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Algebra in a Topos

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Contents

1 Introduction **2**

2 Toposes **4**
2.1 Subobject classifiers 4
2.2 Cartesian closed categories 5
2.3 Elementary toposes 5
2.4 Toposes of presheaves and sheaves 7

3 Mathematics in a topos **17**
3.1 Types 17
3.2 Internal logic 18
3.3 Relations 19
3.4 Natural numbers 20
3.5 Real numbers 21

4 Algebra in Sh(X) **23**
4.1 Internal logic revisited 23
4.2 Integers, rationals, reals 26
4.3 Rings and modules 29
4.4 Finitely generated modules 32
4.5 Free modules 33
4.6 Projective modules 34
4.7 Kaplansky vs. Swan 35

5 Conclusion **38**

Bibliography **40**

Chapter 1

Introduction

An interesting aspect of topos theory is that it establishes a bridge between two seemingly distinct mathematical subjects: on one hand, topology and algebraic geometry, and, on the other, logic and set theory.

In this work we try to play the game of passing mathematical ideas from one side to the other by using this idea. What we do is to think of a topos as a "generalized universe of sets" and use its language to define algebraic structures that we already know.

The ideas used in the present work can be traced back to Lawvere [Law71] and Tierney [Tie72] and in particular we draw much of our inspiration from [Mul73].

In particular, we choose the topos of sheaves, on a topological space X, and try to figure out how to deal with aspects algebra in it. The passage from Set to $\mathrm{Sh}(X)$ is actually a generalization, since, if X is a one point space, these categories are actually the same.

Simply passing mathematical structures such as relations, rings, fields and modules to $\mathrm{Sh}(X)$ is interesting in itself but rather incomplete. So we will concern ourselves with what "rules" we must obey so that propositions and theorems may be transfered as well. One such rule is to use the subobject classifier "logic", which in general is intuitionistic.

Let us make an overview of the following chapters:

- Toposes - Here we make a small introduction to toposes. We define their basic properties, e.g., existence of small limits and colimits, subobject classifiers and exponentials. We also provide some basic examples of toposes. Finally we prove that any category of sheaves or presheaves is a topos.

- Mathematics in a topos - In this chapter we explore how a topos can be used in defining mathematical objects. We begin by defining an abstract formal type language. Next we see how to find the internal language of a given topos and how we can interpret it. We use this language in order to define mathematical objects such as relations, functions, natural numbers, integers, reals. All this work is done for any topos at this stage. If one is thinking of Set then the introduced notions are precisely those we are used to. Naturally one can think of interpreting these concepts in another topos. This leads us to the next chapter.

- Algebra in $\mathrm{Sh}(X)$ - In this chapter we take a look at what the previous definitions turn out to be in $\mathrm{Sh}(X)$. We start by giving a more intuitive interpretation of formulae, i.e., we try to expose the structure of the logic we are dealing with. It will be shown that it is actually intuitionistic logic. Next we deal with the interpretation of relations, natural numbers, integers, rational and real numbers. Here we notice that the classical constructions of the reals do not yield the same result from an intuitionistic point of view. After these first constructions, we introduce a few concepts like rings and modules, and head towards interpreting in $\mathrm{Sh}(X)$ the well known theorem of Kaplansky [Ros94] that states that any finitely generated projective module over a commutative local ring is free. We shall see, that this can be identified certain sense with the theorem of Swan that establishes an equivalence between the category of finitely generated projective modules over $C^{\mathbb{R}}(X)$ and that of locally finite vector bundles on X.

- Conclusion - finally we draw a few conclusions about this work.

The main goal is to provide, by means of very suggestive example, the geometrical significance involved in the passage from classical to intuitionistic logic.

Chapter 2

Toposes

2.1 Subobject classifiers

Suppose that S is a subset of a given set X, i.e. $S \subseteq X$. We can express this fact by means of the characteristic function $\psi_S : X \to \{0, 1\}$ defined as follows:

$$\psi_S(x) = \begin{cases} 0, & x \in S \\ 1, & x \notin S \end{cases}$$

Here we take the values of ψ_S in the typical 2-point set {0,1}, which we are used to thinking about as being the set of "truth values", where "true" is identified with 0.

In *Set*, the simplest non-trivial subset is $\{0\} \subseteq \{0, 1\}$. The inclusion function associated will be named $true : \{0\} \to \{0, 1\}$ and is defined as $true(0) = 0$. It turns out that any arbitrary subset $S \subseteq X$ can be mapped into this simple subset by ψ_S in the following pullback square

$$\begin{array}{ccc} S & \longrightarrow & \{0\} \\ \downarrow & & \downarrow{\scriptstyle true} \\ X & \xrightarrow{\psi_S} & \{0,1\} \end{array}$$

The *true* monomorphism is a "subobject classifier" for the category of sets. The fact that a category has a subobject classifier can be useful for various applications. We are interested in using it for logic, as a "set" of truth values.

In general, given any category, a subobject classifier can be defined as follows:

Definition 2.1.1 In a category C with finite limits, a *subobject classifier* is a monic, $true : 1 \to \Omega$, such that, for every monic $S \to X$ in C, there is a unique arrow ψ_S which, with the given monic, forms a pullback square

$$\begin{array}{ccc} S & \longrightarrow & 1 \\ \downarrow & & \downarrow{\scriptstyle true} \\ X & \xrightarrow{\psi_S} & \Omega \end{array}$$

In the pullback square above, the top horizontal arrow is the unique map to the terminal object 1 and ψ_S acts as the "characteristic function".

2.2 Cartesian closed categories

Definition 2.2.1 A category C is *cartesian closed* if C has all finite products and each of the following functors have a right adjoint:

- $C \to 1$, given as $c \mapsto 0$, where 1 is the category with unique object 0;
- $C \to C \times C$, given as $c \mapsto \langle c, c \rangle$;
- $C \xrightarrow{\_ \times b} C$, given as $a \mapsto a \times b$, for every object b of C.

These adjoints can be written as follows, and have the following interpretation:

- $0 \mapsto t$, which specifies a terminal object in C;
- $\langle a, b \rangle \mapsto a \times b$, which specifies a product for any pair of objects a, b;
- $c \mapsto c^b$, which leads to the notion of exponentials in a category C.

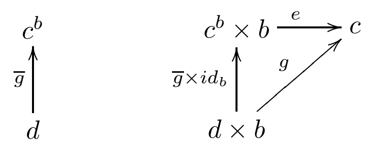
In particular the latter adjunction is the following bijection,

$$\hom(a \times b, c) \cong \hom(a, c^b)$$

natural in a and c, for any given b in C. It can be proved that specifying this adjunction amounts to specifying, for each b and c, an arrow e as follows:

$$e : c^b \times b \to c$$

which is natural in c and universal from $\_ \times b$ to c, i.e. such that, for any object d in C and every $g : d \times b \to c$, there is a unique arrow $\overline{g} : d \to c^b$ such that the following diagram commutes:



2.3 Elementary toposes

Definition 2.3.1 An *elementary topos* is a category E with the following properties:

- E has all the finite limits and colimits;
- E has a subobject classifier;
- E is cartesian closed.

Let us look at some examples.

Example 2.3.2 *Set*, the category of all (small) sets and functions between them, is a topos.

Proposition 2.3.3 *Set has all the finite limits*

Proof.

- The terminal object is the one point set.
- The product is the cartesian product.
- The equalizer is computed by making a simple restriction of the morphisms' domain.

∎

Proposition 2.3.4 *Set has all the finite colimits*

Proof.

- The initial object is the empty set.
- The coproduct is the disjoint union of sets.
- The coequalizer is the quotient set of the equivalence relation $f(x) = g(x)$, for all x.

∎

Proposition 2.3.5 *Set has a subobject classifier.*

Proof. This was seen in section 2.1. ∎

Proposition 2.3.6 *Set has exponentials.*

Proof.
Given two sets A and B we define $A^B = \hom_{Set}(B, A)$.
The evaluation $e : A^B \times B \to A$ is precisely an evaluation, i.e., $e(\langle f, b \rangle) = f(b)$.
∎

The following categories are also toposes:

- $Set \times Set$, the category of all pairs of sets, with morphisms pairs of functions;
- $Set^{\downarrow}$, the category whose objects are all functions between sets, and whose morphisms are commutative squares.

We will now look at two main examples of toposes.

2.4 Toposes of presheaves and sheaves

This section aims at establishing that the category of sheaves $\mathrm{Sh}(X)$ on a topological space X is a topos.

In order to achieve our goal, we will prove that the category of presheaves is also a topos, since sheaves are a particular case of presheaves.

A sheaf is a way of defining a class of functions on X that have some "good" properties, such as being continuous. The notion of sheaf refers to a class of functions that can be defined in on open set U of X, by gluing together a family of matching functions defined on an open covering of U.

We will show that the category $\mathrm{Sh}(X)$ of all sheaves of sets on a given space X is a topos by providing explicit descriptions of the categorical properties required.

For the remainder of this work, we will consider X an arbitrary, but fixed, topological space.

We begin by defining a presheaf, which gives a formal way of talking about a family of "good functions" and of restricting a function. We will treat the topology $\mathcal{O}(X)$ as a category, where the open sets are the objects and inclusions of sets are the morphisms.

Definition 2.4.1 A presheaf of sets is just a functor $P : \mathcal{O}(X)^{\mathrm{OP}} \to Set$.

We can also take a look at a very simple class of presheaves.

Example 2.4.2 Given an open $U \subseteq X$ we can define the following presheaf:

$$F_U(V) = \begin{cases} \{*\}, & V \subseteq U \\ \emptyset, & V \nsubseteq U \end{cases}$$

Each inclusion $V \subseteq U$ of open sets in X determines an obvious function

$$P(V \subseteq U) : P(U) \to P(V)$$

If $t \in P(U)$, one can write $t|_V$ as being short for $P(V \subseteq V)(t)$ and speak of t restricted to V.

Let us look at another example that makes this clear.

Example 2.4.3 The following functor B is the pre-sheaf of bounded real functions.

- $B(U) = \{f | f : U \to \mathbb{R} \text{ bounded}\}$
- $B(V \subseteq U)(f) = f|_V$

We can verify that B is a presheaf by observing the following properties of restriction:

- if $f \in B(U)$ then $f|_U = f$
- if $f \in B(U)$ and $W \subseteq V \subseteq U$ then $(f|_V)|_W = f|_W$

Actually proving that the category $Set^{\mathcal{O}(X)^{\mathrm{OP}}}$ is a topos is as hard as proving that the category $Set^{C^{\mathrm{OP}}}$, for any fixed small category, is a topos.

Proposition 2.4.4 *The functor category $Set^{C^{OP}}$ has all small limits.*

Proof. It suffices to check that *Set* has all small limits, since the limits in $Set^{C^{\mathrm{OP}}}$ are computed pointwise [Mac97].

