# Fibring Logics with Topos Semantics

M. E. Coniglio[1] A. Sernadas[2] C. Sernadas[2]

[1] Departamento de Filosofia, Universidade Estadual de Campinas, Brazil
[2] CLC, Departamento de Matemática, IST, Portugal

**Abstract**

The concept of fibring is extended to higher-order logics with arbitrary modalities and binding operators. A general completeness theorem is established for such logics including *HOL* and with the meta-theorem of deduction. As a corollary, completeness is shown to be preserved when fibring such rich logics. This result is extended to weaker logics in the cases where fibring preserves conservativeness of *HOL*-enrichments. Soundness is shown to be preserved by fibring without any further assumptions.

*Key words*: Modal higher-order logic, categorical logic, completeness, conservative extensions.

## 1 Introduction

Given the interest in the topic of combination of logics [2] and the significance of fibring [5, 6, 10, 13] among the combination mechanisms, we have been following a research program directed at establishing preservation results on fibring. In [16] we established the preservation of completeness when fibring propositional based logics endowed with an algebraic semantics. This transference result was later extended to first-order based logics in [12], albeit at the expense of a quite complex semantics.

It seems worthwhile to pursue also the study of fibring of higher-order based logics. Indeed, the applications that have been motivating our work in combination of logics require sometimes the use of higher-order arbitrary binding operators (like quantifiers but not only), besides arbitrary modal-like operators. Typical applications of powerful techniques for combining logics include (but are not restricted to) multiformalism (read multilogic) approaches in software engineering, security, knowledge representation and linguistics. In such approaches the use of several logics in the same project is the rule, rather than the exception. In short, a specification/theory is build with fragments scattered over several logics. Fibring can help here by allowing the use of a unique logic where all fragments can be expressed and, more importantly, related and mixed. On a more theoretical vein, we also had hopes that working with higher-order logic would allow a simpler, more abstract semantics for binding operators, when compared with the technical difficulties we faced in [12] where we worked only with first-order structures. It turned out to be the case.

In this paper, we start by defining in Section 2 a wide class of logic systems endowed with topos semantics and with Hilbert calculi possibly including rules with provisos. This class is shown to encompass many commonly used logics, such as propositional logics, modal logics, quantification logics, typed lambda calculi and higher-order logics. Arbitrary modalities and binding operators are allowed, as well as any choice of rigid and flexible functions.

The proposed semantics is a generalization of the traditional topos semantics of higher-order logic. This generalization preserves the simplicity and elegance of the traditional one, while being able to cope with arbitrary modalities, quantifiers and other binding operators. Two entailments are defined: the local entailment as usually used in categorical logic and the global entailment necessary to deal with necessitation and generalization. Examples are given of common logics for which it is possible to lift the original semantics to the topos semantics level, while preserving the denotation of terms (and formulae). This shows that no great generality is lost by assuming that we are working with logics endowed with the proposed topos semantics.

The proposed deduction mechanism is in the Hilbert calculus style, but allowing rules with provisos. A formal treatment of provisos was first proposed in [11]. Such a formal treatment is necessary when combining logics since we must be able to know how to apply rules from a given logic in the environment of the combined logic. In this paper, we develop a quite simple notion of universal proviso that is enough for the purpose at hand of fibring deduction systems using only provisos related to binding.

In Subsection 2.4, a general completeness theorem is established by first showing that every consistent deduction system including *HOL* and with the meta-theorem of deduction (*MTD*) has a canonical model. The construction of the canonical model follows the traditional approach in topos semantics of higher-order logic (see for instance [1]), but with the adaptations made necessary by the richer language (arbitrary modalities and binding operators) and the need to work with two entailments.

In Section 3, we start by defining both unconstrained and constrained fibring of logic systems in the sense of Section 2 and show that soundness is preserved by fibring. Then, as a direct corollary of the completeness theorem of Subsection 2.4, we obtain a first completeness preservation result: the fibring of full logic systems endowed with deduction systems including *HOL* and with *MTD* is also complete. We also show that, under some weak conditions, a full logic system is complete iff it can be conservatively enriched with *HOL*. Then, as a direct corollary of this result, we obtain a second completeness preservation result: the fibring of two full, complete logic systems is complete provided that conservativeness of *HOL*-enrichment is preserved. We leave as an open problem finding sufficient conditions for the preservation of *HOL*-enrichment.

## 2 Higher-order based logics

We make precise in this section the notion of logic system (signature, class of models, deduction rules) that we need. We conclude the section with a general completeness theorem.

The envisaged notion of logic system was chosen carefully having in mind two main goals: (i) it should be general enough to encompass many commonly used logics, namely with arbitrary modalities and (possibly higher-order) binding operators like quantifiers; (ii) it should be possible to construct a logic system (in our sense) from any given such logic preserving the denotation of terms/formulae at each model.

The first desideratum is motivated by the *homogeneous scenario* that we want to set up for fibring. Indeed, in such a scenario, when combining logics one assumes that all of them are presented in the same style (with signatures of the same form, with models of the same kind and with deduction systems of the same nature). That is, in the homogeneous scenario, when combining two logics we assume that they are objects in the chosen category of logics. Therefore, the first part of this section is dedicated to setting up the category **Log** of logic systems where in the next section fibrings are to be defined as (universal) constructions.

It should be mentioned that the *homogeneous scenario* is obviously much more tractable than the heterogeneous one. In the heterogeneous case, we may want to combine, for instance, a logic endowed with a tableaux deduction system and a logic endowed with a Hilbert (axiomatic) deduction system. At the semantic level logics can also be presented in dramatically different styles. The applications we have in mind (mentioned in the Introduction) require that, in the future, the problem of heterogeneous fibring should be addressed and solved.

Meanwhile, fibring in the homogeneous scenario can be put to useful work if we are able to provide a means to convert any given logic to a logic system in our sense. That is, before combining two logics we first present them as logic systems in the category **Log** defined in this section. Then, we are able to proceed with the fibring in a homogeneous situation.

The second desideratum above addresses this *preparation step*. The least we would like to have is to make sure that the conversion step preserves the entailments of the given logic. Indeed, we can claim the logic was not changed if the entailments are the same.

But we go further at the semantic level. We would like to recognize each model of the original logic in the corresponding logic system. A similar requirement should of course apply to the language. We would like to be able to recognize each symbol of the original logic in the corresponding logic system, while preserving the language. On the other hand, we refrain to address the problem of converting the deduction rules and we assume that the logics are given from the beginning with a Hilbert axiomatic system.

## 2.1 Language

Assume given once and for all the set $S$ with distinguished elements $\mathbf{1}$ and $\mathbf{\Omega}$. We denote by $\Theta(S)$ the set inductively defined as follows: (i) $s \in \Theta(S)$ whenever $s \in S$; (ii) $(\theta_1 \times \cdots \times \theta_n) \in \Theta(S)$ whenever $\theta_1, \ldots, \theta_n \in \Theta(S)$ for integer $n \geq 2$; (iii) $(\theta \to \theta') \in \Theta(S)$ whenever $\theta, \theta' \in \Theta(S)$. As usual, we write $\theta^n$ for the $n$-th power of $\theta$ (the product of $\theta$ with itself $n$ times) and by convention $\theta^0$ is $\mathbf{1}$ and $\theta^1$ is $\theta$. The elements of $S$ are known as *sorts* or *base types*. The elements of $\Theta(S)$ are known as *types* over $S$. Base types $\mathbf{1}$ and $\mathbf{\Omega}$ are called the *unit sort* and the *truth value sort*, respectively. Assume also as given once and for all the families

- $\Xi = \{\Xi_\theta\}_{\theta \in \Theta(S)}$ where each $\Xi_\theta$ is a denumerable set;
- $X = \{X_\theta\}_{\theta \in \Theta(S)}$ where each $X_\theta$ is a denumerable set.

The elements of each $\Xi_\theta$ and $X_\theta$ are called *schema variables* and *variables*, respectively, of type $\theta$.

**Definition 2.1** A *signature* is a triple $\Sigma = \langle R, F, Q \rangle$ such that:

- $R = \{R_{\theta\theta'}\}_{\theta,\theta' \in \Theta(S)}$ where each $R_{\theta\theta'}$ is a set;
- $F = \{F_{\theta\theta'}\}_{\theta,\theta' \in \Theta(S)}$ where each $F_{\theta\theta'}$ is a set;
- $Q = \{Q_{\theta\theta'\theta''}\}_{\theta,\theta',\theta'' \in \Theta(S)}$ where each $Q_{\theta\theta'\theta''}$ is a set. △

The elements of each $R_{\theta\theta'}$ are called *rigid function symbols* of type $\theta\theta'$. The elements of each $F_{\theta\theta'}$ are called *flexible function symbols* of type $\theta\theta'$. The elements of each $Q_{\theta\theta'\theta''}$ are called (binding) *operator symbols* of type $\theta\theta'\theta''$, like quantifiers but also lambda-abstraction and set comprehension.

**Definition 2.2** The family $ST(\Sigma) = \{ST(\Sigma)_\theta\}_{\theta \in \Theta(S)}$ is inductively defined as follows:

- $\xi \in ST(\Sigma)_\theta$ whenever $\xi \in \Xi_\theta$;
- $x \in ST(\Sigma)_\theta$ whenever $x \in X_\theta$;
- $\xi^x_{\xi'} \in ST(\Sigma)_\theta$ whenever $\xi \in \Xi_\theta$, $x \in X_{\theta'}$ and $\xi' \in \Xi_{\theta'}$;
- $(r\, t) \in ST(\Sigma)_{\theta'}$ whenever $r \in R_{\theta\theta'}$ and $t \in ST(\Sigma)_\theta$;
- $(f\, t) \in ST(\Sigma)_{\theta'}$ whenever $f \in F_{\theta\theta'}$ and $t \in ST(\Sigma)_\theta$;
- $(qx\, t) \in ST(\Sigma)_{\theta''}$ whenever $q \in Q_{\theta\theta'\theta''}$, $x \in X_\theta$ and $t \in ST(\Sigma)_{\theta'}$;
- $\langle t_1, \dots, t_n \rangle \in ST(\Sigma)_{\theta_1 \times \cdots \times \theta_n}$ whenever $t_i \in ST(\Sigma)_{\theta_i}$ for $i = 1, \dots, n$ with $n \neq 1$;
- $(t)_i \in ST(\Sigma)_{\theta_i}$ whenever $t \in ST(\Sigma)_{\theta_1 \times \cdots \times \theta_n}$ for $1 \leq i \leq n$ with $n \geq 2$. △

The elements of each $ST(\Sigma)_\theta$ are called *schema terms* of type $\theta$. Schema terms of type $\mathbf{\Omega}$ are also known as *schema formulae*. Schema terms without schema variables are called *terms*: $T(\Sigma)_\theta$ denotes the set of terms of type $\theta$. Schema formulae without schema variables are called *formulae*. We write $SL(\Sigma)$ and $L(\Sigma)$ for $ST(\Sigma)_{\mathbf{\Omega}}$ and $T(\Sigma)_{\mathbf{\Omega}}$, respectively.

The traditional concepts associated to binding operators are assumed to be carried over to this language. For instance, an occurrence of a variable $x$ in a term $t$ is said to be bound iff it appears within the scope of some (binding) operator $q$ applied to $x$. Note that $x$ is bound in $\xi^x_{\xi'}$.

The following examples show that the proposed notion of signature is rich enough to encompass a wide variety of logics. More importantly, the generated language is not changed in any significant way.

**Example 2.3** *Modal propositional logic.*
Given a traditional propositional signature $\mathcal{P}$ (the set of propositional variables):

- The members of the families $R$ and $F$ are empty, except:
  - $R_{\mathbf{1}\mathbf{\Omega}} = \{\mathbf{f}, \mathbf{t}\}$;
  - $R_{\mathbf{\Omega}\mathbf{\Omega}} = \{\neg\}$;
  - $R_{\mathbf{\Omega}^2\mathbf{\Omega}} = \{\wedge, \vee, \Rightarrow, \Leftrightarrow\}$;
  - $F_{\mathbf{1}\mathbf{\Omega}} = \mathcal{P}$;
  - $F_{\mathbf{\Omega}\mathbf{\Omega}} = \{\Diamond, \Box\}$.
- All members of the family $Q$ are empty. △

**Example 2.4** *Propositional logic.*
As in Example 2.3, except $F_{\mathbf{\Omega}\mathbf{\Omega}} = \emptyset$. Note that it is useful to keep the propositional symbols as flexible for the purpose of fibring as will be explained in Section 3. △

**Example 2.5** *First-order predicate logic.*
Given a first-order signature $\langle \mathcal{G}, \mathcal{P} \rangle$ where $\mathcal{G} = \{\mathcal{G}_n\}_{n\in\mathbb{N}}$ and $\mathcal{P} = \{\mathcal{P}_n\}_{n\in\mathbb{N}^+}$ (the families of sets of fol function symbols and predicate symbols, respectively, of different arities) and a base sort $\mathbf{i}$ different from $\mathbf{1}$ and $\mathbf{\Omega}$:

- All members of the families $R$ and $F$ are empty, except:
  - $R_{\mathbf{i}^n\mathbf{i}} = \mathcal{G}_n$ for $n \in \mathbb{N}$;
  - $R_{\mathbf{\Omega}\mathbf{\Omega}} = \{\neg\}$;
  - $R_{\mathbf{\Omega}^2\mathbf{\Omega}} = \{\wedge, \vee, \Rightarrow, \Leftrightarrow\}$;
  - $F_{\mathbf{i}^n\mathbf{\Omega}} = \mathcal{P}_n$ for $n \in \mathbb{N}^+$.
- All members of the family $Q$ are empty, except:
  - $Q_{\mathbf{i}\mathbf{\Omega}\mathbf{\Omega}} = \{\exists, \forall\}$.

Note that, again having in mind interesting fibrings, here we chose functions to be rigid and predicates to be flexible. △

**Example 2.6** *Pure typed lambda-calculus.*

- The members of the families $R$ and $F$ are empty, except:
  - $R_{((\theta\to\theta')\times\theta)\theta'} = \{\mathbf{app}_{\theta\theta'}\}$;
  - $R_{\theta^2\mathbf{\Omega}} = \{=_\theta\}$.
- All members of the family $Q$ are empty, except:
  - $Q_{\theta\theta'(\theta\to\theta')} = \{\lambda_{\theta\theta'}\}$.

Note that here we chose not to include flexible elements in the signature. △

**Example 2.7** *Higher-order intuitionistic logic.*

- The members of the families $R$ and $F$ are empty, except:
  - $R_{((\theta\to\theta')\times\theta)\theta'} = \{\mathbf{app}_{\theta\theta'}\}$;
  - $R_{\theta^2\mathbf{\Omega}} = \{=_\theta\}$.
- All members of the family $Q$ are empty, except:
  - $Q_{\theta\mathbf{\Omega}(\theta\to\mathbf{\Omega})} = \{\mathbf{set}_\theta\}$.

Note that we chose not to include flexible elements in this signature that will be denoted by $\Sigma_{HOL}$ in the sequel. Other logical operations (true, false, the propositional connectives and the traditional higher-order quantifiers) can be introduced through abbreviations as can be seen in any standard book on higher-order logic (for instance [1]). △

In subsequent examples, we may omit the typing of the variables and other symbols when no confusion arises, writing $=$ for $=_\theta$ and so on. We may also use traditional infix notation, writing $(\gamma_1 \wedge \gamma_2)$ for $(\wedge\langle\gamma_1, \gamma_2\rangle)$, $\{x : \gamma\}$ for $(\mathbf{set} x\, \gamma)$, $t(t')$ for $(\mathbf{app}\langle t, t'\rangle)$ and so on. Finally, we may write simply $f$ instead of $(f\langle\rangle)$ whenever $f \in F_{\mathbf{1}\theta}$.

## 2.2 Semantics

When working with higher-order based logics, it is natural to adopt a topos theoretic semantics in the style of categorical logic (see for instance [9]).

However, a slight generalization is needed in order to fulfill the second desideratum discussed a the beginning of this section. Indeed, given a (possibly general) Kripke semantics of an arbitrary modality we would like to be able to generate the corresponding topos semantics while preserving the original models in a precise sense: each of the original models should be converted into a topos model with the same denotation of terms/formulae.

