma \quad \text{iff} \quad \{\nu{:}\phi \mid \phi \text{ in } \Phi\} \vDash \nu{:}\gamma$$

where ν is a label schema variable.

Proof Assume $\Phi \propto \gamma$, and let s be a structure in $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$ and α an assignment over s such that $s, \alpha \Vdash \{\nu{:}\phi \mid \phi \text{ in } \Phi\}$. Then, we have to consider two cases, (i) s is a structure induced by a relational model m for basic relevance logic, or (ii) s is a structure equivalent in terms of satisfaction with a structure $s^{\cdot}_m$ induced by a relational model m for the $\wedge$-$\Rightarrow$ fragment of relevance logic. If (i) holds then, using Lemma 5.4.14, we have that $m, \omega(\nu\alpha_l) \Vdash \Phi$. Thus, using the fact that $\Phi \propto \gamma$ we have that $m, \omega(\nu\alpha_l) \Vdash \gamma$. Hence, using again Lemma 5.4.14 we can conclude that $s, \alpha \Vdash \nu{:}\gamma$ as we wanted to show. If (ii) holds then there is a assignment $\alpha^{\cdot}$ over $s^{\cdot}_m$ such that $s^{\cdot}_m, \alpha^{\cdot} \Vdash \{\nu{:}\phi \mid \phi \text{ in } \Phi\}$. Therefore, using Lemma 5.4.14, we have that $m, \omega(\nu\alpha^{\cdot}_l) \Vdash \Phi$. Thus, using the fact that $\Phi \propto \gamma$ we have that $m, \omega(\nu\alpha^{\cdot}_l) \Vdash \gamma$. Hence, using again Lemma 5.4.14 we can conclude that $s^{\cdot}_m, \alpha^{\cdot} \Vdash \nu{:}\gamma$. So, since s is a structure equivalent in terms of satisfaction with $s^{\cdot}_m$ we have that $s, \alpha \Vdash \nu{:}\gamma$ as we wanted to show.

Assume $\{\nu{:}\phi \mid \phi \text{ in } \Phi\} \vDash \nu{:}\gamma$ and let m be a relational model for the basic relevance logic over $\Sigma^{\cdot}$ and w a world of m such that $m, w \Vdash \Phi$. Then, by Lemma 5.4.14, we have that $s_m, \alpha \Vdash \{\nu{:}\phi \mid \phi \in \Phi\}$ where α is such that $\omega(\nu\alpha_l) = w$. So, $s_m, \alpha \Vdash \nu{:}\gamma$ since $\{\nu{:}\phi \mid \phi \text{ in } \Phi\} \vDash \nu{:}\gamma$ and s_m is in $\breve{m}$ and m is in M. Therefore using again Lemma 5.4.14, we have that $m, w \Vdash \gamma$, as we wanted to show. *QED*

Proof (Labelled presentation Theorem 5.4.12) According to Definition 2.4.10 it is sufficient to show that $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$ is a labelled presentation for $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}{}^{\cdot}$ with respect to formulae without schema variables. Recall Definition 2.4.9 where it is said what is a lfob logic system be a labelled presentation for a non-labelled logic system with respect to a set of non-labelled schema formulae. Note that the set of formulae of $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$ coincide with the set of formulae of $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}{}^{\cdot}$. So, the lfob logic system $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$ is a labelled presentation for $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}{}^{\cdot}$ with respect to formulae without schema variables because:

- $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$ is sound and complete. See Theorem 5.4.7 for completeness and Theorem 5.4.3 for soundness.

- $X^{\cdot}$ and $\Xi^{\cdot}_t$ are empty sets and $\Xi^{\cdot}_f$ is equal to Ξ_f.

- $F^{\cdot}$ is equal to F, $P^{\cdot}$ is equal to P, and $C^{\cdot}$ is equal to $C \cup O \cup Q$, by definition of $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$, see Example 5.4.1, and of $\mathcal{L}^{\cdot}_{\mathrm{B}_{\wedge,\Rightarrow}}$, see Example 5.4.13.

- $\Phi \propto \gamma$ iff $\{\nu{:}\phi \mid \phi \in \Phi\} \vDash \nu{:}\gamma$, for any set Φ of formulae and formula γ, where ν is a label schema variable. This result follows by Lemma 5.4.15. *QED*

5.5 Common definitions and results

In this section we present definitions and results commonly used through the four examples presented in this chapter.

Remark 5.5.1 Given a deductively closed and E canonical set Ψ, we denote by

$$\iota^{\Psi}_{D^X}$$

the map from $T_{\mathrm{lab},E}$ to ${D_\Psi}^X$ defined as $\iota^{\Psi}_{D^X}(v)(x) = [v{:}x]^{\Psi,E}_g$. Moreover, we denote by

$$[v]^{\Psi,E}_{\omega\times\iota}$$

the pair $\langle [v]_\omega^{\Psi,E}, \iota^\Psi_{D^X}(v)\rangle$, where v is a label schema term.

Proposition 5.5.2 Given any E appropriate set Ψ we have that

$$\text{if} \quad v_1 \equiv_\alpha v_2 \text{ is in } \Psi \quad \text{then} \quad \iota^\Psi_{D^X}(v_1) \text{ is equal to } \iota^\Psi_{D^X}(v_2),$$

for any label schema terms v_1 and v_2.

The proof of the previous proposition is omitted since it follows straightforwardly.

Prop/Definition 5.5.3 Given an E appropriate set Ψ, the families

- ${\_}^{F_\Psi} = \{{\_}^{F_\Psi}{}_{k,w}\}_{k\in I\!N, w\in W_\Psi}$ of maps from F_k to ${D_\Psi}^k \to D_\Psi$ defined such that

$$f^{F_\Psi}{}_{k,[v]_\omega^{\Psi,E}}([v{:}t_1]_g^{\Psi,E}, \ldots, [v{:}t_k]_g^{\Psi,E}) = [v{:}f(t_1, \ldots, t_k)]_g^{\Psi,E}$$

- ${\_}^{P_\Psi} = \{{\_}^{P_\Psi}{}_{k,w}\}_{k\in I\!N, w\in W_\Psi}$ of maps from P_k to ${D_\Psi}^k \to 2$ defined such that

$$p^{P_\Psi}{}_{k,[v]_\omega^{\Psi,E}}([v{:}t_1]_g^{\Psi,E}, \ldots, [v{:}t_k]_g^{\Psi,E}) = 1 \quad \text{iff} \quad v{:}p(t_1, \ldots, t_k) \in \Psi$$

are families of well defined maps.

Proof We only show that $f^{F_\Psi}{}_{k,w}$ is well defined since the proof that $p^{P_\Psi}{}_{k,w}$ is well defined follows analogously. So, suppose there are $v^{\cdot}$ and $t^{\cdot}_1, \ldots, t^{\cdot}_k$ with $v \equiv_\omega v^{\cdot}$ and $v{:}t_1 =_g v^{\cdot}{:}t^{\cdot}_1, \ldots, v{:}t_k =_g v^{\cdot}{:}t^{\cdot}_k$ in Ψ. Then, using rule $=_{g_f}$, common to all lfob deduction systems, $v{:}f(t_1, \ldots, t_k) =_g v^{\cdot}{:}f(t^{\cdot}_1, \ldots, t^{\cdot}_k)$ is in Ψ, as we wanted to show. *QED*

In the following we say that $f^{F_\Psi}{}_{k,w}$ is the *denotation of* f *induced by* Ψ *at world* w, and similarly for $p^{P_\Psi}{}_{k,w}$.

5.5.1 Uniform deduction systems

We now study a class of lfob deduction systems, that we call uniform, characterized by the fact that formulae without free variables are inherited by labels in the same world. Note that all the lfob logic systems presented as examples in this chapter are uniform.

Definition 5.5.4 A lfob deduction system is *uniform* whenever, (i.) for any formula ϕ without free variables, and label schema terms v and $v^{\cdot}$,

$$v \equiv_\omega v^{\cdot}, v{:}\phi \vdash v^{\cdot}{:}\phi,$$

and (ii.) it is connected, see Definition 4.1.2.

Definition 5.5.5 Given an appropriate set Ψ, a map f is Ψ *uniform* whenever f is from $\mathcal{B}_\Psi$ to $\wp(W_\Psi \times {D_\Psi}^X)$ and

$$\langle w, a\rangle \text{ is in } f(|\phi|^{s_\Psi}) \qquad \text{iff} \quad \text{exists } v \text{ in } |\phi|^{s_\Psi} \text{ with } w \text{ equal to } [v]_\omega^{\Psi,E},$$

for any formula ϕ without free variables, world w and assignment a.

Proposition 5.5.6 In the context of a uniform lfob deduction system, given an appropriate set Ψ and a Ψ uniform map f,

$$v \text{ is in } \llbracket \phi \rrbracket_{\alpha}^{s_\Psi} \quad \text{iff} \quad [v]_{\omega\times\iota}^{\Psi,E} \text{ is in } f(\llbracket \phi \rrbracket_{\alpha}^{s_\Psi})$$

for any formula ϕ without free variables and label schema term v.

Proof Let ϕ be a formula without free variables and v a label schema term, and suppose $[v]_{\omega\times\iota}^{\Psi,E}$ is in $f(\llbracket \phi \rrbracket_{\alpha}^{s_\Psi})$. Then, by Proposition 3.1.3, $[v]_{\omega\times\iota}^{\Psi,E}$ is in $f(|\phi|^{s_\Psi})$ and so, by the definition of uniform map, Definition 5.5.5, there is $v^{\cdot}$ in $|\phi|^{s_\Psi}$ with $v \equiv_\omega v^{\cdot}$ in Ψ. Hence $v{:}\phi$ is in Ψ since we are in the context of a uniform lfob deduction system, ϕ is a formula without free variables, and Ψ is deductively closed. Thus v is in $|\phi|^{s_\Psi}$ and by Proposition 3.1.3, v is in $\llbracket \phi \rrbracket_{\alpha}^{s_\Psi}$, as we would like to show. The proof of the other direction is immediate, so we omit it. *QED*

We now prove a proposition that will be very useful in the context of a uniform lfob deduction system, for instance in Lemma 5.4.10, when proving the equivalence between the satisfaction of a universal modal formula without free variables in the canonical structure over an appropriate set, and in a structure induced by the canonical model over that set.

Proposition 5.5.7 In the context of a uniform lfob deduction system, given an appropriate set Ψ and a Ψ uniform map f, we have that,

$$\text{for all } v_1,\ldots,v_n \quad \text{if } r\,vv_1\ldots v_n \text{ and } v_1{:}\phi_1,\ldots,v_{n-1}{:}\phi_{n-1} \text{ are in } \Psi \text{ then } v_n{:}\phi_n \text{ is in } \Psi$$

iff

$$\text{for all } a_1,\ldots,a_n \text{ in } {D_\Psi}^X \text{ and } v^{\cdot}_1,\ldots,v^{\cdot}_n$$

$$\begin{aligned}
&\text{if} \quad \text{there exists } v^{\cdot} \text{ with}\\
&v \equiv_\omega v^{\cdot} \text{ and } r\,v^{\cdot}v^{\cdot}_1\ldots v^{\cdot}_n \text{ in } \Psi, \text{ and}\\
&\langle [v^{\cdot}_1]_{\omega}^{\Psi,E}, a_1\rangle \text{ in } f(\llbracket \phi_1 \rrbracket_{\alpha}^{s_\Psi}) \quad \text{and} \quad \ldots \quad \text{and} \quad \langle [v^{\cdot}_{n-1}]_{\omega}^{\Psi,E}, a_{n-1}\rangle \text{ in } f(\llbracket \phi_{n-1} \rrbracket_{\alpha}^{s_\Psi})\\
&\text{then} \quad \langle [v^{\cdot}_n]_{\omega}^{\Psi,E}, a_n\rangle \text{ is in } f([\llbracket \phi_n \rrbracket_{\alpha}^{s_\Psi}).
\end{aligned}$$

for any label schema term v, formulae $\phi_1,\ldots,\phi_n$ without free variables, and relation r such that the lfob deduction system contains the rule r_{α}^{lgen}.

Proof Suppose that, for any assignments $a_1,\ldots,a_n$ in ${D_\Psi}^X$ and label schema terms $v^{\cdot}_1,\ldots,v^{\cdot}_n$ if there exists a label schema term $v^{\cdot}$ with $v \equiv_\omega v^{\cdot}$ and $r\,v^{\cdot}v^{\cdot}_1\ldots v^{\cdot}_n$ in Ψ, and $\langle [v^{\cdot}_i]_{\omega}^{\Psi,E}, a_i\rangle$ in $f(\llbracket \phi_i \rrbracket_{\alpha}^{s_\Psi})$ for each $i = 1,\ldots,n-1$, then $\langle [v^{\cdot}_n]_{\omega}^{\Psi,E}, a_n\rangle$ is in $f(\llbracket \phi_n \rrbracket_{\alpha}^{s_\Psi})$. Let $v_1,\ldots,v_n$ be label schema terms such that $r\,vv_1\ldots v_n$ and $v_1{:}\phi_1,\ldots,v_{n-1}{:}\phi_{n-1}$ are in Ψ. Then, for each $i = 1,\ldots,n-1$, v_i is in $\llbracket \phi_i \rrbracket_{\alpha}^{s_\Psi}$ by Proposition 3.1.3, and so, $[v_i]_{\omega\times\iota}^{\Psi,E}$ is in $f(\llbracket \phi_i \rrbracket_{\alpha}^{s_\Psi})$, by Proposition 5.5.6. Thus, by the initial assumption, $[v_n]_{\omega\times\iota}^{\Psi,E}$ is in $f(\llbracket \phi_n \rrbracket_{\alpha}^{s_\Psi})$. Hence, by Proposition 5.5.6, v_n is in $\llbracket \phi_n \rrbracket_{\alpha}^{s_\Psi}$, and by Proposition 3.1.3, $v_n{:}\phi_n$ is in Ψ, as we wanted to show.

Suppose that for all label schema terms $v_1,\ldots,v_n$ if $r\,vv_1\ldots v_n$ and $v_1{:}\phi_1\ldots v_{n-1}{:}\phi_{n-1}$ are in Ψ then $v_n{:}\phi_n$ is in Ψ. Let $a_1,\ldots,a_n$ be assignments in ${D_\Psi}^X$ and $v^{\cdot}_1,\ldots,v^{\cdot}_n$ label schema terms such that exist a label schema term v' with $v \equiv_\omega v'$ and $r\, v' v_1' \dots v_n'$ in Ψ, and $\langle [v_i']_\omega^{\Psi,E}, a_i\rangle$ in $f(\llbracket\phi_i\rrbracket_\alpha^{s_\Psi})$ for each $i = 1,\dots,n-1$. Then, for each $i = 1,\dots,n-1$, $\langle [v_i']_\omega^{\Psi,E}, a_i\rangle$ is in $f(|\phi_i|^{s_\Psi})$ and so, by the definition of uniform map, Definition 5.5.5, exists v_i'' in $|\phi_i|^{s_\Psi}$ with $v_i'' \equiv_\omega v_i'$ in Ψ. Hence, for each $i = 1,\dots,n-1$, $v_i'{:}\phi_i$ is in Ψ since we are in the context of a uniform lfob deduction system, ϕ_i is a formula without free variables, and Ψ is deductively closed. Note that $r\, v v_1' \dots v_n'$ is in Ψ by rule r_α^{lgen}. So, we can use the initial assumption to conclude $v_n'{:}\phi_n$ is in Ψ, i.e. v_n' is in $|\phi_n|^{s_\Psi}$. So, since ϕ is a formula without free variables, and by the definition of uniform map, Definition 5.5.5, $\langle [v_n']_\omega^{\Psi,E}, a_n\rangle$ is in $f(|\phi_n|^{s_\Psi})$, i.e., by Proposition 3.1.3, in $f(\llbracket\phi_n\rrbracket_\alpha^{s_\Psi})$, as we wanted to show. *QED*

5.5.2 Uniform deduction systems with universal quantifiers

The purpose of this subsection is to study a class of lfob deduction systems that we call uniform with universal quantifiers, which are a special case of uniform systems. For instance, the lfob logic systems for first-order modal logic with decreasing domains and for the $\wedge$ fragment of first-order modal logic T are uniform and with a universal quantifier.

Definition 5.5.8 A uniform lfob deduction system is with a *universal quantifier* iff (i.) for any formula ϕ with $x_1,\dots,x_k$ as free variables, and for any label schema terms v and v',

$$v \equiv_\omega v', v{:}x_1 =_g v'{:}x_1, \dots, v{:}x_k =_g v'{:}x_k, v{:}\phi \vdash v'{:}\phi.$$

and, (ii.) there is a connective $\forall$ in Q_1, where, for each variable x, $\forall_x$ is a universal connective over a relation $\equiv_x$, docked in $\equiv_\omega$, satisfying

$$v_1 \equiv_x v_2 \vdash v_1{:}y =_g v_2{:}y \text{ for any } y \text{ distinct of } x, \qquad \text{and} \qquad \vdash v \equiv_x v,$$

for any label schema terms v, v_1 and v_2, and if $\forall_x$ is constrained then the constraint formula is $\varepsilon(x)$ where ε is in P_1.

Proposition 5.5.9 In the context of a uniform lfob deduction system with a universal quantifier, given any E appropriate set Ψ, we have that

$$\text{if} \quad v_1 \equiv_x v_2 \text{ is in } \Psi \quad \text{then} \quad \iota_{D^X}^{\Psi}(v_1) \text{ and } \iota_{D^X}^{\Psi}(v_2) \text{ are } x \text{ co-equivalent},$$

for any label schema terms v_1 and v_2.

The proof of the previous proposition is omitted since it follows straightforwardly.

Definition 5.5.10 In the context of a uniform lfob deduction system with a universal quantifier and given an E appropriate set Ψ, a map f is Ψ *strongly uniform* whenever f is from $\mathcal{B}_\Psi$ to $\wp(W_\Psi \times {D_\Psi}^X)$ and, for any formula ϕ,

$$f(|\phi|^{s_\Psi})$$

is the least set containing

- $[v]_{\omega\times\iota}^{\Psi,E}$, if v is in $|\phi|^{s_\Psi}$,
- $\langle [v]_\omega^{\Psi,E}, a\rangle$, if a is a assignment x co-equivalent to $\iota_{D^X}^{\Psi}(v)$, $[v]_x^{\Psi,E}$ is contained in $|\phi|^{s_\Psi}$ and $\forall_x$ is not constrained, for some variable x,

- $\langle [v]^{\Psi,E}_{\omega}, a\rangle$, if a is a assignment x co-equivalent to $\iota^{\Psi}_{DX}(v)$, $[v]^{\Psi,E}_{x} \cap |\varepsilon(x)|^{s_\Psi}$ is contained in $|\phi|^{s_\Psi}$, $a(x) = [v_1{:}t]^{\Psi,E}_{g}$, $v_1{:}\varepsilon(t)$ and $v_1 \equiv_\omega v$ are in Ψ, for some v_1 and t, and $\forall_x$ is constrained, for some variable x.

Proposition 5.5.11 In the context of a uniform lfob deduction system with a universal quantifier, given an appropriate set Ψ and a Ψ strongly uniform map f,

$$v \text{ is in } [\![\phi]\!]^{s_\Psi}_{\alpha} \quad \text{iff} \quad [v]^{\Psi,E}_{\omega\times\iota} \text{ is in } f([\![\phi]\!]^{s_\Psi}_{\alpha}),$$

for any formula ϕ and label schema term v.

Proof Suppose $[v]^{\Psi,E}_{\omega\times\iota}$ is in $f([\![\phi]\!]^{s_\Psi}_{\alpha})$. Then, by Proposition 3.1.3, $[v]^{\Psi,E}_{\omega\times\iota}$ is in $f(|\phi|^{s_\Psi})$. So, by definition of strongly uniform map, Definition 5.5.10, one of the following three cases happens:

i. there is $v^{\cdot}$ in $|\phi|^{s_\Psi}$ with $v \equiv_\omega v^{\cdot}$ in Ψ, and for any variable x, $v{:}x =_g v^{\cdot}{:}x$ is in Ψ. So, by Proposition 3.1.3, $v^{\cdot}{:}\phi$ is in Ψ. Then $v{:}\phi$ is in Ψ since we are in the context of a uniform lfob deduction system with a universal quantifier, ϕ is a formula, and Ψ is deductively closed. Hence v is in $|\phi|^{s_\Psi}$ and by Proposition 3.1.3, v is in $[\![\phi]\!]^{s_\Psi}_{\alpha}$, as we would like to show.

ii. there is $v^{\cdot}$ with $v \equiv_\omega v^{\cdot}$ in Ψ, and there is a variable x such that $\iota^{\Psi}_{DX}(v)$ is x co-equivalent to $\iota^{\Psi}_{DX}(v^{\cdot})$, $[v^{\cdot}]^{\Psi,E}_{x}$ is contained in $|\phi|^{s_\Psi}$, and $\forall_x$ is not constrained. Then, for all $v^{\cdot\cdot}$ if $v^{\cdot} \equiv_x v^{\cdot\cdot}$ is in Ψ then $v^{\cdot\cdot}{:}\phi$ is in Ψ. Hence, since Ψ is an appropriate set, $v^{\cdot}{:}\forall_x\phi$ is in Ψ. So, $v{:}\forall_x\phi$ is in Ψ since we are in the context of a uniform lfob deduction system, $\forall_x\phi$ is a formula where x is not free, and $v{:}y =_g v^{\cdot}{:}y$ is in Ψ for any y distinct of x. Thus, $v{:}\phi$ is in Ψ by rule $\forall_{xE}$ since $v \equiv_x v$ is in Ψ because we are in the context of a uniform lfob deduction system, and so, by Proposition 3.1.3 v is in $[\![\phi]\!]^{s_\Psi}_{\alpha}$, as we wanted to show.

iii. there is $v^{\cdot}$ with $v \equiv_\omega v^{\cdot}$ in Ψ, and there is a variable x such that $\iota^{\Psi}_{DX}(v)$ is x co-equivalent to $\iota^{\Psi}_{DX}(v^{\cdot})$, $[v^{\cdot}]^{\Psi,E}_{x} \cap |\varepsilon(x)|^{s_\Psi}$ is contained in $|\phi|^{s_\Psi}$, for some v_1 and t, $\iota^{\Psi}_{DX}(v)(x) = [v_1{:}t]^{\Psi,E}_{g}$, and $v_1{:}\varepsilon(t)$ and $v_1 \equiv_\omega v$ are in Ψ, and $\forall_x$ is constrained. Note that, for all $v^{\cdot\cdot}$ if $v^{\cdot} \equiv_x v^{\cdot\cdot}$ and $v^{\cdot\cdot}{:}\varepsilon(x)$ are in Ψ then $v^{\cdot\cdot}{:}\phi$ is in Ψ. Hence, since Ψ is an appropriate set, $v^{\cdot}{:}\forall_x\phi$ is in Ψ. So, $v{:}\forall_x\phi$ is in Ψ since we are in the context of a uniform lfob deduction system with a universal connective, $\forall_x\phi$ is a formula where x is not free, and $v{:}y =_g v^{\cdot}{:}y$ is in Ψ for any y distinct of x. Observe now that $v{:}x =_g v_1{:}t$ is in Ψ, and since $v_1{:}\varepsilon(t)$ and $v_1 \equiv_\omega v$ are in Ψ we have that $v{:}\varepsilon(x)$ is in Ψ, by rule $=_{g_\varepsilon}$, see Definition 2.3.10. Thus, $v{:}\phi$ is in Ψ by rule $\forall_{xE}$ since $v{:}\varepsilon(x)$ is in Ψ and $v \equiv_x v$ is in Ψ because we are in the context of a uniform lfob deduction system with a universal connective, and so, by Proposition 3.1.3, v is in $[\![\phi]\!]^{s_\Psi}_{\alpha}$, as we wanted to show.