∎

Proposition 2.4.5 *The functor category $Set^{C^{OP}}$ has all small colimits.*

Proof. It also suffices to check that *Set* has all small colimits. The category $Set^{C^{\mathrm{OP}}}$ has the very special property that its colimits are computed pointwise. This is not true in general for colimits. [MM92]

∎

In order to look at the subobject classifier of $Set^{C^{\mathrm{OP}}}$, we must be familiar with the notion of subobject in $Set^{C^{\mathrm{OP}}}$.

Definition 2.4.6 For any small category C, a *subfunctor* of the functor $P : C^{\mathrm{OP}} \to Set$ is defined to be another functor $Q : C^{\mathrm{OP}} \to Set$, where each $Q(c)$ is a subset of $P(c)$ and such that each $Q(f) : Q(d) \to Q(c)$ is the restriction of $P(f)$, for all arrows $f : c \to d$ of C.

The inclusion $Q \to P$ is then a monic arrow in the functor category $Set^{C^{\mathrm{OP}}}$, so that each subfunctor Q is a subobject. Conversely, all subobjects are given by subfunctors; if a natural transformation $\theta : R \to P$ is monic in the functor category, then each function $\theta(c) : R(c) \to P(c)$ is an injection (monics, like limits in the functor category, are taken pointwise). For each c let $Q(c)$ be the image of $R(c) \to P(c)$; thus Q is manifestly a subfuntor of P, and the given R is equivalent (as a subobject) to Q.

Subfunctors of $\hom_C(-, c)$ play an important role in defining Ω, and we shall call them *sieves*, following well established terminology [MM92], [Bor94], [Joh].

Proposition 2.4.7 *Given an object c in a category C, a sieve on c is equivalent to a set of arrows S with codomain c, such that if $f \in S$ and $dom(f) = cod(g)$ then $f \circ g \in S$, for any other g in C.*

Proposition 2.4.8 *The functor category $Set^{C^{OP}}$ has a subobject classifier.*

Proof. For an arbitrary presheaf category $Set^{C^{\mathrm{OP}}}$, if there is a subobject classifier Ω, it must, in particular, classify the subobjects of each representable presheaf $\hom_C(-, c) : C^{\mathrm{OP}} \to Set$, leading to

$$Sub_{Set^{C^{\mathrm{OP}}}}(\hom_C(-, c)) \cong \hom_{Set^{C^{\mathrm{OP}}}}(\hom_C(-, c), \Omega) = Nat(\hom_C(-, c), \Omega)$$

By the Yoneda lemma, the set on the right is up to isomorphism $\Omega(c)$.Thus the subobject classifier Ω, if it exists, must be the functor $\Omega : C^{\mathrm{OP}} \to Set$ given by

$$\Omega(c) = Sub(\hom_C(-, c)) = \{S | S \text{ a subfunctor of } \hom_C(-, c)\}$$

Given an arrow $g : b \to c$ and a subfunctor Q of the functor $\hom_C(-, c)$, the pullback of Q along g determines a subfunctor of $\hom_C(-, b)$.

Applying this idea to a given sieve S, one can define the following sieve:

$$S.g = \{h | g \circ h \in S\}$$

Finally for a presheaf category, the subobject classifier is given by the functor Ω such that:

- $\Omega(c) = \{S|S \text{ is a sieve on c} \in C\}$
- if $g : c' \to c$ then $(-).g : \Omega(c) \to \Omega(c')$, $S.g = \{h|g \circ h \in S\}$

For an object c of C, the set $t(c)$ of all arrows into c is a sieve, called the maximal sieve on c. These maximal sieves patch together to give a morphism

$$true : 1 \to \Omega$$

in the presheaf category $Set^{C^{\mathrm{OP}}}$. All that is left to do is to check that the given presheaf is indeed the subobject classifier. For any subfuntor Q of a given functor $P : C^{OP} \to Set$, each morphism $F : a \to c$ in C determines a function $P(f) : P(c) \to P(a)$ in Set which might not take a given $x \in P(c)$ into $Q(a) \subseteq P(a)$. For a given $x \in P(c)$, we define

$$\phi_c(x) = \{f|x.f \in Q(dom(f))\},$$

where f ranges over all morphisms in C with domain c. Obviously, $\phi_c(x)$ is a sieve on c and $\phi : P \to \Omega$ is natural. Moreover, $\phi_c(x)$ is the maximal sieve $t(c)$ iff $x \in Q(c)$. So the given subfunctor $Q \subseteq P$ is the pullback along ϕ of the map "true" just defined. This shows that ϕ is indeed a possible characteristic map for the subfunctor Q. But this ϕ is also the unique natural transformation from P to Ω that turns this diagram into a pullback. Indeed, given $x \in P(c)$ and $f : a \to c$, the pullback condition means that $x.f \in Q(a)$ iff $\theta_a(x.f) = true_a$. By naturality of θ, this is equivalent to $\theta_c(x).f = true_a$ which means, by the definition of Ω, that $f \in \theta_c(x)$. The elements $f \in \theta_c(x)$ are thus exactly those such that $x.f \in Q(a)$, i.e., those that coincide with the given definition of $\phi_c(x)$. Hence, the definition of ϕ is imposed upon us in order to guarantee the pullback property.
∎

Proposition 2.4.9 *The functor category $Set^{C^{OP}}$ is cartesian closed.*

Proof. Since $Set^{C^{\mathrm{OP}}}$ has all small limits, all we have to do is to deal with exponentials. For Q^P to be an exponential, it must satisfy $\hom(R \times P, Q) \cong \hom(R, Q^P)$, for any presheaves Q, P and R, in particular, for each representable functor $R = \hom_C(-, c)$. The expected isomorphism composed with the Yoneda isomorphism yields:

$$Q^P(c) \cong \hom_{Set^{C^{\mathrm{OP}}}}(\hom_C(-, c), Q^P) \cong \hom_{Set^{C^{\mathrm{OP}}}}(\hom_C(-, c) \times P, Q)$$

Hence we shall define Q^P as $Q^P = \hom_{Set^{C^{\mathrm{OP}}}}(\hom_C(-, c) \times P, Q)$, i.e., the set of all natural transformations θ from $\hom_C(-, c) \times P$ to Q. This clearly defines a functor $Q^P : C^{\mathrm{OP}} \to Set$. Associated with this definition, we have the following evaluation map

$$e : Q^P \times P \to Q$$

$$e_c(\theta, y) = \theta_c(1_c, y) \in Q(c),$$

$$\text{for all } c \in C,\ \theta : \hom_c(-, c) \times P \to Q \text{ and } y \in P(c).$$

It follows that e is a natural transformation. Moreover, for every natural transformation $\phi : R \times P \to Q$, there is a unique $\phi' : R \to Q^P$ such that the required diagram commutes. ∎

We proved that the category $Set^{C^{\mathrm{OP}}}$ is a topos for any small category C. If C is taken to be $\mathcal{O}(X)$, we are stating that the category of presheaves on X is a topos.

Next let us move on to sheaves.

Definition 2.4.10 A sheaf of sets F on a topological space X is a presheaf F, such that each open covering $\bigcup_i U_i = U, i \in I$, of an open set U of X yields an equalizer diagram

$$F(U) \dashrightarrow^{e} \prod_i F(U_i) \overset{p}{\underset{q}{\rightrightarrows}} \prod_{i,j} F(U_i \cap U_j)$$

such that, for $t \in F(U)$, $e(t) = \{t|_{U_i} | i \in I\}$ and for a family $\tilde{t} = (t_i)_i \in F(U_i)$

$p(\tilde{t}) = \{t_i|_{U_i \cap U_j}\}, \ q(\tilde{t}) = \{t_j|_{U_i \cap U_j}\}$.

A morphism $F \to G$ of sheaves is a natural transformation of functors. So $\mathrm{Sh}(X)$, the category of sheaves, is a full subcategory of the functor category $Set^{\mathcal{O}(X)^{\mathrm{OP}}}$.

We remark that the definition of sheaf implies that every sheaf F must send the empty set $\emptyset$ into a one-point set $\{*\}$, for, since in any space X the empty open $\emptyset$ has an empty covering ($I = \emptyset$), and a product $\prod_i$ over an empty index set I is the one-point set $\{*\}$, the equalizer above becomes

$F(\emptyset) \to \{*\} \rightrightarrows \{*\}$, and thus $F(\emptyset) \cong *$.

Let us take a look at some examples.

Example 2.4.11 For a topological space X, the functor $C : \mathcal{O}(X)^{\mathrm{OP}} \to Set$ defined by

- $C(U) = \{f | f : U \to \mathbb{R} \text{ continuous}\}$
- $C(V \subseteq U)(f) = f|_V$

is the sheaf of continuous real-valued functions on X.

Given an open set U, $C(U)$ is the set of continuous real valued functions on U. Given $V \subseteq U$ and f continuous on U, we can restrict the domain of f to V, and C is a presheaf just like in 2.4.3.

Indeed C is a sheaf, because if U is covered by open sets U_i, and there are functions $f_i : U_i \to \mathbb{R}$ continuous for all $i \in I$, then there is at most one continuous $f : U \to \mathbb{R}$ with restrictions $f|_{U_i} = f_i$ for all i. Moreover, such an f exists if and only if the various given f_i match on all overlaps $U_i \bigcap U_j$. This property makes the map e given by $f \mapsto \{f|_{U_i}\}$ an equalizer of the maps p and q.

Example 2.4.12 Let $id_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ be the identity function, i.e., $id_{\mathbb{R}}(x) = x$. We can consider the following family:

$$\{id_{\mathbb{R}}|_{\{x \in \mathbb{R} :\ |x| < n\}}\}_{n \in \mathbb{N}}$$

The pre-sheaf in 2.4.3 is not a sheaf, because the previous family of bounded functions matches and still it does not yield a bounded function.

Let us take a look at some more examples provided by functions.

Example 2.4.13 The functor of all functions to a fixed set R is a sheaf.

$$F(U) = \{f | f : U \to R\}$$

Example 2.4.14 The functor of all locally constant functions (constant only if X is connected) to a fixed set R is a sheaf.

$$F(U) = \{f | f : U \to R \ f \ is \ constant\}$$

Example 2.4.15 For a given continuous function $p : Y \to X$, the functor Γ_p defined by

- $\Gamma_p(U) = \{s | s : U \to Y$ continuous and $p \circ s = i : (U \subseteq X)\}$
- $\Gamma_p(V \subseteq U)(s) = s|_V$

is the sheaf of the cross-sections of p.

Γ_p is a presheaf just like in 2.4.3. Indeed Γ_p is a sheaf because of the same reason as 2.4.11 and also due to the fact that $p \circ s = i$ can be checked locally.

A subsheaf of a sheaf F on X is defined to be a subfunctor of F which is itself a sheaf.

Proposition 2.4.16 *If F is a sheaf on X, the following statements are equivalent:*

- *A subfunctor $S \subseteq F$ is a sheaf (hence, a subsheaf).*
- *for every open set U, every element $f \in F(U)$ and every open covering $U = \bigcup U_i$, $f \in S(U)$ if and only if $f|_{U_i}$ for all i.*

Let us start by looking at small limits for which we have the following very favorable situation.