We tried to achieve this objective with the traditional topos semantics in categorical logic. The obvious approach led us to consider the (modal) complete Heyting algebra of truth values induced by each Kripke model. Then, each such algebra $H$ induced a topos by the well known technique of $H$-sets (see for instance [15]). But the approach failed badly for non-S4 modalities. With hindsight, this should be expected by the essentially intuitionistic nature of the approach. Indeed, the internal modality of traditional topos semantics is of course the intuitionistic box and, therefore, a S4 box (see for instance [7]).

Since we wanted to be able to cope with arbitrary modal-like operators we were led to a more general topos semantics achieved by endowing each model with an extra parameter (an object $W$ of the topos) playing the role of the world space. As shown in the examples at the end of this subsection, this generalization was effective with respect to the issue at hand: denotation of terms/formulae is preserved when obtaining a topos model from a model in a given logic. Hence, entailment is also preserved. Furthermore, the extra parameter also allows an explicit distinction between modal and other operators that helps a reader better acquainted with more traditional semantics.

However, it should be stressed that the extra parameter is not necessary for achieving completeness (as proved in Subsection 2.4). It is only necessary for being able to generate a topos model from each given Kripke model preserving the denotation of terms/formulae. As explained before, this is essential in order to make the homogeneous scenario of fibring more useful in applications.

From now on, we use the following notation in the context of topos theory. If $f : A \times B \to C$ then $trn(f, B) : A \to C^B$ is the exponential transpose of $f$ obtained from the definition of $C^B$. If $g : A \to C^{B\times D}$ then $ctr(g, D) : A \times D \to C^B$ is the exponential cotranspose of $g$ obtained from the definition of $(C^B)^D$ and the isomorphism between $(C^B)^D$ and $C^{B\times D}$. Given an object $A$ of a topos $\mathcal{E}$, we denote by $Sub(A)$ the lattice of (equivalence classes of) subobjects of $A$. Finally, the extent (or support) of an object $A$, denoted by $E(A)$, is the (domain of the) subobject of 1 given by $\exists_{!_A}(id_A)$ (where $!_A$ is the unique morphism from $A$ to the terminal object 1).

**Definition 2.8** A $\Sigma$-*structure* is a triple $M = \langle \mathcal{E}, W, \cdot_M \rangle$ where $\mathcal{E}$ is a topos, $W$ an object of $\mathcal{E}$ such that $E(W) = 1$, and $\cdot_M$ an interpretation map such that:

- for $\theta \in \Theta(S)$, $\theta_M$ is an object of $\mathcal{E}$ such that:
  - $\mathbf{1}_M$ is terminal;
  - $\mathbf{\Omega}_M$ is subobject classifier $\Omega$;
  - $(\theta_1 \times \cdots \times \theta_n)_M = \theta_{1M} \times \cdots \times \theta_{nM}$;
  - $(\theta \to \theta')_M = (\theta'_M)^{\theta_M}$;
- for $r \in R_{\theta\theta'}$,
  - $r_M = \{r_{M\tau}\}_{\tau\in\Theta(S)}$ where $r_{M\tau} \in \mathcal{E}((\theta_M)^{\tau_M}, (\theta'_M)^{\tau_M})$ (the set of morphisms from $(\theta_M)^{\tau_M}$ to $(\theta'_M)^{\tau_M}$). The family $r_M$ must be natural in the following sense: given $\tau, \tau' \in \Theta(S)$ and $m \in \mathcal{E}(W\times\tau_M, W\times\tau'_M)$, $n \in \mathcal{E}(W\times\tau'_M, \theta_M)$, then $ctr(r_{M\tau'}\circ trn(n, \tau'_M), \tau'_M)\circ m = ctr(r_{M\tau}\circ trn(n \circ m, \tau_M), \tau_M)$;
- for $f \in F_{\theta\theta'}$,

- $f_M = \{f_{M\tau}\}_{\tau \in \Theta(S)}$ where $f_{M\tau} \in \mathcal{E}((\theta_M)^{W \times \tau_M}, (\theta'_M)^{W \times \tau_M})$. The family $f_M$ must be natural in the following sense: given $\tau, \tau' \in \Theta(S)$ and $m \in \mathcal{E}(W \times \tau_M, W \times \tau'_M)$, $n \in \mathcal{E}(W \times \tau'_M, \theta_M)$, then $ctr(f_{M\tau'} \circ trn(n, W \times \tau'_M), W \times \tau'_M) \circ m = ctr(f_{M\tau} \circ trn(n \circ m, W \times \tau_M), W \times \tau_M)$;

- for $q \in Q_{\theta\theta'\theta''}$,
  - $q_M = \{q_{M\tau}\}_{\tau \in \Theta(S)}$ where $q_{M\tau} \in \mathcal{E}((\theta'_M)^{\tau_M \times \theta_M}, (\theta''_M)^{\tau_M})$. The family $q_M$ must be natural in the following sense: given $\tau, \tau' \in \Theta(S)$ and $m \in \mathcal{E}(W \times \tau_M, W \times \tau'_M)$, $n \in \mathcal{E}(W \times \tau'_M \times \theta_M, \theta'_M)$, then $ctr(q_{M\tau'} \circ trn(n, \tau'_M \times \theta_M), \tau'_M) \circ m = ctr(q_{M\tau} \circ trn(n \circ (m \times id_{\theta_M}), \tau_M \times \theta_M), \tau_M)$. △

We denote by $Str(\Sigma)$ the class of all $\Sigma$-structures. We now turn our attention to the definition of the denotation of terms in a given $\Sigma$-structure. To this end, we need to recall the notion of context.

By a *context* we mean a finite sequence $\vec{x} = x_1 \dots x_n$ of distinct variables. We denote by $[]$ the *empty context*. Given a context $\vec{x} = x_1 \dots x_n$ where the variables $x_1, \dots, x_n$ are of type $\theta_1, \dots, \theta_n$, respectively, we write $\theta_{\vec{x}}$ for $\theta_1 \times \cdots \times \theta_n$ and say that $\theta_{\vec{x}}$ is the type of the context $\vec{x}$. This convention is obviously extended to the empty context: $\theta_{[]}$ is $\mathbf{1}$.

Given a set of terms using a finite number of free variables, we may refer to its *canonical context* formed exclusively by those free variables (this canonical context is unique once we fix a total ordering of the variables).

In the sequel we shall need to use $ST(\Sigma, \vec{x})$, $SL(\Sigma, \vec{x})$, $T(\Sigma, \vec{x})$ and $L(\Sigma, \vec{x})$ with the obvious meanings: we use only variables in the indicated context.

The following definition is a slight generalization (made necessary by the modal dimension) of the traditional notion in categorical logic.

**Definition 2.9** Let $\vec{x} = x_1, \dots, x_n$ be a context with type $\theta_{\vec{x}} = \theta_1 \times \cdots \times \theta_n$, and $\theta_{\vec{x}M} = \theta_{1M} \times \cdots \times \theta_{nM}$. Then, the *denotation*

$$[\![\cdot]\!]^M_{\vec{x}} : T(\Sigma, \vec{x})_{\theta'} \to \mathcal{E}(W \times \theta_{\vec{x}M}, \theta'_M)$$

of terms of type $\theta'$ with free variables in $\vec{x}$ is inductively defined as follows:

- $[\![x_i]\!]^M_{\vec{x}} = p_i$ where $p_i$ is the projection from $W \times \theta_{\vec{x}M}$ to $\theta_{iM}$;
- $[\![(r\, t)]\!]^M_{\vec{x}} = ctr(r_{M\theta_{\vec{x}}} \circ trn([\![t]\!]^M_{\vec{x}}, \theta_{\vec{x}M}), \theta_{\vec{x}M})$;
- $[\![(f\, t)]\!]^M_{\vec{x}} = ctr(f_{M\theta_{\vec{x}}} \circ trn([\![t]\!]^M_{\vec{x}}, W \times \theta_{\vec{x}M}), W \times \theta_{\vec{x}M})$;
- $[\![(qx\, t)]\!]^M_{\vec{x}} = ctr(q_{M\theta_{\vec{x}}} \circ trn([\![t^x_y]\!]^M_{\vec{x}y}, \theta_{\vec{x}M} \times \theta_M), \theta_{\vec{x}M})$ where $y$ is of the same type $\theta$ as $x$ and does not occur in $\vec{x}$;
- $[\![\langle\rangle]\!]^M_{\vec{x}} = !_{W \times \theta_{\vec{x}M}}$;
- $[\![\langle t_1, \dots, t_k\rangle]\!]^M_{\vec{x}} = ([\![t_1]\!]^M_{\vec{x}}, \cdots, [\![t_k]\!]^M_{\vec{x}})$ for $k \geq 2$;

- $[\![(t)_i]\!]^M_{\vec{x}} = p_i \circ [\![t]\!]^M_{\vec{x}}$ where $p_i$ is the projection from $\theta_{1M} \times \cdots \times \theta_{kM}$ to $\theta_{iM}$. △

Let $M$ be a $\Sigma$-structure. Since, according to Definition 2.8, the families $r_M$ (for $r \in R_{\theta\theta'}$), $f_M$ (for $f \in F_{\theta\theta'}$) and $q_M$ (for $q \in Q_{\theta\theta'\theta''}$) are natural, then it is straightforward to prove by induction the Substitution Lemma for $\Sigma$-structures:

**Proposition 2.10** Let $t'$ be a term free for a variable $x$ in a term $t$. Then, for any $\Sigma$-structure $M$ and appropriate contexts $\vec{y}$ and $\vec{z}$, it holds:

$$[\![t^x_{t'}]\!]^M_{\vec{y}\vec{z}} = [\![t]\!]^M_{\vec{y}x} \circ (\pi, [\![t']\!]^M_{\vec{y}\vec{z}})$$

where $\pi : W \times \Theta_{\vec{y}M} \times \Theta_{\vec{z}M} \to W \times \Theta_{\vec{y}M}$ is the canonical projection.

Finally, we go for the definition of entailment (actually, of two entailments) for a given class of $\Sigma$-structures.

**Definition 2.11** An *interpretation system* is a pair $\mathcal{S} = \langle \Sigma, \mathcal{M} \rangle$ where $\Sigma$ is a signature and $\mathcal{M}$ is a class of $\Sigma$-structures. △

In the sequel, we shall need more notation from topos theory. Given $\chi : A \to \Omega$ we denote by $mon(\chi) : dom(mon(\chi)) \to A$ the monomorphism obtained in the pullback of the diagram determined by $\{\chi, true\}$. The order in $Sub(A)$ is given as follows: $[f] \leq [g]$ iff there exists an arrow $h : dom(f) \to dom(g)$ in $\mathcal{E}$ such that $f = g \circ h$. If $\chi_1, \chi_2 : A \to \Omega$ then we define $\chi_1 \leq \chi_2$ iff $[mon(\chi_1)] \leq [mon(\chi_2)]$. Given $[f], [g] \in Sub(A)$ and an object $B$ then: $[f] \leq [g]$ iff $[f \times id_B] \leq [g \times id_B]$, provided that $E(B) = 1$ (see [3]). Finally, for each objet $A$, $true_A$ denotes the arrow $true \circ !_A : A \to \Omega$ and $\bigwedge$ denotes the infimum in the lattice $Sub(A)$.

**Definition 2.12** Given an interpretation system $\mathcal{S}$, $\Psi \subseteq L(\Sigma, \vec{x})$ finite and $\varphi \in L(\Sigma, \vec{x})$, we say:

- $\Psi$ *globally $\vec{x}$-entails* $\varphi$ within $\mathcal{S}$, written $\Psi \vDash^{\mathcal{S}}_{\mathrm{p}\vec{x}} \varphi$, iff, for every $M \in \mathcal{M}$, $[\![\varphi]\!]^M_{\vec{x}} = true_{W \times \theta_{\vec{x}M}}$ whenever $\bigwedge_{\psi \in \Psi} [\![\psi]\!]^M_{\vec{x}} = true_{W \times \theta_{\vec{x}M}}$;

- $\Psi$ *globally entails* $\varphi$ within $\mathcal{S}$, written $\Psi \vDash^{\mathcal{S}}_{\mathrm{p}} \varphi$, iff $\Psi \vDash^{\mathcal{S}}_{\mathrm{p}\vec{x}} \varphi$ choosing for $\vec{x}$ the canonical context of $\Psi \cup \{\varphi\}$;

- $\Psi$ *locally $\vec{x}$-entails* $\varphi$ within $\mathcal{S}$, written $\Psi \vDash^{\mathcal{S}}_{\mathrm{d}\vec{x}} \varphi$, iff, for every $M \in \mathcal{M}$, $\bigwedge_{\psi \in \Psi} [\![\psi]\!]^M_{\vec{x}} \leq [\![\varphi]\!]^M_{\vec{x}}$;

- $\Psi$ *locally entails* $\varphi$ within $\mathcal{S}$, written $\Psi \vDash^{\mathcal{S}}_{\mathrm{d}} \varphi$, iff $\Psi \vDash^{\mathcal{S}}_{\mathrm{d}\vec{x}} \varphi$ choosing for $\vec{x}$ the canonical context of $\Psi \cup \{\varphi\}$. △

It is then straightforward to prove for any $\Psi \cup \{\varphi\} \subseteq L(\Sigma, \vec{x})$: $\Psi \vDash^{\mathcal{S}}_{\mathrm{p}} \varphi$ implies $\Psi \vDash^{\mathcal{S}}_{\mathrm{p}\vec{x}} \varphi$; and $\Psi \vDash^{\mathcal{S}}_{\mathrm{d}} \varphi$ implies $\Psi \vDash^{\mathcal{S}}_{\mathrm{d}\vec{x}} \varphi$. The converses are not necessarily true, because we allow empty (initial) domains in the interpretation of types.

For these entailments, we may drop the reference to the assumptions when $\Psi = \emptyset$. And we may also omit the reference to the interpretation system.

The local entailment defined above coincides with the traditional notion of entailment in categorical logic. The global entailment proposed above brings to the topos setting the notion of global entailment already common in modal logic.

As already mentioned, the following two examples (modal propositional logic and first-order logic) show that for many common logics it is possible to lift the original semantics given to those logics to the topos semantics level, while preserving the two entailments. For such logics, working with the original semantics or with the proposed topos semantics is equivalent. For this reason, not much generality is lost by assuming from now that the logics we are working with are endowed with the topos semantics.

### 2.2.1 Modal propositional logic

Let $\Sigma$ be a signature as described in Example 2.3. Assume we are given a class $\mathcal{K}$ of general Kripke structures for $\Sigma$ of the form $K = \langle W, R, \mathcal{B}, V\rangle$. This class defines the global and local entailments as usual. Very briefly, recall that $[\![\varphi]\!]^K \in \mathcal{B}$ (the admissible set of worlds where $\varphi$ holds) and:

- $\Gamma \vDash^{\mathcal{K}}_{\mathrm{p}} \varphi$ iff, for every $K \in \mathcal{K}$, $[\![\varphi]\!]^K = W$ whenever $\displaystyle\bigcap_{\gamma\in\Gamma} [\![\gamma]\!]^K = W$;
- $\Gamma \vDash^{\mathcal{K}}_{\mathrm{d}} \varphi$ iff, for every $K \in \mathcal{K}$, $\displaystyle\bigcap_{\gamma\in\Gamma} [\![\gamma]\!]^K \subseteq [\![\varphi]\!]^K$.

The idea is to generate from $\mathcal{K}$ a class of $\Sigma$-structures $\mathcal{M}_{\mathcal{K}}$ and check whether we recover the original entailments with the topos semantics.

For each $K \in \mathcal{K}$, let $M_K$ be the $\Sigma$-structure $\langle \mathbf{Set}, W, ._{M_K}\rangle$ where, for instance:

- $\mathbf{f}_{M_K\tau} = \lambda h.\, (\lambda\vec{a}.\, 0)$;
- $\neg_{M_K\tau} = \lambda h.\, (\lambda\vec{a}.\, h(\vec{a})^{\mathrm{c}})$;
- $\wedge_{M_K\tau} = \lambda h.\, (\lambda\vec{a}.\, (p_1(h(\vec{a})) \sqcap p_2(h(\vec{a}))))$;
- $p_{M_K\tau} = \lambda h.\, (\lambda u\vec{a}.\, V_p(u))$ for any $p \in \mathcal{P}$;
- $\Diamond_{M_K\tau} = \lambda h.\, (\lambda u\vec{a}.\, \displaystyle\bigsqcup_{v\in W:\, uRv} h(v, \vec{a}))$.