For the left to right direction the proof is immediate so we omit it. *QED*

We now show a proposition that will be very useful, in the context of a uniform lfob deduction system with a universal quantifier, for instance in Lemma 5.2.11, for proving the equivalence between the satisfaction in the canonical structure induced by an appropriate set and in the structure induced by the canonical model over that set, of a universal quantifier formula without schema variables. Similarly for the remaining propositions in this subsection.

Proposition 5.5.12 In the context of a uniform lfob deduction system with a universal quantifier $\forall$, non-constrained, given an appropriate set Ψ and a Ψ strongly uniform map f,

$$v{:}\forall_x \phi_1 \text{ is in } \Psi$$

iff

$$\text{for all } a_1,\ \text{if} \quad a_1 \text{ is } x \text{ co-equivalent to } \iota^{\Psi}_{DX}(v) \quad \text{then} \quad \langle [v]^{\Psi,E}_{\omega}, a_1\rangle \text{ is in } f([\![\phi_1]\!]^{s_\Psi}_{\alpha}),$$

for any label schema term v, variable x, and formula ϕ_1.

Proof Suppose $v{:}\forall_x\phi_1$ is in Ψ and let a_1 be a assignment x co-equivalent to $\iota^{\Psi}_{DX}(v)$. Note that if $v^{\cdot}$ is such that $v \equiv_x v^{\cdot}$ is in Ψ then $v^{\cdot}{:}\phi_1$ is in Ψ by rule $\forall_{xE}$. Thus, $[v]^{\Psi,E}_{x}$ is contained in $|\phi_1|^{s_\Psi}$ and so, since a_1 is x co-equivalent to $\iota^{\Psi}_{DX}(v)$, we have that $\langle [v]^{\Psi,E}_{\omega}, a_1\rangle$ is in $f(|\phi_1|^{s_\Psi})$, since f is a Ψ strongly uniform map, see Definition 5.5.10. So, by Proposition 3.1.3, $\langle [v]^{\Psi,E}_{\omega}, a_1\rangle$ is in $f([\![\phi_1]\!]^{s_\Psi}_{\alpha})$, as we wanted to show.

Suppose that, for all a_1 if a_1 is x co-equivalent to $\iota^{\Psi}_{DX}(v)$ then $\langle [v]^{\Psi,E}_{\omega}, a_1\rangle$ is in $f([\![\phi_1]\!]^{s_\Psi}_{\alpha})$. Note that $v{:}\forall_x\phi_1$ is in Ψ iff for any label schema term v_1 if $v \equiv_x v_1$ is in Ψ then $v_1{:}\phi_1$ is in Ψ, since Ψ is an appropriate set. So we now show that for any label schema term v_1 if $v \equiv_x v_1$ is in Ψ then $v_1{:}\phi_1$ is in Ψ. Let v_1 be a label schema term such that $v \equiv_x v_1$ is in Ψ. Then, by Proposition 5.5.9, $\iota^{\Psi}_{DX}(v_1)$ is x co-equivalent to $\iota^{\Psi}_{DX}(v)$ and $v \equiv_\omega v_1$ is in Ψ since we are in the context of a uniform lfob deduction system. So, by the initial assumption, $[v_1]^{\Psi,E}_{\omega\times\iota}$ is in $f([\![\phi_1]\!]^{s_\Psi}_{\alpha})$. Hence, by Proposition 5.5.11, v_1 is in $[\![\phi_1]\!]^{s_\Psi}_{\alpha}$. Therefore, by Proposition 3.1.3, $v_1{:}\phi_1$ is in Ψ, as we wanted to show. *QED*

Proposition 5.5.13 In the context of a uniform lfob deduction system with a constrained universal quantifier $\forall$, given an appropriate set Ψ and a Ψ strongly uniform map f,

$$v{:}\forall_x \phi_1 \text{ is in } \Psi$$

iff

$$\text{for all } a_1,$$

$$\begin{aligned} &\text{if} \quad a_1 \text{ is } x \text{ co-equivalent to } \iota^{\Psi}_{DX}(v), \text{ and} \\ &\text{for some } v_1 \text{ and } t, v_1{:}\varepsilon(t) \text{ and } v_1 \equiv_\omega v \text{ are in } \Psi \text{ and } a_1(x) = [v_1{:}t]^{\Psi,E}_{g}, \\ &\text{then} \quad \langle [v]^{\Psi,E}_{\omega}, a_1\rangle \text{ is in } f([\![\phi_1]\!]^{s_\Psi}_{\alpha}) \end{aligned}$$

for any label schema term v, variable x, and formula ϕ_1.

Proof Suppose $v{:}\forall_x\phi_1$ is in Ψ and let a_1 be a assignment x co-equivalent to $\iota^{\Psi}_{DX}(v)$, such that exists v_1 and t, with $v_1{:}\varepsilon(t)$ and $v_1 \equiv_\omega v$ in Ψ and $a_1(x) = [v_1{:}t]^{\Psi,E}_{g}$. Note that if $v^{\cdot}$ is such that $v \equiv_x v^{\cdot}$ and $v^{\cdot}{:}\varepsilon(x)$ are in Ψ then $v^{\cdot}{:}\phi_1$ is in Ψ by rule $\forall_{xE}$. Thus, $[v]^{\Psi,E}_{x} \cap |\varepsilon(x)|^{s_\Psi}$ is contained in $|\phi_1|^{s_\Psi}$ and moreover exists v_1 and t, with $v_1{:}\varepsilon(t)$ and $v_1 \equiv_\omega v$ in Ψ and $a_1(x) = [v_1{:}t]^{\Psi,E}_{g}$. So, since a_1 is x co-equivalent to $\iota^{\Psi}_{DX}(v)$, we have that $\langle [v]^{\Psi,E}_{\omega}, a_1\rangle$ is in $f(|\phi_1|^{s_\Psi})$, see the definition of a strongly uniform map in Definition 5.5.10. Hence, by Proposition 3.1.3, $\langle [v]^{\Psi,E}_{\omega}, a_1\rangle$ is in $f([\![\phi_1]\!]^{s_\Psi}_{\alpha})$, as we wanted to show.

Suppose that, for all a_1 if a_1 is x co-equivalent to $\iota^{\Psi}_{DX}(v)$, and $v_1{:}\varepsilon(t)$ and $v_1 \equiv_\omega v$ are in Ψ and $a_1(x) = [v_1{:}t]^{\Psi,E}_g$, for some v_1 and t, then $\langle [v]^{\Psi,E}_\omega, a_1\rangle$ is in $f(\llbracket \phi_1 \rrbracket^{s_\Psi}_\alpha)$. Note that $v{:}\forall_x \phi_1$ is in Ψ iff for any label schema term v_1 if $v \equiv_x v_1$ and $v_1{:}\varepsilon(x)$ are in Ψ then $v_1{:}\phi_1$ is in Ψ, since Ψ is an appropriate set. So we now show that for any label schema term v_1 if $v \equiv_x v_1$ and $v_1{:}\varepsilon(x)$ are in Ψ then $v_1{:}\phi_1$ is in Ψ. Let v_1 be a label schema term such that $v \equiv_x v_1$ and $v_1{:}\varepsilon(x)$ are in Ψ. Then $\iota^{\Psi}_{DX}(v_1)(x) = [v_1{:}x]^{\Psi,E}_g$, $v_1{:}\varepsilon(x)$ is in Ψ, $\iota^{\Psi}_{DX}(v_1)$ is x co-equivalent to $\iota^{\Psi}_{DX}(v)$, $v \equiv_\omega v_1$ is in Ψ, since we are in the context of a uniform lfob deduction system. So, by the initial assumption, $[v_1]^{\Psi,E}_{\omega\times\iota}$ is in $f(\llbracket \phi_1 \rrbracket^{s_\Psi}_\alpha)$, and thus, by Proposition 5.5.11, v_1 is in $\llbracket \phi_1 \rrbracket^{s_\Psi}_\alpha$. Hence, by Proposition 3.1.3, $v_1{:}\phi_1$ is in Ψ, as we wanted to show. *QED*

Proposition 5.5.14 In the context of a uniform lfob deduction system with a universal quantifier, given a strong appropriate set Ψ and a Ψ strongly uniform map f,

for all $v_1, \ldots, v_n$ if $r\, v v_1 \ldots v_n$ and $v_1{:}\phi_1, \ldots, v_{n-1}{:}\phi_{n-1}$ are in Ψ then $v_n{:}\phi_n$ is in Ψ

iff

for all $a_1, \ldots, a_n$ in ${D_\Psi}^X$ and $v^{\cdot}_1, \ldots, v^{\cdot}_n$

if there exists $v^{\cdot}$ with

$v \equiv_\omega v^{\cdot}$ and $r\, v^{\cdot} v^{\cdot}_1 \ldots v^{\cdot}_n$ in Ψ,

$\iota^{\Psi}_{DX}(v) = a_1, \ldots, \iota^{\Psi}_{DX}(v) = a_n$,

$\langle [v^{\cdot}_1]^{\Psi,E}_\omega, a_1\rangle$ in $f(\llbracket \phi_1 \rrbracket^{s_\Psi}_\alpha)$ and ... and $\langle [v^{\cdot}_{n-1}]^{\Psi,E}_\omega, a_{n-1}\rangle$ in $f(\llbracket \phi_{n-1} \rrbracket^{s_\Psi}_\alpha)$

then $\langle [v^{\cdot}_n]^{\Psi,E}_\omega, a_n\rangle$ is in $f(\llbracket \phi_n \rrbracket^{s_\Psi}_\alpha)$,

for any label schema term v, formulae $\phi_1, \ldots, \phi_n$, and relation r such that the lfob deduction system contains r^{gen}_α and the docked rules for r in $\equiv_\alpha$ for all the components.

Proof Suppose that, for any assignments $a_1, \ldots, a_n$ in ${D_\Psi}^X$ and label schema terms $v^{\cdot}_1, \ldots, v^{\cdot}_n$ if exists a label schema term $v^{\cdot}$ with $v \equiv_\omega v^{\cdot}$ and $r\, v^{\cdot} v^{\cdot}_1 \ldots v^{\cdot}_n$ in Ψ, and $\iota^{\Psi}_{DX}(v) = a_i$ for each $i = 1, \ldots, n$, and $\langle [v^{\cdot}_i]^{\Psi,E}_\omega, a_i\rangle$ in $f(\llbracket \phi_i \rrbracket^{s_\Psi}_\alpha)$ for each $i = 1, \ldots, n-1$, then $\langle [v^{\cdot}_n]^{\Psi,E}_\omega, a_n\rangle$ is in $f(\llbracket \phi_n \rrbracket^{s_\Psi}_\alpha)$. Let $v_1, \ldots, v_n$ be label schema terms such that $r\, v v_1 \ldots v_n$ and $v_1{:}\phi_1, \ldots, v_{n-1}{:}\phi_{n-1}$ are in Ψ. Then, since r is a docked rule in $\equiv_\alpha$, we have, for each $i = 1, \ldots, n$, that $v \equiv_\alpha v_i$ is in Ψ, and so, by Proposition 5.5.2, that $\iota^{\Psi}_{DX}(v) = \iota^{\Psi}_{DX}(v_i)$. Moreover, for each $i = 1, \ldots, n-1$, v_i is in $\llbracket \phi_i \rrbracket^{s_\Psi}_\alpha$ by Proposition 3.1.3, and so, $[v_i]^{\Psi,E}_{\omega\times\iota}$ is in $f(\llbracket \phi_i \rrbracket^{s_\Psi}_\alpha)$, by definition of a strongly uniform map, Definition 5.5.10. Then, by the initial assumption, $[v_n]^{\Psi,E}_{\omega\times\iota}$ is in $f(\llbracket \phi_n \rrbracket^{s_\Psi}_\alpha)$, and so, by Proposition 5.5.11, v_n is in $\llbracket \phi_n \rrbracket^{s_\Psi}_\alpha$, i.e., $v_n{:}\phi_n$ is in Ψ, by Proposition 3.1.3, as we wanted to show.

Suppose that for all label schema terms $v_1, \ldots, v_n$ if $r\, v v_1 \ldots v_n$ and $v_1{:}\phi_1 \ldots v_{n-1}{:}\phi_{n-1}$ are in Ψ then $v_n{:}\phi_n$ is in Ψ. Let $a_1, \ldots, a_n$ be assignments in ${D_\Psi}^X$ and $v^{\cdot}_1, \ldots, v^{\cdot}_n$ label schema terms such that exist a label schema term $v^{\cdot}$ with $v \equiv_\omega v^{\cdot}$ and $r\, v^{\cdot} v^{\cdot}_1 \ldots v^{\cdot}_n$ in Ψ, $\iota^{\Psi}_{DX}(v) = a_i$ for each $i = 1, \ldots, n$, and $\langle [v^{\cdot}_i]^{\Psi,E}_\omega, a_i\rangle$ is in $f(\llbracket \phi_1 \rrbracket^{s_\Psi}_\alpha)$ for each $i = 1, \ldots, n-1$. For each $i = 1, \ldots, n$ let v_i be the label schema term such that $v_i \equiv_\omega v^{\cdot}_i$ and $v_i \equiv_\alpha v$ are in Ψ, which exists since Ψ is a strongly appropriate set. Then, for each $i = 1, \ldots, n$, $[v_i]^{\Psi,E}_{\omega\times\iota} = \langle [v^{\cdot}_i]^{\Psi,E}_\omega, a_i\rangle$

by Proposition 5.5.2 and using the fact that $\iota^{\Psi}_{D^X}(v) = a_i$. So, by Proposition 5.5.11, for each $i = 1, \ldots, n-1$, v_i is in $\llbracket \phi_i \rrbracket^{s_\Psi}_{\alpha}$, and so, by Proposition 3.1.3, $v_i{:}\phi_i$ is in Ψ. Moreover, by rule r^{gen}_{α}, and since Ψ is deductively closed, we have that $r\, v v_1 \ldots v_n$ is in Ψ. Thus, by the initial assumption, $v_n{:}\phi_n$ is in Ψ, and so, by Proposition 5.5.11 and Proposition 3.1.3, $[v_n]^{\Psi,E}_{\omega\times\iota}$, i.e., $\langle [v^{\cdot}_n]^{\Psi,E}_{\omega}, a_n\rangle$ is in $f(\llbracket \phi_n \rrbracket^{s_\Psi}_{\alpha})$, as we wanted to show. *QED*

Proposition 5.5.15 In the context of a uniform lfob deduction system with a universal quantifier, given a strong appropriate set Ψ and a Ψ strongly uniform map f,

$$\text{exist } v_1, \ldots, v_n \text{ with } r\, v v_1 \ldots v_n \text{ and } v_1{:}\phi_1, \ldots, v_n{:}\phi_n \text{ in } \Psi$$

iff

$$\text{exist } a_1, \ldots, a_n \text{ in } {D_\Psi}^X \text{ and } v^{\cdot}, v^{\cdot}_1, \ldots, v^{\cdot}_n \text{ with}$$

$$v \equiv_\omega v^{\cdot} \text{ and } r\, v^{\cdot} v^{\cdot}_1 \ldots v^{\cdot}_n \text{ in } \Psi,$$

$$\iota^{\Psi}_{D^X}(v) = a_1, \ldots, \iota^{\Psi}_{D^X}(v) = a_n,$$

$$\langle [v^{\cdot}_1]^{\Psi,E}_{\omega}, a_1\rangle \text{ in } f(\llbracket \phi_1 \rrbracket^{s_\Psi}_{\alpha}) \quad \text{and} \quad \ldots \quad \text{and} \quad \langle [v^{\cdot}_n]^{\Psi,E}_{\omega}, a_n\rangle \text{ in } f(\llbracket \phi_n \rrbracket^{s_\Psi}_{\alpha}),$$

for any label schema term v, formulae $\phi_1, \ldots, \phi_n$, and relation r such that the lfob deduction system contains r^{gen}_{α} and docked rules for r in $\equiv_\alpha$ for all the components.

Proof Suppose exist $a_1, \ldots, a_n$ in ${D_\Psi}^X$ and label schema terms $v^{\cdot}, v^{\cdot}_1, \ldots, v^{\cdot}_n$ with $v \equiv_\omega v^{\cdot}$ and $r\, v^{\cdot} v^{\cdot}_1 \ldots v^{\cdot}_n$ in Ψ, $\iota^{\Psi}_{D^X}(v) = a_i$ for each $i = 1, \ldots, n$, and $\langle [v^{\cdot}_i]^{\Psi,E}_{\omega}, a_i\rangle$ in $f(\llbracket \phi_i \rrbracket^{s_\Psi}_{\alpha})$. For each $i = 1, \ldots, n$, let v_i be the label schema term such that $v_i \equiv_\omega v^{\cdot}_i$ is in Ψ and $v_i \equiv_\alpha v$ is in Ψ, which exists since Ψ is a strong appropriate set. Then, for each $i = 1, \ldots, n$, $[v_i]^{\Psi,E}_{\omega\times\iota} = \langle [v^{\cdot}_i]^{\Psi,E}_{\omega}, a_i\rangle$, by Proposition 5.5.2. So, by Proposition 5.5.11, for each $i = 1, \ldots, n$, v_i is in $\llbracket \phi_i \rrbracket^{s_\Psi}_{\alpha}$, and so, by Proposition 3.1.3, $v_i{:}\phi_i$ is in Ψ. Moreover, by rule r^{gen}_{α}, and since Ψ is deductively closed, we have that $r\, v v_1 \ldots v_n$ is in Ψ, and we are done.

Suppose exist $v_1, \ldots, v_n$ with $r\, v v_1 \ldots v_n$ and $v_1{:}\phi_1, \ldots, v_n{:}\phi_n$ in Ψ. Then, by the docked rules of r in $\equiv_\alpha$, see Definition 4.1.2 of connected lfob deduction system, we have that $v \equiv_\alpha v_i$ is in Ψ for any $i = 1, \ldots, n$. So, by Proposition 5.5.2, we have that $\iota^{\Psi}_{D^X}(v) = \iota^{\Psi}_{D^X}(v_1), \ldots, \iota^{\Psi}_{D^X}(v) = \iota^{\Psi}_{D^X}(v_n)$. Moreover, for each $i = 1, \ldots, n$, by Proposition 3.1.3, v_i is in $\llbracket \phi_i \rrbracket^{s_\Psi}_{\alpha}$, and so, by Proposition 5.5.11, $[v_i]^{\Psi,E}_{\omega\times\iota}$ is in $f(\llbracket \phi_i \rrbracket^{s_\Psi}_{\alpha})$, and we are done. *QED*

5.5.3 Uniform deduction systems with universal and existential quantifiers

The purpose of this subsection is to study a class of lfob deduction systems, that we call uniform with universal and existential quantifiers, which are a special case of uniform systems with a universal quantifier. For instance, the lfob logic system for the $\wedge$ fragment of first-order modal logic T is uniform and with a universal and an existential quantifier.

Definition 5.5.16 A uniform lfob deduction system with a universal quantifier is with an *existential quantifier* whenever there is a connective $\exists$ in Q_1, where for each variable x, $\exists_x$ is an existential connective over $\equiv_x$ which is constrained iff $\forall_x$ is constrained, and in this case also by $\varepsilon(x)$.

We now show a proposition that will be very useful, in the context of a uniform lfob deduction system with a universal and an existential quantifier, for instance in Lemma 5.2.11, for proving the equivalence between the satisfaction in the canonical structure induced by an appropriate set and in the structure induced by the canonical model over that set, of an existential formula without schema variables.

Proposition 5.5.17 In the context of a uniform lfob deduction system with non-constrained universal and existential quantifiers, and given an appropriate set Ψ and a Ψ strongly uniform map f, $v{:}\exists_x\phi_1$ is in Ψ iff

$$\text{exists } a_1 \text{ with } \quad a_1\, x \text{ co-equivalent to } \iota^{\Psi}_{DX}(v) \quad \text{and} \quad \langle [v]^{\Psi,E}_{\omega}, a_1\rangle \text{ in } f([\![\phi_1]\!]^{s_\Psi}_{\alpha})$$

for any label schema term v, variable x and formula ϕ_1.