Proposition 2.4.17 *For any space X, the category Sh(X) has all the small limits and the inclusion of sheaves in presheaves preserves all these limits.*

This means that the product of sheaves is actually their product as presheaves which is pointwise.

The category of sheaves has a subobject classifier, Ω. Since sheaves are dependent on the topology $\mathcal{O}$(X), the natural ways for a sheaf to be a subsheaf of another object should be somehow related to the ways an open set can be a subset of another open. We can think of open sets as truth values. We can think of evaluating a statement by finding the largest open set where it is true.

Hence, consider the following functor:

- $\Omega(U) = \{U' \in \mathcal{O}(X) | U' \subseteq U\}$
- $\Omega(U'' \subseteq U)(U') = U'' \cap U'$ where $U' \in \Omega(U)$

The functor $\Omega : \mathcal{O}(X) \to Set$ is a sheaf. Let us look at the inclusion $(U' \subseteq U)$ as the characteristic function of $U' \subseteq U$ i.e.$f : U \to \{0,1\}$

$$f(x) = \begin{cases} 0, & x \in U' \\ 1, & x \notin U \end{cases}$$

Hence, Ω is a sheaf (cf. example 2.4.14).

The sheaf Ω has precisely the structure of $\mathcal{O}(X)$. This can be checked by observing that any subsheaf of Ω is given by

- $\Omega_V(U) = \{U' \in \mathcal{O}(X) | U' \subseteq U \cap V\}$
- $\Omega_V(U'' \subseteq U)(U') = U'' \cap U'$ where $U' \in \Omega_V(U)$

for some V. The sheaves Ω_V are obviously subsheaves of Ω. Furthermore, given any S subsheaf of Ω, we take V to be the "largest" open in $S(X)$.This open must exist, since S is a sheaf.

The open sets of the topological space X are therefore the truth values in the interpretation of a given formula as described above.

Proposition 2.4.18 *The sheaf Ω with the true morphism is the subobject classifier of Sh(X), where*

- $true_U : * \to \Omega(U)$
- $true_U(*) = U$

Proof. Let F and F' be two sheaves and m a monomorphism between them. Our goal is to find a unique morphism ψ_F between F' and Ω such that the following diagram is a pullback

$$\begin{array}{ccc} F & \longrightarrow & 1 \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle true} \\ F' & \overset{\psi_F}{\dashrightarrow} & \Omega \end{array}$$

ψ_F is a family of morphisms in Set, $\{\psi_{F,U}\}_{U \in \mathcal{O}(X)}$, such that $\psi_{F,U} : F'(U) \to \Omega(U)$.

If we are thinking about sheaves of functions, the image of a given $f \in F'(U)$ by $\psi_{F,U}$ should be the "largest" open subset U' of U such that the restriction of f to that set is in $F(U')$ i.e.

$$\psi_{F,U}(f) = \bigcup_{U' \subseteq U, F'(U' \subseteq U)(f) \in m(F(U'))} U'$$

It is easy to see that for the diagram to commute if suffices to observe that U is the "largest" open subset of itself.

In order to check that the diagram is a pullback, let F'' be another sheaf equipped with the morphism n and the unique morphism to the terminal object, such that the following diagram commutes:

$$\begin{array}{ccc} F'' & \longrightarrow & 1 \\ \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle true} \\ F' & \xrightarrow{\psi_F} & \Omega \end{array}$$

Our aim is to prove that there is a unique morphism l, between F'' and F' such that the following diagram commutes:

$$\begin{array}{ccccc} & & F'' & & \\ & \swarrow^{n} & \downarrow{\scriptstyle l} & \searrow & \\ F' & \xleftarrow[m]{} & F & \longrightarrow & 1 \end{array}$$

How should l be defined? Let g be any element of $F''(U)$. If $n(g)$ happens to belong to $m(F(U))$ then $l(g)$ should be the (unique, because m is a mono) element in $F(U)$ such that $m(l(g)) = n(g)$.

By observing the diagram above, the definition of ψ_F and the fact that F is a sheaf, it is clear that $n(F''(U)) \subseteq m(F(U))$. Since the choice of the image of l is imposed on us for all elements of its domain, l is uniquely determined. Also it is clearly natural.

Finally it is necessary to verify that there can be no other morphism between F' and Ω that can turn our diagram into a pullback. First of all let us observe that we wish the diagram to commute for those elements of $F'(U)$ that are in the image of m. Hence the following condition must be verified:

$$\psi_{F,U}(f) \supseteq \bigcup_{U' \subseteq U, f|_{U'} \in m(F(U'))} U'$$

We will prove that if $U' \subseteq U$ and $f|_{U'} \in m(F(U'))$ then $U' \subseteq \psi_{F,U}(f)$, since the following diagram must commute

$$\begin{array}{ccc} F(U') & \longrightarrow & 1(U') \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle true} \\ F'(U') & \xrightarrow{\psi_{F,U'}} & \Omega(U') \end{array}$$

The given condition and the previous argument ensure that $\psi_{F,U'} = U'$. But now since we want ψ_F to be natural between F' and Ω, our result follows.

Finally we will prove that

$$\psi_{F,U}(f) \subseteq \bigcup_{U' \subseteq U, f|_{U'} \in m(F(U'))} U'$$

Assume that ψ_F is a little "larger", i.e.
$\bigcup_{U' \subseteq U, f|_{U'} \in m(F(U'))} U' \subset U''$ where $U'' = \psi_{F,U}(f)$. Let us consider the following sheaf F''

$$F''(\mathcal{O}^*) = \begin{cases} \emptyset, & \mathcal{O}^* \notin U'' \\ \{*\}, & \text{otherwise} \end{cases}$$

and let n be such that $n(*) = f$, or $n(*) = F'((O^*) \subseteq U)(f)$. We are then led to a contradiction by considering what $l(*)$ in U'' is. It should be an element e such that $m(e) = f$ for the triangle to commute, but in fact there should be no such e in U''. ∎

Next let us look at exponentials.

Since the product of sheaves is actually just their product as presheaves, one might look for the exponential of sheaves in the exponential of presheaves (See 2.4). This turns out to be just the case.

Now one has only to check whether Q^P is a sheaf when P and Q are sheaves. This is verified by the following proposition.

Proposition 2.4.19 *If F is a sheaf and P a presheaf, then the (presheaf) exponential F^P is a sheaf.*

Proof. Due to the structure of y, where by we mean the functor given by $yc = \hom(-, c)$, when referring to F^P, one can concentrate only on the opens which are subsets of U, i.e., we can write:

$$F^P(U) \cong \hom_{Sh(U)}(P|_U, F|_U)$$

where $P|_U$ and $F|_U$ are the functors P and F restricted to $\mathcal{O}(U)^{\mathrm{OP}} \subseteq \mathcal{O}(X)^{\mathrm{OP}}$. Also for a morphism $U' \subseteq U$, the functor F^P works as follows:

$$F^P(U' \subseteq U) : \hom_{Sh(U)}(P|_U, F|_U) \to \hom_{Sh(U')}(P|'_U, F|'_U)$$

i.e., a function that maps natural transformations into natural transformations, $F^P(U' \subseteq U)(\alpha : P|_U \to F|_U) = \alpha|'_U$, where $\alpha|'_U : P|'_U \to F|'_U$, is the restriction of α to U'.

Finally we check that F^P is a sheaf. Let U be an open set of $\mathcal{O}(X)$ and $\bigcup_i U_i = U$ a covering of U.
Also let $\tau_i : P|_{U_i} \to F|_{U_i}$ be natural transformations for all i which match. We need to prove that there is a unique way of extending τ to a natural transformation $\tau : P|_U \to F|_U$. What should τ'_U of a given $U' \subseteq U$ be ? If $U' \subseteq U_i$ for a given i, then $\tau'_U = \tau_{i,U'}$. If there exists another U_j such that $U' \subseteq U_j$ then the above definition is correct, because τ_i and τ_j match. Let $U' \subseteq U$ be a generic inclusion. We would like τ to be a natural transformation. So, for a given $f \in P|_U(U')$, we have that if $\mathcal{O}^* \subseteq U'$ then $\tau(f)|_{\mathcal{O}^*} = \tau_{\mathcal{O}^*}(f|_{\mathcal{O}^*})$. This means $\tau(f)$ should have restrictions for all open subsets of U'. Since $F|_U$ is a sheaf, $\tau(f)$ is forced to be only one thing: the element that $F|_U$ returns from the matching family $\{\tau_{i,\mathcal{U}_i}(f|_{\mathcal{U}_i}) | \mathcal{U}_i \subseteq U_i\}$. ∎

Let us take a look at small colimits. The category Sh(X) has small colimits, since the *sheafification functor* (see below) is left adjoint to the inclusion functor

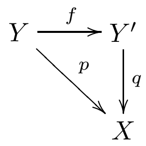
$$Sh(X) \hookrightarrow Set^{\mathcal{O}(X)^{\mathrm{OP}}}.$$

The sheafification functor or associated sheaf functor, $L : Set^{\mathcal{O}(X)^{\mathrm{OP}}} \to Sh(X)$, carries a presheaf P in X to the "best approximation" of P by a sheaf.If F is already a sheaf then $L(F) \cong F$. Hence, to compute the coproduct of two sheaves F and G, one needs only to calculate their coproduct as presheaves, $F \coprod G$ (their pointwise disjoint union), and apply L, obtaining $L(F \coprod G)$.

Let us now take a look at how the sheafification functor is defined. In order to do so, we will introduce two new functors $\Lambda : Set^{\mathcal{O}(X)^{\mathrm{OP}}} \to Top/X$ and $\Gamma : Top/X \to Sh(X)$. The category Top/X is the *slice category* over X. Let us take a closer look at Top/X.

Definition 2.4.20 A *bundle over X* is a continuous map $p : Y \to X$.

Definition 2.4.21 A morphism of bundles $p : Y \to X$ and $q : Y' \to X$ is a continuous map $f : Y \to Y'$ such that the following triangle commutes.



The category Top/X is the category whose objects are bundles over X and morphisms are bundle morphisms.

Let us start by taking a look at Γ.

Definition 2.4.22 The functor Γ turns a bundle into a sheaf of cross-sections (cf. example 2.4.15), i.e., given a bundle $p : Y \to X$ we define

$$\Gamma_p(U) = \{s | s : U \to Y \text{ continuous and } p \circ s = i : (U \subseteq X)\}$$

Γ is clearly a functor.

Now let us look at Λ. First we must introduce the notions of germ and stalk.

Definition 2.4.23 Given any presheaf P, a point $x \in X$ and two open neighborhoods U and V of x, two elements $f \in P(U)$ and $g \in P(V)$ are said to have the same germ at x, $germ_x f = germ_x g$, if there is an open W such that :

- $x \in W$
- $W \subseteq U \cap V$
- $f|_W = g|_W$

Definition 2.4.24 If P is a presheaf then we define, for all $x \in X$, the stalk of P at x to be the set

$$P_x = \lim_{\rightarrow\, x \in U} P(U)$$

We are now ready to build a bundle from a presheaf.