It is easy to verify that each of these families is natural in the sense of Definition 2.8.

Letting $\mathcal{M}_{\mathcal{K}} = \{M_K : K \in \mathcal{K}\}$ and $\mathcal{S}_{\mathcal{K}} = \langle \Sigma, \mathcal{M}_{\mathcal{K}}\rangle$, it is straightforward to prove:

**Proposition 2.13** Let $\mathcal{K}$ be a class of Kripke structures for $\Sigma$. Then:

- $\Gamma \vDash_{\mathrm{p}}^{\mathcal{K}} \varphi$ iff $\Gamma \vDash_{\mathrm{p}}^{\mathcal{S}_{\mathcal{K}}} \varphi$;
- $\Gamma \vDash_{\mathrm{d}}^{\mathcal{K}} \varphi$ iff $\Gamma \vDash_{\mathrm{d}}^{\mathcal{S}_{\mathcal{K}}} \varphi$.

Recall that every monomorphism $f : \mathit{dom}(f) \to A$ has associated an unique morphism $\mathit{char}(f) : A \to \Omega$ such that $\langle \mathit{dom}(f), \{f, !_{\mathit{dom}(f)}\}\rangle$ is the pullback of the diagram determined by $\{\mathit{char}(f), \mathit{true}\}$ (see [8]). Then, the result is a direct consequence of $\mathit{char}([\![\varphi]\!]^K) = [\![\varphi]\!]^{M_K}$ which is proved by induction identifying $W$ with $W \times 1$.

### 2.2.2 First-order predicate logic

Let $\Sigma$ be a signature as described in Example 2.5. Assume we are given a class $\mathcal{I}$ of fol structures for $\Sigma$ of the form $I = \langle D, \cdot_I\rangle$. This class defines the global and local entailments as usual. Very briefly, recall $[\![\varphi]\!]^I \subseteq D^X$ (the set of assignments that make $\varphi$ true) and:

- $\Gamma \vDash_{\mathrm{p}}^{\mathcal{I}} \varphi$ iff, for every $I \in \mathcal{I}$, $[\![\varphi]\!]^I = D^X$ whenever $\displaystyle\bigcap_{\gamma\in\Gamma} [\![\gamma]\!]^I = D^X$;
- $\Gamma \vDash_{\mathrm{d}}^{\mathcal{I}} \varphi$ iff, for every $I \in \mathcal{I}$, $\displaystyle\bigcap_{\gamma\in\Gamma} [\![\gamma]\!]^I \subseteq [\![\varphi]\!]^I$.

Again, we envisage to generate from $\mathcal{I}$ a class of $\Sigma$-structures $\mathcal{M}_{\mathcal{I}}$ and check whether we recover the original entailments.

For each $I \in \mathcal{I}$, let $M_I$ be the $\Sigma$-structure $\langle \mathbf{Set}, 1, ._{M_I}\rangle$ where, for instance:

- $\mathbf{i}_{M_I} = D$;
- $g_{M_I\tau} = \lambda h.\, (\lambda\vec{a}.\, g_I(p_1(h(\vec{a})), \ldots, p_n(h(\vec{a}))))$ for any $g \in \mathcal{G}_n$;
- $\neg_{M_I\tau} = \lambda h.\, (\lambda\vec{a}.\, (h(\vec{a}))^{\mathrm{c}})$;
- $\pi_{M_I\tau} = \lambda h.\, (\lambda u\vec{a}.\, \pi_I(p_1(h(u,\vec{a})), \ldots, p_n(h(u,\vec{a}))))$ for any $\pi \in \mathcal{P}_n$;
- $\exists_{M_I\tau} = \lambda h.\, (\lambda\vec{a}.\, \displaystyle\bigsqcup_{d\in D} h(\vec{a}, d))$.

Again, it is easy to check that each of these families is natural in the sense of Definition 2.8.

Letting $\mathcal{M}_{\mathcal{I}} = \{M_I : I \in \mathcal{I}\}$ and $\mathcal{S}_{\mathcal{I}} = \langle \Sigma, \mathcal{M}_{\mathcal{I}}\rangle$, it is again straightforward to prove:

**Proposition 2.14** Let $\mathcal{I}$ be a class of fol structures for $\Sigma$. Then:

- $\Gamma \vDash_{\mathrm{p}}^{\mathcal{I}} \varphi$ iff $\Gamma \vDash_{\mathrm{p}}^{\mathcal{S}_{\mathcal{I}}} \varphi$;
- $\Gamma \vDash_{\mathrm{d}}^{\mathcal{I}} \varphi$ iff $\Gamma \vDash_{\mathrm{d}}^{\mathcal{S}_{\mathcal{I}}} \varphi$.

### 2.2.3 Higher-order intuitionistic logic

Higher-order logic is usually defined with topos-theoretic semantics. However, it is worthwhile to show how that semantics is adapted to our more general setting. Let $\Sigma_{HOL}$ be the signature described in Example 2.7. We establish the semantics of this logic by endowing it with the class $\mathcal{M}^0_{HOL}$ of all $\Sigma$-structures of the form $M = \langle \mathcal{E}, W, \cdot_M \rangle$ such that:

- $\mathbf{app}_{\theta\theta' M\tau} = \mathit{trn}(\mathit{eval}(\theta_M, \theta'_M) \circ \mathit{eval}(\tau_M, (\theta'_M)^{\theta_M} \times \theta_M), \tau_M)$ (where the arrow $\mathit{eval}(B, A) : B \times A^B \to A$ is the evaluation map obtained from the definition of exponential $A^B$);

- $=_{\theta M\tau} = \mathit{trn}(\mathit{char}(\mathit{diag}(\theta_M)) \circ \mathit{eval}(\tau_M, \theta_M \times \theta_M), \tau_M)$ (where the diagonal map $\mathit{diag}(A) : A \to A \times A$ is the monomorphism $(\mathit{id}_A, \mathit{id}_A)$);

- $\mathbf{set}_{\theta M\tau} = \mathit{trn}(\mathit{trn}(\mathit{eval}(\tau_M \times \theta_M, \Omega) \circ \mathit{can}, \theta_M), \tau_M)$ where $\mathit{can}$ is the canonical isomorphism from $(\Omega^{\tau_M \times \theta_M} \times \tau_M) \times \theta_M$ to $\Omega^{\tau_M \times \theta_M} \times (\tau_M \times \theta_M)$.

Clearly, this adaptation leaves the entailments unchanged since the extra $W$ has extent 1 (see [3]) and we have no flexible symbols. Observe that these morphisms are natural by construction.

## 2.3 Deduction

We now leave for a moment semantic concerns and concentrate on making precise the notion of deduction system we want to work with. Such systems include in general inference rules with provisos. For instance, $\delta\rho \Rightarrow \forall x\, \delta\rho$ can be inferred from $\delta \Rightarrow \forall x\, \delta$ *provided that* "$x$ is not free in $\delta\rho$". So, we start by defining what we mean by a proviso as a "predicate" on substitutions.

By a $\Sigma$-*substitution* $\rho$ we mean a $\Theta(S)$-indexed family of maps from $\Xi_\theta$ to $T(\Sigma)_\theta$. As usual we write $t\rho$ instead of $\hat{\rho}(t)$ for any $t \in ST(\Sigma)$, where $\hat{\rho} : ST(\Sigma) \to T(\Sigma)$ is defined inductively from $\rho$ as expected. It is only worthwhile to mention that $\hat{\rho}(\xi^x_{\xi'}) = (\xi\rho)^x_{\xi'\rho}$, where the right-side expression is the $\Sigma$-term obtained from $\xi\rho$ by replacing every free occurrence of $x$ by $\xi'\rho$. We denote by $\mathit{Sbs}(\Sigma)$ the set of all $\Sigma$-substitutions.

Analogously, we define a *schema* $\Sigma$-*substitution* $\sigma$ as a $\Theta(S)$-indexed family of maps from $\Xi_\theta$ to $ST(\Sigma)_\theta$, as well as the induced map $\hat{\sigma}$. And we denote by $\mathit{SSbs}(\Sigma)$ the set of all schema $\Sigma$-substitutions.

Let $\Sigma$ be a signature in the sense of Definition 2.1. By a *local* $\Sigma$-*proviso* we mean a map $\pi : \mathit{Sbs}(\Sigma) \to 2$. Intuitively, $\pi(\rho) = 1$ iff the $\Sigma$-substitution $\rho$ is allowed. Recall the example above: $\delta\rho \Rightarrow \forall x\, \delta\rho$ can be inferred from $\delta \Rightarrow \forall x\, \delta$ *provided that* "$x$ is not free in $\delta\rho$". In this case the envisaged proviso is defined by: $\pi(\rho) = 1$ iff $x$ is not free in $\delta\rho$.

But this local notion of proviso is not enough. Indeed, for the purpose of fibring, we shall need to be able to translate rules from one signature to another. So, we need a universal notion of proviso that may be evaluated at any signature.

To this end, simplifying from [11], we start by setting up the *category* **Sig** of signatures with the obvious morphisms. In this category, the signature $\Sigma_1$ with three families of singletons is terminal. We denote by $!_\Sigma$ the unique signature morphism from $\Sigma$ to $\Sigma_1$.

Observe that any given local $\Sigma_1$-proviso $\pi$ can easily be extended to another signature $\Sigma$ as follows:

$$\pi_\Sigma(\rho) = \pi(\hat{!}_\Sigma \circ \rho).$$

(Here, as usual, $\hat{h}$ denotes the unique extension of $h : \Sigma \to \Sigma'$ to the language of schema terms.) Therefore, we are led to the following definition:

**Definition 2.15** A *(universal) proviso* is a map $\Pi : Sbs(\Sigma_1) \to 2$. △

Observe that provisos about binding are universal in this sense. That is, they can be defined in the terminal signature and extended as above to other signatures. For instance, that is the case of the proviso "$x$ is not free in $\delta\rho$" taken as motivating example.

We denote by *Prov* the set of all (universal) provisos which includes the unit proviso $\mathbf{U}$ such that $\mathbf{U}(\rho) = 1$ for every $\Sigma_1$-substitution.

We are finally ready to state what we mean by an inference rule (premises, conclusion, proviso) and, afterwards, to introduce the notion of (Hilbert) deduction system.

**Definition 2.16** A $\Sigma$-*rule* is a triple $\langle \Gamma, \delta, \Pi \rangle$ where $\Gamma \cup \{\delta\} \subseteq SL(\Sigma)$ and $\Pi$ is a (universal) proviso. △

When $\Gamma = \emptyset$ the conclusion $\delta$ of the rule is also known as an *axiom*. When $\Gamma$ is finite the rule is said to be *finitary*.

**Definition 2.17** A *deduction system* is a triple $\mathcal{D} = \langle \Sigma, \mathcal{R}_\mathrm{d}, \mathcal{R}_\mathrm{p} \rangle$ where $\Sigma$ is a signature and both $\mathcal{R}_\mathrm{d}$ and $\mathcal{R}_\mathrm{p}$ are sets of finitary $\Sigma$-rules and $\mathcal{R}_\mathrm{d} \subseteq \mathcal{R}_\mathrm{p}$. △

The elements of $\mathcal{R}_\mathrm{p}$ are called *proof rules* and those of $\mathcal{R}_\mathrm{d}$ are known as *derivation rules*. As we shall see, the former reflect global entailments and the latter local entailments. Naturally, deductions appear also in two forms: proofs and derivations.

But, before defining proof and derivation, we need some further notation about provisos. If $\Pi, \Pi' \in Prov$ then the proviso $\Pi \sqcap \Pi' \in Prov$ is defined as follows: $(\Pi \sqcap \Pi')(\rho) = 1$ iff $\Pi(\rho) = \Pi'(\rho) = 1$. Furthermore, we say that $\Pi \leq \Pi'$ iff $\Pi = \Pi \sqcap \Pi'$. Finally, given $\Pi \in Prov$ and a schema $\Sigma$-substitution $\sigma$, we denote by $(\Pi\sigma)$ the (universal) proviso defined as follows:

$$(\Pi\sigma)(\rho) = \Pi(\hat{\rho} \circ \hat{!}_\Sigma \circ \sigma).$$

**Definition 2.18** A $\vec{x}$-*proof* within a deduction system $\mathcal{D}$ of $\delta \in SL(\Sigma, \vec{x})$ from $\Gamma \subseteq SL(\Sigma, \vec{x})$ with proviso $\Pi$ is a sequence of pairs $\langle \delta_1, \Pi_1 \rangle, \ldots, \langle \delta_n, \Pi_n \rangle$ in $SL(\Sigma, \vec{x}) \times Prov$ such that $\delta_n$ is $\delta$, $\Pi_n$ is $\Pi$ and for each $i = 1, \ldots, n$:

- either $\delta_i \in \Gamma$ and $\Pi_i$ is arbitrary;

- or there is a rule $\langle \{\gamma'_1, \ldots, \gamma'_k\}, \delta', \Pi' \rangle \in \mathcal{R}_\mathrm{p}$ and a schema $\Sigma$-substitution $\sigma$ such that:
  1. for each $j = 1, \ldots, k$, there is a $i_j \in \{1, \ldots, i-1\}$ such that $\delta_{i_j} = \gamma'_j\sigma$;
  2. $\delta_i = \delta'\sigma$;
  3. $\Pi_i \leq \Pi_{i_1} \sqcap \cdots \sqcap \Pi_{i_k} \sqcap (\Pi'\sigma)$.

When there is such a $\vec{x}$-proof in $\mathcal{D}$ of $\delta$ from $\Gamma$ with proviso $\Pi$, we write $\Gamma \vdash^{\mathcal{D}}_{\mathrm{p}\vec{x}} \delta \lhd \Pi$. And when there is a context $\vec{x}$ such that $\Gamma \vdash^{\mathcal{D}}_{\mathrm{p}\vec{x}} \delta \lhd \Pi$ we write $\Gamma {\vdash_\mathrm{p}}^{\mathcal{D}} \delta \lhd \Pi$. △

**Definition 2.19** A $\vec{x}$-*derivation* within a deduction system $\mathcal{D}$ of $\delta \in SL(\Sigma, \vec{x})$ from $\Gamma \subseteq SL(\Sigma, \vec{x})$ with proviso $\Pi$ is a sequence of pairs $\langle \delta_1, \Pi_1 \rangle, \ldots, \langle \delta_n, \Pi_n \rangle$ in $SL(\Sigma, \vec{x}) \times Prov$ such that $\delta_n$ is $\delta$, $\Pi_n$ is $\Pi$ and for each $i = 1, \ldots, n$:

- either $\delta_i \in \Gamma$ and $\Pi_i$ is arbitrary;
- or $\emptyset \vdash^{\mathcal{D}}_{\mathrm{p}\vec{x}} \delta_i \lhd \Pi_i$;
- or there is a rule $\langle \{\gamma'_1, \ldots, \gamma'_k\}, \delta', \Pi' \rangle \in \mathcal{R}_\mathrm{d}$ and a schema $\Sigma$-substitution $\sigma$ such that:
  1. for each $j = 1, \ldots, k$, there is a $i_j \in \{1, \ldots, i-1\}$ such that $\delta_{i_j} = \gamma'_j\sigma$;
  2. $\delta_i = \delta'\sigma$;
  3. $\Pi_i \leq \Pi_{i_1} \sqcap \cdots \sqcap \Pi_{i_k} \sqcap (\Pi'\sigma)$.