Proof Suppose exists a_1 with a_1 x co-equivalent to $\iota^{\Psi}_{DX}(v)$, and $\langle [v]^{\Psi,E}_{\omega}, a_1\rangle$ in $f([\![\phi_1]\!]^{s_\Psi}_{\alpha})$. Then $\langle [v]^{\Psi,E}_{\omega}, a_1\rangle$ is in $f(|\phi_1|^{s_\Psi})$, by Proposition 3.1.3, and we can consider two cases

i. there is v_1 with v_1 in $|\phi_1|^{s_\Psi}$, $v \equiv_\omega v_1$ in Ψ, and with $[v_1{:}y]^{\Psi,E}_g = a_1(y)$, for any variable y. So $v_1{:}\exists_x\phi_1$ is in Ψ by rule $\exists_{xI}$ since $v_1 \equiv_x v_1$ is also in Ψ because we are in the context of a uniform lfob deduction system with a universal and an existential quantifier. Note that $[v{:}y]^{\Psi,E}_g = [v_1{:}y]^{\Psi,E}_g$ for any variable y distinct of x. Hence, $v{:}\exists_x\phi_1$ is in Ψ, since we are in the context of a uniform lfob deduction system with a universal quantifier.

ii. there is a variable z and a label schema term v_1 with $[v_1]^{\Psi,E}_z$ contained in $|\phi_1|^{s_\Psi}$, $v_1 \equiv_\omega v$ in Ψ and a_1 z co-equivalent to $\iota^{\Psi}_{DX}(v_1)$. Then, $v_1{:}\forall_z\phi_1$ is in Ψ, since Ψ is an appropriate set. Moreover $v_1{:}\exists_x\forall_z\phi_1$ is in Ψ, by rule $\exists_{xI}$ since $v_1 \equiv_x v_1$ is in Ψ because we are in the context of a uniform lfob deduction system with a universal and an existential quantifier. Note that $[v_1{:}y]^{\Psi,E}_g = [v{:}y]^{\Psi,E}_g$ for any variable y distinct of x and z. So, $v{:}\exists_x\forall_z\phi_1$ is in Ψ since we are in the context of a uniform lfob deduction system with a universal and an existential quantifier, $v \equiv_\omega v_1$ and Ψ is deductively closed. Hence, there is v_2 with $v \equiv_x v_2$ and $v_2{:}\forall_z\phi_1$ in Ψ since Ψ is an appropriate set. Thus, $v_2{:}\phi_1$ is in Ψ by rule $\forall_{zE}$ since $v_2 \equiv_z v_2$, and so $v{:}\exists_x\phi_1$ is in Ψ by rule $\exists_{xI}$, as we wanted to show.

Suppose $v{:}\exists_x\phi_1$ is in Ψ. Then exists v_1 with $v \equiv_x v_1$ and $v_1{:}\phi_1$ in Ψ, since Ψ is an appropriate set. Note that $\iota^{\Psi}_{DX}(v_1)$ is x co-equivalent to $\iota^{\Psi}_{DX}(v)$ by Proposition 5.5.9. Moreover, by Proposition 5.5.11 and Proposition 3.1.3, we have that $[v_1]^{\Psi,E}_{\omega\times\iota}$ is in $f([\![\phi_1]\!]^{s_\Psi}_{\alpha})$, as we wanted to show. *QED*

5.6 Conclusions

In the previous sections we proved that the lfob logic system $\mathcal{L}_{\mathrm{R}}$ is a labelled presentation for the relevance logic R, $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$ is a labelled presentation for the $\wedge$-$\Rightarrow$ fragment of basic relevance logic B with respect to formulae without schema variables, $\mathcal{L}_{\mathrm{FOMdD}}$ is a labelled presentation for first-order modal logic with decreasing domains with respect to formulae without schema variables, and $\mathcal{L}_{\mathrm{FOT}_\wedge}$ is a labelled presentation for the $\wedge$ fragment of first-order modal logic T with respect to formulae without schema variables.

We want to explain now, why in certain cases we proposed a labelled presentation for a logic with respect to all the formulae, schematic or not schematic, and in another cases the labelled presentation proposed is with respect to formulae without schema variables. The distinction is caused by the possibility, or not, that, for any label schema terms v and v', schema formula φ, and set of labelled schema formulae Ψ, we prove $\Psi \vdash v'{:}\varphi$ whenever

$$\Psi \vdash v{:}x =_g v'{:}x \text{ for all variables x,} \qquad \Psi \vdash v \equiv_\omega v' \quad \text{and} \quad \Psi \vdash v{:}\varphi.$$

This inheritance is required in order to show that formulae, schematic or not schematic, whose main connective is universal or existential, are satisfied by a canonical structure over an appropriate set if and only if they are satisfied by the structure induced by the canonical model over that set. So, if the inheritance of schema formulae is possible, then we can show that the lfob logic system proposed is a labelled presentation for that logic with respect to all the formulae, schematic or not schematic. If only the inheritance of formulae without schema variables is possible, then we can only show that the lfob logic system proposed is a labelled presentation for that logic with respect only to formulae without schema variables.

The point is that in general, in any lfob deduction system, given two label schema terms v_1 and v_2, we can not conclude $v_1 \equiv_\alpha v_2$ from $\{v_1{:}x =_g v_2{:}x \mid \text{ for all variables } x\}$. For this to happen, the lfob deduction system would either have the infinitary rule,

$$\frac{\nu_1{:}x =_g \nu_2{:}x \text{ for all variables x}}{\nu_1 \equiv_\alpha \nu_2}$$

which is not permitted, or it would have the rules

$$\frac{}{\nu_1{:}\theta_1 =_g \nu_2{:}\theta_2}\ \textit{1 ind} \qquad\qquad \frac{}{\nu_1 \equiv_\alpha \nu_2}\ \textit{1 assg}$$

saying that any label schema term is in the same assignment relation, $\equiv_\alpha$, with any other label schema term, and that any schema term is equal to any other schema term independently of the label schema terms prefixing them. The problem with this last solution is that, semantically, it would imply that all the structures of the lfob logic system have a singleton as the set of individuals. But this is not admissible in lfob logic systems which are labelled presentations of first-order based logics. So, the lfob logic systems $\mathcal{L}_{\mathrm{FOMdD}}$ and $\mathcal{L}_{\mathrm{FOT}_\wedge}$ do not have those rules, which caused that they are labelled presentations for, first-order modal logic with decreasing domains and the $\wedge$ fragment of first-order modal logic T, respectively, with respect only to non-schematic formulae.

Nevertheless, in the context of propositional based logic systems, the fact that in all the structures the set of individuals is a singleton, is not a problem, and is certainly desired. This is the case of Example 5.3.1, where we propose a labelled presentation for relevance logic R with respect to all schema formulae. The only reason we did not used it when we proposed a labelled presentation for the $\wedge$-$\Rightarrow$ fragment of basic relevance logic B, was because we want to fibre this lfob logic system with the lfob logic system for first-order modal logic with decreasing domains. As you will see in Chapter 6 the structures present in the lfob logic system resulting from the fibring are obtained from the similar structures of the lfob logic systems to which fibring is applied, and so with the same individuals, the same truth values and the same concepts. Hence, if $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$ have those rules, then the only structures that could be considered in the fibring would be structures with a singleton set of individuals. But, in our case this is hardly admissible since the lfob logic system resulting from the fibring is first-order based. Hence, we preferred not to include the rules

1 ind and *1 assg* in $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$, and so that $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$ is a labelled presentation for the $\wedge$ fragment of basic relevance logic B with respect to only non-schematic formulae.

Finally, note that, when proposing labelled presentations for positive logics, i.e., logics without negation, or for positive fragments of logics, i.e., for fragments of logics not including a negation, we had to restrict to logics and to fragments which can be expressed by lfob logic systems with locality and do not have a disjunction and a implication. Recall Proposition 4.4.9 and the analysis of this result in Remark 4.4.8. Nevertheless there are a broad range of positive fragments of logics that can be captured and expressed by labelled presentations as we illustrated in two of these four relevant examples.

Chapter 6

Fibring

In this chapter we study the fibring of lfob logic systems and investigate under which conditions are properties like soundness, completeness, consequence and entailment preserved. Note that by the fibring of lfob logics we mean the collection of all lfob logic systems resulting from the fibring of the suitable pairs of lfob logic systems for each logic. So the important notion is the notion of fibring of lfob logic systems. In the following, without loss of generality, we consider the fibring of lfob logics simply as the fibring of a suitable pair of their lfob logic systems, whenever from that single case, it is straightforward to infer the fibring of all the other suitable pairs of lfob logic systems for that logics. So, we start by defining what is the fibring of a suitable pair of lfob logic systems. The suitability condition serves to guarantee that the provisos are set in such a way that the fibring is well defined. Note that the consequence relation of a lfob logic system depends only on the components of the provisos over the signature of the system. So, when we want to fiber two lfob logic systems, we can adjust the components of the provisos over their joint signature without changing the entailment and consequence of the system. After introducing fibring and making some remarks and considerations, we show that the lfob logic system resulting from the fibring of a suitable pair of lfob logic systems is always sound. This is due to the suitability condition. Then, we study completeness preservation, and prove that the fibring of a suitable pair of full and connected lfob logic systems, where one is with a classical negation, is complete. Moreover we also show completeness results for lfob logic systems with locality and without disjunction and implication, under reasonable conditions. These results rely on the preservation by fibring of fullness and of appropriateness. Before concluding the chapter we prove that fibring also preserves entailment and consequence, and we illustrate the results obtained, by studying the fibring of the lfob logic system for first-order modal logic with decreasing domains with the $\wedge$-$\Rightarrow$ fragment of basic relevance logic B. We show first that they constitute a suitable pair and then, relying on the preservation results, we show that the fibring is sound and complete, capitalizing on the fact that the components are sound and complete.

Fibring of first-order based logics was also studied in [74] but in the different context of non-labelled systems endowed with Hilbert style deduction systems. In that work it was established a completeness preservation theorem for the fibring of full, uniform, Qp- and Op-persistent logics with implication, equivalence, strong equality and inequality, provided that both implication and equivalence are shared at each signature. This result is similar to our completeness preservation theorems in the sense that it is obtained by preservation by fibring of conditions implying the sufficient conditions established in the completeness theorem.

6.1 General definition

In this section we define what is the fibring of a suitable pair of lfob logic systems. In order to define fibring, we need the concept of *reduct of a structure*, which is defined in the following way. Given signatures Σ and $\Sigma^{\cdot}$, and $\Sigma \cup \Sigma^{\cdot}$ structure $\langle U, A, W, \alpha, \omega, D, \mathcal{E}, \mathcal{B}, [.] \rangle$ named s, the reduct of s to Σ, denoted by s_Σ or $s_{|_\Sigma}$, is the Σ structure $\langle U, A, W, \alpha, \omega, D, \mathcal{E}, \mathcal{B}, [.]_\Sigma \rangle$, where $[.]_\Sigma$ is the restriction of $[.]$ to the connectives in Σ.

Definition 6.1.1 The *fibring* of a suitable pair of lfob logic systems $\langle \Sigma_1, R_1, M_1, \check{\cdot} \rangle$ and $\langle \Sigma_2, R_2, M_2, \check{\cdot} \rangle$, is the lfob logic system $\langle \Sigma, R, M, \hat{\cdot} \rangle$, where

- Σ is $\Sigma_1 \cup \Sigma_2$,
- R is $R_1 \cup R_2$,
- M is the class of all Σ structures s such that

  there exists m_1 in M_1 with $s_{|_{\Sigma_1}}$ in $\check{m}_1$ and
  there exists m_2 in M_2 with $s_{|_{\Sigma_2}}$ in $\check{m}_2$,

- and $\hat{\cdot}$ associates to each structure s in M a singleton with that same structure s.

Note that the fibring operator, denoted in the following by $\oplus$, is only defined for pairs of lfob logic systems that are suitable. So, before we can apply the fibring operator to a pair of lfob logic systems we have to check if it is suitable. But, when we say that two lfob logic systems are suitable? It is the purpose of next definition to answer this question.

Definition 6.1.2 The pair of lfob logic systems $\langle \Sigma_1, R_1, M_1, \check{\cdot} \rangle$ and $\langle \Sigma_2, R_2, M_2, \check{\cdot} \rangle$ is *suitable*, whenever,

- any $\Sigma_1 \cup \Sigma_2$ structure s such that

  there exists m_1 in M_1 with $s_{|_{\Sigma_1}}$ in $\check{m}_1$ and
  there exists m_2 in M_2 with $s_{|_{\Sigma_2}}$ in $\check{m}_2$,

  satisfies each rule of R_1 and of R_2, (with respect to the $\Sigma_1 \cup \Sigma_2$ component of the provisos),

- any rule r in R_i is such that, for any Σ_i schema substitution σ,
  - $(\pi_{\Sigma_1 \cup \Sigma_2} \sigma_{\mathrm{nl}})$ is 1 whenever $(\pi_{\Sigma_i} \sigma_{\mathrm{nl}})$ is 1 for each π in P_f,
  - $\pi_{\Sigma_1 \cup \Sigma_2; E}(\sigma)$ is 1 whenever $\pi_{\Sigma_i; E}(\sigma)$ is 1 for each π in P_d

  for any i equal to 1 or 2,

- if the systems are connected then, any shared relation r, i.e., any r in S_{1k} and in S_{2k}, is such that

- r appears in an introduction or elimination rule of a connective in O_{1k-1} iff r appears in an introduction or elimination rule of a connective in O_{2k-1}, and
- there is a laxed generalization rule for r over $\equiv_\alpha$ in R_1 iff there is a laxed generalization rule for r over $\equiv_\alpha$ in R_2, and
- there is a generalization rule for r over $\equiv_\alpha$ in R_1 iff there is a generalization rule for r over $\equiv_\alpha$ in R_2.

As we will see in the next section, the fact that the pair of lfob logic systems to be fibred is suitable is important to show the soundness of the lfob logic system resulting from the fibring, due to the first condition in the definition of suitable, is important for the preservation of fullness, due to the second condition in the definition of suitable, and is important for the preservation of connectedness due to the third condition.

The suitability condition imposes a careful setting of the component over the fibred signature, of the provisos in the rules of the lfob logic systems to be fibred. Note that it is even possible to define the provisos in each lfob logic system, so that their components over the fibred signature only allow substitutions with values in the language of the lfob logic system. For a better understanding of the role of the provisos in the fibring of lfob logic systems and in the suitability condition, we refer the reader to the comments and remarks provided in next section, and also to the example of fibring presented in Section 6.5, where a particular case of adjustment of the provisos can be appreciated.

Remark 6.1.3 *Constrained fibring by sharing.* The fibring can be constrained by sharing connectives. This happens when the signatures of the lfob logic systems participating in the fibring share some connective, i.e., when $\Sigma_1 \cap \Sigma_2$ is not a tuple of families only with $\emptyset$.

6.2 Considerations

In this section we make some comments and remarks about practical aspects of fibring lfob logic systems.

The connectives $\equiv_\omega$, $\equiv_\alpha$, $=_g$ and $=$ in the fibring. Recall that the connective $=_g$, the relations $\equiv_\omega$ and $\equiv_\alpha$, and the predicate $=$ are not present in the signature of a lfob logic system, and so, these components are not inherited from the components. By definition they are always in the language of any lfob logic system, so they are also in the language of the lfob logic resulting from the fibring. This agrees with the fact that the models resulting from the fibring are based on structures of the fibred lfob logic systems with the same worlds, assignments, individuals, truth values, and term values.

The consequence relation of a lfob logic system over a signature Σ depends only of the Σ component of the provisos of its rules. This can be straightforwardly proven just by looking into the definition of deduction. Thus, it is possible to consider lfob logic systems differing only in the other signature components of the provisos, but with the same consequence relation. The question that prompts now is: if the Σ' components of a proviso, where Σ' is not the signature of the logic system, have no influence for the consequence relation of that lfob logic system, why do they appear in the rules. And the answer is: because they have an important role in the fibring. We try to illustrate this fact in the paragraphs below. We stress again that their presence is only possible since they do not affect the consequence relation associated to the logic system.

Importance of the component over the fibred signature of the provisos present in the rules, for the suitability of a pair of lfob logic systems. Consider the lfob logic system $\mathcal{L}^{\times}_{\text{B}_{\wedge,\Rightarrow}}$ equal to $\mathcal{L}_{\text{B}_{\wedge,\Rightarrow}}$ except that $\text{p-mon}_{R_{\Rightarrow 0}}(\xi)_{\Sigma_{\text{FOMdD}}\cup\Sigma_{\text{B}_{\wedge,\Rightarrow}}}(\rho_{\text{nl}}) = 1$ for any substitution ρ. Then, $\mathcal{L}^{\times}_{\text{B}_{\wedge,\Rightarrow}}$ is equivalent in terms of consequence relation and entailment to $\mathcal{L}_{\text{B}_{\wedge,\Rightarrow}}$. The same happens with the lfob logic system $\mathcal{L}^{\oplus}_{\text{B}_{\wedge,\Rightarrow}}$ introduced in Example 6.5.1. Nevertheless $\mathcal{L}^{\oplus}_{\text{B}_{\wedge,\Rightarrow}}$ and $\mathcal{L}^{\oplus}_{\text{FOMdD}}$ are suitable as we show in Proposition 6.5.5, and $\mathcal{L}^{\times}_{\text{B}_{\wedge,\Rightarrow}}$ and $\mathcal{L}^{\oplus}_{\text{FOMdD}}$ are not, as it is straightforward to see.

Lfob logic systems and fibring. Assume we are interested in the fibring of two sound and complete lfob logic systems $\mathcal{L}_1$ and $\mathcal{L}_2$. Then, we are interested that the pair $\mathcal{L}_1$ and $\mathcal{L}_2$ is suitable, and that the resulting lfob logic system is complete, since we have for granted that it is sound (see Theorem 6.3.1). So, if one of these cases does not hold, i.e. the pair of lfob logic systems is not suitable or the resulting lfob logic system is not complete, we do not mind, actually we consider, instead of $\mathcal{L}_i$, for simplicity suppose $i = 1$, other lfob logic system $\mathcal{L}'_1$, as long as (i) $\mathcal{L}'_1$ and $\mathcal{L}_1$ have the same entailment and consequence relation, (ii) the pair $\mathcal{L}'_1$ and $\mathcal{L}_2$ is suitable, and (iii) $\mathcal{L}'_1 \oplus \mathcal{L}_2$ have better properties than $\mathcal{L}_1 \oplus \mathcal{L}_2$. In many cases, it is sufficient that $\mathcal{L}'_1$ differ from $\mathcal{L}_1$ only either at the $\Sigma_1 \cup \Sigma_2$ component of the provisos of its rules, or because $\mathcal{L}'{}_1$ is the fullness closure of $\mathcal{L}_1$, in order for the pair $\mathcal{L}'_1$ and $\mathcal{L}_2$ to be suitable, or $\mathcal{L}'_1 \oplus \mathcal{L}_2$ to be complete.

6.3 Soundness and completeness preservation

We start by showing that the lfob logic system resulting from the fibring of a suitable pair of lfob logic systems is sound.

Theorem 6.3.1 The fibring of a suitable pair of lfob logic systems is sound.

Proof Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be a pair of suitable lfob logic systems. Then, to prove that $\mathcal{L}_1 \oplus \mathcal{L}_2$ is sound, and taking into account the soundness theorem, Theorem 2.4.6, it is sufficient to show that all the structures of $\mathcal{L}_1 \oplus \mathcal{L}_2$ satisfy all its rules, i.e., taking into account the definition of fibring, Definition 6.1.1, it is sufficient to show that any $\Sigma_1 \cup \Sigma_2$ structure $\langle U, A, W, \alpha, \omega, D, \mathcal{E}, \mathcal{B}, [.]\rangle$ such that there exists m_1 in M_1 with $\langle U, A, W, \alpha, \omega, D, \mathcal{E}, \mathcal{B}, [.]|_{\Sigma_1}\rangle$ in $\breve{m}_1$ and there exists m_2 in M_2 with $\langle U, A, W, \alpha, \omega, D, \mathcal{E}, \mathcal{B}, [.]|_{\Sigma_2}\rangle$ in $\breve{m}_2$, satisfies all rules of $R_1 \cup R_2$ (with respect to the $\Sigma_1 \cup \Sigma_2$ component of the proviso). But this happens since the pair $\mathcal{L}_1$ and $\mathcal{L}_2$ is suitable, Definition 6.1.2. *QED*

So, when the fibring of two lfob logic systems is defined, i.e., when that pair of lfob logic systems is suitable, it is sound. Note that, in the proof of Theorem 6.3.1, the condition that guarantees the soundness of the fibring is precisely the fact that the pair of lfob logic systems is suitable.

We now establish several completeness preservation theorems. In an attempt to simplify the presentation, we present first the completeness preservation theorems, then we prove some auxiliary lemmas and propositions, and in the end of the section we prove the theorems.

Theorem 6.3.2 The fibring of a suitable pair of full and connected lfob logic systems, where one is with a classical negation, is complete.

Theorem 6.3.3 The fibring of a suitable pair of full and connected lfob logic systems with locality and without disjunction and implication is complete whenever it is connected with locality.

Relying on the previous completeness preservation theorem, Theorem 6.3.3, we now establish two corollaries.

Corollary 6.3.4 The fibring of a suitable pair of full and connected lfob logic systems with locality and without disjunction and implication is complete whenever both lfob logic systems have at least a universal connective c, constrained or not, based on a relation s, where either c is not unary or s is not reflexive, and no relation symbol, universal connective or existential connective are shared.

Corollary 6.3.5 The fibring of a suitable pair of full and connected lfob logic systems with locality and without disjunction and implication is complete whenever both lfob logic systems do not have any universal connective c, constrained or not, based on a relation s, where either c is not unary or s is not reflexive.

The strategy to prove Theorem 6.3.2 is the following. We show that the fibring of a suitable pair of full and connected lfob logic systems where one is with a classical negation is also a full and connected lfob logic system with a classical negation. So, by Theorem 4.3.10, the lfob logic system resulting from the fibring is appropriate, and thus, we can invoke Corollary 3.3.7 to conclude that it is complete. To implement this strategy we show first that fibring preserves fullness and then that the lfob logic system resulting from the fibring is a connected lfob logic system with a classical negation whenever the lfob logic systems participating in the fibring are also connected and one of them is with a classical negation. With respect to Theorem 6.3.3 we follow a different strategy since we left as a sufficient condition that the lfob logic system resulting from the fibring be with locality. Note that there are many possible combinations of conditions that can be imposed to the two clfob logic systems with locality in order for the fibring of their deduction systems be also a deduction system with locality. So, we preferred to state a general theorem first, and then present particular corollaries of it, in which we specify characteristics of the two deduction systems and requirements of sharing/not sharing of connectives, in order for the fibring of the two deduction systems be also with locality. We present two corollaries with sufficient conditions for the system resulting from the fibring being with locality, Corollary 6.3.4 and Corollary 6.3.5. Please note that there are several more sets of conditions that can be imposed to clfob logic systems, in the spirit of the mentioned corollaries, in order for their fibring to be with locality and so for completeness to be preserved.