Definition 2.4.25 The functor $\Lambda : Set^{\mathcal{O}(X)^{\mathrm{OP}}} \to Top/X$ is given as follows for a presheaf P.

$$p : \Lambda_P \to X$$

where

$$\Lambda_P = \coprod_x P_x = \{germ_x s | x \in X, s \in P(U) \text{ for some } U\}$$

is considered to be a topological space by taking $\{germ_x s | x \in U\}$, for each open $U \subseteq X$ and each $s \in P(U)$, to be a basic open set. Thus the map p defined as $p(germ_x s) = x$ is a continuous function.

Definition 2.4.26 The *sheafification functor*, $L : Set^{\mathcal{O}(X)^{\mathrm{OP}}} \to Sh(X)$, is now simply defined as $\Gamma \circ \Lambda$.

The following theorems can be found in [MM92], chapter II, section 5, and reveal the importance of the sheafification functor.

Theorem 2.4.27 *Let $\eta : P \to \Gamma \circ \Lambda_P$ be the natural transformation given as $\eta_U(s) : U \to \Lambda_P$, $\eta_U(s)(x) = germ_x s$, for any $x \in U$ and any U open subset of X.*

If the presheaf P is a sheaf, then η is an isomorphism.

In order words, every sheaf is a sheaf of cross-sections.

Theorem 2.4.28 *For any presheaf P, the corresponding morphism $\eta : P \to \Gamma\Lambda_P$ of presheaves is universal from P to sheaves.*

Corollary 2.4.29 *For any topological space X, the category Sh(X) of sheaves of sets on X is reflexive in the category $Set^{\mathcal{O}(X)^{OP}}$ of presheaves on X. By reflexive we mean that the sheafification functor is left adjoint to the inclusion Sh(X) $\to$ $Set^{\mathcal{O}(X)^{OP}}$.*

Chapter 3

Mathematics in a topos

3.1 Types

Definition 3.1.1 A *type theory* is a formal theory given by the following data:

- a class of types, including a special type Ω;
- a class of terms, including countably many variables of each type;
- for each finite set X of variables, a binary relation $\vdash$ of entailment between terms of type Ω, all free variables of which are elements of X.

These data are subject to the following conditions:

- The class of types is closed under the inductive clauses:
  - 1 and Ω are types;
  - if A and B are types, then so are $A \times B$ and $\mathcal{P}A$.

  Additional types besides those specified by clauses above are allowed. We interpret 1 as a one-element type and Ω as the type of truth values or propositions.

- The class of terms is freely generated from certain basic terms by certain operations. Among the terms of type A are countably many variables $x_1^A, x_2^A, \ldots$. The set of terms also contains a number of specific terms and is closed under operations, as follows.
  - $*$ is a term of type 1;
  - if a is a term of type A and b is a term of type B, then $\langle a, b\rangle$ is a term of type $A \times B$;
  - if a is a term of type A and α is a term of type $\mathcal{P}A$, then $a \in \alpha$ is a term of type Ω;
  - if $\varphi(x)$ is a term of type Ω (possibly containing the free variable x of type A), then $\{x \in A|\varphi(x)\}$ is a term of type $\mathcal{P}A$ not containing a free occurrence of x;
  - if a and b are both terms of type A then $a = b$ is a term of type Ω.

Based on the symbols above it is possible to introduce the following definitions:

- $\top \equiv * = *$, representing the logical value true;
- $p \wedge q \equiv \langle p, q \rangle = \langle \top, \top \rangle$
- $p \rightarrow q \equiv p \wedge q = p$
- $\forall_{x \in A} \varphi(x) \equiv \{x \in A | \varphi(x)\} = \{x \in A | \top\}$

Terms of type Ω are also called formulae.

3.2 Internal logic

Our use of semantics will turn a formula of the given formal language into a statement in ordinary (naive) language. The variables that occur in a formula may be divided into free and non-free, as can be seen by the relation given in the syntax. A formula will be interpreted as a morphism in the topos.

Definition 3.2.1 The internal language L($\mathcal{T}$) of a topos $\mathcal{T}$ has the objects of $\mathcal{T}$ as types.

Definition 3.2.2 The interpretation of terms of the internal language of a topos $\mathcal{T}$ turns each term t in $\mathcal{T}$ into a morphism from the product of the types corresponding to the free variables of t into the type of t. The interpretation is given by the following inductive clauses:

- variables of type A are the identity arrows $A \rightarrow A$
- * is $1 \rightarrow 1$;
- $\langle a, b \rangle$ is $A_1 \times ... \times A_i \times B_1 \times ... \times B_j \stackrel{f \times g}{\rightarrow} A \times B$, i.e., the product arrow of $A_1 \times ... \times A_i \stackrel{f}{\rightarrow} A$ and $B_1 \times ... \times B_j \stackrel{g}{\rightarrow} B$, where f and g are the interpretation of a and b respectively;
- $a = a'$ is $A_1 \times ... \times A_i \times B_1 \times ... \times B_j \stackrel{f \times g}{\rightarrow} A \times A \stackrel{\psi_{A \times A}}{\rightarrow} \Omega$, where $A_1 \times ... \times A_i \stackrel{f}{\rightarrow} A$, $B_1 \times ... B_j \stackrel{g}{\rightarrow} A$ are the interpretation of a and b, respectively, and $\psi_{A \times A}$ is the characteristic morphism of $A \stackrel{id_A \times id_A}{\rightarrow} A \times A$;
- $a \in \alpha$ is $A_1 \times ... \times A_i \times B_1 \times ... \times B_j \stackrel{f \times g}{\rightarrow} A \times \mathcal{P}A \stackrel{e}{\rightarrow} \Omega$, where $A_1 \times ... \times A_i \stackrel{f}{\rightarrow} A$, $B_1 \times ... \times B_j \stackrel{g}{\rightarrow} \mathcal{P}A$ are the interpretation of a and α respectively. Furthermore, $\mathcal{P}A$ is interpreted as Ω^A;
- $\{x \in A | \varphi(x)\}$ is the result of applying the right adjoint of the product to the interpretation of $\varphi(x)$, which is a morphism $A \times A_1 \times ... \times A_n \rightarrow \Omega$, yielding a morphism $A_1 \times ... \times A_n \rightarrow \Omega^A$.

3.3 Relations

We are now ready to interpret axioms relative to elementary mathematical constructions in a topos. Let us start by developing some constructions which lie at the foundations of analysis.

First we look at relations.

Definition 3.3.1 A relation between objects c, c' is a subobject of the product $c \times c'$. Hence the object of relations between c and c' is $\Omega^{c \times c'}$.

A partial order is a special kind of relation that can be described by first order logic.

Definition 3.3.2 A relation $\leq$ on c is a partial order provided that the conjunction of the following formulae is valid:

- reflexivity, $\forall_{a \in c}(a \leq a)$
- anti-symmetry, $\forall_{a \in c}\forall_{a' \in c}(a \leq a' \wedge a' \leq a \Rightarrow a = a')$
- transitivity, $\forall_{a \in c}\forall_{a' \in c}\forall_{a'' \in c}(a \leq a' \wedge a' \leq a'' \Rightarrow a \leq a'')$

where $a \leq a'$ is short for the formula $(a, a') \in \leq$

Example 3.3.3 The inclusion relation on the object Ω^c of the subobjects of any object c defined as

$$\forall_{a \in c}(a \in c' \Rightarrow a \in c'')$$

is an example of a partial order. This formula can be shortened to $c' \subseteq c''$.

We will next deal with the notion of functional relation.

Definition 3.3.4 A relation ϕ from an object c to an object c' is a functional relation or mapping from c to c', provided the conjunction of the following conditions is a valid formula:

- $\forall_{a \in c}\exists_{b \in c'}(a, b) \in \phi$
- $\forall_{a \in c}\forall_{b \in c'}\forall_{b' \in c'}((a, b) \in \phi \wedge (a, b') \in \phi \Rightarrow b = b')$

Definition 3.3.5 An equivalence relation on an object c is a subobject $\cong$ of $c \times c$ for which the conjunction of the following formulae is valid.

- reflexivity, $\forall_{a \in c}(a \cong a)$
- symmetry, $\forall_{a \in c}\forall_{a' \in c}(a \cong a' \Rightarrow a' \cong a)$
- transitivity, $\forall_{a \in c}\forall_{a' \in c}\forall_{a'' \in c}(a \cong a' \wedge a' \cong a'' \Rightarrow a \cong a'')$

The next natural step is to consider the object of equivalence classes of a given equivalence relation.

Definition 3.3.6 The object of equivalence classes of a given equivalence relation, $c_{/\cong}$, is the subobject of Ω^c obtained by interpreting the following formula:
$\forall_{a \in c}\forall_{a' \in c}(a \in c \wedge a' \in c \Leftrightarrow a \cong a')$

The map assigning to each element its equivalence class in $c_{/\cong}$ can be described as the coequalizer of the following maps in the topos:

$$\cong \hookrightarrow c \times c \rightrightarrows c$$

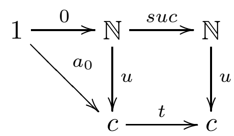
where the maps considered are the projections from the product $c \times c$ and the inclusion map from $\cong$ to $c \times c$.

3.4 Natural numbers

In this section we aim to describe what an object of natural numbers in a topos should be. We define a natural number object by resorting to a natural universal property.

Definition 3.4.1 A *natural number object* is given by

- an element $0 : 1 \to \mathbb{N}$;
- an arrow $suc : \mathbb{N} \to \mathbb{N}$, called the successor arrow;
- a universal property; for any object c, element $a_0 : 1 \to c$ and map $t : c \to c$ there is a unique map $u : \mathbb{N} \to c$ such that the following diagram commutes



It should be noted that not all toposes must have a natural number object.

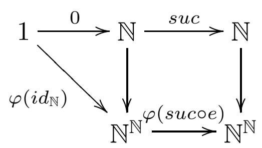
This property is very useful because it allows us to define a new morphism by primitive recursion. This can be done by using the commuting triangle to define a recursion base and the commuting square to define a recursive step. For example, let us define an addition map:

$$\mathbb{N} \times \mathbb{N} \xrightarrow{+} \mathbb{N}$$

This is defined by its exponential adjoint

$$\mathbb{N} \to \mathbb{N}^{\mathbb{N}}$$

which is obtained from the universal property



where $\varphi(id_{\mathbb{N}})$ is the exponential adjoint of the identity of $\mathbb{N}$ and $\varphi(suc \circ e)$ is the exponential adjoint of the morphism $suc \circ e$:

$$\mathbb{N} \times \mathbb{N}^{\mathbb{N}} \xrightarrow{e} \mathbb{N} \xrightarrow{suc} \mathbb{N}$$

Multiplication can be defined in a similar fashion.

Now we are interested in defining the order relation on $\mathbb{N}$.