When there is such a $\vec{x}$-derivation in $\mathcal{D}$ of $\delta$ from $\Gamma$ with proviso $\Pi$, we write $\Gamma \vdash^{\mathcal{D}}_{\mathrm{d}\vec{x}} \delta \lhd \Pi$. And when there is a context $\vec{x}$ such that $\Gamma \vdash^{\mathcal{D}}_{\mathrm{d}\vec{x}} \delta \lhd \Pi$ we write $\Gamma {\vdash_\mathrm{d}}^{\mathcal{D}} \delta \lhd \Pi$. △

As usual, with respect to both proofs and derivations, we may drop the reference to the assumptions when $\Gamma = \emptyset$ and the reference to the deduction system. Furthermore, when $\Pi = \mathbf{U}$ we may also omit the reference to the proviso.

By putting together an interpretation system in the sense of Subsection 2.2 and a deduction system we are led to the following notion of logic system.

**Definition 2.20** A *logic system* is a tuple $\mathcal{L} = \langle \Sigma, \mathcal{M}, \mathcal{R}_\mathrm{d}, \mathcal{R}_\mathrm{p} \rangle$ provided that $\mathcal{S}_\mathcal{L} = \langle \Sigma, \mathcal{M} \rangle$ is an interpretation system and $\mathcal{D}_\mathcal{L} = \langle \Sigma, \mathcal{R}_\mathrm{d}, \mathcal{R}_\mathrm{p} \rangle$ is a deduction system. △

Given a logic system $\mathcal{L}$ and $\mathrm{o} \in \{\mathrm{p}, \mathrm{d}\}$, we write $\Psi \vDash^{\mathcal{L}}_{\mathrm{o}\vec{x}} \varphi$ for $\Psi \vDash^{\mathcal{S}_\mathcal{L}}_{\mathrm{o}\vec{x}} \varphi$ and $\Gamma \vdash^{\mathcal{L}}_{\mathrm{o}\vec{x}} \delta \lhd \Pi$ for $\Gamma \vdash^{\mathcal{D}_\mathcal{L}}_{\mathrm{o}\vec{x}} \delta \lhd \Pi$.

**Definition 2.21** A logic system $\mathcal{L}$ is said to be:

- *sound* iff, for each $\mathrm{o} \in \{\mathrm{p}, \mathrm{d}\}$, any context $\vec{x}$, and finite $\Psi \cup \{\varphi\} \subseteq L(\Sigma, \vec{x})$, the entailment $\Psi \vDash^{\mathcal{L}}_{\mathrm{o}\vec{x}} \varphi$ holds whenever $\Psi \vdash^{\mathcal{L}}_{\mathrm{o}\vec{x}} \varphi$;

- *complete* iff, for each $\mathrm{o} \in \{\mathrm{p}, \mathrm{d}\}$ and $\Psi \cup \{\varphi\} \subseteq L(\Sigma)$, the deduction $\Psi {\vdash_{\mathrm{o}}}^{\mathcal{L}} \varphi$ holds whenever $\Psi \vDash_{\mathrm{o}}^{\mathcal{L}} \varphi$. △

**Remark 2.22** A few words about the definition of soundness stated above. The intended definition of soundness of a logic system $\mathcal{L}$ is, for $\mathrm{o} \in \{\mathrm{p}, \mathrm{d}\}$,

$$\Psi {\vdash_{\mathrm{o}}}^{\mathcal{L}} \varphi \text{ implies } \Psi \vDash_{\mathrm{o}}^{\mathcal{L}} \varphi.$$

Unfortunately, this definition is not correct in the realm of logic systems, because of the (possibly) empty domains interpreting the types. In fact, from $\Psi, \psi \vDash_{\mathrm{o}}^{\mathcal{L}} \varphi$ and $\Psi \vDash_{\mathrm{o}}^{\mathcal{L}} \psi$ we cannot infer $\Psi \vDash_{\mathrm{o}}^{\mathcal{L}} \varphi$, for $\mathrm{o} \in \{\mathrm{p}, \mathrm{d}\}$ (see, for instance, [1]). On the other hand, it is obvious that any logic system $\mathcal{L}$ satisfies the following property: from $\Psi, \psi {\vdash_{\mathrm{o}}}^{\mathcal{L}} \varphi$ and $\Psi {\vdash_{\mathrm{o}}}^{\mathcal{L}} \psi$ we infer $\Psi {\vdash_{\mathrm{o}}}^{\mathcal{L}} \varphi$, for $\mathrm{o} \in \{\mathrm{p}, \mathrm{d}\}$. Therefore the standard definition of soundness must be changed, and we must live with the fact that it is possible to have

$$\Psi {\vdash_{\mathrm{o}}}^{\mathcal{L}} \varphi \text{ but } \Psi \nvDash_{\mathrm{o}}^{\mathcal{L}} \varphi$$

even in a sound logic system $\mathcal{L}$. △

The following semantic concepts associated to how well structures fit rules are used in the sequel. A $\Sigma$-structure $M$ is said to be *appropriate* for a deduction system $\mathcal{D}$ iff, for $\mathrm{o} \in \{\mathrm{p}, \mathrm{d}\}$,

$$\Psi \vDash_{\mathrm{o}\vec{x}}^{\langle \Sigma, \{M\} \rangle} \varphi \text{ whenever } \Psi \vdash_{\mathrm{o}\vec{x}}^{\mathcal{D}} \varphi.$$

For each $\mathrm{o} \in \{\mathrm{p}, \mathrm{d}\}$, a $\Sigma$-structure $M$ is said to be *o-appropriate* for a $\Sigma$-rule $\langle \Gamma, \delta, \Pi \rangle$ iff for every $\Sigma$-substitution $\rho$ such that $\Pi(\hat{!}_\Sigma \circ \rho) = 1$,

$$\Gamma\rho \vDash_{\mathrm{o}}^{\langle \Sigma, \{M\} \rangle} \delta\rho.$$

Clearly, a $\Sigma$-structure is appropriate for a deduction system $\mathcal{D} = \langle \Sigma, \mathcal{R}_{\mathrm{d}}, \mathcal{R}_{\mathrm{p}} \rangle$ iff it is d-appropriate for every derivation rule in $\mathcal{R}_{\mathrm{d}}$ and p-appropriate for every proof rule in $\mathcal{R}_{\mathrm{p}}$.

For each $\mathrm{o} \in \{\mathrm{p}, \mathrm{d}\}$, given a set $\mathcal{R}$ of $\Sigma$-rules, we denote by $Ap_{\mathrm{o}}(\mathcal{R})$ the class of all $\Sigma$-structures that are o-appropriate for the rules in $\mathcal{R}$. Moreover, given a deduction system $\mathcal{D}$, we define $Ap(\mathcal{D}) = Ap_{\mathrm{d}}(\mathcal{R}_{\mathrm{d}}) \cap Ap_{\mathrm{p}}(\mathcal{R}_{\mathrm{p}})$.

Clearly, a logic system $\mathcal{L} = \langle \Sigma, \mathcal{M}, \mathcal{R}_{\mathrm{d}}, \mathcal{R}_{\mathrm{p}} \rangle$ is sound iff $\mathcal{M} \subseteq Ap(\mathcal{D}_{\mathcal{L}})$. Finally, a logic system $\mathcal{L}$ is said to be *full* iff $\mathcal{M} = Ap(\mathcal{D}_{\mathcal{L}})$.

We conclude this subsection with two interesting examples.

### 2.3.1 Modal propositional logic

Let $\Sigma$ be a signature as described in Example 2.3. We establish the deductive component of the modal logic $K$ by endowing it with the set $\mathcal{R}_{\mathrm{d}}$ composed of the following rules:

**taut1:** $\langle \emptyset, \xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_1), \mathbf{U} \rangle$;

**taut2:** $\langle \emptyset, (\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_3)) \Rightarrow ((\xi_1 \Rightarrow \xi_2) \Rightarrow (\xi_1 \Rightarrow \xi_3)), \mathbf{U}\rangle$;

**taut3:** $\langle \emptyset, (\neg\, \xi_1 \Rightarrow \neg\, \xi_2) \Rightarrow ((\neg\, \xi_1 \Rightarrow \xi_2) \Rightarrow \xi_1)), \mathbf{U}\rangle$;

**norm:** $\langle \emptyset, ((\Box(\xi_1 \Rightarrow \xi_2)) \Rightarrow ((\Box\xi_1) \Rightarrow (\Box\xi_2))), \mathbf{U}\rangle$;

**MP:** $\langle \{\xi_1, (\xi_1 \Rightarrow \xi_2)\}, \xi_2, \mathbf{U}\rangle$;

and the set $\mathcal{R}_\mathrm{p}$ containing the rules in $\mathcal{R}_\mathrm{d}$ plus necessitation:

**NEC:** $\langle \{\xi_1\}, (\Box\xi_1), \mathbf{U}\rangle$.

By endowing this deduction system with the class of all structures appropriate for it we obtain a full logic system that we call $MPL_K$.

### 2.3.2 Higher-order intuitionistic logic

We need some additional notation for provisos:

- $x \prec \delta$ denotes the (universal) proviso that, for each $\rho \in Sbs(\Sigma_1)$, returns the value of the assertion "$x$ occurs free in $\delta\rho$";
- $x \not\prec \delta$ denotes the (universal) proviso that, for each $\rho \in Sbs(\Sigma_1)$, returns the value of the assertion "$x$ does not occur free in $\delta\rho$";
- $\delta_1 \rhd x : \delta_2$ denotes the (universal) proviso that, for each $\rho \in Sbs(\Sigma_1)$, returns the value of the assertion "$\delta_1\rho$ is free for $x$ in $\delta_2\rho$".

Let $\Sigma_{HOL}$ be the signature in Example 2.7 and recall the usual set-theoretic abbreviations in the context of higher-order logic. For instance, $U_\theta$ stands for $\{x^\theta \;:\; \mathbf{t}\}$ and $t_2^{t_1}$ for $\{h \subseteq t_1 \times t_2 \;:\; \forall x(x \in t_1 \Rightarrow \exists! y((y \in t_2) \wedge (\langle x, y\rangle \in h))\}$.

The deductive component of the envisaged higher-order logic is as follows (omitting the types of schema variables, variables and other symbols and assuming that $i \in \mathbb{N}$, $k \geq 2$ and $\theta$, $\theta_1$, ..., $\theta_k$ are types):

- $\mathcal{R}_\mathrm{d}$ is the set composed by:

  **taut1:** $\langle \emptyset, \xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_1), \mathbf{U}\rangle$;
  **taut2:** $\langle \emptyset, (\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_3)) \Rightarrow ((\xi_1 \Rightarrow \xi_2) \Rightarrow (\xi_1 \Rightarrow \xi_3)), \mathbf{U}\rangle$;
  **taut3:** $\langle \emptyset, (\xi_1 \Rightarrow \xi_2) \Rightarrow ((\xi_1 \Rightarrow \xi_3) \Rightarrow (\xi_1 \Rightarrow (\xi_2 \wedge \xi_3))), \mathbf{U}\rangle$;
  **taut4:** $\langle \emptyset, \xi_1 \Rightarrow (\xi_2 \Rightarrow (\xi_1 \wedge \xi_2)), \mathbf{U}\rangle$;
  **uni:** $\langle \emptyset, \forall x_1(x_1 = \langle\rangle), \mathbf{U}\rangle$;
  **equa**$_{i,\theta}$**:** $\langle \emptyset, (\xi_1 = \xi_2) \Rightarrow ({\xi_3}_{\xi_1}^{x_i} \Rightarrow {\xi_3}_{\xi_2}^{x_i}), (\xi_1 \rhd x_i : \xi_3) \sqcap (\xi_2 \rhd x_i : \xi_3)\rangle$;
  **ref**$_\theta$**:** $\langle \emptyset, \forall x_1(x_1 = x_1), \mathbf{U}\rangle$;
  **proj**$_{k,\theta_1,\ldots,\theta_k,i}$**:** $\langle \emptyset, \forall x_1 \cdots \forall x_k((\langle x_1, \ldots, x_k\rangle)_i = x_i), \mathbf{U}\rangle$ for $1 \leq i \leq k$;
  **prod**$_{k,\theta_1,\ldots,\theta_k}$**:** $\langle \emptyset, \forall x_1(x_1 = \langle (x_1)_1, \ldots, (x_1)_k\rangle), \mathbf{U}\rangle$;
  **comph**$_\theta$**:** $\langle \emptyset, \forall x_1(x_1 \in \{x_1 : \xi_1\} \Leftrightarrow \xi_1), \mathbf{U}\rangle$;
  **subs**$_{i,\theta}$**:** $\langle \emptyset, (\forall x_i \xi_2) \Rightarrow {\xi_2}_{\xi_1}^{x_i}, (\xi_1 \rhd x_i : \xi_2) \sqcap (x_i \prec \xi_2)\rangle$;

**$\mathbf{fun}_{\theta,\theta'}$:** $\langle\emptyset, \forall x_1(x_1 \in U_{\theta'}^{U_\theta} \Rightarrow \exists! x_2 \forall x_3 \forall x_4(\langle x_3, x_4\rangle \in x_1 \Leftrightarrow x_2(x_3) = x_4)), \mathbf{U}\rangle$;

**equiv:** $\langle\emptyset, (\xi_1 \Rightarrow \xi_2) \Rightarrow ((\xi_2 \Rightarrow \xi_1) \Rightarrow (\xi_1 \Leftrightarrow \xi_2)), \mathbf{U}\rangle$;

**MP:** $\langle\{\xi_1, \xi_1 \Rightarrow \xi_2\}, \xi_2, \mathbf{U}\rangle$;

- $\mathcal{R}_\mathrm{p}$ is obtained by adding to $\mathcal{R}_\mathrm{d}$ the following rules:

  **$\mathbf{GEN}_{i,\theta}$:** $\langle\{\xi_1 \Rightarrow \xi_2\}, \xi_1 \Rightarrow (\forall x_i \xi_2), x_i \not\prec \xi_1\rangle$.

In [4] it is shown that $x_1 = x_1, \ldots, x_n = x_n, \Psi \vdash_{\mathrm{d}\vec{x}} \varphi$ in the sequent calculus for higher-order logic presented for instance in [1] iff $\Psi \vdash_{\mathrm{d}\vec{x}} \varphi$ in the deduction system above (for $\Psi, \varphi$ in the language of [1]).

By endowing the deduction system presented above with the class $\mathcal{M}$ of all appropriate structures we obtain a full logic system that we call *HOL*. It is worth noting that this *HOL* contains all the traditional models $\mathcal{M}^0_{HOL}$ described in Paragraph 2.2.3. Indeed:

**Proposition 2.23** $\mathcal{M}^0_{HOL} \subseteq Ap(\mathcal{D}_{HOL})$.

**Proof:** Let $M \in \mathcal{M}^0_{HOL}$. We need to prove that $M$ is o-appropriate for every rule in $\mathcal{R}_\mathrm{o}$ of *HOL* for $\mathrm{o} \in \{\mathrm{p}, \mathrm{d}\}$. Note that $M$ is d-appropriate for every axiom of $\mathcal{R}_\mathrm{d}$ of *HOL* other than $\mathbf{fun}_{\theta\theta'}$. This is a consequence of the equivalence, stated in [4], of $\mathcal{D}_{HOL}$ restricted to the language of Local Set Theory, and the sequent calculus of Bell [1]. On the other hand, $M$ is d-appropriate for axiom $\mathbf{fun}_{\theta\theta'}$. In fact, it is well-known that, in any topos, using the properties of the subobject classifier $\Omega$, finite limits and exponentials of the form $\Omega^A$, it is possible to construct arbitrary exponentials (see, for instance, [8]). The fact that exponentials appear as pullbacks of certain diagrams guarantees the d-appropriateness of $M$ for $\mathbf{fun}_{\theta\theta'}$. It is straightforward to prove the d-appropriateness of $M$ for **MP** and the p-appropriateness of $M$ for **MP** and **GEN**, because the interpretation of the universal quantifier is the usual in categorical semantics. QED

## 2.4 Completeness

We proceed now to establish a general completeness theorem about full logic systems containing *HOL* and with the meta-theorem of deduction.

Let $\mathcal{D}$ be a deduction system. We say that: (i) $\mathcal{D}$ includes *HOL* iff the deductive system $\mathcal{D}_{HOL}$ defined in Paragraph 2.3.2 is embedded in $\mathcal{D}$; and (ii) $\mathcal{D}$ has the meta-theorem of deduction (*MTD*) iff it includes *HOL* and the following condition holds: $\Psi, \psi \vdash_\mathrm{d} \varphi$ iff $\Psi \vdash_\mathrm{d} (\psi \Rightarrow \varphi)$.