We now show some lemmas needed to prove the completeness preservation theorems and corollaries. Other lemmas and results used by the lemmas we show now are proved in Section 6.6.

Lemma 6.3.6 Fullness is preserved by fibring.

Proof Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be a pair of suitable full lfob logic systems with signatures Σ_1 and Σ_2, respectively. Suppose s is a $\Sigma_1 \cup \Sigma_2$ structure satisfying all the rules of $\mathcal{L}_1 \oplus \mathcal{L}_2$. Then, by Lemma 6.6.3, we have that $s|_{\Sigma_1}$ satisfies all the rules of $\mathcal{L}_1$ and $s|_{\Sigma_2}$ satisfies all the rules of $\mathcal{L}_2$. So, there exists m_1 in M_1 with $s|_{\Sigma_1}$ in $\breve{m}_1$ and there exists m_2 in M_2 with $s|_{\Sigma_2}$ in $\breve{m}_2$, because both $\mathcal{L}_1$ and $\mathcal{L}_2$ are full. Then, by definition of fibring, Definition 6.1.1, s is a model of $\mathcal{L}_1 \oplus \mathcal{L}_2$, and $\breve{s}$ is the singleton with s. Hence s is a structure in $\mathcal{L}_1 \oplus \mathcal{L}_2$ as we wanted to show. *QED*

Lemma 6.3.7 The fibring of a suitable pair of connected lfob logic systems where one is with a classical negation, is also connected and with a classical negation.

Proof Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be a suitable pair of connected lfob logic systems where one is with a classical negation. Therefore, by Lemma 6.6.4, we have that the fibring is also a connected lfob logic system. Recall Definition 4.3.3 of a connected lfob logic system with a classical negation. Then, it is straightforward to see that the fibring is also with a classical negation. *QED*

Lemma 6.3.8 The fibring of a suitable pair of connected lfob logic systems with locality is also connected and with locality whenever both lfob logic systems have at least a universal connective c, constrained or not, based on a relation s, where either c is not unary or s is not reflexive, and no relation symbol, universal connective or existential connective are shared.

Proof We can use Lemma 6.6.4 to see that the fibring of a suitable pair of connected lfob logic systems is connected. Moreover, it is straightforward to see that the fibring is with locality whenever both components are with locality, and the conditions mentioned in the lemma are satisfied. *QED*

Finally we can show the preservation theorems and corollaries.

Proof (Completeness preservation Theorem 6.3.2) Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be a suitable pair of full and connected lfob logic systems where one is with a classical negation. Then, by Lemma 6.3.6, the lfob logic system resulting from the fibring is full and by Lemma 6.3.7 it is connected and with a classical negation. So, by Theorem 4.3.10, it is also appropriate. Henceforth, by Corollary 3.3.7 it is complete as we wanted to show. *QED*

Proof (Completeness preservation Theorem 6.3.3) Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be a suitable pair of full and connected lfob logic systems with locality and without disjunction and implication such that the lfob logic system resulting from the fibring is connected with locality. Since it is also without disjunction and implication, we can use Theorem 4.4.10 to conclude that it is appropriate. Moreover, by Lemma 6.3.6 it is full. So, by Corollary 3.3.7 it is complete as we wanted to show. *QED*

Proof (Completeness preservation Corollary 6.3.4) Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be a suitable pair of full and connected lfob logic systems with locality and without disjunction and implication such that both lfob logic systems have a universal connective c, constrained or not, based on a relation s, where either c is not unary or s is not reflexive, and no relation symbol, universal connective or existential connective are shared. Then by Lemma 6.3.8 the lfob logic system resulting from the fibring is connected and with locality. Moreover it is straightforward to see that it is without disjunction and implication. Then, by Theorem 6.3.3, it is complete, as we wanted to show. *QED*

The proof of Corollary 6.3.5 is identical to the proof of Corollary 6.3.4, just presented, so we decided to omit it.

6.4 Entailment and consequence preservation

Theorem 6.4.1 Given a suitable pair of lfob logic systems $\mathcal{L}_1$ and $\mathcal{L}_2$, we have

$$\Psi \vDash_{\mathcal{L}_1 \oplus \mathcal{L}_2} \eta \quad \text{whenever} \quad \Psi \vDash_{\mathcal{L}_i} \eta$$

for any Ψ contained in the language of $\mathcal{L}_i$, η also in that language, and i equal to 1 or 2.

Proof Suppose without loss of generality that i is equal to 1. Assume $\Psi \vDash_{\mathcal{L}_1} \eta$ and let s be a structure in $\mathcal{L}_1 \oplus \mathcal{L}_2$ and α a structure over s such that $s, \alpha \Vdash \Psi$. Note that s_{Σ_1} is in $\mathcal{L}_1$ by definition of fibring, Definition 6.1.1. Additionally, let E_l, E_t and E_f be the finite sets of all label, term, and formula, respectively, schema variables used in Ψ and in η, ι_l, ι_t and ι_f bijections from $T_{\mathrm{lab},E}(\Sigma_1 \cup \Sigma_2, \Xi_l)$ to $\Xi_l \setminus E_l$, from $L(\Sigma_1 \cup \Sigma_2, \Xi_t)$ to $\Xi_t \setminus E_t$, and from $L(\Sigma_1 \cup \Sigma_2, \Xi_t, \Xi_f)$ to $\Xi_f \setminus E_f$, respectively, and α_1^ι a schema assignment over s_{Σ_1} such that $\nu\alpha_1^\iota = \nu\alpha$, $\theta\alpha_1^\iota = \theta\alpha$ and $\xi\alpha_1^\iota = \xi\alpha$ for any label, term and formula schema variables ν, θ and ξ in $E_l \cup E$, E_t or E_f, respectively, and $\nu\alpha_1^\iota = [\![\iota_t^{-1}(\nu)]\!]_\alpha^s$, $\theta\alpha_1^\iota = [\![\iota_t^{-1}(\theta)]\!]_\alpha^s$ and $\xi\alpha_1^\iota = [\![\iota_f^{-1}(\xi)]\!]_\alpha^s$ for any label, term and formula schema variables ν, θ and ξ in $(\Xi_l \cup E) \setminus E_l$, $\Xi_t \setminus E_t$ or $\Xi_f \setminus E_f$. Then, by Proposition 6.6.2, $s_{\Sigma_1}, \alpha_1^\iota \Vdash \mathrm{tr}_\Sigma^\iota(\psi)$ for any ψ in Ψ. So, $s_{\Sigma_1}, \alpha_1^\iota \Vdash \psi$ for any ψ in Ψ since, by Definition 6.6.1, $\mathrm{tr}_\Sigma^\iota(\psi) = \psi$ for any ψ in Ψ because Ψ is contained in the language of $\mathcal{L}_1$. Hence $s_{\Sigma_1}, \alpha_1^\iota \Vdash \eta$, and so $s_{\Sigma_1}, \alpha_1^\iota \Vdash \mathrm{tr}_\Sigma^\iota(\eta)$ because η is also in that language. Thus, again by Proposition 6.6.2, we have that $s, \alpha \Vdash \eta$ as we wanted to show. *QED*

Theorem 6.4.2 Given a suitable pair of lfob logic systems $\mathcal{L}_1$ and $\mathcal{L}_2$, we have

$$\Psi \vdash_{\mathcal{L}_1 \oplus \mathcal{L}_2} \eta \quad \text{whenever} \quad \Psi \vdash_{\mathcal{L}_i} \eta$$

for any Ψ contained in the language of $\mathcal{L}_i$, η also in that language, and i equal to 1 or 2.

Proof Suppose without loss of generality that i is equal to 1. Assume $\Psi \vdash_{\mathcal{L}_1} \eta$. We show by induction on the length of a sober deduction sequence ϖ for $\Psi \vdash_{\mathcal{L}_1} \eta$ that it is also a sober deduction sequence for $\Psi \vdash_{\mathcal{L}_1 \oplus \mathcal{L}_2} \eta$. Base of induction: suppose $\langle \eta, \Psi_1, 1 \rangle$ is a sober deduction sequence for $\Psi \vdash_{\mathcal{L}_1} \eta$. We consider two cases:

either Ψ_1 is $\{\eta\}$. So, it is also a deduction sequence for $\Psi \vdash_{\mathcal{L}_1 \oplus \mathcal{L}_2} \eta$.

or there is a rule $\langle \emptyset, \eta'', P_f, P_d \rangle$ that we denote by r, and Σ_1 schema substitution σ such that Ψ_1 is $\emptyset$, $\eta''\sigma$ is η, $(\pi_{\Sigma_1}\sigma_{\mathrm{nl}})$ is 1, for each π in P_f, and $\pi_{\Sigma_1;E}(\sigma)$ is 1 for each π in P_d. Note that σ is also a $\Sigma_1 \cup \Sigma_2$ schema substitution, r is also in $\mathcal{L}_1 \oplus \mathcal{L}_2$ and, by the second condition in the definition of suitability, Definition 6.1.2, that, $(\pi_{\Sigma_1 \cup \Sigma_2}\sigma_{\mathrm{nl}})$ is 1, for each π

in P_f, and $\pi_{\Sigma_1 \cup \Sigma_2}(\sigma)$ is 1 for each π in P_d. So, we can use this same rule and substitution to conclude that $\langle \eta, \Psi_1, 1 \rangle$ is a sober deduction sequence for $\Psi \vdash_{\mathcal{L}_1 \oplus \mathcal{L}_2} \eta$, as we wanted to show.

Assume as induction hypothesis that, for any set Ψ of labelled schema formulae contained in the language of $\mathcal{L}_1$, and labelled schema formula η in that language, if there is a sober deduction sequence ϖ of length less than or equal to n for $\Psi \vdash_{\mathcal{L}_1} \eta$, then, ϖ is also a sober deduction sequence for $\Psi \vdash_{\mathcal{L}_1 \oplus \mathcal{L}_2} \eta$. Let ϖ be a sober deduction sequence with length $n+1$ for $\Psi \vdash_{\mathcal{L}_1} \eta$ ending in $\langle \eta, \Psi_{n+1}, 1 \rangle$. Then, there are $\langle \{ \langle \vartheta'_1, \Psi'_1, \eta'_1 \rangle, \dots, \langle \vartheta'_k, \Psi'_k, \eta'_k \rangle \}, \eta', P'_f, P'_d \rangle$ in $\mathcal{L}_1$, Σ_1 schema substitution σ, and $i_1, \dots, i_k$ in $\{1, \dots, n\}$, with $\eta_{i_j} = \eta'_j \sigma$, $\Psi_{i_j} = \vartheta'_j \sigma$ and $\pi_{i_j} = 1$ for each $j = 1, \dots, k$, $\Psi_{n+1} = \Psi_{i_1} \setminus \Psi'_1 \sigma \cup \dots \cup \Psi_{i_k} \setminus \Psi'_k \sigma$, $(\pi_{\Sigma_1} \sigma_{\mathrm{nl}}) = 1$ for each π in P'_f, $\pi_{\Sigma_1}(\sigma) = 1$ for each π in P'_d, and $\eta = \eta' \sigma$.

For each $j = 1, \dots, k$ let ϖ_j denote the sober deduction subsequence of ϖ ending in $\langle \eta'_j \sigma, \vartheta'_j \sigma, 1 \rangle$ for $\Psi, \Psi'_j \sigma \vdash_{\mathcal{L}_1} \eta'_j \sigma$. Then, by induction hypothesis, ϖ_j is also a deduction sequence in $\mathcal{L}_1 \oplus \mathcal{L}_2$.

Moreover, note that σ is also a $\Sigma_1 \cup \Sigma_2$ schema substitution, r is also in $\mathcal{L}_1 \oplus \mathcal{L}_2$ and, by the second condition in the definition of suitability, Definition 6.1.2, that, $(\pi_{\Sigma_1 \cup \Sigma_2} \sigma_{\mathrm{nl}})$ is 1, for each π in P_f, and $\pi_{\Sigma_1 \cup \Sigma_2}(\sigma)$ is 1 for each π in P_d.

So, we can use the deduction sequences $\varpi_1, \dots, \varpi_j$, this same rule and the substitution σ to conclude that ϖ is a sober deduction sequence for $\Psi \vdash_{\mathcal{L}_1 \oplus \mathcal{L}_2} \eta$, as we wanted to show. *QED*

6.5 Example

According to the discussion made in section 6.2, this section starts by presenting the lfob logic systems $\mathcal{L}^{\oplus}_{\mathrm{B}_{\wedge,\Rightarrow}}$ and $\mathcal{L}^{\oplus}_{\mathrm{FOMdD}}$, which are equal in terms of the consequence relation and entailment with the lfob logic systems $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$ of Example 5.4.1, and $\mathcal{L}_{\mathrm{FOMdD}}$ of Example 5.1.1, respectively. So $\mathcal{L}^{\oplus}_{\mathrm{B}_{\wedge,\Rightarrow}}$ and $\mathcal{L}^{\oplus}_{\mathrm{FOMdD}}$ are also sound, complete and are labelled presentations for, respectively, the $\wedge$-$\Rightarrow$ fragment of basic relevance logic B, and first-order modal logic with decreasing domains. Moreover, they constitute a suitable pair as we show in Proposition 6.5.5, and their fibring is sound, Proposition 6.5.7, and complete, Proposition 6.5.8. Note that $\mathcal{L}^{\oplus}_{\mathrm{B}_{\wedge,\Rightarrow}}$ and $\mathcal{L}^{\oplus}_{\mathrm{FOMdD}}$ are obtained from $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$ and $\mathcal{L}_{\mathrm{FOMdD}}$, just by properly adjusting some provisos and doing their fullness closure.

6.5.1 $\wedge$-$\Rightarrow$ fragment of basic relevance logic B

Example 6.5.1 $\wedge$-$\Rightarrow$ *fragment of basic relevance logic B lfob logic system.* Denote by

$$\mathcal{L}^{\oplus}_{\mathrm{B}_{\wedge,\Rightarrow}}$$

the lfob logic system equal to the lfob logic system resulting from the fullness closure of $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$, except in the $\Sigma_{\mathrm{FOMdD}} \cup \Sigma_{\mathrm{B}_{\wedge,\Rightarrow}}$ component of the provisos p-gmon$_{\equiv_\omega}(\xi)$ and p-mon$_{R_{\Rightarrow}0}(\xi)$ defined such that p-gmon$_{\equiv_\omega}(\xi)_{\Sigma_{\mathrm{FOMdD}} \cup \Sigma_{\mathrm{B}_{\wedge,\Rightarrow}}}(\rho_{\mathrm{nl}}) = 1$ iff $\xi \rho_f$ is a formula in $L(\Sigma_{\mathrm{B}_{\wedge,\Rightarrow}}, X)$ without variables, and p-mon$_{R_{\Rightarrow}0}(\xi)_{\Sigma_{\mathrm{FOMdD}} \cup \Sigma_{\mathrm{B}_{\wedge,\Rightarrow}}}(\rho_{\mathrm{nl}}) = 1$ iff $\xi \rho$ is a formula in $L(\Sigma_{\mathrm{B}_{\wedge,\Rightarrow}}, X)$ without variables.

Proposition 6.5.2 The lfob logic system $\mathcal{L}^{\oplus}_{\mathrm{B}_{\wedge,\Rightarrow}}$ is sound, complete and a labelled presentation for the $\wedge$-$\Rightarrow$ fragment of basic relevance logic B with respect to formulae without schema variables.

Proof Denote by ${\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}}^c$ the lfob logic system resulting from the fullness closure of $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$ introduced in Example 5.1.1. Then, by Proposition 3.4.2, the consequence and the entailment relations of ${\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}}^c$ and $\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}$ coincide. So, ${\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}}^c$ is also sound, complete and a labelled presentation for the $\wedge$-$\Rightarrow$ fragment of basic relevance logic B with respect to formulae without schema variables (see section 5.4). So, $\mathcal{L}^{\oplus}_{\mathrm{B}_{\wedge,\Rightarrow}}$ is also sound, complete and a labelled presentation for the $\wedge$-$\Rightarrow$ fragment of basic relevance logic B with respect to formulae without schema variables, since its consequence and entailment relations coincide with those of ${\mathcal{L}_{\mathrm{B}_{\wedge,\Rightarrow}}}^c$, because the only difference between them are in the components of some provisos over signatures different of $\Sigma_{\mathrm{B}_{\wedge,\Rightarrow}}$. *QED*

6.5.2 First-order modal logic with decreasing domains

Example 6.5.3 *First-order modal logic with decreasing domains lfob logic system.* Denote by

$$\mathcal{L}^{\oplus}_{\mathrm{FOMdD}}$$

the lfob logic system equal to the lfob logic system resulting from the fullness closure of $\mathcal{L}_{\mathrm{FOMdD}}$, except in the $\Sigma_{\mathrm{FOMdD}} \cup \Sigma_{\mathrm{B}_{\wedge,\Rightarrow}}$ component of the provisos $\xi' \lhd \xi^x_\theta$, $\mathrm{vardif}(x,\theta)$, and $\text{p-gmon}_{\equiv_\omega}(\xi,\theta_1,\ldots,\theta_k)$ which are defined as follows,

- $\mathrm{vardif}(x,\theta)_{\Sigma_{\mathrm{FOMdD}}\cup\Sigma_{\mathrm{B}_{\wedge,\Rightarrow}}}(\rho_{\mathrm{nl}}) = 1$ iff $\theta\rho_t$ is a variable different of x,
- $\text{p-gmon}_{\equiv_\omega}(\xi,\theta_1,\ldots,\theta_k)_{\Sigma_{\mathrm{FOMdD}}\cup\Sigma_{\mathrm{B}_{\wedge,\Rightarrow}}}(\rho_{\mathrm{nl}}) = 1$ iff $\xi\rho_f$ is a formula in $L(\Sigma_{\mathrm{FOMdD}}, X)$ whose free variables are $\theta_1\rho_t,\ldots,\theta_k\rho_t$,
- $\xi' \lhd {\xi^x_\theta}_{\Sigma_{\mathrm{FOMdD}}\cup\Sigma_{\mathrm{B}_{\wedge,\Rightarrow}}}(\rho_{\mathrm{nl}}) = 1$ iff if a connective of $\Sigma_{\mathrm{B}_{\wedge,\Rightarrow}} \setminus \Sigma_{\mathrm{FOMdD}}$ appears in $\xi\rho_f$ then it appears under the scope of a $\forall_x$ or of a $\exists_x$ and $\xi'\rho_f$ is obtained by replacing the free occurrences of x in $\xi\rho_f$ by a term $\theta\rho_t$ equal to a variable whenever some free x in $\xi\rho_f$ is in the scope of a modality, such that no variable in $\theta\rho_t$ is captured by a quantifier.

Proposition 6.5.4 The lfob logic system $\mathcal{L}^{\oplus}_{\mathrm{FOMdD}}$ is sound, complete and a labelled presentation for first-order modal logic with decreasing domains with respect to formulae without schema variables.

Proof Denote by ${\mathcal{L}_{\mathrm{FOMdD}}}^c$ the lfob logic system resulting from the fullness closure of $\mathcal{L}_{\mathrm{FOMdD}}$ introduced in Example 5.1.1. Then, by Proposition 3.4.2, the consequence and the entailment relations of ${\mathcal{L}_{\mathrm{FOMdD}}}^c$ and $\mathcal{L}_{\mathrm{FOMdD}}$ coincide. So, ${\mathcal{L}_{\mathrm{FOMdD}}}^c$ is also sound, complete and a labelled presentation for first-order modal logic with decreasing domains with respect to formulae without schema variables (see section 5.1). So, $\mathcal{L}^{\oplus}_{\mathrm{FOMdD}}$ is also sound, complete and a labelled presentation for first-order modal logic with decreasing domains with respect to formulae without schema variables, since its consequence and entailment relations coincide with those of ${\mathcal{L}_{\mathrm{FOMdD}}}^c$, because the only difference between them are in the components of some provisos over signatures different of Σ_{FOMdD}. *QED*

6.5.3 Suitability

Proposition 6.5.5 The pair of lfob logic systems $\mathcal{L}^{\oplus}_{\mathrm{FOMdD}}$ and $\mathcal{L}^{\oplus}_{\mathrm{B}_{\wedge,\Rightarrow}}$ is suitable.

Proof In this proof we abbreviate Σ_{FOMdD} to Σ_{Q}, and $\Sigma_{\mathrm{B}_{\wedge,\Rightarrow}}$ to Σ_{B}. The general idea to prove that the pair Σ_{Q} and Σ_{B} satisfies the first condition of suitability is to show that a structure over the joint signature satisfies a rule with mixed formulae and terms, relying on the fact that the reduct of that structure over one of the signatures satisfies that rule instantiated by the translation of the formulae and terms to that signature. So we use the translation maps $\mathrm{ltr}^{\iota_l}_{\Sigma_{\mathrm{Q}}}$, $\mathrm{ttr}^{\iota_l}_{\Sigma_{\mathrm{Q}}}$ and $\mathrm{ftr}^{\iota_l}_{\Sigma_{\mathrm{Q}}}$, introduced in Definition 6.6.1, and the satisfaction result of Proposition 6.6.2. The second condition follows straightforwardly by analysis of the possible rules and provisos, as well as the third condition.