Definition 3.4.2 The order relation $\leq$ in $\mathbb{N}$ is given by the interpretation of the following formula:

$$\exists_{k\in\mathbb{N}}(m+k=n)$$

which can be shortened to $m \leq n$.

We can also define the object of positive natural numbers $\mathbb{N}^+$ by the interpretation of $\exists_{m\in\mathbb{N}}(n = suc(m))$.

3.5 Real numbers

We are now in possession of all the tools needed to define the object of integers.

The object of integers is constructed from the object of natural numbers by taking the equivalence relation on the sheaf $\mathbb{N}\times\mathbb{N}$ defined by the interpretation of the formula

$$m + n' = m' + n$$

which we will write as $(m, n) \cong (m', n')$.

Definition 3.5.1 The object $\mathbb{Z}$ of integers is defined to be the object of equivalence classes of the above relation.

We can define addition, multiplication and the order relation in the usual way.

The object of rational numbers $\mathbb{Q}$ is constructed in a similar way. We need only to use the equivalence relation $m.n' = m'.n$, as a subobject of $\mathbb{Z}\times\mathbb{N}^+$. Once more, defining the addition, multiplication and order relation, is done by extending those of the object of integers.

A not so trivial construction is that of the object of real numbers. The first obstacle one faces is that the available constructions are not necessarily equivalent under the "topos logic". As so, they might not yield the same result. This will be dealt with by following a construction due to Tierney, which takes Dedekind cuts on the object of rationals.

Definition 3.5.2 A subobject c of $\mathbb{Q}$ is a Dedekind lower cut if it satisfies the formula

$$c = \{q \in \mathbb{Q} | \exists_{q'\in c}(q < q')\}$$

A Dedekind upper cut c is a subobject of $\mathbb{Q}$ which satisfies the condition

$$c = \{q \in \mathbb{Q} | \exists_{q'\in c}(q' < q)\}$$

Next one considers the pairs of subobjects (c_1, c_2) of $\mathbb{Q}$, where c_1 is a Dedekind lower cut and c_2 a Dedekind upper cut, which satisfy the conditions

$$\forall_{q_1\in c_1}\forall_{q_2\in c_2} q_1 < q_2$$

and

$$\forall_{n\in\mathbb{N}^+}\exists_{q_1\in c_1}\exists_{q_2\in c_2}(q_2 - q_1 < 1/n)$$

These formulae simply state that the upper cut lies above the lower cut at zero distance from it. This construction will yield the object of real numbers.

Definition 3.5.3 The object $\mathbb{R}$ is defined to be the subobject of $\Omega^{\mathbb{Q}} \times \Omega^{\mathbb{Q}}$ satisfying the Dedekind cut's conditions and the conjunction of the previous formulae.

We could also have tried to construct the object of real numbers through Cauchy sequences. Later on we shall check that this does not yield the same result.

Chapter 4

Algebra in Sh(X)

4.1 Internal logic revisited

The semantics previously given can be made more explicit when dealing specifically with Sh(X), instead of an arbitrary topos.

First let us start by interpreting terms. A term will be interpreted as a morphism in Sh(X) as follows:

Definition 4.1.1 Interpretation of a term

- if the term is a constant c of type A then c is interpreted as a selected morphism $1 \to A$;
- if the term is a variable x^A then it is interpreted as the id_A;
- if the term is $f(t_1, ..., t_n)$ of type A then it is interpreted as $A_{1,1} \times ... \times A_{1,m_1} \times ... \times A_{n,m_n} \overset{f_1 \times ... \times f_n}{\to} A_1 \times ... \times A_n \overset{\bar{f}}{\to} A$, where $\bar{f}$ is a morphism that interprets the symbol f and $A_{i,j}$ are types of free variables of A_i.

A formula with free variables $x^A, x^B, ..., x^C$ will be interpreted as a subsheaf of the product $A \times B \times ... \times C$. The subsheaves play the part of the subsets in classical set theory.

Definition 4.1.2 Interpretation of elementary formulae in the topos Sh(X):

- $x^A = x'^A$ is interpreted as the subsheaf of $A \times A$ consisting of the diagonal elements, which is the same as to interpret $x^A = x'^A$ pointwise, as in set theory;
- $x^A \in A'$ is interpreted as the subsheaf A' of the sheaf A.
- $x^A \in x^{\Omega^A}$ is the subsheaf of $A \times \Omega^A$, whose sections for a given open U are those pairs consisting of an element $a \in A(U)$ and a subsheaf A' of $A|_U$ for which $a \in A'(U)$.

The formula $x^A \in x^{\Omega^A}$ can be shortened to $x^A \in A'$, thus causing ambiguity. This will always be made clear by the context.

General formulae might have a very much complex structure than the ones given above, in particular because the terms they involve might not be as simple as variables, they might have a lot more structure. To interpret a more general formula, i.e. one which is obtained by replacing a variable by a term t of the same type, simply pullback the interpretation of the original formula along the interpretation of the given term. It is easy to check that this returns a subsheaf of the product sheaf of all free variables of t.

The construction amounts to understanding the effect of replacing the variable for every open subset of X.

Definition 4.1.3 Suppose that ϕ' and ϕ'' are interpreted as subsheaves A' and A'' of a given sheaf A. Then we now interpret the following formulae:

- $\phi' \wedge \phi''$ as the largest subsheaf of A contained in both A' and A'';
- $\phi' \vee \phi''$ as the smallest subsheaf of A containing both A' and A'';
- $\phi' \Rightarrow \phi''$ as the largest subsheaf of A whose intersection with A' is contained in A'';
- $\neg\phi'$ as the largest subsheaf of A whose intersection with A' is the empty subsheaf of A.

In order to get a clearer notion of the sheaves obtained this way, one can look at their sections:

- the sections of $\phi' \wedge \phi''$ over U are those sections of $A(U)$ that belong to both subsets $A'(U)$ and $A''(U)$;
- the sections of $\phi' \vee \phi''$ are the sections of $A(U)$ for which there is an open covering $\bigcup_\alpha U_\alpha = U$ such that for all α, restricting the section to U_α one gets an element of $A'(U_\alpha)$ or an element of $A''(U_\alpha)$;
- the sections of $\phi' \Rightarrow \phi''$ are those that whose restriction to an arbitrary $U' \subseteq U$ is such that if it belongs to $A'(U')$ then it also belongs to $A''(U')$

Finally we will look at quantifiers.

Definition 4.1.4 Suppose $\phi(x^A, x^B, ..., x^C)$ is a formula whose free variables are $x^A, x^B, ..., x^C$.

- The interpretation of $\exists_{x^A \in A}\phi(x^A, x^B, ..., x^C)$ is the smallest subsheaf of $B \times ... \times C$ whose product with A contains the subsheaf that is the interpretation of ϕ.
- The interpretation of $\forall_{x^A \in A}\phi(x^A, x^B, ..., x^C)$ is the largest subsheaf of $B \times ... \times C$ whose product with A is contained in the subsheaf that is the interpretation of ϕ.

Let us also take a look at what happens at a section level for quantifiers.

- The sections of the subsheaf $B \times .... \times C$ that is the interpretation of $\exists_{x^A \in A} \phi(x^A, x^B, ..., x^C)$ are obtained, for every open U, from the sections $b \in B(U), ..., c \in C(U)$ for which there exists both a covering $\bigcup_\alpha U_\alpha = U$ and a section $a_\alpha \in A(U_\alpha)$, for each U_α, such that $(a_\alpha, b|_{U_\alpha}, ..., c|_{U_\alpha})$ is a section over U_α in the interpretation of the formula ϕ.

- The sections of the subsheaf $B \times .... \times C$ that is the interpretation of $\forall_{x^A \in A} \phi(x^A, x^B, ...x^C)$ are obtained, for every open U, from the sections $b \in B(U), ..., c \in C(U)$ for which, given any open $U' \subseteq U$ and any $a' \in A(U')$, then $(a', b|'_U, ..., c|'_U)$ is a section over U' in the interpretation of the formula ϕ.

The interpretation of a formula as described above coincides with the notion given in 3.2.2. This can be verified in proposition 10.5, in part II of [LS86]. We now know that the interpretation of a formula will yield a subsheaf of the product of the sheaves corresponding to the types of the free variables. We shall consider a formula to be valid, provided that its interpretation is the product of the sheaves themselves, i.e., that its interpretation is the largest subsheaf possible. On the opposite end, a formula will be referred to as invalid if its interpretation is the empty subsheaf of the product. Since we have already taken a look at the subobject classifier and are aware of its close relation to the topology of X, it is clear that a formula has to be neither valid nor invalid, even if there are no free variables in it. The interpretation of any formula with no free variables determines a subsheaf of the sheaf 1, which is the product indexed by the empty set of variables. So one can think of the truth values as being the open sets of the topological space X which are the subsheaves of 1, indicating the largest open set where our formula is valid.

We get a feeling of propositional intuitionistic logic by checking out how the interpretation of the logical operations is performed when dealing with closed formulae. Suppose U and U' are the interpretations of ϕ and ϕ', respectively:

- $\phi \vee \phi'$ is interpreted as $U \cup U'$.
- $\phi \wedge \phi'$ is interpreted as $U \cap U'$.
- $\neg\phi$ is interpreted as $int(X \backslash U)$.
- $\phi \Rightarrow \phi'$ is interpreted as $int((U \cap U') \cup (X \backslash U))$

In set theory it is natural to have a way of representing functions other than by fixed interpretation of constants. In fact, one can have a variable representing a function. Such an accomplishment is achieved here by resorting to the notion of exponential in the topos of sheaves. Finally one can define the following formulae

$$\exists_{x^A \in A'} \phi$$

and

$$\forall_{x^A \in A'} \phi$$

as being short for

$$\exists_{x^A \in A} ((x^A \in A') \wedge \phi)$$

and

$$\forall_{x^A \in A} ((x^A \in A') \Rightarrow \phi).$$

4.2 Integers, rationals, reals

Now let us give more explicit descriptions of the constructions in the previous chapter.

Simply restating 3.3.1, we get the following definition of a relation in $\mathrm{Sh}(X)$.

Definition 4.2.1 A relation from sheaf a A to a sheaf B is a subsheaf of the product $A \times B$. The sheaf $\Omega^{A \times B}$ is therefore the sheaf of relations from A to B.

This is the same as stating that a relation in $\mathrm{Sh}(X)$ is just a family of relations, indexed by open sets, which respects restrictions.

Likewise, definition 3.3.2 is equivalent to giving a partial order on each set of the sections of sheaf that is compatible with the restriction maps.

Functional relations, definition 3.3.4, in $\mathrm{Sh}(X)$, determine a map from the sheaf A to a sheaf B. Let us take a closer look.

The interpretation of the first condition is satisfied when the interpretation of $\exists_{b \in B}(a, b) \in \phi$ is the sheaf A, i.e., for any open U and $a \in A(U)$, there exist both an open covering $\bigcup_\alpha U_\alpha = U$ and $b_\alpha \in B(U_\alpha)$ such that $(a|_{U_\alpha}, b_\alpha)$ belongs to the subsheaf ϕ. This amounts to the idea that ϕ is everywhere defined.