In the sequel, we use the following notation. Let $\Gamma$ be a finite subset of $SL(\Sigma)$. Then $(\bigwedge \Gamma)$ denotes a schema formula obtained from $\Gamma$ by taking the conjunction of all the schema formulae in $\Gamma$ in an arbitrary order and association (if $\Gamma = \emptyset$ then we take $(\bigwedge \Gamma)$ to be $\mathbf{t}$). Let $\vec{x} = x_1 \ldots x_n$ be a context. Then $(\vec{x} = \vec{x})$ denotes a formula $(\bigwedge \{x_1 = x_1, \ldots, x_n = x_n\})$.

**Lemma 2.24** Every deduction system $\mathcal{D}$ including *HOL* and with *MTD* has a canonical model $M_\mathcal{D} = \langle \mathcal{E}_\mathcal{D}, W_\mathcal{D}, \cdot_\mathcal{D}\rangle$.

**Proof:** In the proof we adopt the usual set-theoretic abbreviations in the context of higher-order logic. For instance, as already used, $U_\theta$ stands for $\{x^\theta \ : \ \mathbf{t}\}$ and $t_2^{t_1}$ for $\{h \subseteq t_1 \times t_2 \ : \ \forall x(x \in t_1 \Rightarrow \exists! y((y \in t_2) \wedge (\langle x, y\rangle \in h))\}$.

• **Part I:** The topos $\mathcal{E}_\mathcal{D}$.
This first part of the proof follows with very light adaptations the standard construction in categorical logic (see for instance [1]). Let $cT(\Sigma)_\theta$ be the set of closed $\Sigma$-terms of sort $\theta \in \Theta(S)$, and consider the collection $\mathfrak{A}_\Sigma = \bigcup_{\theta\in\Theta(S)} cT(\Sigma)_{(\theta\to\mathbf{\Omega})}$. We define in $\mathfrak{A}_\Sigma$ the following relation:

$$t_1 \sim_\mathcal{D} t_2 \text{ iff } \vdash_{\rm d}{}^\mathcal{D}(t_1 = t_2).$$

Note that $t_1 \sim_\mathcal{D} t_2$ implies that $t_1, t_2 \in cT(\Sigma)_{(\theta\to\mathbf{\Omega})}$ for some sort $\theta$, and $\sim_\mathcal{D}$ is an equivalence relation. The equivalence class of $t \in \mathfrak{A}_\Sigma$ will be denoted by $[t]_\mathcal{D}$, or $[t]$ whenever $\mathcal{D}$ is obvious.

We define the category $\mathcal{E}_\mathcal{D}$ as follows: the objects of $\mathcal{E}_\mathcal{D}$ are equivalence classes $[t]$. We use the letters $A$, $B$, $C$, etc. to denote the objects of $\mathcal{E}_\mathcal{D}$. Given $A = [t_1]$ and $B = [t_2]$, a morphism in $\mathcal{E}_\mathcal{D}$ from $A$ to $B$ is an equivalence class $[t]$ such that $\vdash_{\rm d}{}^\mathcal{D} t \in t_2^{t_1}$. Note that the notion of morphism is well-defined. As usual we write $g : A \to B$ whenever $g = [t]$ satisfies $\vdash_{\rm d}{}^\mathcal{D} t \in t_2^{t_1}$. Given $[t] : [t_1] \to [t_2]$ and $[t'] : [t_2] \to [t_3]$ then the composition map $[t'] \circ [t]$ in $\mathcal{E}_\mathcal{D}$ is defined as

$$[\{\langle x, z\rangle \ : (x \in t_1) \wedge (z \in t_3) \wedge (\exists y((y \in t_2) \wedge (\langle x, y\rangle \in t) \wedge (\langle y, z\rangle \in t')))\}].$$

Note that $[t'] \circ [t]$ is well-defined and $[t'] \circ [t] : [t_1] \to [t_3]$. The composition $\circ$ is associative. Given $t \in \mathfrak{A}_\Sigma$ let $id_{[t]} = [\{\langle x, x\rangle \ : \ x \in t\}]$. Then, $id_{[t]} : [t] \to [t]$ and $id_{[t]} \circ g = g$ and $h \circ id_{[t]} = h$ for $g : A \to [t]$ and $h : [t] \to B$. Thus $\mathcal{E}_\mathcal{D}$ is a category.

If $x_i \in X_{\theta_i}$ $(i = 1, \ldots, n)$ such that $x_i \neq x_j$ for $i \neq j$ then $\bar{x}$ will denote the term $\langle x_1, \ldots, x_n\rangle$. Let $t, t' \in \mathfrak{A}_\Sigma$ such that $t$ and $t'$ have sort $((\theta_1 \times \cdots \times \theta_n) \to \mathbf{\Omega})$ and $(\theta \to \mathbf{\Omega})$, respectively, and let $\delta \in T(\Sigma, \vec{x})_\theta$. If $\bar{x} \in t \vdash_{\rm d}{}^\mathcal{D} \delta \in t'$ then $[\{\langle \bar{x}, \delta\rangle \ : \ \bar{x} \in t\}]$ is a morphism from $[t]$ to $[t']$, which will be denoted by

$$(\bar{x} \mapsto \delta).$$

Observe that this notation is somewhat inaccurate; maybe $(\bar{x} \in t \mapsto \delta \in t')$ would be better but, for the sake of simplicity, we prefer to maintain it.

If $\delta_i$ is free for $y_i$ in $\delta$ then

$$(\bar{y} \mapsto \delta) \circ (\bar{x} \mapsto \langle \delta_1, \ldots \delta_m\rangle) = (\bar{x} \mapsto \delta^{y_1 \ldots y_m}_{\delta_1 \ldots \delta_m}).$$

Now we will prove that $\mathcal{E}_\mathcal{D}$ is a topos. Consider $1_\mathcal{D} = [U_\mathbf{1}]$. For any object $A$ there exists an unique morphism from $A$ to $1_\mathcal{D}$ given by $(x \mapsto \langle\rangle)$, therefore $1_\mathcal{D}$ is terminal in $\mathcal{E}_\mathcal{D}$.

The product $[t_1] \times [t_2]$ of $[t_1]$ and $[t_2]$ in $\mathcal{E}_\mathcal{D}$ is $[t_1 \times t_2]$ with canonical projections $(\langle x, y\rangle \mapsto x)$ and $(\langle x, y\rangle \mapsto y)$.

Consider $\Omega_\mathcal{D} = [U_\mathbf{\Omega}]$ and $true : 1_\mathcal{D} \to \Omega_\mathcal{D}$ given by $(z^\mathbf{1} \mapsto \mathbf{t})$. Then $\langle \Omega_\mathcal{D}, true\rangle$ is the subobject classifier in $\mathcal{E}_\mathcal{D}$. If $[t] : A \to B$ is a monic in $\mathcal{E}_\mathcal{D}$ then $char([t]) : B \to \Omega_\mathcal{D}$ is given by

$$(y \mapsto \exists x(\langle x, y\rangle \in t)).$$

Let $[t_1]$ and $[t_2]$. Then the exponential $[t_2]^{[t_1]}$ is given by $[t_2^{t_1}]$. The morphism $\mathit{eval}([t_1],[t_2]) : [t_2^{t_1}] \times [t_1] \to [t_2]$ is given by

$$[\{\langle\langle h,x\rangle,y\rangle \ : \ (h \in t_2^{t_1}) \wedge (x \in t_1) \wedge (\langle x,y\rangle \in h)\}].$$

If $[t] : A \times [t_1] \to [t_2]$ then $\mathit{trn}([t],[t_1]) : A \to [t_2^{t_1}]$ is given by

$$(x \mapsto \{\langle y,z\rangle \ : \ \langle\langle x,y\rangle,z\rangle \in t\}).$$

If $[t] : A \to [t_2^{t_1}]$ then $\mathit{ctr}([t],[t_1]) : A \times [t_1] \to [t_2]$ is given by

$$[\{\langle\langle x,y\rangle,z\rangle \ : \ (x \in A) \wedge (y \in t_1) \wedge (\forall u(\langle x,u\rangle \in t \ \Rightarrow \ \langle y,z\rangle \in u))\}].$$

It is easy to show that $[U_{\theta'}]^{[U_\theta]}$ is isomorphic to $[U_{(\theta\to\theta')}]$ and $\mathit{eval}([U_\theta],[U_{\theta'}])$ is $(\langle h,x\rangle \mapsto \mathbf{app}\langle h,x\rangle)$ in this case. On the other hand, $[U_\theta] \times [U_{\theta'}] = [U_{(\theta\times\theta')}]$.

• **Part II:** The $\Sigma$-structure $M_{\mathcal{D}} = \langle \mathcal{E}_{\mathcal{D}}, W_{\mathcal{D}}, \cdot_{\mathcal{D}}\rangle$.
Now we define a $\Sigma$-structure $M_{\mathcal{D}} = \langle \mathcal{E}_{\mathcal{D}}, W_{\mathcal{D}}, \cdot_{\mathcal{D}}\rangle$ as follows: $W_{\mathcal{D}}$ is $1_{\mathcal{D}}$ and

- $\theta_{M_{\mathcal{D}}}$ is $[U_\theta]$ for every $\theta \in \Theta(S)$;
- $r_{M_{\mathcal{D}}\tau}$ is $(h \mapsto \{\langle \bar{x},(ry)\rangle \ : \ \langle \bar{x},y\rangle \in h\})$ whenever $r \in R_{\theta\theta'}$, $r \neq \mathbf{app}_{\theta''\theta'}$, $r \neq =_{\theta'''}$;
- $f_{M_{\mathcal{D}}\tau}$ is $(h \mapsto \{\langle\langle z^{\mathbf{1}},\bar{x}\rangle,(fy)\rangle \ : \ \langle\langle z^{\mathbf{1}},\bar{x}\rangle,y\rangle \in h\})$ whenever $f \in F_{\theta\theta'}$;
- $q_{M_{\mathcal{D}}\tau}$ is $(h \mapsto \{\langle \bar{x},(qxu)\rangle \ : \ \langle\langle \bar{x},x\rangle,u\rangle \in h\})$ whenever $q \in Q_{\theta\theta'\theta''}$, $q \neq \mathbf{set}_\theta$.

The interpretation in $M_{\mathcal{D}}$ of $\mathbf{app}_{\theta\theta'}$, $=_\theta$ and $\mathbf{set}_\theta$ is as in Subsection 2.2.3. In what follows, we will briefly analyze them. Considering $[U_{\theta'}]^{[U_\theta]}$ as $[U_{(\theta\to\theta')}]$ let

$$g = \mathit{eval}([U_\theta],[U_{\theta'}]) \circ \mathit{eval}([U_\tau],[U_{(\theta\to\theta')}] \times [U_\theta]).$$

Using the results stated in Part I, we get that

$$g = (\langle h',x\rangle \mapsto \mathbf{app}\langle h',x\rangle) \circ [\{\langle\langle h,\bar{x}\rangle,u\rangle \ : \ \langle \bar{x},u\rangle \in h\}].$$

From this follows easily that

$$g = [\{\langle\langle h,\bar{x}\rangle,\mathbf{app}\, u\rangle \ : \ \langle \bar{x},u\rangle \in h\}]$$

and then

$$\mathbf{app}_{\theta\theta' M_{\mathcal{D}}\tau} = \mathit{trn}(g,[U_\tau]) = (h \mapsto \{\langle \bar{x},\mathbf{app}\, u\rangle \ : \ \langle \bar{x},u\rangle \in h\}).$$

Now consider the diagonal map

$$\mathit{diag}([U_\theta]) = (x \mapsto \langle x,x\rangle) : [U_\theta] \to [U_\theta] \times [U_\theta].$$

Using Part I and the deduction rules of *HOL* we get

$$\mathit{char}(\mathit{diag}([U_\theta])) = (y \mapsto \exists x(\langle x,y\rangle \in \mathit{diag}([U_\theta]))) = (y \mapsto ((y)_1 = (y)_2)).$$

Let

$$g = char(diag([U_\theta])) \circ eval([U_\tau], [U_\theta] \times [U_\theta]).$$

Then $g = [\{\langle\langle h, \bar{x}\rangle, ((y)_1 = (y)_2)\rangle \ : \ \langle \bar{x}, y\rangle \in h\}]$, therefore

$$=_{\theta M_{\mathcal{D}}\tau} = trn(g, [U_\tau]) = (h \mapsto \{\langle \bar{x}, ((y)_1 = (y)_2)\rangle \ : \ \langle \bar{x}, y\rangle \in h\}).$$

Finally, considering $[U_{\mathbf{\Omega}}]^{[U_\theta]}$ as $[U_{(\theta\to\mathbf{\Omega})}]$ let

$$g = eval([U_\tau] \times [U_\theta], [U_{\mathbf{\Omega}}]) \circ can,$$

where

$$can : ([U_{\mathbf{\Omega}}]^{[U_\tau]\times[U_\theta]} \times [U_\tau]) \times [U_\theta] \to [U_{\mathbf{\Omega}}]^{[U_\tau]\times[U_\theta]} \times ([U_\tau] \times [U_\theta])$$

is the canonical isomorphism. Using Part I and the rules of *HOL* it is easy to prove that

$$\begin{aligned} g &= [\{\langle\langle h', \langle \bar{x}, x\rangle\rangle, v\rangle \ : \langle\langle \bar{x}, x\rangle, v\rangle \in h'\}] \circ [\{\langle\langle\langle h, \bar{x}\rangle, x\rangle, \langle h, \langle \bar{x}, x\rangle\rangle\rangle \ : \ \mathbf{t}\}] \\ &= [\{\langle\langle\langle h, \bar{x}\rangle, x\rangle, v\rangle \ : \langle\langle \bar{x}, x\rangle, v\rangle \in h\}]. \end{aligned}$$

Let $g_1 = trn(g, [U_\theta])$. Then

$$g_1 = [\{\langle\langle h, \bar{x}\rangle, u\rangle \ : \ \forall x \forall v((x \in u) = v \ \Leftrightarrow \ \langle\langle \bar{x}, x\rangle, v\rangle \in h)\}].$$

Therefore $\mathbf{set}_{\theta M_{\mathcal{D}}\tau} = trn(g_1, [U\tau])$ is given by

$$\mathbf{set}_{\theta M_{\mathcal{D}}\tau} = (h \mapsto \{\langle \bar{x}, u\rangle \ : \ \forall x \forall v((x \in u) = v \ \Leftrightarrow \ \langle\langle \bar{x}, x\rangle, v\rangle \in h)\}).$$

It only remains to prove the naturality of morphisms $r_{M_{\mathcal{D}}\tau}$ (where $r \neq \mathbf{app}_{\theta''\theta'}$ and $r \neq =_{\theta'''}$), $f_{M_{\mathcal{D}}\tau}$ and $q_{M_{\mathcal{D}}\tau}$ (where $q \neq \mathbf{set}_\theta$). Observe that the other families of morphisms are natural by construction. We only consider the case of $r_{M_{\mathcal{D}}\tau}$, because the proof for the other cases is similar. Since $W = 1$ then

$$r_{M_{\mathcal{D}}\tau} = (h \mapsto \{\langle \bar{x}, (ry)\rangle : \langle \bar{x}, y\rangle \in h\})$$

for every $\tau$. Let $m = [t] : U_\tau \to U_{\tau'}$ and $n = [t'] : U_{\tau'} \to U_\theta$. Then

$$r_{M\tau'} \circ \mathrm{trn}(n, {\tau'}_M) = (z^{\mathbf{1}} \mapsto \{\langle \bar{x}, (ry)\rangle : \langle \bar{x}, y\rangle \in t'\}).$$

Using Part I and the rules of *HOL* we obtain

$$ctr(r_{M\tau'} \circ \mathrm{trn}(n, {\tau'}_M), {\tau'}_M) \circ m = [\{\langle \bar{y}, (ry)\rangle : \exists \bar{x}(\langle \bar{y}, \bar{x}\rangle \in t \wedge \langle \bar{x}, y\rangle \in t')\}].$$