We start by showing the first condition of the definition of suitability of a pair of lfob logic systems, Definition 6.1.2. Let s be a $\Sigma_{\mathrm{Q}} \cup \Sigma_{\mathrm{B}}$ structure such that exists m_1 in $M^{\oplus}_{\mathrm{FOMdD}}$ with $s_{\Sigma_{\mathrm{Q}}}$ in $\breve{m}_1$ and exists m_2 in $M^{\oplus}_{\mathrm{B}_{\wedge,\Rightarrow}}$ with $s_{\Sigma_{\mathrm{B}}}$ in $\breve{m}_2$. Observe that $s_{\Sigma_{\mathrm{Q}}}$ and $s_{\Sigma_{\mathrm{B}}}$ satisfy all the rules of $\mathcal{L}^{\oplus}_{\mathrm{FOMdD}}$ and $\mathcal{L}^{\oplus}_{\mathrm{B}_{\wedge,\Rightarrow}}$, respectively, since all the structures in these lfob logic systems satisfy their rules. The proof follows by case analysis on the possible rules in $\mathcal{L}^{\oplus}_{\mathrm{FOMdD}}$ and in $\mathcal{L}^{\oplus}_{\mathrm{B}_{\wedge,\Rightarrow}}$. Note also that according to Proposition 2.4.7, it is sufficient to show that the structures satisfy the specific rules since every structure satisfies the rules specified in Definition 2.3.10 common to all lfob deduction systems. So, let $r = \langle\{\langle\vartheta_1, \Psi_1, \eta_1\rangle, \ldots, \langle\vartheta_k, \Psi_k, \eta_k\rangle\}, \eta, P_f, P_d\rangle$ be a rule in $R^{\oplus}_{\mathrm{FOMdD}}$ that is not a rule common to all lfob logic systems, i.e., that was not specified in Definition 2.3.10, E a set of label schema variables, σ a $\Sigma_{\mathrm{Q}} \cup \Sigma_{\mathrm{B}}$ schema substitution within L_E such that $(\pi_{\Sigma_{\mathrm{Q}}\cup\Sigma_{\mathrm{B}}}\sigma_{\mathrm{nl}})$ is 1 for each π in P_f, and $\pi_{\Sigma_{\mathrm{Q}}\cup\Sigma_{\mathrm{B}};E}(\sigma)$ is 1 for each π in P_d, and α a $\Sigma_{\mathrm{Q}} \cup \Sigma_{\mathrm{B}}$ schema assignment over s and E. Let E_l, E_t and E_f be the sets of all label, term, and formula, respectively, schema variables used in $\Psi_i\sigma$ and in $\eta_i\sigma$ for $i = 1, \ldots, k$. Note that E_l, E_t and E_f are finite. Moreover, consider bijections ι_l from $T_{\mathrm{lab},E}(\Sigma_{\mathrm{Q}} \cup \Sigma_{\mathrm{B}}, \Xi_l)$ to $\Xi_l \setminus E_l$, ι_t from $L(\Sigma_{\mathrm{Q}} \cup \Sigma_{\mathrm{B}}, \Xi_t)$ to $\Xi_t \setminus E_t$, and ι_f from $L(\Sigma_{\mathrm{Q}} \cup \Sigma_{\mathrm{B}}, \Xi_t, \Xi_f)$ to $\Xi_f \setminus E_f$. Finally, let $\alpha^{\iota}_{\mathrm{Q}}$ be a schema assignment over $s_{\Sigma_{\mathrm{Q}}}$ such that $\nu\alpha^{\iota}_{\mathrm{Q}} = \nu\alpha$, $\theta\alpha^{\iota}_{\mathrm{Q}} = \theta\alpha$ and $\xi\alpha^{\iota}_{\mathrm{Q}} = \xi\alpha$ for any label, term and formula schema variables ν, θ and ξ in $E_l \cup E$, E_t or E_f, respectively, and $\nu\alpha^{\iota}_{\mathrm{Q}} = [\![\iota_t^{-1}(\nu)]\!]^s_\alpha$, $\theta\alpha^{\iota}_{\mathrm{Q}} = [\![\iota_t^{-1}(\theta)]\!]^s_\alpha$ and $\xi\alpha^{\iota}_{\mathrm{Q}} = [\![\iota_f^{-1}(\xi)]\!]^s_\alpha$ for any label, term and formula schema variables ν, θ and ξ in $(\Xi_l \cup E) \setminus E_l$, $\Xi_t \setminus E_t$ or $\Xi_f \setminus E_f$. We define exactly in the same way a schema assignment $\alpha^{\iota}_{\mathrm{B}}$ over $s_{\Sigma_{\mathrm{B}}}$. Thus,

suppose r is $gmon^{k}_{\equiv_\omega}$, and that $s, \alpha \Vdash \nu\sigma \equiv_\omega \nu'\sigma$, $s, \alpha \Vdash \nu\sigma{:}\theta_i\sigma =_g \nu'\sigma{:}\theta_i\sigma$ for each $i = 1, \ldots, k$, and $s, \alpha \Vdash \nu\sigma{:}\xi\sigma$, and that $\xi\sigma$ is a formula in $L(\Sigma_{\mathrm{Q}}, X)$ whose free variables are precisely $\theta_1\sigma, \ldots, \theta_k\sigma$. Note that by Proposition 6.6.2 and taking into account the definition of $\mathrm{tr}^{\iota}_{\Sigma_{\mathrm{Q}}}$ in Definition 6.6.1 we have that $s_{\Sigma_{\mathrm{Q}}}, \alpha^{\iota}_{\mathrm{Q}} \Vdash \mathrm{ltr}^{\iota_l}_{\Sigma_{\mathrm{Q}}}(\nu\sigma) \equiv_\omega \mathrm{ltr}^{\iota_l}_{\Sigma_{\mathrm{Q}}}(\nu'\sigma)$, $s_{\Sigma_{\mathrm{Q}}}, \alpha^{\iota}_{\mathrm{Q}} \Vdash \mathrm{ltr}^{\iota_l}_{\Sigma_{\mathrm{Q}}}(\nu\sigma){:}\mathrm{ttr}^{\iota_t}_{\Sigma_{\mathrm{Q}}}(\theta_i\sigma) =_g \mathrm{ltr}^{\iota_l}_{\Sigma_{\mathrm{Q}}}(\nu'\sigma){:}\mathrm{ttr}^{\iota_t}_{\Sigma_{\mathrm{Q}}}(\theta_i\sigma)$ for each $i = 1, \ldots, k$, and that $s_{\Sigma_{\mathrm{Q}}}, \alpha^{\iota}_{\mathrm{Q}} \Vdash \mathrm{ltr}^{\iota_l}_{\Sigma_{\mathrm{Q}}}(\nu\sigma){:}\mathrm{ftr}^{\iota_f}_{\Sigma_{\mathrm{Q}}}(\xi\sigma)$. Note that $\mathrm{ftr}^{\iota_f}_{\Sigma_{\mathrm{Q}}}(\xi\sigma)$ is $\xi\sigma$ since $\xi\sigma$ is a formula in $L(\Sigma_{\mathrm{Q}}, X)$. Then, since $s_{\Sigma_{\mathrm{Q}}}$ satisfies all the rules of $\mathcal{L}^{\oplus}_{\mathrm{FOMdD}}$, by rule $gmon^{k}_{\equiv_\omega}$ we have that $s_{\Sigma_{\mathrm{Q}}}, \alpha^{\iota}_{\mathrm{Q}} \Vdash \mathrm{ltr}^{\iota_l}_{\Sigma_{\mathrm{Q}}}(\nu'\sigma){:}\mathrm{ftr}^{\iota_f}_{\Sigma_{\mathrm{Q}}}(\xi\sigma)$. Henceforth, by Proposition 6.6.2, $s, \alpha \Vdash \nu'\sigma{:}\xi\sigma$, as we wanted to show.

suppose r is $\forall_x{}^{\mathrm{spc}}$, and $s, \alpha \Vdash \nu\sigma{:}\forall_x\xi\sigma$, $s, \alpha \Vdash \nu\sigma{:}\varepsilon(\theta\sigma)$ and that $\xi\sigma$ is a schema formula where all schema variables appear under the scope of a $\forall_x$ or a $\exists_x$, if a connective of $\Sigma_{\mathrm{B}} \setminus \Sigma_{\mathrm{Q}}$

appears in $\xi\sigma$ then it appears under the scope of a $\forall_x$ or of a $\exists_x$, and $\xi'\sigma$ is $\xi\sigma$ if there are no free x's in $\xi\sigma$, otherwise $\xi'\sigma$ is obtained by replacing the free occurrences of x in $\xi\sigma$ by a schema term $\theta\sigma$ either equal to a variable whenever some free x in $\xi\sigma$ is in the scope of a modality, or not having schema variables whenever some free x in $\xi\sigma$ is in the scope of a quantifier, such that no variable in $\theta\sigma$ is captured by a quantifier. Note that $\mathrm{ftr}^{\iota f}_{\Sigma_{\mathrm{Q}}}(\xi\sigma)$, $\mathrm{ftr}^{\iota f}_{\Sigma_{\mathrm{Q}}}(\xi'\sigma)$ and $\mathrm{ttr}^{\iota t}_{\Sigma_{\mathrm{Q}}}(\theta\sigma)$ satisfies also the conditions of proviso $\xi' \lhd \xi^x_\theta$. Moreover by Proposition 6.6.2 and taking into account the definition of $\mathrm{tr}^{\iota}_{\Sigma_{\mathrm{Q}}}$ in Definition 6.6.1 we have that $s_{\Sigma_{\mathrm{Q}}}, \alpha^{\iota}_{\mathrm{Q}} \Vdash \mathrm{ltr}^{\iota l}_{\Sigma_{\mathrm{Q}}}(\nu\sigma){:}\forall_x \mathrm{ftr}^{\iota f}_{\Sigma_{\mathrm{Q}}}(\xi\sigma)$ and $s_{\Sigma_{\mathrm{Q}}}, \alpha^{\iota}_{\mathrm{Q}} \Vdash \mathrm{ltr}^{\iota l}_{\Sigma_{\mathrm{Q}}}(\nu\sigma){:}\varepsilon(\mathrm{ttr}^{\iota t}_{\Sigma_{\mathrm{Q}}}(\theta\sigma))$. Then, since $s_{\Sigma_{\mathrm{Q}}}$ satisfies all the rules of $\mathcal{L}^{\oplus}_{\mathrm{FOMdD}}$, by rule ${\forall_x}^{\mathrm{spc}}$ we have that $s_{\Sigma_{\mathrm{Q}}}, \alpha^{\iota}_{\mathrm{Q}} \Vdash \mathrm{ltr}^{\iota l}_{\Sigma_{\mathrm{Q}}}(\nu\sigma){:}\mathrm{ftr}^{\iota f}_{\Sigma_{\mathrm{Q}}}(\xi'\sigma)$. Henceforth, by Proposition 6.6.2, $s, \alpha \Vdash \nu\sigma{:}\xi'\sigma$, as we wanted to show.

suppose r is *exh*, and that for any α' $\{\nu'\sigma\}$ co-equivalent to α if $s, \alpha' \Vdash \nu_1\sigma \equiv_\omega \nu'\sigma$ and $s, \alpha' \Vdash \nu'\sigma \equiv_\alpha \nu_2\sigma$ then $s, \alpha' \Vdash \nu''\sigma{:}\xi''\sigma$. We now show that for every α'_{Q} $\{\nu'\sigma\}$ co-equivalent to $\alpha^{\iota}_{\mathrm{Q}}$, if $s_{\Sigma_{\mathrm{Q}}}, \alpha'_{\mathrm{Q}} \Vdash \mathrm{ltr}^{\iota l}_{\Sigma_{\mathrm{Q}}}(\nu_1\sigma) \equiv_\omega \nu'\sigma$ and $s_{\Sigma_{\mathrm{Q}}}, \alpha'_{\mathrm{Q}} \Vdash \nu'\sigma \equiv_\alpha \mathrm{ltr}^{\iota l}_{\Sigma_{\mathrm{Q}}}(\nu_2\sigma)$ then $s_{\Sigma_{\mathrm{Q}}}, \alpha'_{\mathrm{Q}} \Vdash \mathrm{ltr}^{\iota l}_{\Sigma_{\mathrm{Q}}}(\nu''\sigma){:}\mathrm{ftr}^{\iota f}_{\Sigma_{\mathrm{Q}}}(\xi''\sigma)$. So, let α'_{Q} be a schema assignment $\{\nu'\sigma\}$ co-equivalent to α_{Q}, with $s_{\Sigma_{\mathrm{Q}}}, \alpha'_{\mathrm{Q}} \Vdash \mathrm{ltr}^{\iota l}_{\Sigma_{\mathrm{Q}}}(\nu_1\sigma) \equiv_\omega \nu'\sigma$ and $s_{\Sigma_{\mathrm{Q}}}, \alpha'_{\mathrm{Q}} \Vdash \nu'\sigma \equiv_\alpha \mathrm{ltr}^{\iota l}_{\Sigma_{\mathrm{Q}}}(\nu_2\sigma)$. Note that $\nu'\sigma$ is in E_l. Consider the schema assignment α' co-equivalent to α with $\nu'\sigma\alpha' = \nu'\sigma\alpha'_{\mathrm{Q}}$. Then, by Proposition 6.6.2, $s, \alpha' \Vdash \nu_1\sigma \equiv_\omega \nu'\sigma$ and $s, \alpha' \Vdash \nu'\sigma \equiv_\alpha \nu_2\sigma$, and so $s, \alpha' \Vdash \nu''\sigma{:}\xi''\sigma$. Thus, using again Proposition 6.6.2 we have $s_{\Sigma_{\mathrm{Q}}}, \alpha'_{\mathrm{Q}} \Vdash \mathrm{ltr}^{\iota l}_{\Sigma_{\mathrm{Q}}}(\nu''\sigma){:}\mathrm{ftr}^{\iota f}_{\Sigma_{\mathrm{Q}}}(\xi''\sigma)$. Then, since $s_{\Sigma_{\mathrm{Q}}}$ satisfies all the rules of $\mathcal{L}^{\oplus}_{\mathrm{FOMdD}}$, by rule *exh* we have that $s_{\Sigma_{\mathrm{Q}}}, \alpha^{\iota}_{\mathrm{Q}} \Vdash \mathrm{ltr}^{\iota l}_{\Sigma_{\mathrm{Q}}}(\nu''\sigma){:}\mathrm{ftr}^{\iota f}_{\Sigma_{\mathrm{Q}}}(\xi''\sigma)$. Henceforth, by Proposition 6.6.2, $s, \alpha \Vdash \nu''\sigma{:}\xi''\sigma$, as we wanted to show.

suppose r is $\forall_{xI}$, and that, for every α' $\{\nu'\sigma\}$ co-equivalent to α, if $s, \alpha' \Vdash \nu\sigma \equiv_x \nu'\sigma$ and $s, \alpha' \Vdash \nu'\sigma{:}\varepsilon(x)$ then $s, \alpha' \Vdash \nu'\sigma{:}\xi\sigma$. We now show that for every α'_{Q} $\{\nu'\sigma\}$ co-equivalent to α_{Q}, if $s_{\Sigma_{\mathrm{Q}}}, \alpha'_{\mathrm{Q}} \Vdash \mathrm{ltr}^{\iota l}_{\Sigma_{\mathrm{Q}}}(\nu\sigma) \equiv_x \nu'\sigma$ and $s_{\Sigma_{\mathrm{Q}}}, \alpha'_{\mathrm{Q}} \Vdash \nu'\sigma{:}\varepsilon(x)$ then $s_{\Sigma_{\mathrm{Q}}}, \alpha'_{\mathrm{Q}} \Vdash \nu'\sigma{:}\mathrm{ftr}^{\iota f}_{\Sigma_{\mathrm{Q}}}(\xi\sigma)$. So, let α'_{Q} be a schema assignment $\{\nu'\sigma\}$ co-equivalent to α_{Q}, with $s_{\Sigma_{\mathrm{Q}}}, \alpha'_{\mathrm{Q}} \Vdash \mathrm{ltr}^{\iota l}_{\Sigma_{\mathrm{Q}}}(\nu\sigma) \equiv_x \nu'\sigma$ and $s_{\Sigma_{\mathrm{Q}}}, \alpha'_{\mathrm{Q}} \Vdash \nu'\sigma{:}\varepsilon(x)$. Note that $\nu'\sigma$ is in E_l. Consider the schema assignment α' co-equivalent to α with $\nu'\sigma\alpha' = \nu'\sigma\alpha'_{\mathrm{Q}}$. Then, by Proposition 6.6.2, $s, \alpha' \Vdash \nu\sigma \equiv_x \nu'\sigma$ and $s, \alpha' \Vdash \nu'\sigma{:}\varepsilon(x)$, and so $s, \alpha' \Vdash \nu'\sigma{:}\xi\sigma$. Thus, using again Proposition 6.6.2 we have $s_{\Sigma_{\mathrm{Q}}}, \alpha'_{\mathrm{Q}} \Vdash \nu'\sigma{:}\mathrm{ftr}^{\iota f}_{\Sigma_{\mathrm{Q}}}(\xi\sigma)$. Then, since $s_{\Sigma_{\mathrm{Q}}}$ satisfies all the rules of $\mathcal{L}^{\oplus}_{\mathrm{FOMdD}}$, by rule $\forall_{xI}$ we have that $s_{\Sigma_{\mathrm{Q}}}, \alpha^{\iota}_{\mathrm{Q}} \Vdash \mathrm{ltr}^{\iota l}_{\Sigma_{\mathrm{Q}}}(\nu\sigma){:}\mathrm{ftr}^{\iota f}_{\Sigma_{\mathrm{Q}}}(\forall_x\xi\sigma)$. Henceforth, by Proposition 6.6.2, $s, \alpha \Vdash \nu\sigma{:}\forall_x\xi\sigma$, as we wanted to show.

suppose r is $\Box_I$. The proof that s satisfies $\Box_I$ is similar to the proof for $\forall_{xI}$, so it is omitted.

suppose r is $\Box_E$, and that $s, \alpha \Vdash \nu\sigma{:}\Box\xi\sigma$ and $s, \alpha \Vdash \nu\sigma R_\Box \nu'\sigma$. Then, by Proposition 6.6.2 and taking into account the definition of $\mathrm{tr}^{\iota}_{\Sigma_{\mathrm{Q}}}$ in Definition 6.6.1 we have that $s_{\Sigma_{\mathrm{Q}}}, \alpha^{\iota}_{\mathrm{Q}} \Vdash \mathrm{ltr}^{\iota l}_{\Sigma_{\mathrm{Q}}}(\nu\sigma){:}\Box\mathrm{ftr}^{\iota f}_{\Sigma_{\mathrm{Q}}}(\xi\sigma)$ and $s_{\Sigma_{\mathrm{Q}}}, \alpha^{\iota}_{\mathrm{Q}} \Vdash \mathrm{ltr}^{\iota l}_{\Sigma_{\mathrm{Q}}}(\nu\sigma) R_\Box \mathrm{ltr}^{\iota l}_{\Sigma_{\mathrm{Q}}}(\nu'\sigma)$. Then, since $s_{\Sigma_{\mathrm{Q}}}$ satisfies all the rules of $\mathcal{L}^{\oplus}_{\mathrm{FOMdD}}$, by rule $\Box_E$ we have that $s_{\Sigma_{\mathrm{Q}}}, \alpha^{\iota}_{\mathrm{Q}} \Vdash \mathrm{ltr}^{\iota l}_{\Sigma_{\mathrm{Q}}}(\nu'\sigma){:}\mathrm{ftr}^{\iota f}_{\Sigma_{\mathrm{Q}}}(\xi\sigma)$. Henceforth, by Proposition 6.6.2, $s, \alpha \Vdash \nu'\sigma{:}\xi\sigma$, as we wanted to show.

suppose r is $\forall_{xE}$ or $\wedge_I$ or $\wedge^1_E$ or $\wedge^2_E$ or $\neg_I$ or $\neg_E$ or $\neg_c$ or $R_{\Box_\alpha}^{\mathrm{gen}}$ or $\equiv_{x\omega}^{\mathrm{gen}}$ or $R_{\Box_{\alpha^{1,2}}}$ or $\equiv_{x\omega^{1,2}}$ or $\equiv_{xg^{1,2}}$ or dD or $\equiv_{xt}$ or $\equiv_{xr}$ or $\equiv_{xs}$. We omit these proofs since they are similar to the one for $\Box_E$.

suppose r is $mon_{R_{\Rightarrow}0}$, and that $s, \alpha \Vdash R_{\Rightarrow}0\nu_1\sigma\nu_2\sigma$, and $s, \alpha \Vdash \nu_1\sigma{:}\xi\sigma$, and that $\xi\sigma$ is a formula in $L(\Sigma_{\mathrm{B}}, X)$ without variables. Note that by Proposition 6.6.2 and taking into account the definition of $\mathrm{tr}^{\iota}_{\Sigma_{\mathrm{B}}}$ in Definition 6.6.1 we have $s_{\Sigma_{\mathrm{B}}}, \alpha^{\iota}_{\mathrm{B}} \Vdash R_{\Rightarrow}0\mathrm{ltr}^{\iota l}_{\Sigma_{\mathrm{B}}}(\nu_1\sigma)\mathrm{ltr}^{\iota l}_{\Sigma_{\mathrm{B}}}(\nu_2\sigma)$, $s_{\Sigma_{\mathrm{B}}}, \alpha^{\iota}_{\mathrm{B}} \Vdash \mathrm{ltr}^{\iota l}_{\Sigma_{\mathrm{B}}}(\nu_1\sigma){:}\mathrm{ftr}^{\iota f}_{\Sigma_{\mathrm{B}}}(\xi\sigma)$. Note that $\mathrm{ftr}^{\iota f}_{\Sigma_{\mathrm{B}}}(\xi\sigma)$ is $\xi\sigma$ since $\xi\sigma$ is a formula in $L(\Sigma_{\mathrm{B}}, X)$. Then, since $s_{\Sigma_{\mathrm{B}}}$ satisfies all the rules of $\mathcal{L}^{\oplus}_{\mathrm{B}_{\wedge,\Rightarrow}}$, by rule $mon_{R_{\Rightarrow}0}$ we have that $s_{\Sigma_{\mathrm{B}}}, \alpha^{\iota}_{\mathrm{B}} \Vdash \mathrm{ltr}^{\iota l}_{\Sigma_{\mathrm{B}}}(\nu_2\sigma){:}\mathrm{ftr}^{\iota f}_{\Sigma_{\mathrm{B}}}(\xi\sigma)$. Henceforth, by Proposition 6.6.2, $s, \alpha \Vdash \nu_2\sigma{:}\xi\sigma$.

suppose r is $gmon_{\equiv_\omega}$. We omit the proof since it is very similar to the proof for rule $mon_{R_{\Rightarrow}0}$.

suppose r is $\Rightarrow_I$. We omit the proof since it is very similar to the proof for rule $\forall_{xI}$.

suppose r is $\Rightarrow_E$ or $\wedge_I$ or $\wedge^1_E$ or $\wedge^2_E$ or $R_{\Rightarrow}{}^{\mathrm{lgen}}_{\alpha}$ or *rmon1* or *rmon2* or *rmon3* or *iden*. The proofs are similar to the proof for $\Box_E$, so they are omitted.