The second condition is valid if $((a, b) \in \phi \wedge (a, b') \in \phi \Rightarrow b = b')$ is the sheaf $A \times B \times B$. This imposes that, for any open U, if $a \in A(U)$ and $b, b' \in B(U)$ then b and b' are equal, provided that (a, b) and (a, b') belong to ϕ.

Hence a relation ϕ respecting both conditions given above determines a map from the sheaf A to the sheaf B.

Now let us take a look at natural numbers in the topos of sheaves. We can observe that the given construction has the expected universal property of natural numbers.

Definition 4.2.2 The sheaf $\mathbb{N}$ is called the sheaf of *natural numbers*. It is the sheaf whose sections over U are the continuous maps from U to the set of natural numbers (equiped with the discrete topology), i.e., the sheaf such that $\mathbb{N}(U)$ is the set of locally constant functions from U to the set of natural numbers.

0 and *suc* can be defined as follows:

- 0 is the map $1 \to \mathbb{N}$ such that $0(U) = \{zero\}$, where $zero : U \to \mathbb{N}$ is constant and equal to zero.
- *suc* is the map $\mathbb{N} \to \mathbb{N}$ which maps each section to the section obtained by taking the successor of each natural number.

In order for the given construction to be the sheaf of natural numbers it must comply with definition 3.4.1.

Proposition 4.2.3 *For any sheaf A together with maps a_0 and t,*

$$1 \xrightarrow{a_0} A \xrightarrow{t} A$$

there is a unique map of sheaves from $\mathbb{N}$ to A such that the following diagram commutes:

$$\begin{array}{ccccc} 1 & \xrightarrow{0} & \mathbb{N} & \xrightarrow{suc} & \mathbb{N} \\ & \searrow^{a_0} & \downarrow & & \downarrow \\ & & A & \xrightarrow{t} & A \end{array}$$

Proof. Let U in X be an open set and n a locally constant function from U to the natural numbers. We have that n determines a covering of U by disjoint open sets U_1, ..., U_n where n is constant. Then n is mapped to the element $a \in A(U)$, defined uniquely by requiring its restriction a_i to U_i to be the $n(U_i)$th iterate of the given map t, for each set of the open covering. ∎

Addition and multiplication are computed pointwise, i.e.

$$(n + m)(x) = n(x) + m(x)$$

$$(n.m)(x) = n(x).m(x)$$

for all $x \in X$.

The order relation is also computed pointwise, i.e. $n \leq m$ if $n(x) \leq m(x)$ for all $x \in X$.

Naturally, n belongs to $\mathbb{N}^+$ if n is nowhere zero. Finally it is interesting to verify that the following formula is valid, even though an individual element of $\mathbb{N}$ does neither have to be globally zero nor globally positive:

$$\forall_{n \in \mathbb{N}}(n = 0 \vee n \in \mathbb{N}^+)$$

Let us check this carefully. By definition 4.1.4, the formula above is valid iff the interpretation of $(n = 0 \vee n \in \mathbb{N}^+)$ is the sheaf $\mathbb{N}$. According to 4.1.3, the interpretation of this formula over U consists of those locally constant integer-valued functions $f_n : U \to \mathbb{N}$, for which there is an open covering $\bigcup_\alpha U_\alpha = U$, such that, for all α, restricting f_n to each U_α yields an element of the interpretation of $n = 0$ over U_α or an element of the interpretation of $n \in \mathbb{N}^+$ over U_α. Every locally constant integer-valued function over any U verifies the previous condition. Given f_n such a function, consider the following covering:

$$U = \bigcup_{m \in \mathbb{N}} f_n^{-1}(m)$$

In such a covering, each restriction turns f_n into actual constant functions $f_n|_{U_\alpha}$. So naturally $f_n|_{U_\alpha}$ is either everywhere zero or nowhere zero.

Now let us look at the sheaf of integers.

Proposition 4.2.4 *The sheaf of integers $\mathbb{Z}$, constructed from $\mathbb{N}$ as in 3.5.1, is the sheaf of locally constant integer valued functions.*

Addition, multiplication and the order relation are also computed pointwise.

For rational numbers we have a similar situation.

Proposition 4.2.5 *The sheaf of rationals $\mathbb{Q}$ is the sheaf of locally constant rational valued functions.*

Again addition, multiplication and the order relation are computed pointwise.

Let us look at the construction of real numbers. A more intuitive notion of the lower Dedekind cuts, defined in 3.5.2, is given by the following proposition.

Proposition 4.2.6 *The Dedekind lower cuts on the sheaf of rationals can be identified with lower semi-continuous functions from X into the extended reals, i.e., reals with both infinities.*

Proof. The interpretation of a lower Dedekind cut is a subsheaf A of $\mathbb{Q}$, such that $A = \{q \in \mathbb{Q} | \exists_{q' \in A}(q < q')\}$ is a valid formula.

Hence, this formula must be, at least, verified at stalk level, for a given $x \in X$, i.e., since $\mathbb{Q}_x$ is the stalk of rational numbers, A_x has to be a subset of the rational numbers that is itself a lower Dedekind cut in a set-like manner.

It is straight forward to check that if A satisfies the previous condition then it is a Dedekind lower cut in sheaf-like manner. Next let us do so. Suppose A is a sheaf such that A_x is a Dedekind lower cut, for all $x \in X$. We must verify that the interpretation of $\exists_{q' \in A}(q < q')$, where q is a variable, is A. This must be checked locally, i.e., for every open $U \subseteq X$. If f_q is a section of A over U then it satisfies the condition $\exists_{q' \in A}(q < q')$, since A_x is a Dedekind lower cut and $f_q(x) \in A_x$. To be precise, these conditions imply that there exists $f_{q'}(x) \in A_x$ over U_x, such that $f_q(x) < f_{q'}(x)$. We can consider U_x small enough for f_q and $f_{q'}$ to be constant. Therefore we have that $q < q'$ over U_x. Now just consider the covering $\bigcup_{x \in U} U_x = U$.

Conversely, if f_q is a section of the interpretation of $\exists_{q' \in A}(q < q')$ over U then there is an open covering $\bigcup_\alpha U_\alpha = U$, such that there is $f_{q'_\alpha}$ for every α where $(q < q'_\alpha)$ over U_α.

As A_x is a lower Dedekind cut and $f_{q'_\alpha}(x) \in A_x$, then f_q is a section of A. In fact, $f_q(x) \in A_x$, for all $x \in U_\alpha$, since $f_q(x) < f_{q'_\alpha}(x)$.

In this section, we interpret formulae as sheaves of locally constant functions, simply by looking at its interpretation over the stalks, which turns out to be equivalent but easier to describe.

Let α be defined as a function between X and the set of extended reals, such that $\alpha(x)$ is the element defined by stalk A_x.

The function α is lower semi-continuous, i.e., for any real number a the subset $\{x \in X | a < \alpha(x)\}$ is an open of X.

Let us choose a rational number q with a representation belonging to A_x, such that $a < q$. Then there exists a neighborhood of x, small enough for q to be represented by a constant section of A. Therefore we can observe that, for any y in such a neighborhood, $q \leq \alpha(y)$, because q lies in the subsheaf A.

Since $q \leq \alpha(y)$, then q is in the Dedekind cut representation of $\alpha(y)$.

As q was supposed to be any rational, we have that $a \leq \alpha(y)$ and hence this neighborhood is contained in the given set.

Finally, given any lower semi-continuous function from the topological space X into the extended reals, then the subsheaf A of $\mathbb{Q}$ whose sections over an open U are the locally constant rational functions that are everywhere less than the given function, satisfy the given Dedekind lower cut condition. ∎

We also have the dual proposition.

Proposition 4.2.7 *The Dedekind upper cuts on the sheaf of rationals can be identified with upper semi-continuous functions from X into the extended reals.*

The interpretation of 3.5.3 implies that the function α_1 of A_1 and the function α_2 of A_2 are equal and have the same image, now restricted to the reals. As such, these pairs can be identified with the continuous real functions on the topological space X.

Theorem 4.2.8 *The sheaf of Dedekind real numbers is isomorphic to the sheaf of the continuous real functions (see 2.4.11), on the topological space X.*

If we had tried to build the sheaf of real numbers through Cauchy sequences we would end up with the sheaf of locally constant real valued functions.

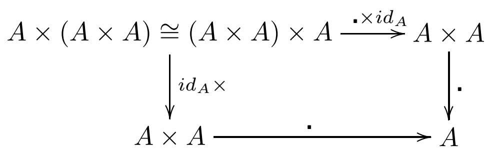
4.3 Rings and modules

If one is interested in algebraic constructions in the topos of sheaves, the next natural step would be to take a look at what the notion of a field should be. But here one comes to an intuitionistic cross-road, since there are several notions of field that are not intuitionistically equivalent. A similar situation has just been met regarding the sheaf of real numbers. Nonetheless, the final description of the sheaf of real numbers is quite nice and can be obtained by the interpretation of more than one construction of the real numbers. For fields, the situation is not so clear. Fortunately, for the concepts of ring and module we have natural definitions.

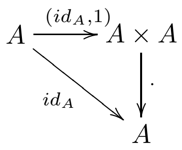
Definition 4.3.1 A *ring* in the category of sheaves on a topological space X is a sheaf A together with elements $1 \xrightarrow{0} A$ and $1 \xrightarrow{1} A$ and operation maps $A \times A \xrightarrow{+} A$ and $A \times A \xrightarrow{\cdot} A$ which satisfy the usual axioms for a ring in the language of the category of sheaves, i.e.

- associativity of addition,
  $\forall_{a \in A} \forall_{a' \in A} \forall_{a'' \in A} (a + (a' + a'') = (a + a') + a'')$
- identity of addition,
  $\forall_{a \in A} ((a + 0 = a) \wedge (0 + a = a))$
- $\forall_{a \in A} \exists_{a' \in A} (a + a' = 0 \wedge a' + a = 0)$
- commutativity of addition,
  $\forall_{a \in A} \forall_{a' \in A} (a + a' = a' + a)$
- associativity of multiplication,
  $\forall_{a \in A} \forall_{a' \in A} \forall_{a'' \in A} (a.(a'.a'') = (a.a').a'')$
- identity of multiplication,
  $\forall_{a \in A} ((a.1 = a) \wedge (1.a = a))$
- distributivity,
  $\forall_{a \in A} \forall_{a' \in A} \forall_{a'' \in A} (a.(a' + a'') = a.a' + a.a'')$

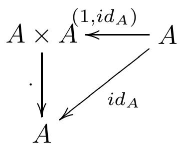
There is also a categorical notion of a ring in the category of sheaves, also by requiring A to have zero and identity elements, the operations mentioned above and a negation operation $A \xrightarrow{-} A$. The idea is to express the axioms above using the commutativity of certain diagrams. For instance, for the associativity of multiplication, one has



Also one can express that the identity element is indeed an identity for the multiplication of the ring by



and



In fact both these definitions are equivalent and correspond to the familiar concept of a sheaf of rings.