On the other hand,it is straightforward to prove that

$$r_{M\tau} \circ \mathrm{trn}(n \circ m, \tau_M) = (z^{\mathbf{1}} \mapsto \{\langle \bar{y}, (ry)\rangle : \exists \bar{x}(\langle \bar{y}, \bar{x}\rangle \in t \wedge \langle \bar{x}, y\rangle \in t')\}).$$

Hence we obtain the desired naturality of the morphism:

$$ctr(r_{M\tau'} \circ \mathrm{trn}(n, {\tau'}_M), {\tau'}_M) \circ m = ctr(r_{M\tau} \circ \mathrm{trn}(n \circ m, \tau_M), \tau_M).$$

• **Part III:** $[\![t]\!]_{\vec{x}}^{M_{\mathcal{D}}} = (\langle z^{\mathbf{1}}, \bar{x}\rangle \mapsto t)$.
Let $t \in T(\Sigma, \vec{x})_\theta$. By induction on the complexity of $t$, and using the fact that $W_{\mathcal{D}}$ is $1_{\mathcal{D}}$ (therefore $[U_\theta] \simeq W_{\mathcal{D}} \times [U_\theta]$ for all $\theta$) we prove now that

$$[\![t]\!]_{\vec{x}}^{M_{\mathcal{D}}} = (\langle z^{\mathbf{1}}, \bar{x}\rangle \mapsto t) : 1_{\mathcal{D}} \times [U_\tau] \to [U_\theta],$$

where $z^{\mathbf{1}} \in X_{\mathbf{1}}$ does not occur in $\vec{x}$. If $t$ is a variable or $t$ is $\langle\rangle$ the result is obvious. If $t$ is $\langle t_1, \ldots t_n\rangle$ or $t$ is $(t')_i$ the conclusion follows easily by induction hypothesis. If $t$ is $\mathbf{app}_{\theta\theta'}\langle t_1, t_2\rangle$ then $[\![\langle t_1, t_2\rangle]\!]_{\vec{x}}^{M_{\mathcal{D}}} = (\langle z^{\mathbf{1}}, \bar{x}\rangle \mapsto \langle t_1, t_2\rangle)$, by induction hypothesis. Let

$$g = \mathbf{app}_{\theta\theta' M_{\mathcal{D}}\tau} \circ trn([\![\langle t_1, t_2\rangle]\!]_{\vec{x}}^{M_{\mathcal{D}}}, [U_\tau]).$$

Using Parts I and II we obtain that

$$\begin{aligned} g &= (h \mapsto \{\langle \bar{x}, \mathbf{app}\, u\rangle \ : \ \langle \bar{x}, u\rangle \in h\}) \circ (z^{\mathbf{1}} \mapsto \{\langle \bar{x}, \langle t_1, t_2\rangle\rangle \ : \ \mathbf{t}\}) \\ &= (z^{\mathbf{1}} \mapsto \{\langle \bar{x}, \mathbf{app}\langle t_1, t_2\rangle\rangle \ : \ \mathbf{t}\}). \end{aligned}$$

From this follows easily that

$$[\![\mathbf{app}\langle t_1, t_2\rangle]\!]_{\vec{x}}^{M_{\mathcal{D}}} = ctr(g, [U_\tau]) = (\langle z^{\mathbf{1}}, \bar{x}\rangle \mapsto \mathbf{app}\langle t_1, t_2\rangle).$$

If $t$ is $(t_1 =_\theta t_2)$ let $g$ be $=_{\theta M_{\mathcal{D}}\tau} \circ trn([\![\langle t_1, t_2\rangle]\!]_{\vec{x}}^{M_{\mathcal{D}}}, [U_\tau])$. Then $g$ is

$$(h \mapsto \{\langle \bar{x}, ((y)_1 = (y)_2)\rangle \ : \ \langle \bar{x}, y\rangle \in h\}) \circ (z^{\mathbf{1}} \mapsto \{\langle \bar{x}, \langle t_1, t_2\rangle\rangle \ : \ \mathbf{t}\}),$$

therefore $g = (z^{\mathbf{1}} \mapsto \{\langle \bar{x}, (t_1 = t_2)\rangle \ : \ \mathbf{t}\})$. From this we infer that

$$[\![(t_1 = t_2)]\!]_{\vec{x}}^{M_{\mathcal{D}}} = ctr(g, [U_\tau]) = (\langle z^{\mathbf{1}}, \bar{x}\rangle \mapsto (t_1 = t_2)).$$

If $t$ is $\mathbf{set}_\theta x\varphi$ let $y$ be a variable free for $x$ in $\varphi$ not occurring in $\vec{x}$. By induction hypothesis,

$$[\![\varphi_y^x]\!]_{\vec{x}y}^{M_{\mathcal{D}}} = (\langle z^{\mathbf{1}}, \langle \bar{x}, y\rangle\rangle \mapsto \varphi_y^x).$$

Let $g = \mathbf{set}_{\theta M_{\mathcal{D}}\tau} \circ trn([\![\varphi_y^x]\!]_{\vec{x}y}^{M_{\mathcal{D}}}, [U_\tau] \times [U_\theta])$. Then $g$ is the composite of

$$(h \mapsto \{\langle \bar{x}, u\rangle \ : \ \forall x \forall v((x \in u) = v \ \Leftrightarrow \ \langle\langle \bar{x}, x\rangle, v\rangle \in h)\})$$

and

$$(z^{\mathbf{1}} \mapsto \{\langle\langle \bar{x}, y\rangle, \varphi_y^x\rangle \ : \ \mathbf{t}\}).$$

Therefore, using the deduction rules of *HOL*,

$$\begin{aligned} g &= (z^{\mathbf{1}} \mapsto \{\langle \bar{x}, u\rangle \ : \ \forall x \forall v((x \in u) = v \ \Leftrightarrow \ v = \varphi)\}) \\ &= (z^{\mathbf{1}} \mapsto \{\langle \bar{x}, u\rangle \ : \ \forall x(x \in u \ \Leftrightarrow \ \varphi)\}) \\ &= (z^{\mathbf{1}} \mapsto \{\langle \bar{x}, \{x \ : \ \varphi\}\rangle \ : \ \mathbf{t}\}). \end{aligned}$$

From this follows that

$$[\![\mathbf{set}_\theta x\varphi]\!]_{\vec{x}}^{M_\mathcal{D}} = [\![\{x\ :\ \varphi\}]\!]_{\vec{x}}^{M_\mathcal{D}} = ctr(g, [U_\tau]) = (\langle z^\mathbf{1}, \bar{x}\rangle \mapsto \{x\ :\ \varphi\}).$$

If $t$ is $(ft')$, with $f \in F_{\theta\theta'}$ and $t \in T(\Sigma)_\theta$ then

$$[\![t']\!]_{\vec{x}}^{M_\mathcal{D}} = (\langle z^\mathbf{1}, \bar{x}\rangle \mapsto t'),$$

by induction hypothesis. Let $g = f_{M_\mathcal{D}\tau} \circ trn([\![t']\!]_{\vec{x}}^{M_\mathcal{D}}, W_\mathcal{D} \times [U_\tau])$ and let $x^\mathbf{1} \in X_\mathbf{1}$ not occurring in $\langle z^\mathbf{1}, \bar{x}\rangle$. Then $g$ is the composite of

$$(h \mapsto \{\langle\langle z^\mathbf{1}, \bar{x}\rangle, (fy)\rangle\ :\ \langle\langle z^\mathbf{1}, \bar{x}\rangle, y\rangle \in h\})$$

and

$$(x^\mathbf{1} \mapsto \{\langle\langle z^\mathbf{1}, \bar{x}\rangle, t'\rangle\ :\ \mathbf{t}\}).$$

Therefore, using the rules of *HOL*, it is easy to see that

$$g = (x^\mathbf{1} \mapsto \{\langle\langle z^\mathbf{1}, \bar{x}\rangle, (ft')\rangle\ :\ \mathbf{t}\}).$$

Thus

$$[\![(ft')]\!]_{\vec{x}}^{M_\mathcal{D}} = ctr(g, W_\mathcal{D} \times [U_\tau]) = (\langle z^\mathbf{1}, \bar{x}\rangle \mapsto (ft'))$$

as desired. The proof for the case $t = (rt')$ with $r \in R_{\theta\theta'}$ is similar. Finally, suppose that $t$ is $qxt'$, where $q \in Q_{\theta\theta'\theta''}$, $x \in X_\theta$ and $t' \in T(\Sigma)_{\theta'}$. Let $y$ free for $x$ in $t'$ not occurring in $\vec{x}$. By induction hypothesis,

$$[\![{t'}_y^x]\!]_{\vec{x}y}^{M_\mathcal{D}} = (\langle z^\mathbf{1}, \langle\bar{x}, y\rangle\rangle \mapsto {t'}_y^x).$$

Let $g = q_{M_\mathcal{D}\tau} \circ trn([\![{t'}_y^x]\!]_{\vec{x}y}^{M_\mathcal{D}}, [U_\tau] \times [U_\theta])$. Then $g$ is the composite of

$$(h \mapsto \{\langle\bar{x}, (qxu)\rangle\ :\ \langle\langle\bar{x}, x\rangle, u\rangle \in h\})$$

and

$$(z^\mathbf{1} \mapsto \{\langle\langle\bar{x}, y\rangle, {t'}_y^x\rangle\ :\ \mathbf{t}\}).$$

Using again the deduction rules of *HOL* we obtain that

$$g = (z^\mathbf{1} \mapsto \{\langle\bar{x}, (qxt')\rangle\ :\ \mathbf{t}\}).$$

Then

$$[\![(qxt')]\!]_{\vec{x}}^{M_\mathcal{D}} = ctr(g, [U_\tau]) = (\langle z^\mathbf{1}, \bar{x}\rangle \mapsto (qxt')).$$

• **Part IV:** $\Psi \vdash_\mathrm{d}{}^\mathcal{D} \varphi$ implies $\Psi \vDash_\mathrm{d}^{\langle\Sigma,\{M_\mathcal{D}\}\rangle} \varphi$.
Let $\varphi \in L(\Sigma)$ with canonical context $\vec{x}$. Then $\vdash_\mathrm{d}{}^\mathcal{D} \varphi$ implies $\vdash_\mathrm{d}{}^\mathcal{D} \varphi = \mathbf{t}$ implies $(\langle z^\mathbf{1}, \bar{x}\rangle \mapsto \varphi) = (\langle z^\mathbf{1}, \bar{x}\rangle \mapsto \mathbf{t})$ implies $[\![\varphi]\!]_{\vec{x}}^{M_\mathcal{D}} = [\![\mathbf{t}]\!]_{\vec{x}}^{M_\mathcal{D}}$ (by Part III) implies $\vDash_\mathrm{d}^{\langle\Sigma,\{M_\mathcal{D}\}\rangle} \varphi$. Now, let $\Psi \cup \{\varphi\}$ be a finite set of $\Sigma$-formulae and let $(\bigwedge \Psi)$ be as introduced at the beginning of this subsection. Then $\Psi \vdash_\mathrm{d}{}^\mathcal{D} \varphi$ implies $\vdash_\mathrm{d}{}^\mathcal{D} (\bigwedge \Psi) \Rightarrow \varphi$, because $\mathcal{D}$ has *MTD*. Then, $\vDash_\mathrm{d}^{\langle\Sigma,\{M_\mathcal{D}\}\rangle} (\bigwedge \Psi) \Rightarrow \varphi$ and so $\Psi \vDash_\mathrm{d}^{\langle\Sigma,\{M_\mathcal{D}\}\rangle} \varphi$.

• **Part V:** $\Psi \vDash_\mathrm{d}^{\langle\Sigma,\{M_\mathcal{D}\}\rangle} \varphi$ implies $\Psi \vdash_\mathrm{d}{}^\mathcal{D} \varphi$.

Now we verify that $M_{\mathcal{D}}$ is a canonical model for $\mathcal{D}$ (w.r.t. derivations). Let $\varphi \in L(\Sigma)$ with canonical context $\vec{x}$. Then $\vDash_{\mathrm{d}}^{\langle\Sigma,\{M_{\mathcal{D}}\}\rangle} \varphi$ implies $[\![\varphi]\!]_{\vec{x}}^{M_{\mathcal{D}}} = [\![\mathbf{t}]\!]_{\vec{x}}^{M_{\mathcal{D}}}$ implies $(\langle z^{\mathbf{1}}, \bar{x}\rangle \mapsto \varphi) = (\langle z^{\mathbf{1}}, \bar{x}\rangle \mapsto \mathbf{t})$ (by Part III) implies $\vdash_{\mathrm{d}}{}^{\mathcal{D}}\varphi = \mathbf{t}$ implies $\vdash_{\mathrm{d}}{}^{\mathcal{D}}\varphi$. Now, let $\Psi \cup \{\varphi\}$ be a finite subset of $L(\Sigma)$. Then $\Psi \vDash_{\mathrm{d}}^{\langle\Sigma,\{M_{\mathcal{D}}\}\rangle} \varphi$ implies $\vDash_{\mathrm{d}}^{\langle\Sigma,\{M_{\mathcal{D}}\}\rangle} (\bigwedge \Psi) \Rightarrow \varphi$ implies $\vdash_{\mathrm{d}}{}^{\mathcal{D}}(\bigwedge \Psi) \Rightarrow \varphi$ implies $\Psi \vdash_{\mathrm{d}}{}^{\mathcal{D}}\varphi$, because $\mathcal{D}$ has *MTD*.

• **Part VI:** $M_{\mathcal{D}}$ is appropriate for $\mathcal{D}$.
By Part IV, it suffices to prove appropriateness of $M_{\mathcal{D}}$ w.r.t. proofs. Let $\Psi \cup \{\varphi\}$ be a finite subset of $L(\Sigma)$. If $\Psi \vdash_{\mathrm{p}\vec{x}}^{\mathcal{D}} \varphi$ and $\bigwedge_{\psi\in\Psi} [\![\psi]\!]_{\vec{x}}^{M_{\mathcal{D}}} = \mathit{true}_{W\times\theta_{\vec{x}M_{\mathcal{D}}}}$ then $\vDash_{\mathrm{d}\vec{x}}^{\langle\Sigma,\{M_{\mathcal{D}}\}\rangle} \psi$, then $\vDash_{\mathrm{d}\vec{x}}^{\langle\Sigma,\{M_{\mathcal{D}}\}\rangle} \psi \wedge (\vec{x} = \vec{x})$, i.e., $\vDash_{\mathrm{d}}^{\langle\Sigma,\{M_{\mathcal{D}}\}\rangle} \psi \wedge (\vec{x} = \vec{x})$ for every $\psi \in \Psi$. By Part V we infer $\vdash_{\mathrm{d}}{}^{\mathcal{D}}\psi \wedge (\vec{x} = \vec{x})$ and so $\vdash_{\mathrm{d}}{}^{\mathcal{D}}\psi$ for every $\psi \in \Psi$. Then $\vdash_{\mathrm{d}}{}^{\mathcal{D}}\varphi$ and so $\vDash_{\mathrm{d}}^{\langle\Sigma,\{M_{\mathcal{D}}\}\rangle} \varphi$, by Part IV. Therefore $[\![\varphi]\!]_{\vec{x}}^{M_{\mathcal{D}}} = \mathit{true}_{W\times\theta_{\vec{x}M_{\mathcal{D}}}}$.
QED

Observe that the reduct to $\Sigma_{HOL}$ of $M_{\mathcal{D}}$ belongs to $\mathcal{M}_{HOL}^0$. That is, $M_{\mathcal{D}}$ is standard with respect to the language of pure *HOL*.

**Theorem 2.25** Every full logic system with deduction system including *HOL* and with *MTD* is complete.

**Proof:** Let $\mathcal{S} = \langle\Sigma, \mathcal{M}\rangle$, where $\mathcal{M}$ is the class of $\Sigma$-structures appropriate for $\mathcal{D}$. Consider a finite set of formulae $\Psi \cup \{\varphi\}$. If $\Psi \vDash_{\mathrm{d}}^{\mathcal{S}} \varphi$ then, since $M_{\mathcal{D}}$ is appropriate for $\mathcal{D}$, we get $\Psi \vDash_{\mathrm{d}}^{\langle\Sigma,\{M_{\mathcal{D}}\}\rangle} \varphi$. Thus $\Psi \vdash_{\mathrm{d}}{}^{\mathcal{D}}\varphi$, by Part V of the proof of Lemma 2.24.