We now show the second condition of the definition of suitability of a pair of lfob logic systems, Definition 6.1.2. Let r be a rule in $\mathcal{L}^{\oplus}_{\mathrm{B}_{\wedge,\Rightarrow}}$, and σ and $\sigma^{\cdot}$ Σ_{B} schema substitutions such that $(\pi_{\Sigma_{\mathrm{B}}}\sigma_{\mathrm{nl}})$ is 1 for each π in P_f, and $\pi_{\Sigma_{\mathrm{B}};E}(\sigma^{\cdot})$ is 1 for each π in P_d. Then we show that $(\pi_{\Sigma_{\mathrm{B}}\cup\Sigma_{\mathrm{Q}}}\sigma_{\mathrm{nl}})$ is 1 for each π in P_f, and $\pi_{\Sigma_{\mathrm{B}}\cup\Sigma_{\mathrm{Q}};E}(\sigma^{\cdot})$ is 1 for each π in P_d by analysis on the possible cases for rule r in $\mathcal{L}^{\oplus}_{\mathrm{B}_{\wedge,\Rightarrow}}$. Consider the case where r is:

$mon_{R_{\Rightarrow}0}$. Then we only have to show that $(\text{p-mon}_{R_{\Rightarrow}0}(\xi)_{\Sigma_{\mathrm{B}}\cup\Sigma_{\mathrm{Q}}}\sigma_{\mathrm{nl}})$ is 1, i.e., that $\xi\sigma$ is a formula in $L(\Sigma_{\mathrm{B}_{\wedge,\Rightarrow}}, X)$ without variables. This happens because $\xi\sigma$ is a formula without variables since $(\text{p-mon}_{R_{\Rightarrow}0}(\xi)_{\Sigma_{\mathrm{B}}}\sigma_{\mathrm{nl}})$ is 1, and because $\xi\sigma$ is in $L(\Sigma_{\mathrm{B}}, X)$ since σ is a Σ_{B} schema substitution;

$gmon_{\equiv_\omega}$. We omit the proof since it is very similar to the proof for rule $mon_{R_{\Rightarrow}0}$.

$\Rightarrow_I$. So, we only have to show $\mathrm{fresh}(\{\nu_1, \nu_2\}, \langle\vartheta_1, \{R_{\Rightarrow}\nu\nu_1\nu_2, \nu_1{:}\xi_1\}, \nu_2{:}\xi_2\rangle)_{\Sigma_{\mathrm{B}}\cup\Sigma_{\mathrm{Q}};E}(\sigma^{\cdot}) = 1$. This happens since $\mathrm{fresh}(\{\nu_1, \nu_2\}, \langle\vartheta_1, \{R_{\Rightarrow}\nu\nu_1\nu_2, \nu_1{:}\xi_1\}, \nu_2{:}\xi_2\rangle)_{\Sigma_{\mathrm{B}};E}(\sigma^{\cdot}) = 1$, by definition of fresh proviso, see Remark 2.3.8.

$\Rightarrow_E$ or $\wedge_I$ or $\wedge^1_E$ or $\wedge^2_E$ or $R_{\Rightarrow}{}^{\mathrm{lgen}}_{\alpha}$ or *rmon1* or *rmon2* or *rmon3* or *iden*. The proofs are straightforward so they are omitted.

Now we treat the case where the rule is in $\mathcal{L}^{\oplus}_{\mathrm{FOMdD}}$. Let r be a rule in $\mathcal{L}^{\oplus}_{\mathrm{FOMdD}}$, and σ and $\sigma^{\cdot}$ Σ_{Q} schema substitutions such that $(\pi_{\Sigma_{\mathrm{Q}}}\sigma_{\mathrm{nl}})$ is 1 for each π in P_f, and $\pi_{\Sigma_{\mathrm{Q}};E}(\sigma^{\cdot})$ is 1 for each π in P_d. Then we show that $(\pi_{\Sigma_{\mathrm{B}}\cup\Sigma_{\mathrm{Q}}}\sigma_{\mathrm{nl}})$ is 1 for each π in P_f, and $\pi_{\Sigma_{\mathrm{B}}\cup\Sigma_{\mathrm{Q}};E}(\sigma^{\cdot})$ is 1 for each π in P_d by analysis on the possible cases for rule r in $\mathcal{L}^{\oplus}_{\mathrm{FOMdD}}$. Consider the case where r is:

$gmon^{k}_{\equiv_\omega}$. We omit the proof since it is very similar to the proof for rule $mon_{R_{\Rightarrow}0}$.

${\forall_x}^{\mathrm{spc}}$. Then we only have to show that $(\xi' \lhd \xi^{x}_{\theta}{}_{\Sigma_{\mathrm{B}}\cup\Sigma_{\mathrm{Q}}}\sigma_{\mathrm{nl}})$ is 1, i.e., $\xi\sigma$ is a schema formula where all schema variables appear under the scope of a $\forall_x$ or a $\exists_x$, if a connective of

$\Sigma_{\mathrm{B}} \setminus \Sigma_{\mathrm{Q}}$ appears in $\xi\sigma$ then it appears under the scope of a $\forall_x$ or of a $\exists_x$, and $\xi'\sigma$ is $\xi\sigma$ if there are no free x's in $\xi\sigma$, otherwise $\xi'\sigma$ is obtained by replacing the free occurrences of x in $\xi\sigma$ by a schema term $\theta\sigma$ either equal to a variable whenever some free x in $\xi\sigma$ is in the scope of a modality, or not having schema variables whenever some free x in $\xi\sigma$ is in the scope of a quantifier, such that no variable in $\theta\sigma$ is captured by a quantifier. This happens because σ is a Σ_{Q} schema substitution, and $(\xi' \lhd \xi^{x}_{\theta\,\Sigma_{\mathrm{Q}}}\sigma_{\mathrm{nl}})$ is 1.

exh or $\forall_{xI}$ or $\Box_I$. We omit these proofs since they are very similar to the proof for $\Rightarrow_I$.

$\Box_E$ or $\forall_{xE}$ or $\wedge_I$ or $\wedge^1_E$ or $\wedge^2_E$ or $\neg_I$ or $\neg_E$ or $\neg_c$ or $R\Box^{\mathrm{gen}}_{\alpha}$ or $\equiv_{x\omega}^{\mathrm{gen}}$ or $R_{\Box\alpha^{1,2}}$ or $\equiv_{x\omega^{1,2}}$ or $\equiv_{xg^{1,2}}$ or dD or $\equiv_{xt}$ or $\equiv_{xr}$ or $\equiv_{xs}$. We omit these proofs since they are straightforward.

It is immediate to see that the third condition of the definition of suitability of a pair of lfob logic systems, Definition 6.1.2, is satisfied by the pair of lfob logic systems $\mathcal{L}^{\oplus}_{\mathrm{FOMdD}}$ and $\mathcal{L}^{\oplus}_{\mathrm{B}_{\wedge,\Rightarrow}}$. *QED*

6.5.4 Fibring

We now consider the lfob logic system $\mathcal{L}^{\oplus}_{\mathrm{FOMdD}} \oplus \mathcal{L}^{\oplus}_{\mathrm{B}_{\wedge,\Rightarrow}}$, that we prove to be sound and complete, resulting from the fibring of the lfob logic system $\mathcal{L}^{\oplus}_{\mathrm{FOMdD}}$ for first-order modal logic with decreasing domains and the lfob logic system $\mathcal{L}^{\oplus}_{\mathrm{B}_{\wedge,\Rightarrow}}$ for the $\wedge$-$\Rightarrow$ fragment of basic relevance logic B. The interested reader can consult [2, 53, 3] for a study on first-order relevance logics.

Example 6.5.6 *Fibring of $\mathcal{L}^{\oplus}_{FOMdD}$ and $\mathcal{L}^{\oplus}_{B_{\wedge,\Rightarrow}}$.* We now briefly describe

$$\mathcal{L}^{\oplus}_{\mathrm{FOMdD}} \oplus \mathcal{L}^{\oplus}_{\mathrm{B}_{\wedge,\Rightarrow}}$$

which, if we denote its components by $\langle \Sigma, R, M, \breve{\cdot} \rangle$ is such that Σ is

- $F^l_0 = \{0\}$, and $F^l_k = \emptyset$ for $k \geq 1$,
- $S_2 = \{R_{\Box}\} \cup \{\equiv_x \mid x \in X\}$, $S_3 = \{R_{\Rightarrow}\}$, and $S_k = \emptyset$ for $k \geq 4$ and $k = 1$,
- $\langle F, P \rangle$ is a first-order alphabet, P_0 is non-empty and ε is in P_1,
- $C_0 = \{\bot\}$, $C_1 = \{\neg\}$, $C_2 = \{\wedge\}$, and $C_k = \emptyset$ for $k \geq 3$ and $k = 1$,
- $Q_1 = \{\forall\}$, and $Q_k = \emptyset$ for $k \geq 2$;
- $O_1 = \{\Box\}$, $O_2 = \{\Rightarrow\}$, and $O_k = \emptyset$ for $k \geq 3$,

and R, besides the rules specified in Definition 2.3.10 common to all lfob deduction systems, contains:

$$\frac{\nu{:}\xi_1 \qquad \nu{:}\xi_2}{\nu{:}\xi_1 \wedge \xi_2}\;\wedge_I \qquad\qquad \frac{\nu{:}\xi_1 \wedge \xi_2}{\nu{:}\xi_1}\;\wedge^1_E \qquad\qquad \frac{\nu{:}\xi_1 \wedge \xi_2}{\nu{:}\xi_2}\;\wedge^2_E$$

$$\frac{\nu{:}\xi \,/\, \nu'{:}\bot}{\nu{:}\neg\xi}\;\neg_I \qquad\qquad \frac{\nu{:}\neg\xi \qquad \nu{:}\xi}{\nu'{:}\bot}\;\neg_E \qquad\qquad \frac{\nu{:}\neg\xi \,/\, \nu'{:}\bot}{\nu{:}\xi}\;\neg_c$$

$$\frac{\nu \equiv_x \nu', \nu':\varepsilon(x) \,/\, \nu':\xi}{\nu:\forall_x\xi}\ \forall_{xI};\ \text{fresh}(\nu', \langle\vartheta_1, \{\nu \equiv_x \nu', \nu':\varepsilon(x)\}, \nu':\xi\rangle)$$

$$\frac{\nu:\forall_x\xi \quad \nu \equiv_x \nu' \quad \nu':\varepsilon(x)}{\nu':\xi}\ \forall_{xE} \qquad\qquad \frac{\nu:\forall_x\xi \quad \nu:\varepsilon(\theta)}{\nu:\xi'}\ {\forall_x}^{\text{spc}};\ \xi' \lhd \xi^x_\theta$$

$$\frac{\nu R_\Box \nu' \quad \nu':\varepsilon(x)}{\nu:\varepsilon(x)}\ dD$$

$$\frac{\nu R_\Box \nu' \,/\, \nu':\xi}{\nu:\Box\xi}\ \Box_I;\ \text{fresh}(\nu', \langle\vartheta_1, \{\nu R_\Box \nu'\}, \nu':\xi\rangle) \qquad\qquad \frac{\nu:\Box\xi \quad \nu R_\Box \nu'}{\nu':\xi}\ \Box_E$$

$$\frac{\nu R_\Box \nu'}{\nu \equiv_\alpha \nu'}\ R_{\Box\alpha^{1,2}} \qquad \frac{\nu R_\Box \nu_1 \quad \nu \equiv_\omega \nu' \quad \nu_1 \equiv_\omega \nu_1' \quad \nu' \equiv_\alpha \nu_1'}{\nu' R_\Box \nu_1'}\ R_{\Box\alpha}^{\text{gen}}$$

$$\frac{\nu \equiv_x \nu'}{\nu \equiv_\omega \nu'}\ \equiv_{x\omega^{1,2}} \qquad \frac{\nu \equiv_x \nu_1 \quad \nu \equiv_\alpha \nu' \quad \nu_1 \equiv_\alpha \nu_1' \quad \nu' \equiv_\omega \nu_1'}{\nu' \equiv_x \nu_1'}\ \equiv_{x\omega}^{\text{gen}}$$

$$\frac{\nu \equiv_x \nu'}{\nu:\theta =_g \nu':\theta}\ \equiv_{xg^{1,2}};\ \text{vardif}(x,\theta)$$

$$\frac{\nu_1 \equiv_\omega \nu', \nu' \equiv_\alpha \nu_2 \,/\, \nu'':\xi''}{\nu'':\xi''}\ \mathit{exh};\ \text{fresh}(\nu', \langle\vartheta_1, \{\nu_1 \equiv_\omega \nu', \nu' \equiv_\alpha \nu_2\}, \nu'':\xi''\rangle)$$

$$\frac{\nu \equiv_x \nu' \quad \nu' \equiv_x \nu''}{\nu \equiv_x \nu''}\ \equiv_{xt} \qquad \frac{}{\nu \equiv_x \nu}\ \equiv_{xr} \qquad \frac{\nu' \equiv_x \nu}{\nu \equiv_x \nu'}\ \equiv_{xs}$$

$$\frac{\nu \equiv_\omega \nu' \qquad \nu:\theta_1 =_g \nu':\theta_1 \quad \dots \quad \nu:\theta_k =_g \nu':\theta_k \qquad \nu:\xi}{\nu':\xi}\ \mathit{gmon}^k_{\equiv_\omega};\ \text{p-gmon}_{\equiv_\omega}(\xi, \theta_1, \dots, \theta_k)$$

$$\frac{R_\Rightarrow \nu\nu_1\nu_2, \nu_1:\xi_1 \,/\, \nu_2:\xi_2}{\nu:\xi_1 \Rightarrow \xi_2}\ \Rightarrow_I;\ \text{fresh}(\{\nu_1, \nu_2\}, \langle\vartheta_1, \{R_\Rightarrow \nu\nu_1\nu_2, \nu_1:\xi_1\}, \nu_2:\xi_2\rangle)$$

$$\frac{\nu:\xi_1 \Rightarrow \xi_2 \qquad \nu_1:\xi_1 \qquad R_\Rightarrow \nu\nu_1\nu_2}{\nu_2:\xi_2}\ \Rightarrow_E$$

$$\frac{R_\Rightarrow 0\nu\nu_1 \quad R_\Rightarrow \nu_1\nu_2\nu_3}{R_\Rightarrow \nu\nu_2\nu_3}\ \mathit{rmon1} \qquad\qquad \frac{R_\Rightarrow 0\nu\nu_2 \quad R_\Rightarrow \nu_1\nu_2\nu_3}{R_\Rightarrow \nu_1\nu\nu_3}\ \mathit{rmon2}$$

$$\frac{R_{\Rightarrow}0\nu_3\nu \quad R_{\Rightarrow}\nu_1\nu_2\nu_3}{R_{\Rightarrow}\nu_1\nu_2\nu}\ \mathit{rmon3}$$

$$\frac{R_{\Rightarrow}\nu_1'\nu_2'\nu_3' \quad \nu_1 \equiv_\omega \nu_1' \quad \nu_2 \equiv_\omega \nu_2' \quad \nu_3 \equiv_\omega \nu_3'}{R_{\Rightarrow}\nu_1\nu_2\nu_3}\ R_{\Rightarrow}{}_{\alpha}^{\mathrm{lgen}} \qquad\qquad \frac{}{R_{\Rightarrow}0\nu\nu}\ \mathit{iden}$$

$$\frac{R_{\Rightarrow}0\nu_1\nu_2 \qquad \nu_1{:}\xi}{\nu_2{:}\xi}\ \mathit{mon}_{R_{\Rightarrow}0};\ \text{p-mon}_{R_{\Rightarrow}0}(\xi) \qquad\qquad \frac{\nu \equiv_\omega \nu' \qquad \nu{:}\xi}{\nu'{:}\xi}\ \mathit{gmon}_{\equiv_\omega};\ \text{p-gmon}_{\equiv_\omega}(\xi)$$

for every x in X, where the provisos $\text{p-gmon}_{\equiv_\omega}(\xi, \theta_1, \dots, \theta_k)$, $\text{vardif}(x, \theta)$, $\text{p-mon}_{R_{\Rightarrow}0}(\xi)$, $\text{p-gmon}_{\equiv_\omega}(\xi)$ and $\xi' \lhd \xi^x_\theta$ are such that

- $\text{p-gmon}_{\equiv_\omega}(\xi)_\Sigma(\rho_{\mathrm{nl}}) = 1$ iff $\xi\rho_f$ is a formula in $L(\Sigma_{\mathrm{B}_{\wedge,\Rightarrow}}, X)$ without variables,
- $\text{p-mon}_{R_{\Rightarrow}0}(\xi)_\Sigma(\rho_{\mathrm{nl}}) = 1$ iff $\xi\rho$ is a formula in $L(\Sigma_{\mathrm{B}_{\wedge,\Rightarrow}}, X)$ without variables,
- $\text{vardif}(x, \theta)_\Sigma(\rho_{\mathrm{nl}}) = 1$ iff $\theta\rho_t$ is a variable different of x,
- $\text{p-gmon}_{\equiv_\omega}(\xi, \theta_1, \dots, \theta_k)_\Sigma(\rho_{\mathrm{nl}}) = 1$ iff $\xi\rho_f$ is a formula in $L(\Sigma_{\mathrm{FOMdD}}, X)$ whose free variables are $\theta_1\rho_t, \dots, \theta_k\rho_t$,
- $\xi' \lhd \xi^x_{\theta}{}_\Sigma(\rho_{\mathrm{nl}}) = 1$ iff if a connective of $\Sigma_{\mathrm{B}_{\wedge,\Rightarrow}} \setminus \Sigma_{\mathrm{FOMdD}}$ appears in $\xi\rho_f$ then it appears under the scope of a $\forall_x$ or of a $\exists_x$ and $\xi'\rho_f$ is obtained by replacing the free occurrences of x in $\xi\rho_f$ by a term $\theta\rho_t$ equal to a variable whenever some free x in $\xi\rho_f$ is in the scope of a modality, such that no variable in $\theta\rho_t$ is captured by a quantifier,

M is composed of all Σ structures $\langle U, A, W, \alpha, \omega, D, \mathcal{E}, \mathcal{B}, [.]\rangle$ such that exists m_1 in $M^{\oplus}_{\mathrm{FOMdD}}$ with $\langle U, A, W, \alpha, \omega, D, \mathcal{E}, \mathcal{B}, [.]|_{\Sigma_1}\rangle$ in $\check{m}_1$, and exists m_2 in $M^{\oplus}_{\mathrm{B}_{\wedge,\Rightarrow}}$ with $\langle U, A, W, \alpha, \omega, D, \mathcal{E}, \mathcal{B}, [.]|_{\Sigma_2}\rangle$ in $\check{m}_2$,

and $\check{\ }$ associates to each structure s in M a singleton with that same structure s.

6.5.5 Completeness and soundness

Proposition 6.5.7 The lfob logic system $\mathcal{L}^{\oplus}_{\mathrm{FOMdD}} \oplus \mathcal{L}^{\oplus}_{\mathrm{B}_{\wedge,\Rightarrow}}$ is sound.

Proof The proposition follows straightforwardly by Theorem 6.3.1 because $\mathcal{L}^{\oplus}_{\mathrm{FOMdD}} \oplus \mathcal{L}^{\oplus}_{\mathrm{B}_{\wedge,\Rightarrow}}$ is the fibring of the suitable pair of lfob logic systems $\mathcal{L}^{\oplus}_{\mathrm{FOMdD}}$ and $\mathcal{L}^{\oplus}_{\mathrm{B}_{\wedge,\Rightarrow}}$, as shown in Proposition 6.5.5. *QED*

Proposition 6.5.8 The lfob logic system $\mathcal{L}^{\oplus}_{\mathrm{FOMdD}} \oplus \mathcal{L}^{\oplus}_{\mathrm{B}_{\wedge,\Rightarrow}}$ is complete.

Proof The proposition follows straightforwardly by the completeness preservation theorem, Theorem 6.3.2, because the pair of lfob logic systems $\mathcal{L}^{\oplus}_{\mathrm{FOMdD}}$ and $\mathcal{L}^{\oplus}_{\mathrm{B}_{\wedge,\Rightarrow}}$ is suitable, as shown in Proposition 6.5.5, $\mathcal{L}^{\oplus}_{\mathrm{FOMdD}}$ and $\mathcal{L}^{\oplus}_{\mathrm{B}_{\wedge,\Rightarrow}}$ are full and connected, and $\mathcal{L}^{\oplus}_{\mathrm{FOMdD}}$ is with a classical negation. *QED*

6.6 Auxiliary definitions and results

In order to establish the suitability of a pair of lfob logic systems, for instance as we did in Proposition 6.5.5, it is essential the translation of formulae and terms over the joint signature, to formulae and terms over one of the signatures. So, in Definition 6.6.1, we introduce the translation maps used in that proposition.

Definition 6.6.1 Given signatures Σ and $\Sigma^{\cdot}$, a set E of label schema variables disjoint of Ξ_l, and a bijection ι_l from $T_{\mathrm{lab},E}(\Sigma \cup \Sigma^{\cdot}, \Xi_l)$ to a set E_l contained in Ξ_l, the map $\mathrm{ltr}_{\Sigma}^{\iota_l}$ from $T_{\mathrm{lab},E}(\Sigma \cup \Sigma^{\cdot}, \Xi_l)$ to $T_{\mathrm{lab},E}(\Sigma, \Xi_l)$ is such that $\mathrm{ltr}_{\Sigma}^{\iota_l}(v)$ is inductively defined on v as follows,

- if v is in F_0^l then $\mathrm{ltr}_{\Sigma}^{\iota_l}(v) = v$,
- if v is in $F_0^{l\cdot} \setminus F_0^l$ then $\mathrm{ltr}_{\Sigma}^{\iota_l}(v) = \iota_l(v)$,
- if v is in $\Xi_l \cup E$ then $\mathrm{ltr}_{\Sigma}^{\iota_l}(v) = v$,
- if v is $f(v_1, \ldots, v_n)$ with f in F_n^l then $\mathrm{ltr}_{\Sigma}^{\iota_l}(v) = f(\mathrm{ltr}_{\Sigma}^{\iota_l}(v_1), \ldots, \mathrm{ltr}_{\Sigma}^{\iota_l}(v_n))$,
- if v is $f(v_1, \ldots, v_n)$ with f in $F_n^{l\cdot} \setminus F_n^l$ then $\mathrm{ltr}_{\Sigma}^{\iota_l}(v) = \iota_l(v)$.