Definition 4.3.2 A *sheaf of rings* FR is defined as in 2.4.10, but being a functor to the category of (small) rings, i.e. $FR : \mathcal{O}(X)^{\mathrm{OP}} \to \mathit{Rng}$.

Example 4.3.3 The example 2.4.11 is a sheaf of rings because functions have pointwise addition and multiplication.

We will now introduce some notation. Thus $unit(A)$ will replace the expression

$$\{x^A \in A | \exists_{x'^A \in A}((x^A x'^A = 1) \wedge (x'^A x^A = 1))\}$$

Example 4.3.4 The formula $a \in unit(A)$, where a is a section of the ring A, denotes the largest open subset U of X, for which the element $a|_U$ is a unit in the ring $A(U)$. In the case where A is the ring of continuous real functions on the topological space, this determines the "largest" open set on which the function is non-zero, i.e., the support of the function.

The first definition coincides with the categorical one due to the fact that the zero, identity and negation of a ring are uniquely determined by the addition and multiplication of the ring. Also, in order to have a categorical definition, we use the fact that the concept of ring is given equationally in terms of the universally defined operations. The notion of field, however, cannot be equationally defined, because inversion exists only for non-zero elements, although we have, in the category of sheaves, the necessary language to make a definition of a field. However here we face the problem mentioned earlier. For the following discussion, let us assume that the condition $\neg(1 = 0)$ is satisfied. A field A could be defined as a ring that verifies the condition

$$\forall_{a \in A}(a \in unit(A) \vee a = 0)$$

The interpretation of this condition yields that a sheaf A is a field if, for each $x \in X$, the stalk A_x is a field. One immediately observes that the sheaf of rational numbers $\mathbb{Q}$ defined above is a field according to this notion; the sheaf of real numbers $\mathbb{R}$ as given above is however not a field in this sense. It should be clear that the stalk of $\mathbb{R}$ at $x \in X$ is the ring of germs of continuous real functions defined on open neighborhoods of $x \in X$. It is a ring but not a field, since the units are considered to be the f_x for each $f(x)$ that is non-zero, which is a different condition from having f_x non-zero in all the neighborhood. This is a somehow undesirable property and suggests that one should look for a more appropriate notion of field.

The sheaf of continuous real functions possesses an important property: if a function f, defined on an open U, is such that $f(x) = 0$ for all $x \in X$, then f is the zero in the stalk.

A ring A should then be a field if the following condition is satisfied:

$$\forall_{a \in A}(\neg(a \in unit(A)) \Rightarrow (a = 0))$$

By interpreting this formula, we have the desired situation that the sheaf of reals numbers $\mathbb{R}$ is a field, the sheaf of rational numbers $\mathbb{Q}$ is also a field and, more generally, any sheaf A that was a field under the previous definition remains a field under this one.

Thus this definition seems to be the "correct" one. Still one might consider another definition where a ring A would be considered a field provided the following formula was satisfied:

$$\forall_{a \in A}(\neg(a = 0) \Rightarrow a \in unit(A))$$

This definition, however, does not make the sheaf of real numbers $\mathbb{R}$ a field, and so we will henceforth consider a sheaf A to be a field if the second condition we stated is verified.

What seems important to grasp from the given exposition is that the problem we faced is due to the fact that the conditions

- $\forall_{a \in A}(a \in unit(A) \vee a = 0)$
- $\forall_{a \in A}(\neg(a \in unit(A)) \Rightarrow (a = 0))$
- $\forall_{a \in A}(\neg(a = 0) \Rightarrow a \in unit(A))$

which are classically equivalent definitions of a field, are not equivalent in an intuitionistic framework. Choosing one formula over another is not done here with the idea that we pick the right one. All the given formulae yield, when interpreted, perfectly valid intuitionistic notions of "field", which are, in this sense, different mathematical objects. The criterion for choosing a definition was that we wanted $\mathbb{Q}$ and $\mathbb{R}$ to be fields.

Following the ideas given above, we will now present some definitions:

Definition 4.3.5 A commutative ring A is defined to be an *integral domain* if the following condition is satisfied

$$\forall_{a\in A}\forall_{a'\in A}((aa'=0 \wedge \neg(a'=0)) \Rightarrow a=0)$$

We have also a natural theory for modules, as can be seen from the following definitions.

Definition 4.3.6 A module M over a ring A is defined to be a sheaf with two operations, given by the maps $M\times M \xrightarrow{+} M$ and $A \times M \xrightarrow{\cdot} M$, for which the usual formulae are satisfied:

- $\forall_{a\in A}\forall_{v_1\in M}\forall_{v_2\in M}(a.(v_1+v_2)=a.v_1+a.v_2)$
- $\forall_{a_1\in A}\forall_{a_2\in A}\forall_{v\in M}((a_1+a_2).v=a_1.v+a_2.v)$
- $\forall_{a_1\in A}\forall_{a_2\in A}\forall_{v\in M}(a_1.(a_2.v)=(a_1a_2).v)$
- $\forall_{a\in A}(1.v=v)$

There is also a description of this structure at a categorical level. Both these definitions are equivalent to the concept of a sheaf of modules over a sheaf of rings.

Definition 4.3.7 A sheaf of modules FM over a sheaf of rings FR is defined as in 2.4.10, but such that $FM(U)$ is a $FR(U)$-module for each open $U \subseteq X$.

4.4 Finitely generated modules

We will now define the notion of sequence in a module M.

Definition 4.4.1 A *sequence* in a module M is a map

$$\mathbb{N} \to M$$

So the sheaf $M^{\mathbb{N}}$ will be called the sheaf of sequences in the module M.

One can now define a map

$$A^{\mathbb{N}} \times M^{\mathbb{N}} \to M^{\mathbb{N}}$$

to be a natural extension of the multiplication map $A \times M \to M$. Its interpretation is as one would expect. It is defined by resorting to the exponential adjoint, the multiplication and the evaluation maps.

Definition 4.4.2 A finite sequence in a module M is an element $y \in M^{\mathbb{N}}$ (the sheaf of sequences in M) for which there exists an element $n \in \mathbb{N}$ such that the following condition is satisfied:

$$\forall_{i\geq n}\ y_i = 0$$

A finite sequence is interpreted as a sequence (y_i) of sections of M, for which there is a locally constant function n to the natural numbers, such that, on any open set, the sequence is zero for all $i \geq n$. Hence this notion of finite sequence is only locally finite because the locally constant function n is not necessarily bounded.

Definition 4.4.3 A module M is generated by a finite sequence if the following formula is satisfied:

$$\forall_{m \in M} \exists_{a \in A^{\mathbb{N}}} \sum_{i=0}^{n} a_i . y_i = m$$

In this formula, the term $\sum_{i=0}^{n} a_i . y_i = m$ denotes the evaluation of the sequence of partial sums of the given product at a natural number, beyond which the finite sequence is zero.

Let us introduce the sequence of partial sums. We will define a map $M^{\mathbb{N}} \to M^{\mathbb{N}}$ that assigns to each sequence in a module M its sequence of partial sums. Consider the map $\mathbb{N} \times M^{\mathbb{N}} \times M^{M^{\mathbb{N}}} \to M$ given by

$$\mathbb{N} \times M^{\mathbb{N}} \times M^{M^{\mathbb{N}}} \xrightarrow{diag_{M^{\mathbb{N}}}} \mathbb{N} \times M^{\mathbb{N}} \times M^{\mathbb{N}} \times M^{M^{\mathbb{N}}} \xrightarrow{e_{\mathbb{N}} \times e_{M^{\mathbb{N}}}} M \times M \xrightarrow{+} M$$

where $diag_{M^{\mathbb{N}}}$ is the diagonal map and $e_{\mathbb{N}}$, $e_{M^{\mathbb{N}}}$ are the evaluation maps. The exponential adjoint of the previous map is a map $\mathbb{N} \times M^{M^{\mathbb{N}}} \to M^{M^{\mathbb{N}}}$. Next consider the map $1 \to \mathbb{N} \times M^{M^{\mathbb{N}}}$ given by the product of the following maps:

$$1 \xrightarrow{0} \mathbb{N}$$

$$1 \xrightarrow{0} \mathbb{N} \longrightarrow M^{M^{\mathbb{N}}}$$

where the second map is the exponential adjoint of $\mathbb{N} \times M^{\mathbb{N}} \xrightarrow{e} M$.

The application of the universal property of $\mathbb{N}$ to the maps $1 \to \mathbb{N} \times M^{M^{\mathbb{N}}}$ and $\mathbb{N} \times M^{M^{\mathbb{N}}} \to M^{M^{\mathbb{N}}}$ yields a map $\mathbb{N} \to \mathbb{N} \times M^{M^{\mathbb{N}}}$. Now we simply need to compose it with the projection to $M^{M^{\mathbb{N}}}$ and apply the exponential adjoint twice to obtain the desired map $M^{\mathbb{N}} \to M^{\mathbb{N}}$.

A finite sequence generates a module if, for each $x \in X$, the sequence $(y_{i,x})$ generates the stalk M_x as an A_x-module.

4.5 Free modules

Definition 4.5.1 A finite sequence in a module M is said to be linearly independent in the module if the following formula is satisfied:

$$\forall_{a \in A^{\mathbb{N}}} (\sum_{i=0}^{n} a_i . y_i = 0 \Rightarrow \forall_{i<n} a_i = 0)$$

Finally one can refer to a finite basis of a module M over the ring A as a finite sequence that generates M and is linearly independent. The given notions of generating a module and being linearly independent actually translate into the following notions.

A finite sequence is linearly independent if, for each $x \in X$, the sequence $(y_{i,x})$ is linearly independent in M_x over the ring A_x.

A finite sequence is a basis if, for each $x \in X$, the sequence $(y_{i,x})$ is a basis for the A_x-module M_x. Hence, for any open set on which the natural number is constant, the sections $y_0, ..., y_{n-1}$ establish an isomorphism between the module M and the direct sum A^n of the ring A, indexed by the natural number n.

Definition 4.5.2 The sheaf of finite sequences in a module M, $\text{fin}(M)$, results as the interpretation of the formula

$$\forall_{i \geq n} y_i = 0$$

as a subsheaf of $\mathbb{N} \times M^{\mathbb{N}}$, where n and y are considered to be variables of types $\mathbb{N}$ and $M^{\mathbb{N}}$, respectively.

Definition 4.5.3 The sheaf of finite sequences which generate the module M, $\text{fgen}(M)$, is the subsheaf of the sheaf $\text{fin}(M)$ obtained by interpreting the following formula, where n and y are taken as variables:

$$\forall_{m \in M} \exists_{a \in A^{\mathbb{N}}} \sum_{i=0}^{n} a_i . y_i = m$$

Definition 4.5.4 The sheaf of finite sequences which are linearly independent in M, $\text{find}(M)$, is the subsheaf of the sheaf $\text{fin}(M)$ obtained by interpreting the following formula, where n and y are taken as variables:

$$\forall_{a \in A^{\mathbb{N}}} (\sum_{i=0}^{n} a_i . y_i = 0 \Rightarrow \forall_{i<n} a_i = 0)$$

Definition 4.5.5 The sheaf of finite bases, $\text{fbasis}(M)$, is the largest subsheaf of $\text{find}(M)$ and $\text{fgen}(M)$.