Finally, suppose that $\Psi \vDash_{\mathrm{p}}^{\mathcal{S}} \varphi$. Let $\vec{x}$ be the canonical context of $\Psi \cup \{\varphi\}$ and let $\mathcal{D}^{\Psi}$ be the deduction system obtained from $\mathcal{D}$ by adding the axiom

$$\langle\emptyset, (\bigwedge \Psi) \wedge (\vec{x} = \vec{x}), \mathbf{U}\rangle,$$

where $(\bigwedge \Psi)$ and $(\vec{x} = \vec{x})$ are as defined above. If $\mathcal{M}^{\Psi}$ is the class of $\Sigma$-structures appropriate for $\mathcal{D}^{\Psi}$ then

$$\mathcal{M}^{\Psi} = \{M \in \mathcal{M} : [\![(\bigwedge \Psi) \wedge (\vec{x} = \vec{x})]\!]_{\vec{x}}^{M} = \mathit{true}_{W\times\theta_{\vec{x}M}}\}.$$

Let $\mathcal{S}^{\Psi} = \langle\Sigma, \mathcal{M}^{\Psi}\rangle$. Since $\vDash_{\mathrm{d}}^{\mathcal{S}^{\Psi}} (\varphi \wedge (\vec{x} = \vec{x}))$, we have $\vDash_{\mathrm{d}}^{\langle\Sigma,\{M_{\mathcal{D}^{\Psi}}\}\rangle} (\varphi \wedge (\vec{x} = \vec{x}))$ by Part VI of the proof of Lemma 2.24. Therefore, $\vdash_{\mathrm{d}}{}^{\mathcal{D}^{\Psi}}(\varphi \wedge (\vec{x} = \vec{x}))$, by Part V of the proof of the same lemma. Hence $\Psi \vdash_{\mathrm{p}}{}^{\mathcal{D}}\varphi$. QED

## 3 Fibring

In this section, we start by defining both unconstrained and constrained forms of fibring as universal constructions in a suitable category of logic systems. Afterwards, we show that soundness is always preserved by fibring. Finally, we address the problem of preservation of completeness, first establishing a result in the case of rich logics (including *HOL* and with the *MTD*) and then extending this result to weaker logics assuming the preservation of the conservativeness of *HOL*-enrichment.

### 3.1 Concept

Let $h : \Sigma \to \Sigma'$ be a signature morphism. Given a set $\mathcal{R}$ of $\Sigma$-rules, the *image* of $\mathcal{R}$ by $h$, denoted by $h(\mathcal{R})$, is the set $\{\langle h(\Gamma), h(\delta), \Pi\rangle : \langle \Gamma, \delta, \Pi\rangle \in \mathcal{R}\}$ of $\Sigma'$-rules. Given a deduction system $\mathcal{D} = \langle \Sigma, \mathcal{R}_{\mathrm{d}}, \mathcal{R}_{\mathrm{p}}\rangle$, let $h(\mathcal{D}) = \langle \Sigma', h(\mathcal{R}_{\mathrm{d}}), h(\mathcal{R}_{\mathrm{p}})\rangle$. And given a $\Sigma'$-structure $M' = \langle \mathcal{E}', W', \cdot_{M'}\rangle$, the *reduct* of $M'$ along $h$, denoted by $M'|_h$, is the $\Sigma$-structure $\langle \mathcal{E}', W', \cdot_{M'} \circ h\rangle$.

Let $\mathcal{L} = \langle \Sigma, \mathcal{M}, \mathcal{R}_{\mathrm{d}}, \mathcal{R}_{\mathrm{p}}\rangle$ and $\mathcal{L}' = \langle \Sigma', \mathcal{M}', \mathcal{R}'_{\mathrm{d}}, \mathcal{R}'_{\mathrm{p}}\rangle$ be logic systems. By a *logic system morphism* $h : \mathcal{L} \to \mathcal{L}'$ we mean a signature morphism $h : \Sigma \to \Sigma'$ such that:

1. $M'|_h \in \mathcal{M}$ whenever $M' \in \mathcal{M}'$;
2. for every $M' \in \mathcal{M}'$, $M' \in Ap(h(\mathcal{D}_{\mathcal{L}}))$ whenever $M'|_h \in Ap(\mathcal{D}_{\mathcal{L}})$;
3. $h(\mathcal{R}_{\mathrm{d}}) \subseteq \mathcal{R}'_{\mathrm{d}}$;
4. $h(\mathcal{R}_{\mathrm{p}}) \subseteq \mathcal{R}'_{\mathrm{p}}$.

Conditions (1), (3) and (4) are to be expected from previous work on the subject of fibring, namely [16, 10]. Condition (2) is a reasonable requirement that will allow the preservation of soundness by fibring. Methodologically, condition (2) should be looked upon more as a requirement on the rules of $\mathcal{D}_{\mathcal{L}}$ than on the models. For instance, consider the usual rule of quantified logic

$$\langle \emptyset, \forall x\, \xi \Rightarrow \xi^x_{\xi'},\ \xi' \rhd x : \xi\rangle$$

where the proviso has the usual meaning that no free variable in $\xi'$ is captured by a quantifier when $x$ is replaced by $\xi'$ in $\xi$. This rule becomes unexpectedly unsound when put in an environment where modalities and flexible symbols are available. More precisely, condition (2) will be violated when this rule is present in $\mathcal{L}$ and modalities and other flexible symbols appear in $\mathcal{L}'$. For instance, choose for $\mathcal{L}'$ a temporal quantified logic. Consider for $\xi$ the formula $(s = x) \Rightarrow (\mathbf{F}(s > x))$ where $s$ is flexible and $\mathbf{F}$ is the "sometime in the future" modality, and for $\xi'$ the term $s$. Then, $\xi^x_{\xi'}$ becomes $(s = s) \Rightarrow (\mathbf{F}(s > s))$.

Fortunately, is is easy to make rules more robust in the sense that they do not bring problems with respect to condition (2). In the example at hand, it is enough to reinforce the proviso with the additional requirement that no flexible symbol in $\xi'$ falls into the scope of a modality when $x$ is replaced by $\xi'$ in $\xi$. This stronger proviso changes nothing in the original quantified logic but it makes all the difference when embedding it into a richer logic such as a fibring.

This remark suggests the following definition which will be used at the end of this section.

**Definition 3.1** A deductive system $\mathcal{D}$ is said to be *robust* iff, for every signature monomorphism $h : \Sigma \to \Sigma'$ and $\Sigma'$-structure $M'$, $M' \in Ap(h(\mathcal{D}))$ whenever $M'|h \in Ap(\mathcal{D})$. A logic system $\mathcal{L}$ is said to be robust iff $\mathcal{D}_{\mathcal{L}}$ is robust. $\triangle$

It is trivial to make robust any given logic system. Indeed, the *brute force* method (including in all rules the additional requirement forbidding foreign categories of symbols) always works. For instance, if a logic system has no flexible symbols then we include in all rules the (proviso) additional requirement that they may not be applied when $\rho$ uses flexible symbols. This changes nothing in the original logic system but makes it much weaker when combined with other logic systems.

From now on, we assume that *HOL* is made robust, namely by interpreting the proviso $\xi' \rhd x : \xi$ as forbidding both (i) capture of free variables in $\xi'$ by binding operators in $\xi$ and (ii) capture of flexible symbols in $\xi'$ by flexible symbols in $\xi$.

It is straightforward to set up the *category* **Log** of logic systems and their morphisms. In this category fibrings appear as universal constructions.

**Definition 3.2** Given two logic systems $\mathcal{L}'$ and $\mathcal{L}''$, their *unconstrained fibring* is the logic system $\mathcal{L} = \mathcal{L}' \oplus \mathcal{L}''$ with:

- $\Sigma = \Sigma' \oplus \Sigma''$ with injections $i'$ and $i''$ (coproduct in **Sig** of $\Sigma'$ and $\Sigma''$);
- $\mathcal{M} = \{M \in Str(\Sigma' \oplus \Sigma'') : M|_{i'} \in \mathcal{M}' \;\&\; M|_{i''} \in \mathcal{M}'' \;\&$
  $M|_{i'} \in Ap(\mathcal{D}')$ implies $M \in Ap(i'(\mathcal{D}')) \;\&$
  $M|_{i''} \in Ap(\mathcal{D}'')$ implies $M \in Ap(i''(\mathcal{D}''))\}$;
- $\mathcal{R}_{\mathrm{d}} = i'(\mathcal{R}'_{\mathrm{d}}) \cup i''(\mathcal{R}''_{\mathrm{d}})$;
- $\mathcal{R}_{\mathrm{p}} = i'(\mathcal{R}'_{\mathrm{p}}) \cup i''(\mathcal{R}''_{\mathrm{p}})$. $\triangle$

Before proving that unconstrained fibring is a coproduct in the category **Log** we need to state a useful result.

**Lemma 3.3** Let $h : \Sigma \to \Sigma'$ be a signature morphism.

1. Let $\sigma$ be a schema $\Sigma$-substitution and consider the schema $\Sigma'$-substitution $\sigma' = \hat{h} \circ \sigma$. Then $\hat{\sigma}' \circ \hat{h} = \hat{h} \circ \hat{\sigma}$.

2. Let $M'$ be a $\Sigma'$-structure, $\rho \in Sbs(\Sigma)$ and $\rho' \in Sbs(\Sigma')$ such that $\rho' = \hat{h} \circ \rho$. Then $[\![t\rho]\!]_{\vec{x}}^{M'|_h} = [\![\hat{h}(t)\rho']\!]_{\vec{x}}^{M'}$ for every $t \in ST(\Sigma, \vec{x})$.

3. Let $\mathcal{D}$ be a $\Sigma$-deduction system and $M' \in Str(\Sigma')$. Then $M' \in Ap(h(\mathcal{D}))$ implies $M'|_h \in Ap(\mathcal{D})$.

**Proof:** 1. It is easy to prove by induction on the complexity of a schema term $t \in ST(\Sigma, \vec{x})$ (where, by convention, $\xi^x_{\xi'}$ has complexity 1) that a variable $x$ occurs free in $t$ iff it occurs free in $\hat{h}(t)$. Then, it is immediate that a schema term $t$ is free for a variable $x$ in a schema term $t'$ iff $\hat{h}(t)$ is free for $x$ in $\hat{h}(t')$.

From these facts the result follows by induction on the complexity of the schema term.
2. Immediate from item 1 and our definitions.
3. Immediate from item 2 and our definitions. QED

**Proposition 3.4** The unconstrained fibring $\mathcal{L}' \oplus \mathcal{L}''$ is the coproduct in **Log** of $\mathcal{L}'$ and $\mathcal{L}''$.

**Proof:** Let $\mathcal{L} = \mathcal{L}' \oplus \mathcal{L}'' = \langle \Sigma, \mathcal{M}, \mathcal{R}_{\mathrm{d}}, \mathcal{R}_{\mathrm{p}} \rangle$ as in Definition 3.2. Then it is immediate that the injections $i' : \Sigma' \to \Sigma$ and $i'' : \Sigma'' \to \Sigma$ are morphisms in **Log**. Let $\check{\mathcal{L}} = \langle \check{\Sigma}, \check{\mathcal{M}}, \check{\mathcal{R}}_{\mathrm{d}}, \check{\mathcal{R}}_{\mathrm{p}} \rangle$ be a logic system and $j' : \mathcal{L}' \to \check{\mathcal{L}}$, $j'' : \mathcal{L}'' \to \check{\mathcal{L}}$ in **Log**. Consider the unique signature morphism $h : \Sigma \to \check{\Sigma}$ such that $h \circ i' = j'$ and $h \circ i'' = j''$. It suffices to show that $h$ is morphism in **Log**. Let $\check{M} \in \check{\mathcal{M}}$. We need to show that $\check{M}|_h \in \mathcal{M}$. Observe that $(\check{M}|_h)|_{i'} = \check{M}|_{j'}$ belongs to $\mathcal{M}'$, because $j'$ is a **Log**-morphism. Analogously we show that $(\check{M}|_h)|_{i''} \in \mathcal{M}''$. Assume that $(\check{M}|_h)|_{i'} = \check{M}|_{j'}$ belongs to $Ap(\mathcal{D}')$. Then $\check{M} \in Ap(j'(\mathcal{D}'))$, that is, $\check{M} \in Ap(h(i'(\mathcal{D}')))$, because $j'$ is a morphism in **Log**. Then $\check{M}|_h \in Ap(i'(\mathcal{D}'))$, by Lemma 3.3(3). Analogously we prove that $(\check{M}|_h)|_{i''} \in Ap(\mathcal{D}')$ implies $\check{M}|_h \in Ap(i''(\mathcal{D}''))$, and so $\check{M}|_h \in \mathcal{M}$.

Suppose now that $\check{M} \in \check{\mathcal{M}}$ is such that $\check{M}|_h \in Ap(\mathcal{D})$. Since $\check{M}|_h \in Ap(i'(\mathcal{D}'))$ then $\check{M}|_{j'} \in Ap(\mathcal{D}')$, by Lemma 3.3(3), and so $\check{M} \in Ap(h(i'(\mathcal{D}')))$ because $j' = h \circ i'$ is a morphism in **Log**. Analogously we prove that $\check{M} \in Ap(h(i''(\mathcal{D}'')))$, because $\check{M}|_h \in Ap(i''(\mathcal{D}''))$. Therefore $\check{M} \in Ap(h(\mathcal{D}))$. By definition of $\mathcal{L}$ and the fact that $j' = h \circ i'$ and $j'' = h \circ i''$ are morphism in **Log** we have that $h(\mathcal{R}_{\mathrm{o}}) \subseteq \check{\mathcal{R}}_{\mathrm{o}}$ for $\mathrm{o} \in \{\mathrm{p}, \mathrm{d}\}$. QED

Let $Sg$ : **Log** $\to$ **Sig** be the obvious *forgetful functor*. Then:

**Proposition 3.5** The forgetful functor $Sg$ admits cocartesian liftings.

**Proof:** Let $h : \Sigma \to \Sigma'$ be a signature morphism and $\mathcal{L} = \langle \Sigma, \mathcal{M}, \mathcal{R}_{\mathrm{d}}, \mathcal{R}_{\mathrm{p}} \rangle$ a logic system. Consider the logic system $h_{Sg}(\mathcal{L}) = \langle \Sigma', \mathcal{M}', h(\mathcal{R}_{\mathrm{d}}), h(\mathcal{R}_{\mathrm{p}}) \rangle$ where

$$\mathcal{M}' = \{M' \in Str(\Sigma') : M'|_h \in \mathcal{M}, \text{ and } M'|_h \in Ap(\mathcal{D}) \text{ implies } M' \in Ap(h(\mathcal{D}))\}.$$

Now we prove that $\langle h_{Sg}(\mathcal{L}), h \rangle$ is a cocartesian lifting of $h$ by $Sg$ at $\mathcal{L}$. Consider a logic system $\check{\mathcal{L}} = \langle \check{\Sigma}, \check{\mathcal{M}}, \check{\mathcal{R}}_{\mathrm{d}}, \check{\mathcal{R}}_{\mathrm{p}} \rangle$, a logic morphism $g : \mathcal{L} \to \check{\mathcal{L}}$ and a signature morphism $f : \Sigma' \to \check{\Sigma}$ such that $f \circ h = g$. In a similar fashion that we prove Proposition 3.4 it can be easily proved that $f$ is the unique morphism in **Log** from $h_{Sg}(\mathcal{L})$ to $\check{\mathcal{L}}$ such that $f \circ h = g$. QED

Following the notation used in the proof above, given a signature morphism $h : \Sigma \to \Sigma'$ and a logic system $\mathcal{L} = \langle \Sigma, \mathcal{M}, \mathcal{R}_{\mathrm{d}}, \mathcal{R}_{\mathrm{p}} \rangle$, we denote by $h_{Sg}(\mathcal{L})$ the codomain of the cocartesian lifting of $h$ by $Sg$ at $\mathcal{L}$. This construction is useful when defining a more complex form of fibring where we allow the sharing of symbols.