Given a bijection ι_t from $L(\Sigma \cup \Sigma^{\cdot}, \Xi_t)$ to a set E_t contained in Ξ_t, the map $\mathrm{ttr}_{\Sigma}^{\iota_t}$ from $L(\Sigma \cup \Sigma^{\cdot}, \Xi_t)$ to $L(\Sigma, \Xi_t)$ is such that $\mathrm{ttr}_{\Sigma}^{\iota_t}(t)$ is inductively defined on t as follows,

- if t is in X then $\mathrm{ttr}_{\Sigma}^{\iota_t}(t) = t$,
- if t is in F_0 then $\mathrm{ttr}_{\Sigma}^{\iota_t}(t) = t$,
- if t is in $F_0^{\cdot} \setminus F_0$ then $\mathrm{ttr}_{\Sigma}^{\iota_t}(t) = \iota_t(t)$,
- if t is in Ξ_t then $\mathrm{ttr}_{\Sigma}^{\iota_t}(t) = t$,
- if t is $f(t_1, \ldots, t_n)$ with f in F_n then $\mathrm{ttr}_{\Sigma}^{\iota_t}(t) = f(\mathrm{ttr}_{\Sigma}^{\iota_t}(t_1), \ldots, \mathrm{ttr}_{\Sigma}^{\iota_t}(t_n))$
- if t is $f(t_1, \ldots, t_n)$ with f in $F_n^{\cdot} \setminus F_n$ then $\mathrm{ttr}_{\Sigma}^{\iota_t}(t) = \iota_t(t)$,

moreover given a bijection ι_f from $L(\Sigma \cup \Sigma^{\cdot}, \Xi_t, \Xi_f)$ to a set E_f contained in Ξ_f, the map $\mathrm{ftr}_{\Sigma}^{\iota_f}$ from $L(\Sigma \cup \Sigma^{\cdot}, \Xi_t, \Xi_f)$ to $L(\Sigma, \Xi_t, \Xi_f)$ is such that $\mathrm{ftr}_{\Sigma}^{\iota_f}(\varphi)$ is inductively defined on φ as follows,

- if φ is $p(t_1, \ldots, t_n)$ with p in P_n then $\mathrm{ftr}_{\Sigma}^{\iota_f}(\varphi) = p(\mathrm{ttr}_{\Sigma}^{\iota_t}(t_1), \ldots, \mathrm{ttr}_{\Sigma}^{\iota_t}(t_n))$,
- if φ is $p(t_1, \ldots, t_n)$ with p in $P_n^{\cdot} \setminus P_n$ then $\mathrm{ftr}_{\Sigma}^{\iota_f}(\varphi) = \iota_f(\varphi)$,
- if φ is $t_1 = t_2$ then $\mathrm{ftr}_{\Sigma}^{\iota_f}(\varphi)$ is $\mathrm{ttr}_{\Sigma}^{\iota_t}(t_1) = \mathrm{ttr}_{\Sigma}^{\iota_t}(t_n)$,
- if φ is in C_0 then $\mathrm{ftr}_{\Sigma}^{\iota_f}(\varphi) = \varphi$
- if φ is in $C_0^{\cdot} \setminus C_0$ then $\mathrm{ftr}_{\Sigma}^{\iota_f}(\varphi) = \iota_f(\varphi)$,
- if φ is in Ξ_f then $\mathrm{ftr}_{\Sigma}^{\iota_f}(\varphi) = \varphi$,
- if φ is $c(\varphi_1, \ldots, \varphi_n)$ with c in C_n then $\mathrm{ftr}_{\Sigma}^{\iota_f}(\varphi) = c(\mathrm{ftr}_{\Sigma}^{\iota_f}(\varphi_1), \ldots, \mathrm{ftr}_{\Sigma}^{\iota_f}(\varphi_n))$,

- if φ is $c(\varphi_1, \dots, \varphi_n)$ with c in $C^{\cdot}_n \setminus C_n$ then $\mathrm{ftr}^{\iota_f}_{\Sigma}(\varphi) = \iota_f(\varphi)$,
- similarly for φ equal to $q_x(\varphi_1, \dots, \varphi_n)$ or $o(\varphi_1, \dots, \varphi_n)$.

Finally, the map $\mathrm{tr}^{\iota}_{\Sigma}$ from $L_E(\Sigma \cup \Sigma^{\cdot}, X, \Xi_l, \Xi_t, \Xi_f)$ to $L_E(\Sigma, X, \Xi_l, \Xi_t, \Xi_f)$ is such that $\mathrm{tr}^{\iota}_{\Sigma}(\eta)$ is defined by case analysis on η as follows,

- if η is $v{:}\varphi$ then $\mathrm{tr}^{\iota}_{\Sigma}(\eta) = \mathrm{ltr}^{\iota_l}_{\Sigma}(v){:}\mathrm{ftr}^{\iota_f}_{\Sigma}(\varphi)$,
- if η is $v_1{:}t_1 =_g v_2{:}t_2$ then $\mathrm{tr}^{\iota}_{\Sigma}(\eta)$ is $\mathrm{ltr}^{\iota_l}_{\Sigma}(v_1){:}\mathrm{ttr}^{\iota_t}_{\Sigma}(t_1) =_g \mathrm{ltr}^{\iota_l}_{\Sigma}(v_2){:}\ \mathrm{ttr}^{\iota_t}_{\Sigma}(t_2)$,
- if η is $r\, v_1 \dots v_n$ then $\mathrm{tr}^{\iota}_{\Sigma}(\eta) = r\, \mathrm{ltr}^{\iota_l}_{\Sigma}(v_1) \dots \mathrm{ltr}^{\iota_l}_{\Sigma}(v_n)$,
- if η is $v_1 \equiv_\omega v_2$ then $\mathrm{tr}^{\iota}_{\Sigma}(\eta) = \mathrm{ltr}^{\iota_l}_{\Sigma}(v_1) \equiv_\omega \mathrm{ltr}^{\iota_l}_{\Sigma}(v_2)$,
- if η is $v_1 \equiv_\alpha v_2$ then $\mathrm{tr}^{\iota}_{\Sigma}(\eta) = \mathrm{ltr}^{\iota_l}_{\Sigma}(v_1) \equiv_\alpha \mathrm{ltr}^{\iota_l}_{\Sigma}(v_2)$.

Note that the above translations are only useful if we could guarantee that a structure over the joint signature satisfies a formula iff the reduct of that structure satisfies the translation of the formulae. This is the purpose of the next proposition. Note that this proposition has a crucial role in the proof of Proposition 6.5.5.

Proposition 6.6.2 We have

1. $[\![v]\!]^{s}_{\alpha} = [\![\mathrm{ltr}^{\iota_l}_{\Sigma}(v)]\!]^{s_\Sigma}_{\alpha_\iota}$
2. $[\![t]\!]^{s}_{\alpha} = [\![\mathrm{ttr}^{\iota_t}_{\Sigma}(t)]\!]^{s_\Sigma}_{\alpha_\iota}$
3. $[\![\varphi]\!]^{s}_{\alpha} = [\![\mathrm{ftr}^{\iota_f}_{\Sigma}(\varphi)]\!]^{s_\Sigma}_{\alpha_\iota}$
4. $s, \alpha \Vdash \eta$ iff $s_\Sigma, \alpha_\iota \Vdash \mathrm{tr}^{\iota}_{\Sigma}(\eta)$

for any

- set E of label schema variables disjoint of Ξ_l,
- signatures Σ and $\Sigma^{\cdot}$,
- finite sets E_l, E_t and E_f contained in $\Xi_l \cup E$, Ξ_t and Ξ_f, respectively,
- bijections ι_l from $T_{\mathrm{lab},E}(\Sigma \cup \Sigma^{\cdot}, \Xi_l)$ to $(\Xi_l \cup E) \setminus E_l$, ι_t from $T(\Sigma \cup \Sigma^{\cdot}, \Xi_t)$ to $\Xi_t \setminus E_t$, and ι_f from $L(\Sigma \cup \Sigma^{\cdot}, \Xi_t, \Xi_f)$ to $\Xi_f \setminus E_f$,
- φ in $L(\Sigma \cup \Sigma^{\cdot}, X, \Xi_t, \Xi_f)$, with its term and formula schema variables contained in E_t and in E_f, respectively,
- η in $L_E(\Sigma \cup \Sigma^{\cdot}, X, \Xi_l, \Xi_t, \Xi_f)$, with its label, term and formula schema variables contained in E_l, E_t, and E_f, respectively, such that if η is $r\, v_1 \dots v_n$ then r is in $S_{\Sigma n}$,
- t in $T(\Sigma \cup \Sigma^{\cdot}, X, \Xi_t)$, with its term schema variables contained in E_t,
- v in $T_{\mathrm{lab},E}(\Sigma \cup \Sigma^{\cdot}, \Xi_l)$, with its label schema variables contained in E_l,

- $\Sigma \cup \Sigma^{\cdot}$ structure s and schema assignments α and α_ι over s and s_Σ such that $\nu\alpha_\iota = \nu\alpha$, $\theta\alpha_\iota = \theta\alpha$ and $\xi\alpha_\iota = \xi\alpha$ for any label, term and formula schema variables, ν in E_l, θ in E_t, and ξ in E_f, respectively, and $\nu\alpha_\iota = \llbracket \iota_l^{-1}(\nu) \rrbracket_\alpha^s$, $\theta\alpha_\iota = \llbracket \iota_t^{-1}(\theta) \rrbracket_\alpha^s$, and $\xi\alpha_\iota = \llbracket \iota_f^{-1}(\xi) \rrbracket_\alpha^s$ for any label, term and formula schema variables, ν in $(\Xi_l \cup E) \setminus E_l$, θ in $\Xi_t \setminus E_t$, and ξ in $\Xi_f \setminus E_f$, respectively.

Proof We show 1. by induction on the structure of the label schema term v:

- v is in F_0^l. Note that $\mathrm{ltr}_\Sigma^{\iota_l}(v) = v$. So $\llbracket v \rrbracket_\alpha^s = [v]^s = [v]^{s_\Sigma} = [\mathrm{ltr}_\Sigma^{\iota_l}(v)]^{s_\Sigma} = \llbracket \mathrm{ltr}_\Sigma^{\iota_l}(v) \rrbracket_{\alpha_\iota}^{s_\Sigma}$.

- v is in $F_0^{l\cdot} \setminus F_0^l$. Note that $\mathrm{ltr}_\Sigma^{\iota_l}(v) = \iota_l(v)$, and so $v = \iota_l^{-1}(\mathrm{ltr}_\Sigma^{\iota_l}(v))$. Therefore $\llbracket v \rrbracket_\alpha^s = \llbracket \iota_l^{-1}(\mathrm{ltr}_\Sigma^{\iota_l}(v)) \rrbracket_\alpha^s = \mathrm{ltr}_\Sigma^{\iota_l}(v)\alpha_\iota = \llbracket \mathrm{ltr}_\Sigma^{\iota_l}(v) \rrbracket_{\alpha_\iota}^{s_\Sigma}$.

- v is in $\Xi_l \cup E$. Note that $\mathrm{ltr}_\Sigma^{\iota_l}(v) = v$. So $\llbracket v \rrbracket_\alpha^s = v\alpha = v\alpha_\iota = \llbracket \mathrm{ltr}_\Sigma^{\iota_l}(v) \rrbracket_{\alpha_\iota}^{s_\Sigma}$.

- v is $f(v_1, \ldots, v_n)$ with f in F_n^l. Note that $\mathrm{ltr}_\Sigma^{\iota_l}(v) = f(\mathrm{ltr}_\Sigma^{\iota_l}(v_1), \ldots, \mathrm{ltr}_\Sigma^{\iota_l}(v_n))$. Then $\llbracket v \rrbracket_\alpha^s = [f]^s(\llbracket v_1 \rrbracket_\alpha^s, \ldots, \llbracket v_n \rrbracket_\alpha^s) = [f]^{s_\Sigma}(\llbracket \mathrm{ltr}_\Sigma^{\iota_l}(v_1) \rrbracket_{\alpha_\iota}^{s_\Sigma}, \ldots, \llbracket \mathrm{ltr}_\Sigma^{\iota_l}(v_n) \rrbracket_{\alpha_\iota}^{s_\Sigma}) = \llbracket \mathrm{ltr}_\Sigma^{\iota_l}(v) \rrbracket_{\alpha_\iota}^{s_\Sigma}$.

- v is $f(v_1, \ldots, v_n)$ with f in $F_n^{l\cdot} \setminus F_n^l$. Note that $\mathrm{ltr}_\Sigma^{\iota_l}(v) = \iota_l(v)$, and so $v = \iota_l^{-1}(\mathrm{ltr}_\Sigma^{\iota_l}(v))$. Thus $\llbracket v \rrbracket_\alpha^s = \llbracket \iota_l^{-1}(\mathrm{ltr}_\Sigma^{\iota_l}(v)) \rrbracket_\alpha^s = \mathrm{ltr}_\Sigma^{\iota_l}(v)\alpha_\iota = \llbracket \mathrm{ltr}_\Sigma^{\iota_l}(v) \rrbracket_{\alpha_\iota}^{s_\Sigma}$.

We show 2. by induction on the structure of the schema term t:

- t is in X. Note that $\mathrm{ttr}_\Sigma^{\iota_t}(t) = t$. So $\llbracket t \rrbracket_\alpha^s(u) = [t]_{\alpha_s(u)}^s = [t]_{\alpha_{s_\Sigma}(u)}^{s_\Sigma} = [\mathrm{ttr}_\Sigma^{\iota_t}(t)]_{\alpha_{s_\Sigma}(u)}^{s_\Sigma} = \llbracket \mathrm{ttr}_\Sigma^{\iota_t}(t) \rrbracket_{\alpha_\iota}^{s_\Sigma}(u)$.

- t is in F_0. Note that $\mathrm{ttr}_\Sigma^{\iota_t}(t) = t$. So $\llbracket t \rrbracket_\alpha^s(u) = [t]_{\omega_s(u)}^s = [t]_{\omega_{s_\Sigma}(u)}^{s_\Sigma} = [\mathrm{ttr}_\Sigma^{\iota_t}(t)]_{\omega_{s_\Sigma}(u)}^{s_\Sigma} = \llbracket \mathrm{ttr}_\Sigma^{\iota_t}(t) \rrbracket_{\alpha_\iota}^{s_\Sigma}(u)$.

- t is in $F_0^{\cdot} \setminus F_0$. Note that $\mathrm{ttr}_\Sigma^{\iota_t}(t) = \iota_t(t)$, and so $t = \iota_t^{-1}(\mathrm{ttr}_\Sigma^{\iota_t}(t))$. Thus $\llbracket t \rrbracket_\alpha^s = \llbracket \iota_t^{-1}(\mathrm{ttr}_\Sigma^{\iota_t}(t)) \rrbracket_\alpha^s = \mathrm{ttr}_\Sigma^{\iota_t}(t)\alpha_\iota = \llbracket \mathrm{ttr}_\Sigma^{\iota_t}(t) \rrbracket_{\alpha_\iota}^{s_\Sigma}$.

- t is in Ξ_t. Note that $\mathrm{ttr}_\Sigma^{\iota_t}(t) = t$. So $\llbracket t \rrbracket_\alpha^s = t\alpha = t\alpha_\iota = \llbracket \mathrm{ttr}_\Sigma^{\iota_t}(t) \rrbracket_{\alpha_\iota}^{s_\Sigma}$.

- t is $f(t_1, \ldots, t_n)$ with f in F_n. Note that $\mathrm{ttr}_\Sigma^{\iota_t}(t) = f(\mathrm{ttr}_\Sigma^{\iota_t}(t_1), \ldots, \mathrm{ttr}_\Sigma^{\iota_t}(t_n))$. Then $\llbracket t \rrbracket_\alpha^s(u) = [f]_{\omega_s(u)}^s(\llbracket t_1 \rrbracket_\alpha^s(u), \ldots, \llbracket t_n \rrbracket_\alpha^s(u)) = [f]_{\omega_{s_\Sigma}(u)}^{s_\Sigma}(\llbracket \mathrm{ttr}_\Sigma^{\iota_t}(t_1) \rrbracket_{\alpha_\iota}^{s_\Sigma}(u), \ldots, \llbracket \mathrm{ttr}_\Sigma^{\iota_t}(t_n) \rrbracket_{\alpha_\iota}^{s_\Sigma}(u)) = \llbracket f(\mathrm{ttr}_\Sigma^{\iota_t}(t_1), \ldots, \mathrm{ttr}_\Sigma^{\iota_t}(t_n)) \rrbracket_{\alpha_\iota}^{s_\Sigma}(u)$.

- t is $f(t_1, \ldots, t_n)$ with f in $F_n^{\cdot} \setminus F_n$. Note that $\mathrm{ttr}_\Sigma^{\iota_t}(t) = \iota_t(t)$, and so $t = \iota_t^{-1}(\mathrm{ttr}_\Sigma^{\iota_t}(t))$. Thus $\llbracket t \rrbracket_\alpha^s = \llbracket \iota_t^{-1}(\mathrm{ttr}_\Sigma^{\iota_t}(t)) \rrbracket_\alpha^s = \mathrm{ttr}_\Sigma^{\iota_t}(t)\alpha_\iota = \llbracket \mathrm{ttr}_\Sigma^{\iota_t}(t) \rrbracket_{\alpha_\iota}^{s_\Sigma}$.

We show 3. by induction on the structure of the schema formula φ:

- φ is $p(t_1, \ldots, t_n)$ with p in P_n. Note that $\mathrm{ftr}_\Sigma^{\iota_f}(\varphi) = p(\mathrm{ttr}_\Sigma^{\iota_t}(t_1), \ldots, \mathrm{ttr}_\Sigma^{\iota_t}(t_n))$. Then $\llbracket \varphi \rrbracket_\alpha^s(u) = [p]_{\omega_s(u)}^s(\llbracket t_1 \rrbracket_\alpha^s(u), \ldots, \llbracket t_n \rrbracket_\alpha^s(u)) = [p]_{\omega_{s_\Sigma}(u)}^{s_\Sigma}(\llbracket \mathrm{ttr}_\Sigma^{\iota_t}(t_1) \rrbracket_{\alpha_\iota}^{s_\Sigma}(u), \ldots, \llbracket \mathrm{ttr}_\Sigma^{\iota_t}(t_n) \rrbracket_{\alpha_\iota}^{s_\Sigma}(u)) = \llbracket \mathrm{ftr}_\Sigma^{\iota_f}(\varphi) \rrbracket_{\alpha_\iota}^{s_\Sigma}(u)$.

- φ is $p(t_1, \ldots, t_n)$ with p in $P_n^{\cdot} \setminus P_n$. Note that $\mathrm{ftr}_\Sigma^{\iota_f}(\varphi) = \iota_f(\varphi)$, and so $\varphi = \iota_f^{-1}(\mathrm{ftr}_\Sigma^{\iota_f}(\varphi))$. Thus $\llbracket \varphi \rrbracket_\alpha^s = \llbracket \iota_f^{-1}(\mathrm{ftr}_\Sigma^{\iota_f}(\varphi)) \rrbracket_\alpha^s = \mathrm{ftr}_\Sigma^{\iota_f}(\varphi)\alpha_\iota = \llbracket \mathrm{ftr}_\Sigma^{\iota_f}(\varphi) \rrbracket_{\alpha_\iota}^{s_\Sigma}$.

- φ is $t_1 = t_2$. Note that $\mathrm{ftr}_\Sigma^{\iota_f}(\varphi)$ is $\mathrm{ttr}_\Sigma^{\iota_t}(t_1) = \mathrm{ttr}_\Sigma^{\iota_t}(t_n)$. Then $\llbracket \varphi \rrbracket_\alpha^s(u) = 1$ iff $\llbracket t_1 \rrbracket_\alpha^s(u) = \llbracket t_2 \rrbracket_\alpha^s(u)$ iff $\llbracket \mathrm{ttr}_\Sigma^{\iota_t}(t_1) \rrbracket_{\alpha_\iota}^{s_\Sigma}(u) = \llbracket \mathrm{ttr}_\Sigma^{\iota_t}(t_2) \rrbracket_{\alpha_\iota}^{s_\Sigma}(u)$ iff $\llbracket \mathrm{ftr}_\Sigma^{\iota_f}(\varphi) \rrbracket_{\alpha_\iota}^{s_\Sigma}(u) = 1$.

- φ is in C_0. Note that $\mathrm{ftr}_\Sigma^{\iota_f}(\varphi) = \varphi$. Then $\llbracket \varphi \rrbracket_\alpha^s(u) = [\varphi]_{\omega_s(u)\alpha_s(u)}^s(u) = [\varphi]_{\omega_{s_\Sigma}(u)\alpha_{s_\Sigma}(u)}^{s_\Sigma}(u) =$

$[\mathrm{ftr}_{\Sigma}^{\iota f}(\varphi)]^{s_\Sigma}_{\omega_{s_\Sigma}(u)\alpha_{s_\Sigma}(u)}(u) = \llbracket \mathrm{ftr}_{\Sigma}^{\iota f}(\varphi)\rrbracket^{s_\Sigma}_{\alpha_\iota}(u)$.