It is now possible to define what it is for a module M to have a finite basis.

Definition 4.5.6 A module M has a finite basis, i.e., M is free, if the following condition is satisfied:

$$\exists_{n \in \mathbb{N}} \exists_{y \in M^{\mathbb{N}}} (n, y) \in \text{fbasis}(M)$$

This formula gets interpreted into the condition that a sheaf of modules has a finite basis if it is locally finitely free over the sheaf of rings.

Definition 4.5.7 A sheaf of modules FM is *locally finitely free* over a sheaf of rings FR if, for any $x \in X$, there is an open U that contains x, over which FM is finitely free.

4.6 Projective modules

In the previous section, the notion of a homomorphism between modules was skipped, but it is now going to be introduced, in order to define a projective module.

Definition 4.6.1 A map $f : M \to M'$ is a module homomorphism between modules M and M' over A if the following conditions are satisfied:

- $\forall_{m_1 \in M} \forall_{m_2 \in M} f(m_1 + m_2) = f(m_1) + f(m_2)$
- $\forall_{m \in M} \forall_{a \in A} f(a.m) = a.f(m)$

One can also define the sheaf of homomorphisms of A-modules, denoted by $\hom_A(M, M')$, as being the interpretation of the above formulae, where f is taken as a variable of type M'^M.

A notable subsheaf of $\hom_A(M, M')$ is the sheaf $Epi_A(M, M')$ described by the formula:

- $\forall_{m' \in M'} \exists_{m \in M} f(m) = m'$

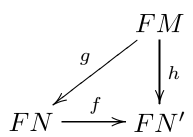
Both these concepts get interpreted into their categorical versions, respectively that of map of sheaves of modules and that of epimorphisms of sheaves of modules.

Definition 4.6.2 A module P over a ring A is called a *projective module* if, for any modules M, M' over A, the following condition is satisfied:

$$\forall_{f \in Epi_A(M,M'} \forall_{h \in \hom_A(P,M')} \exists_{g \in \hom_A(P,M)} fg = h$$

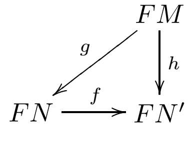
where by fg we mean the composite homomorphism of f and g.

Proposition 4.6.3 *The interpretation of a projective module M is a sheaf of modules FM that verifies the following: given an epimorphism f and a morphism h, there is a covering of X such that, on each of its opens, there is a morphism g that makes the following diagram commute:*



Apparently this definition does not correspond to the usual notion of projective sheaf of modules, since in our definition there is a covering that depends on the morphisms f and h. The usual definition for a projective sheaf of modules is the following:

Definition 4.6.4 A sheaf of modules FM is *projective* if, given an epimorphism f and a morphism h, there is a morphism g that makes the following diagram commute:



4.7 Kaplansky vs. Swan

In this chapter we made a systematic interpretation of some natural concepts of algebra and analysis in the topos Sh(X), i.e., we took a look at what a natural syntactical representation of these concepts was interpreted into. This interpretation process introduced us to which objects in the category Sh(X) play the role of our notions of natural numbers, real numbers, modules, etc. We now take a step further in this analogy and will take a look at what a theorem of Kaplansky (Any finitely generated projective module over a local ring admits a finite basis) will translate into. But first let us try to clarify which relations we are uncovering.

Suppose we were willing to prove that Kaplansky's theorem is valid over Sh(X). We can try to achieve this in two different ways: by a semantical approach, interpreting the theorem and proving that the subsheaf obtained is the sheaf itself, or by a syntactical approach, using a correct deductive system for intuitionistic first order logic and finding a derivation of the theorem in it.

An informal proof of Kaplansky's theorem that is intuitionistically valid is given in [Mul73]. We are however interested in relating the interpretation of Kaplansky's theorem with familiar theorems in Sh(X).

Now let us just introduce some more concepts in order to reach Kaplansky's theorem.

Definition 4.7.1 A ring A will be called a *local ring*, provided that the following condition is satisfied

- $\forall_{a \in A} a \in unit(A) \vee 1 - a \in unit(A)$

This condition is actually equivalent to the following one

- $\forall_{a \in A} \forall_{a' \in A} a + a' \in unit(A) \Rightarrow a \in unit(A) \vee a' \in unit(A)$

These conditions are verified whenever each stalk of the sheaf is a local ring.

Now we try to make clear what the interpretation of Kaplansky's theorem yields.

As explained, all we have to do is to restate the theorem internally in Sh(X).

Proposition 4.7.2 (Kaplansky) *Any locally finitely generated projective sheaf of modules FM, over a locally local sheaf of rings A, is a locally free sheaf of modules.*

As was mentioned above, there is an intuitionistically valid proof of Kaplansky's theorem [Mul73]. So this proposition must hold in Sh(X).

A direct proof of a similar result in terms of sheaves can also be found in [Mor02]. It is similar, since it is almost the same statement but, instead of locally projective sheaf of modules, one uses projective sheaf of modules.

Next we will try to establish a relation between this proposition about sheaves of modules and some more familiar concepts in Sh(X), like $\mathbb{R}$-vector bundles.

Let us assume we are working with $C^{\mathbb{R}}(X)$, the sheaf of rings of continuous real valued functions. We can in fact do this because the sheaf of continuous real functions is a local ring, since all of its stalks are local rings.

First we need the concept of (locally trivial) real vector bundles over the topological space $\mathcal{O}(X)$. Let us take a detour, in order to make a clearer introduction of this structure.

Definition 4.7.3 Let $\mathcal{O}(X)$ be a topological space (assumed to be compact Hausdorff). A $\mathbb{R}$-vector bundle consists of a continuous open surjective map $p : E \to X$ with the following properties:

- each fiber $p^{-1}(x)$ of p, for each $x \in X$, is a finite-dimensional vector space over $\mathbb{R}$;
- there are continuous maps $E \times E \to E$ and $\mathbb{R} \times E \to E$ which restrict to vector addition and scalar multiplication on each fiber.

Naturally one can form the category of $\mathbb{R}$-vector bundles $p : E \to X$, in which the morphisms are the commutative diagrams

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ X & = & X \end{array}$$

where f is linear on each fiber.

For each n, a notable element of this category is the trivial $\mathbb{R}$-vector bundle of rank n, which is the map $X \times \mathbb{R}^n \overset{\pi_1}{\to} X$, where π_1 is the projection on the first factor.

The notion of locally trivial $\mathbb{R}$-vector bundle is a restriction of the above notion, where we also request that, for each $x \in X$, there is a neighborhood U of x and an isomorphism (in the category of $\mathbb{R}$-vector bundles) from $p^{-1}(U) \overset{p|_{p^{-1}(U)}}{\to} U$ to a trivial bundle, of the same rank, over U.

A similar result to proposition 4.7.2 is then the following theorem due to Swan, a proof of which can be seen at [Ros94], chapter 1, section 6.

Theorem 4.7.4 (Swan) *For any compact Hausdorff space X, the category of finite dimensional (locally trivial)* $\mathbb{R}$*-vector bundles is equivalent to the category of finitely generated projective* $C^{\mathbb{R}}(X)$*-modules.*

What we now must do is to check that there is a relation between $\mathbb{R}$-vector bundles and locally free sheaves and also a relation between finitely generated projective $C^{\mathbb{R}}(X)$-modules and locally finitely generated projective modules.

The relation between $\mathbb{R}$-vector bundles and locally free sheaves of $C^{\mathbb{R}}(X)$-modules can be seen at [WJ80], theorem 1.13, chapter II.

The link between finitely generated projective $C^{\mathbb{R}}(X)$-modules and locally finitely generated projective $C^{\mathbb{R}}(X)$-modules is not, to the best of my knowledge, established. Nevertheless if it were indeed obtained, it would proof an equivalence between the following categories:

- $\mathbb{R}$-vector bundles;
- finitely generated projective $C^{\mathbb{R}}(X)$-modules;
- locally finitely generated projective $C^{\mathbb{R}}(X)$-modules;
- locally free sheaves of $C^{\mathbb{R}}(X)$-modules

Chapter 5

Conclusion

Let us analyse what we have done up to this point and draw some final conclusions.

We set out to define algebraic constructions in a topos. The idea was to think of a topos as a universe, as a "universe of sets", and check out what our usual algebraic constructions are in this generalized universe. We started by formally defining a topos and providing some examples of toposes in chapter 2. This provided a clear notion of what we mean by a "universe of sets".

Then we associated to a topos its internal language and a topos semantics. We did this in order to have a formal language in which we could define our constructions. This actually just added a layer of syntax over our topos that got interpreted in a very natural way.

Nonetheless this is indeed a crucial step, since it enlightened us about the logical structure we were dealing with. This is not immediately realized in chapter 3. Still the subobject classifier of any topos is a Heyting algebra and therefore we are in the realm of intuitionistic logic. This raises interesting questions about how to deal with the construction of real numbers.

Finally we described explicitly in the topos $\mathrm{Sh}(X)$ the concepts previously given solely by the formal language. In chapter 4 we also introduced the concepts of ring and module in a topos, which indeed could have been done in a more general way.

At this stage, we are very close to our final objective: to establish a relation between Kaplansky's theorem and Swan's Theorem. The basic idea was to pick a theorem in the language of a topos of sheaves which could be proved in intuitionistic logic and interpret it in the topos. An intuitionistic proof of Kaplansky's theorem can be seen in [Mul73]. Then comes the task of interpreting Kaplansky's theorem and relating its interpretation with Swan's theorem. In our case it was relatively simple to interpret Kaplansky's theorem. What turned out be rather difficult was to relate what we had just proved with familiar concepts in $\mathrm{Sh}(X)$.

Nonetheless this technique is rather interesting and provides us with a way of establishing theorems internally in the topos.

Problems with this technique are that we must be very careful to ensure that our propositions are valid in intuitionistic logic and that finding a relation between the interpretation of the structures we are working with, and familiar constructions might reveal itself to be rather hard. It might even be the case that there are no familiar constructions. In that case the methods we have seen give us plenty of objects to work in a topos. Nonetheless the author is not aware of such a case.

Intuitionistic logic is by itself harder to work with than classical logic. Since it allows neither the rule of the excluded middle nor the axiom of choice. Mathematicians are in general very used to theses laws and it might prove to be a real challenge to elaborate proofs without them.

This technique is not the easiest way to establish Swan's theorem. Indeed it turned out to be rather hard. Also some of the propositions used in the process are proved in a similar fashion to Swan's theorem. Still this technique allows us to establish a connection between theorems in rings and modules and theorems in sheaves. Uncovering these relations is by itself interesting.

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