- φ is in $C_0^{\cdot} \setminus C_0$. Note that $\mathrm{ftr}_{\Sigma}^{\iota f}(\varphi) = \iota_f(\varphi)$, and so $\varphi = \iota_f^{-1}(\mathrm{ftr}_{\Sigma}^{\iota f}(\varphi))$. Therefore $\llbracket\varphi\rrbracket^s_\alpha = \llbracket\iota_f^{-1}(\mathrm{ftr}_{\Sigma}^{\iota f}(\varphi))\rrbracket^s_\alpha = \mathrm{ftr}_{\Sigma}^{\iota f}(\varphi)\alpha_\iota = \llbracket\mathrm{ftr}_{\Sigma}^{\iota f}(\varphi)\rrbracket^{s_\Sigma}_{\alpha_\iota}$.
- φ is in Ξ_f. Note that $\mathrm{ftr}_{\Sigma}^{\iota f}(\varphi) = \varphi$. Then $\llbracket\varphi\rrbracket^s_\alpha = \varphi\alpha = \varphi\alpha_\iota = \llbracket\mathrm{ftr}_{\Sigma}^{\iota f}(\varphi)\rrbracket^{s_\Sigma}_{\alpha_\iota}$.
- φ is $c(\varphi_1, \ldots, \varphi_n)$ with c in C_n. Note that $\mathrm{ftr}_{\Sigma}^{\iota f}(\varphi) = c(\mathrm{ftr}_{\Sigma}^{\iota f}(\varphi_1), \ldots, \mathrm{ftr}_{\Sigma}^{\iota f}(\varphi_n))$. Then $\llbracket\varphi\rrbracket^s_\alpha(u) = [c]^s_{\omega_s(u)\alpha_s(u)}(\llbracket\varphi_1\rrbracket^s_\alpha \cap U^s_{\omega_s(u)\alpha_s(u)}, \ldots, \llbracket\varphi_n\rrbracket^s_\alpha \cap U^s_{\omega_s(u)\alpha_s(u)})(u) = [c]^{s_\Sigma}_{\omega_{s_\Sigma}(u)\alpha_{s_\Sigma}(u)}(\llbracket\mathrm{ftr}_{\Sigma}^{\iota f}(\varphi_1)\rrbracket^{s_\Sigma}_{\alpha_\iota} \cap U^{s_\Sigma}_{\omega_\Sigma(u)\alpha_\Sigma(u)}, \ldots, \llbracket\mathrm{ftr}_{\Sigma}^{\iota f}(\varphi_n)\rrbracket^{s_\Sigma}_{\alpha_\iota} \cap U^{s_\Sigma}_{\omega_{s_\Sigma}(u)\alpha_{s_\Sigma}(u)})(u) = \llbracket\mathrm{ftr}_{\Sigma}^{\iota f}(\varphi)\rrbracket^{s_\Sigma}_{\alpha_\iota}(u)$.
- φ is $c(\varphi_1, \ldots, \varphi_n)$ with c in $C_n^{\cdot} \setminus C_n$. Note that $\mathrm{ftr}_{\Sigma}^{\iota f}(\varphi) = \iota_f(\varphi)$, and so $\varphi = \iota_f^{-1}(\mathrm{ftr}_{\Sigma}^{\iota f}(\varphi))$. Thus $\llbracket\varphi\rrbracket^s_\alpha = \llbracket\iota_f^{-1}(\mathrm{ftr}_{\Sigma}^{\iota f}(\varphi))\rrbracket^s_\alpha = \mathrm{ftr}_{\Sigma}^{\iota f}(\varphi)\alpha_\iota = \llbracket\mathrm{ftr}_{\Sigma}^{\iota f}(\varphi)\rrbracket^{s_\Sigma}_{\alpha_\iota}$.
- φ is $q_x(\varphi_1, \ldots, \varphi_n)$ or $o(\varphi_1, \ldots, \varphi_n)$. These cases are similar to the preceding ones so we omit them.

We show 4. by case analysis on the labelled schema formula η:

- η is $v{:}\varphi$. Note that $\mathrm{tr}_{\Sigma}^{\iota}(\eta) = \mathrm{ltr}_{\Sigma}^{\iota l}(v){:}\mathrm{ftr}_{\Sigma}^{\iota f}(\varphi)$. Then $s, \alpha \Vdash \eta$ iff $\llbracket\varphi\rrbracket^s_\alpha(\llbracket v\rrbracket^s_\alpha) = 1$ iff (by items 1. and 3.) $\llbracket\mathrm{ftr}_{\Sigma}^{\iota f}(\varphi)\rrbracket^{s_\Sigma}_{\alpha_\iota}(\llbracket\mathrm{ltr}_{\Sigma}^{\iota l}(v)\rrbracket^{s_\Sigma}_{\alpha_\iota}) = 1$ iff $s_\Sigma, \alpha_\iota \Vdash \mathrm{ltr}_{\Sigma}^{\iota l}(v){:}\ \mathrm{ftr}_{\Sigma}^{\iota f}(\varphi)$ iff $s_\Sigma, \alpha_\iota \Vdash \mathrm{tr}_{\Sigma}^{\iota}(\eta)$.
- η is $v_1{:}t_1 =_g v_2{:}t_2$. Note that $\mathrm{tr}_{\Sigma}^{\iota}(\eta)$ is $\mathrm{ltr}_{\Sigma}^{\iota l}(v_1){:}\mathrm{ttr}_{\Sigma}^{\iota t}(t_1) =_g \mathrm{ltr}_{\Sigma}^{\iota l}(v_2){:}\ \mathrm{ttr}_{\Sigma}^{\iota t}(t_2)$. Then $s, \alpha \Vdash \eta$ iff $\llbracket t_1\rrbracket^s_\alpha(\llbracket v_1\rrbracket^s_\alpha) = \llbracket t_2\rrbracket^s_\alpha(\llbracket v_2\rrbracket^s_\alpha)$ iff (by items 1. and 2.) $\llbracket\mathrm{ttr}_{\Sigma}^{\iota t}(t_1)\rrbracket^{s_\Sigma}_{\alpha_\iota}(\llbracket\mathrm{ltr}_{\Sigma}^{\iota l}(v_1)\rrbracket^{s_\Sigma}_{\alpha_\iota}) = \llbracket\mathrm{ttr}_{\Sigma}^{\iota t}(t_2)\rrbracket^{s_\Sigma}_{\alpha_\iota}(\llbracket\mathrm{ltr}_{\Sigma}^{\iota l}(v_2)\rrbracket^{s_\Sigma}_{\alpha_\iota})$ iff $s_\Sigma, \alpha_\iota \Vdash \mathrm{ltr}_{\Sigma}^{\iota l}(v_1){:}\mathrm{ttr}_{\Sigma}^{\iota t}(t_1) =_g \mathrm{ltr}_{\Sigma}^{\iota l}(v_2){:}\ \mathrm{ttr}_{\Sigma}^{\iota t}(t_2)$ iff $s_\Sigma, \alpha_\iota \Vdash \mathrm{tr}_{\Sigma}^{\iota}(\eta)$.
- η is $r\, v_1 \ldots v_n$. Note that $\mathrm{tr}_{\Sigma}^{\iota}(\eta) = r\, \mathrm{ltr}_{\Sigma}^{\iota l}(v_1) \ldots \mathrm{ltr}_{\Sigma}^{\iota l}(v_n)$. Therefore we have $s, \alpha \Vdash \eta$ iff $\langle\llbracket v_1\rrbracket^s_\alpha, \ldots, \llbracket v_n\rrbracket^s_\alpha\rangle \in [r]^s$ iff (r is in $S_{\Sigma n}$ and by item 1.) $\langle\llbracket\mathrm{ltr}_{\Sigma}^{\iota l}(v_1)\rrbracket^{s_\Sigma}_{\alpha_\iota}, \ldots, \llbracket\mathrm{ltr}_{\Sigma}^{\iota l}(v_n)\rrbracket^{s_\Sigma}_{\alpha_\iota}\rangle \in [r]^{s_\Sigma}$ iff $s_\Sigma, \alpha_\iota \Vdash r\, \mathrm{ltr}_{\Sigma}^{\iota l}(v_1) \ldots\ \mathrm{ltr}_{\Sigma}^{\iota l}(v_n)$ iff $s_\Sigma, \alpha_\iota \Vdash \mathrm{tr}_{\Sigma}^{\iota}(\eta)$.
- η is $v_1 \equiv_\omega v_2$. Note that $\mathrm{tr}_{\Sigma}^{\iota}(\eta) = \mathrm{ltr}_{\Sigma}^{\iota l}(v_1) \equiv_\omega \mathrm{ltr}_{\Sigma}^{\iota l}(v_2)$. Therefore we have $s, \alpha \Vdash \eta$ iff $\omega_s(\llbracket v_1\rrbracket^s_\alpha) = \omega_s(\llbracket v_n\rrbracket^s_\alpha)$ iff (by item 1.) $\omega_{s_\Sigma}(\llbracket\mathrm{ltr}_{\Sigma}^{\iota l}(v_1)\rrbracket^{s_\Sigma}_{\alpha_\iota}) = \omega_{s_\Sigma}(\llbracket\mathrm{ltr}_{\Sigma}^{\iota l}(v_2)\rrbracket^{s_\Sigma}_{\alpha_\iota})$ iff $s_\Sigma, \alpha_\iota \Vdash \mathrm{ltr}_{\Sigma}^{\iota l}(v_1) \equiv_\omega\ \mathrm{ltr}_{\Sigma}^{\iota l}(v_2)$ iff $s_\Sigma, \alpha_\iota \Vdash \mathrm{tr}_{\Sigma}^{\iota}(\eta)$.
- η is $v_1 \equiv_\alpha v_2$. We omit the proof for this case because it is similar to the preceding one. *QED*

The next lemma was used in the proof of Lemma 6.3.6, that establishes the preservation by fibring of fullness, for proving that if a structure satisfies a rule then its reduct over the signature of the rule satisfies also that rule. Note that it is important that we are in the context of a suitable pair of lfob logic systems.

Lemma 6.6.3 For any suitable pair of lfob logic systems $\mathcal{L}_1$ and $\mathcal{L}_2$, $\Sigma_1 \cup \Sigma_2$ structure s, i equal to 1 or 2, and r in $\mathcal{L}_i$, we have that s_{Σ_i} satisfies r whenever s satisfies r.

Proof Let E be a set of label schema variables, i be equal to 1 or 2, and suppose r, in $\mathcal{L}_i$, is the rule $\langle\{\langle\vartheta_1, \Psi_1, \eta_1\rangle, \ldots, \langle\vartheta_k, \Psi_k, \eta_k\rangle\}, \eta, P_f, P_d\rangle$, σ is a Σ_i schema substitution within L_E such that $(\pi_{\Sigma_i}\sigma_{\mathrm{nl}})$ is 1 for each π in P_f, and $\pi_{\Sigma_i;E}(\sigma)$ is 1 for each π in P_d, and α_i is a schema assignment over s_{Σ_i} and E. Suppose also that, for any $j = 1, \ldots, k$ and any $\alpha_i^{\cdot}$ over s_{Σ_i} and E $\Upsilon_j\sigma$ co-equivalent to α_i, $s_{\Sigma_i}, \alpha_i^{\cdot} \Vdash \Psi_j\sigma$ implies $s_{\Sigma_i}, \alpha_i^{\cdot} \Vdash \eta_j\sigma$.

Additionally, let E_l, E_t and E_f be the finite sets of all label, term, and formula, respectively, schema variables used in $\eta\sigma$ and for each $j = 1, \ldots, k$ in $\Psi_j\sigma$ and in $\eta_j\sigma$, ι_l, ι_t and ι_f bijections from $T_{\mathrm{lab},E}(\Sigma_1 \cup \Sigma_2, \Xi_l)$ to $\Xi_l \setminus E_l$, from $L(\Sigma_1 \cup \Sigma_2, \Xi_t)$ to $\Xi_t \setminus E_t$, and from $L(\Sigma_1 \cup \Sigma_2, \Xi_t, \Xi_f)$ to $\Xi_f \setminus E_f$, respectively.

Let α be a $\Sigma_1 \cup \Sigma_2$ schema assignment over s and E equal to α_i. Note that, since $\mathcal{L}_1$ and $\mathcal{L}_2$ is a pair of suitable lfob logic systems, see Definition 6.1.2, we have that $(\pi_{\Sigma_1 \cup \Sigma_2}\sigma_{\mathrm{nl}})$ is 1 for each π in P_f, and $\pi_{\Sigma_1 \cup \Sigma_2;E}(\sigma)$ is 1 for each π in P_d. We now show that for any $j = 1, \ldots, k$ and any $\alpha^{\cdot}$ over s and E $\Upsilon_j\sigma$ co-equivalent to α, $s, \alpha^{\cdot} \Vdash \Psi_j\sigma$ implies $s, \alpha^{\cdot} \Vdash \eta_j\sigma$. Let $\alpha^{\cdot}$ be a schema assignment over s and E $\Upsilon_j\sigma$ co-equivalent to α such that $s, \alpha^{\cdot} \Vdash \Psi_j\sigma$.

Let $\alpha^{\cdot}_{\iota i}$ be a schema assignment over s_{Σ_i} and E such that $\nu\alpha^{\cdot}_{\iota i} = \nu\alpha^{\cdot}$, $\theta\alpha^{\cdot}_{\iota i} = \theta\alpha^{\cdot}$ and $\xi\alpha^{\cdot}_{\iota i} = \xi\alpha^{\cdot}$ for any label, term and formula schema variables ν, θ and ξ in $E_l \cup E$, E_t or E_f, respectively, and $\nu\alpha^{\cdot}_{\iota i} = \llbracket \iota_t^{-1}(\nu) \rrbracket^s_\alpha$, $\theta\alpha^{\cdot}_{\iota i} = \llbracket \iota_t^{-1}(\theta) \rrbracket^s_\alpha$ and $\xi\alpha^{\cdot}_{\iota i} = \llbracket \iota_f^{-1}(\xi) \rrbracket^s_\alpha$ for any label, term and formula schema variables ν, θ and ξ in $(\Xi_l \cup E) \setminus E_l$, $\Xi_t \setminus E_t$ or $\Xi_f \setminus E_f$. Then, by Proposition 6.6.2, $s_{\Sigma_i}, \alpha^{\cdot}_{\iota i} \Vdash \mathrm{tr}^{\iota}_{\Sigma}(\Psi_j\sigma)$ which is equivalent to $s_{\Sigma_i}, \alpha^{\cdot}_{\iota i} \Vdash \Psi_j\sigma$, since by Definition 6.6.1, $\mathrm{tr}^{\iota}_{\Sigma_i}(\psi) = \psi$ for any ψ in $\Psi_j\sigma$ because $\Psi_j\sigma$ is contained in the language of $\mathcal{L}_i$.

Let $\alpha^{\cdot}_i$ be a schema assignment over s_{Σ_i} and E $\Upsilon_j\sigma$ co-equivalent to α_i with $\nu\alpha^{\cdot}_i = \nu\alpha^{\cdot}_{\iota i}$. Note that, for any label, term and formula schema variables, ν, θ and ξ, in E_l, E_t and E_f, respectively, we have that $\nu\alpha^{\cdot}_i = \nu\alpha^{\cdot}_{\iota i}$, $\theta\alpha^{\cdot}_i = \theta\alpha^{\cdot}_{\iota i}$ and $\xi\alpha^{\cdot}_i = \xi\alpha^{\cdot}_{\iota i}$. To see this, let ν be a label schema variable in E_l but not in $\Upsilon_j\sigma$, (the case where ν is in $\Upsilon_j\sigma$ follows immediately by definition of $\alpha^{\cdot}_i$), θ in E_t and ξ in E_f. Then $\nu\alpha^{\cdot}_i = \nu\alpha_i = \nu\alpha = \nu\alpha^{\cdot} = \nu\alpha^{\cdot}_{\iota i}$, $\theta\alpha^{\cdot}_i = \theta\alpha_i = \theta\alpha = \theta\alpha^{\cdot} = \theta\alpha^{\cdot}_{\iota i}$ and $\xi\alpha^{\cdot}_i = \xi\alpha_i = \xi\alpha = \xi\alpha^{\cdot} = \xi\alpha^{\cdot}_{\iota i}$, as we wanted to show.

So, by Proposition 2.4.4, we have that $s_{\Sigma_i}, \alpha^{\cdot}_i \Vdash \Psi_j\sigma$ and so $s_{\Sigma_i}, \alpha^{\cdot}_i \Vdash \eta_j\sigma$. Thus, again by Proposition 2.4.4, we have that $s_{\Sigma_i}, \alpha^{\cdot}_{\iota i} \Vdash \eta_j\sigma$, and so $s_{\Sigma_i}, \alpha^{\cdot}_{\iota i} \Vdash \mathrm{tr}^{\iota}_{\Sigma_i}(\eta_j\sigma)$. Hence, by Proposition 6.6.2, $s, \alpha^{\cdot} \Vdash \eta_j\sigma$ as we wanted to show. Therefore, since s satisfies r we have that $s, \alpha \Vdash \eta\sigma$.

Let $\alpha_{\iota i}$ be a schema assignment over s_{Σ_i} and E such that $\nu\alpha_{\iota i} = \nu\alpha (= \nu\alpha_i)$, $\theta\alpha_{\iota i} = \theta\alpha (= \theta\alpha_i)$ and $\xi\alpha_{\iota i} = \xi\alpha (= \xi\alpha_i)$ for any label, term and formula schema variables ν, θ and ξ in $E_l \cup E$, E_t or E_f, respectively, and $\nu\alpha_{\iota i} = \llbracket \iota_t^{-1}(\nu) \rrbracket^s_\alpha$, $\theta\alpha_{\iota i} = \llbracket \iota_t^{-1}(\theta) \rrbracket^s_\alpha$ and $\xi\alpha_{\iota i} = \llbracket \iota_f^{-1}(\xi) \rrbracket^s_\alpha$ for any label, term and formula schema variables ν, θ and ξ in $(\Xi_l \cup E) \setminus E_l$, $\Xi_t \setminus E_t$ or $\Xi_f \setminus E_f$. Then, by Proposition 6.6.2, $s_{\Sigma_i}, \alpha_{\iota i} \Vdash \mathrm{tr}^{\iota}_{\Sigma}(\eta\sigma)$ which is equivalent to $s_{\Sigma_i}, \alpha_{\iota i} \Vdash \eta\sigma$, since by Definition 6.6.1, $\mathrm{tr}_{\Sigma_i}(\eta\sigma) = \eta\sigma$ because $\eta\sigma$ is contained in the language of $\mathcal{L}_i$.

Henceforth $s_{\Sigma_i}, \alpha_i \Vdash \eta\sigma$ by Proposition 2.4.4 since α_i and $\alpha_{\iota i}$ coincide in all the schema variables present in $\eta\sigma$. So s_{Σ_i} satisfies rule r as we wanted to show. *QED*

Finally we present our last auxiliary lemma, which establishes the preservation of connectedness by fibring suitable clfob logic systems. The proof of the lemma is omitted since it is straightforward given that the pair of clfob logic systems is suitable. Recall Definition 4.1.2 of a connected lfob logic system and Definition 6.1.2 of suitability of a pair of clfob logic systems.

Lemma 6.6.4 The fibring of a suitable pair of connected lfob logic systems is also connected.

Chapter 7

Conclusion

7.1 Final remarks

In this dissertation we studied the fibring of labelled first-order based logic systems. We started by developing the labelled deduction and labelled semantics for first-order based logics, taking into account fibring, extending, after substantial re-formulation, our previous work on labelled deduction for propositional based logics, and getting inspiration from a similar work for the semantics of non-labelled first-order based logics, [61, 74]. This work was materialized in Chapter 2. Then we studied sufficient conditions for completeness that would be preserved under suitable requirements by fibring, and that were sufficient general to be satisfied by a large class of lfob logic systems. This investigation gave rise to Chapter 3 and Chapter 4. After that, our goal was the development of labelled presentations for interesting logics, involving first-order, relevance and modal logic, satisfying the conditions identified during the study of completeness. That research originated Chapter 5. Finally we studied the fibring of labelled first-order based logic, and we investigated the preservation of properties like soundness, completeness, entailment and consequence. We dedicated Chapter 6 to these tasks. During this process several achievements were obtained, which were summarized in Section 1.5, but also some questions and conjectures remained opened, and some not expected characteristics were found and analyzed. We summarize them in the next paragraphs.

We start by mentioning the fact that, without assuming any kind of negation in a connected lfob logic system, we were not able to construct in general an appropriate set extending a consistent one, in the context of that clfob logic system, satisfying the appropriateness condition for the rules of natural deduction for disjunction and implication. Although with this limitation, we identified the class of connected lfob logic systems with locality and proved that, if the system do not have disjunction and implication, we could construct for any consistent set an appropriate set extending it. Note that, if we assume that a classical negation, local or non-local, is present in the connected lfob logic system, then for any consistent set we can construct an appropriate set extending it. So, following this line of investigation it would be very interesting:

- to study, in the context of clfob logic systems in which we assume only certain weak forms of negation, the extension of any consistent set to an appropriate one, and investigate its effects on the rules the system may have,

- to conceive a construction different from the one we developed in Proposition 4.4.7, and see if with this new construction it is possible to construct for any consistent set an appropriate set extending it, satisfying the appropriateness condition for the rules of disjunction and implication,

- to develop new rules for disjunction and implication, such that the appropriate set extending a consistent set, according to our construction, satisfies the appropriateness conditions for disjunction and implication. As we conjectured, in the analysis we did to this fact in Remark 4.4.8, we would have to introduce a generalized disjunction connective, which in our opinion, would make our systems closer to hybrid systems than to labelled deduction systems.

Another characteristic are that the lfob logic systems we propose for first-order based logics are labelled presentations with respect to only non-schematic formulae, and not to schema formulae. Recall the discussion we made about this fact in Section 5.6. Finally a last note to recall that the consequence and the entailment considered in this dissertation are local. It would be very interesting to study this subject for a global consequence and entailment.

7.2 Future work

There are several promising and interesting lines of future work. One such line is the extension of our work on modulated fibring [75] to first-order based logics endowed with labelled deduction. Modulated fibring is a refinement of the fibring technique that avoids, in some well identified cases, the collapse of the two logics being combined.

Another promising line of research, in the context of the fibring of first-order based logic systems endowed with labelled deduction, is the study of the preservation by fibring of properties like decidability, interpolation, normalization and weak completeness.

Heterogeneous fibring is a challenging line of research. It could be investigated from a semantics point of view or from a deduction point of view. It would definitely be very interesting, for instance, the study of the fibring of logics endowed with labelled deduction, with logics endowed with Hilbert style deduction systems. Even for propositional based logics. Once that were settled than it could be undertaken for first-order based ones.

Another important line of future work is the application of the results obtained in this dissertation. Since labelled deduction and fibring has been widely used for instance to develop specification and linguistic logics [15, 16, 57, 40, 23], and given the importance and application of first-order based logics and of its fragments [49, 39, 78], we expect that our results be used, for instance, in the specification and verification of security protocols and in the specification and reasoning of linguistic properties.

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