Given two signatures $\Sigma'$ and $\Sigma''$, a *sharing constraint* over $\Sigma'$ and $\Sigma''$ is a source diagram $\mathcal{G}$ in **Sig** of the form

$$\Sigma' \xleftarrow{h'} \check{\Sigma} \xrightarrow{h''} \Sigma''$$

for some signature $\check{\Sigma}$ and signature monomorphisms $h'$ and $h''$. In this situation, we denote by

$$\Sigma' \overset{\mathcal{G}}{\oplus} \Sigma''$$

the pushout of the diagram $\mathcal{G}$.

**Definition 3.6** Given two logic systems $\mathcal{L}'$ and $\mathcal{L}''$ and a sharing constraint $\mathcal{G}$ over $\Sigma'$ and $\Sigma''$, their *$\mathcal{G}$-constrained fibring by sharing symbols* is the logic system

$$\mathcal{L}' \overset{\mathcal{G}}{\oplus} \mathcal{L}''$$

given by $q_{Sg}(\mathcal{L}' \oplus \mathcal{L}'')$ where $q$ is the coequalizer of $i' \circ h'$ and $i'' \circ h''$. △

Observe that we recover the unconstrained fibring as a special case of the constrained fibring by choosing an appropriate sharing constraint: it is enough to take $\check{\Sigma}$ as the initial signature.

As an illustration of constrained fibring by sharing symbols, consider the following useful example where we also explain the impact of choosing symbols as flexible or as rigid even in a logic (like *HOL*) where no modalities are available.

**Example 3.7** *Modal higher-order logic*
Consider the fibring of $MPL_K$ (defined in 2.3.1) and *HOL* (defined in 2.3.2) while sharing the propositional signature (defined in Example 2.4) for obtaining a modal higher-order logic. Choosing a symbol as flexible or rigid in *HOL* changes nothing in that logic system. But, when $HOL$ is combined with another logic system with modalities rigid and flexible symbols will have quite different properties. For instance, in the resulting logic we shall have as a theorem $(r = x) \Rightarrow (\Box\, (r = x))$ for any rigid symbol $r$, but not for a flexible symbol. △

Sometimes, besides sharing symbols, we may also want to share deduction rules. This form of combination appears as a *colimit* in **Log**.

In order to show that **Log** is small cocomplete it is sufficient to show that **Log** has small coproducts (the proof is similar to the proof of Proposition 3.4) and coequalizers. Observe that, given $h_1, h_2 : \Sigma \to \Sigma'$ in **Sig**, their coequalizer is $h : \Sigma' \to \Sigma''$ where $\Sigma'' = \langle R'', F'', Q'' \rangle$ with $R'' = R'/\approx_{R'}$, $F'' = F'/\approx_{F'}$ and $Q'' = Q'/\approx_{Q'}$, $\approx_{R'} \subseteq {R'}^2$ is the least equivalence relation generated from $\{\langle h_1(r), h_2(r) \rangle : r \in R\}$, $\approx_{F'}$ and $\approx_{Q'}$ are defined in a similar way and $h(r') = [r']$ (for $r' \in R'$), $h(f') = [f']$ (for $f' \in F'$) and $h(q') = [q']$ (for $q' \in Q'$).

**Proposition 3.8** The category **Log** has coequalizers.

**Proof:** Let $h_1, h_2 : \mathcal{L} \to \mathcal{L}'$ be logic system morphisms. The coequalizer $h : Sg(\mathcal{L}') \to \Sigma''$ in **Sig** of $Sg(h_1), Sg(h_2) : Sg(\mathcal{L}) \to Sg(\mathcal{L}')$ is the coequalizer $h : \mathcal{L}' \to \langle \Sigma'', \mathcal{M}'', \mathcal{R}''_{\mathrm{d}}, \mathcal{R}''_{\mathrm{p}} \rangle$ in **Log** of $h_1, h_2$, where $\mathcal{M}'' = \{[M'] : M' \in \mathcal{M}'_0\}$, $\mathcal{M}'_0$ is the class of all models $M' \in \mathcal{M}'$ such that $r'_{1M'} = r'_{2M'}$ whenever $r'_1 \approx_{R'} r'_2$ for every $r'_1, r'_2 \in R'$, $f'_{1M'} = f'_{2M'}$ whenever $f'_1 \approx_{F'} f'_2$ for every $f'_1, f'_2 \in F'$, $q'_{1M'} = q'_{2M'}$ whenever $q'_1 \approx_{Q'} q'_2$ for every $q'_1, q'_2 \in Q'$; and for every $M' = \langle \mathcal{E}', W', \cdot_{M'} \rangle \in Str(\Sigma')$, $[M'] = \langle \mathcal{E}', W', \cdot_{[M']} \rangle$ with $h(r')_{[M']} = {r'}_{M'}$ for every $r' \in R'$, $h(f')_{[M']} = {f'}_{M'}$ for every $f' \in F'$, $h(q')_{[M']} = {q'}_{M'}$ for every $q' \in Q'$; $\mathcal{R}''_{\mathrm{d}} = h(\mathcal{R}'_{\mathrm{d}})$ and $\mathcal{R}''_{\mathrm{p}} = h(\mathcal{R}'_{\mathrm{p}})$. QED

**Corollary 3.9** The category **Log** is small cocomplete.

Pushouts are specially useful for combining two logics while sharing a common sublogic.

Note also that the forgetful functor $Sg$ : **Log** $\to$ **Sig** has a left adjoint. Consider $G$ : **Sig** $\to$ **Log** such that $G(\Sigma) = \langle \Sigma, Str(\Sigma), \emptyset, \emptyset \rangle$ and $G(h) = h$. Using this functor, it is possible to provide an alternative characterization of constrained fibring by sharing symbols. Given two logic systems $\mathcal{L}'$ and $\mathcal{L}''$ and a sharing constraint $\mathcal{G} = \Sigma' \xleftarrow{h'} \check{\Sigma} \xrightarrow{h''} \Sigma''$, their $\mathcal{G}$-constrained fibring by sharing symbols is the pushout in **Log** of $\mathcal{L}' \xleftarrow{h'} G(\check{\Sigma}) \xrightarrow{h''} \mathcal{L}''$.

Therefore, all forms of fibring appear also as colimits in **Log**. So, from now on we shall establish results about colimits that will also apply to fibrings. To this end, observe that, for every signature $\Sigma$, the logic system $G(\Sigma)$ is full and, thus, sound.

Finally, note that the category **Log** might have been obtained as the flattening of the obvious indexed category **Sig** $\to$ **Cat**. We refrained to analyze here the properties of this indexed category because we were interested only in the flat category of logic systems. However, many properties of **Log** would be derivable from interesting properties of the indexed category (see [14] for relevant results about indexed categories).

## 3.2 Preservation of soundness

All forms of combination of logic systems considered above preserve soundness thanks to condition (2) in the definition of logic system morphism.

**Proposition 3.10** Sound logic systems are closed under colimits in **Log**.

**Proof:** We start by showing that soundness is preserved by coproducts. For simplification, we just prove that the coproduct of two sound logic systems is sound. The general case is proved analogously.

Let $\mathcal{L} = \langle \Sigma, \mathcal{M}, \mathcal{R}_{\mathrm{d}}, \mathcal{R}_{\mathrm{p}} \rangle$ be the coproduct of $\mathcal{L}'$ and $\mathcal{L}''$, and let $M \in \mathcal{M}$. Since $M|_{i'} \in \mathcal{M}'$ and $\mathcal{M}' \subseteq Ap(\mathcal{D}')$ (by hypothesis) then $M|_{i'} \in Ap(\mathcal{D}')$ and so $M \in Ap(i'(\mathcal{D}'))$. Analogously we get $M \in Ap(i''(\mathcal{D}''))$ and so $M \in Ap(\mathcal{D})$.

Finally, we show that the codomain of the coequalizer is sound whenever the logic systems in the given diagram are sound.

Consider morphisms $h_1, h_2 : \mathcal{L} \to \mathcal{L}'$ where $\mathcal{L}$ and $\mathcal{L}'$ are sound and let $h : \mathcal{L}' \to \mathcal{L}''$ be the coequalizer as in the proof of Proposition 3.8. We have to show that $\mathcal{M}'' \subseteq Ap(\mathcal{D}'')$. Take $[M'] \in \mathcal{M}''$. Observe that $[M']|_h = M'$ and $M' \in Ap(\mathcal{D}')$. Since $h$ is a morphism in **Log** then $[M'] \in Ap(\mathcal{D}'')$. QED

**Corollary 3.11** Both forms of fibring preserve soundness.

### 3.3 Preservation of completeness

It is easy to find examples of complete logic systems that by fibring result in a logic system that is not complete.

**Example 3.12** *Completeness is not always preserved.*
For instance, consider the full logic systems $\mathcal{L}'$ and $\mathcal{L}''$ defined as follows.

- $\mathcal{L}' = \langle \Sigma', \mathcal{M}', \mathcal{R}'_{\mathrm{d}}, \mathcal{R}'_{\mathrm{p}} \rangle$:
  - $\Sigma' = \langle R', F', Q' \rangle$ such that:
    - All members of the families $R'$ and $F'$ are empty, except:
      - $R'_{\mathbf{i}\boldsymbol{\Omega}} = \{p'\}$.
    - All members of the family $Q'$ are empty.
  - $\mathcal{R}'_{\mathrm{d}} = \mathcal{R}'_{\mathrm{p}} = \{\langle \emptyset, p'(x), \mathbf{U} \rangle : x \in X_{\mathbf{i}}\}$.
- $\mathcal{L}'' = \langle \Sigma'', \mathcal{M}'', \mathcal{R}''_{\mathrm{d}}, \mathcal{R}''_{\mathrm{p}} \rangle$:
  - $\Sigma'' = \langle R'', F'', Q'' \rangle$ such that:
    - All members of the families $R''$ and $F''$ are empty, except:
      - $R''_{\mathbf{1i}} = \{c''\}$;
      - $R''_{\mathbf{1}\boldsymbol{\Omega}} = \{\mathbf{t}''\}$.
    - All members of the family $Q''$ are empty.
  - $\mathcal{R}'_{\mathrm{d}} = \mathcal{R}'_{\mathrm{p}} = \{\langle \emptyset, \mathbf{t}'', \mathbf{U} \rangle\}$.

The two logic systems are obviously complete. However, their unconstrained fibring $\mathcal{L}$ is not complete. Indeed, the resulting (still full) logic system is as follows:

- $\mathcal{L} = \langle \Sigma, \mathcal{M}, \mathcal{R}_{\mathrm{d}}, \mathcal{R}_{\mathrm{p}} \rangle$:
  - $\Sigma = \langle R, F, Q \rangle$:
    - All members of the families $R$ and $F$ are empty, except:
      - $R_{\mathbf{1i}} = \{c''\}$;
      - $R_{\mathbf{1}\boldsymbol{\Omega}} = \{\mathbf{t}''\}$;
      - $R_{\mathbf{i}\boldsymbol{\Omega}} = \{p'\}$.
    - All members of the family $Q$ are empty.
  - $\mathcal{R}_{\mathrm{d}} = \mathcal{R}_{\mathrm{p}} = \{\langle \emptyset, p'(x), \mathbf{U} \rangle : x \in X_{\mathbf{i}}\} \cup \{\langle \emptyset, \mathbf{t}'', \mathbf{U} \rangle\}$.

In every $\Sigma$-structure $M \in \mathcal{M}$, we have $[\![p'(c'')]\!]^M_{[]} = true_W$. Hence, $\vDash^{\mathcal{L}}_{\mathrm{o}} p'(c'')$, but, obviously, $\nvdash_{\mathrm{o}}{}^{\mathcal{L}} p'(c'')$. $\triangle$

However, following the idea of [16], it is possible to take advantage of a general completeness theorem in order to obtain a sufficient condition for the preservation of completeness by fibring. In the case at hand, we can use the very general completeness theorem obtained at the end of Subsection 2.4. To this end, we need the following lemmas.

**Lemma 3.13** Let $h : \mathcal{L} \to \mathcal{L}'$ be a logic system morphism. Then for every $\Gamma \cup \{\delta\} \subseteq SL(\Sigma, \vec{x})$, proviso $\Pi$ and $\mathrm{o} \in \{\mathrm{p}, \mathrm{d}\}$:

$$\Gamma \vdash^{\mathcal{L}}_{\mathrm{o}\vec{x}} \delta \lhd \Pi \text{ implies } h(\Gamma) \vdash^{\mathcal{L}'}_{\mathrm{o}\vec{x}} h(\delta) \lhd \Pi.$$

**Proof:** We start by proving the following:
**Fact:** Let $\Pi$ be a proviso, $\sigma$ a $\Sigma$-substitution and $\sigma'$ the $\Sigma'$-substitution given by $\sigma' = \hat{h} \circ \sigma$. Then $(\Pi\sigma') = (\Pi\sigma)$.
Let $\rho \in Sbs(\Sigma_1)$. Since $\hat{!}_{\Sigma'} \circ \hat{h} = \hat{!}_{\Sigma}$ then $\hat{!}_{\Sigma'} \circ \sigma' = \hat{!}_{\Sigma'} \circ (\hat{h} \circ \sigma) = (\hat{!}_{\Sigma'} \circ \hat{h}) \circ \sigma = \hat{!}_{\Sigma} \circ \sigma$ and so $(\Pi\sigma')(\rho) = \Pi(\hat{\rho} \circ \hat{!}_{\Sigma'} \circ \sigma') = \Pi(\hat{\rho} \circ \hat{!}_{\Sigma} \circ \sigma) = (\Pi\sigma)(\rho)$.
We are ready to prove the result. Assume that $\Gamma \vdash^{\mathcal{L}}_{\mathrm{p}\vec{x}} \delta \lhd \Pi$. By induction on the length $n$ of a $\vec{x}$-proof of $\delta$ from $\Gamma$ with proviso $\Pi$ we show that $h(\Gamma) \vdash^{\mathcal{L}'}_{\mathrm{p}\vec{x}} h(\delta) \lhd \Pi$.
Base $n = 1$. If $\delta \in \Gamma$ then the conclusion is obvious. If $\delta$ is obtained from an axiom $\langle \emptyset, \gamma, \Pi' \rangle$ in $\mathcal{R}_{\mathrm{p}}$ using a $\Sigma$-substitution $\sigma$ then $\delta$ is $\gamma\sigma$, $\Pi \leq (\Pi'\sigma)$ and $\langle \emptyset, h(\gamma), \Pi' \rangle$ is an axiom in $\mathcal{R}'_{\mathrm{p}}$. Consider the $\Sigma'$-substitution $\sigma' = \hat{h} \circ \sigma$. Then $h(\gamma)\sigma' = h(\gamma\sigma)$ and $(\Pi'\sigma') = (\Pi'\sigma)$ by Lemma 3.3(1) and by the Fact proved above. This shows the result for this case.
Step: Assume that there are a rule $\langle \{\gamma'_1, \dots, \gamma'_k\}, \delta', \Pi' \rangle$ in $\mathcal{R}_{\mathrm{p}}$ and a $\Sigma$-schema substitution $\sigma$ such that $\delta = \delta'\sigma$ and $\delta_{i_j} = \gamma'_j\sigma$ for $j = 1, \dots, k$ and some $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n-1\}$ and additionally $\Pi \leq \Pi_{i_1} \sqcap \dots \sqcap \Pi_{i_k} \sqcap (\Pi'\sigma)$. So $\Gamma \vdash^{\mathcal{L}}_{\mathrm{p}\vec{x}} \delta_{i_j} \lhd \Pi_{i_j}$ for $j = 1, \dots, k$. Therefore, by the induction hypothesis, $h(\Gamma) \vdash^{\mathcal{L}'}_{\mathrm{p}\vec{x}} h(\delta_{i_j}) \lhd \Pi_{i_j}$ for $j = 1, \dots, k$. But $\langle \{h(\gamma'_1), \dots, h(\gamma'_k)\}, h(\delta'), \Pi' \rangle$ is in $\mathcal{R}'_{\mathrm{p}}$. Hence considering $\sigma' = \hat{h} \circ \sigma$ we obtain $\{h(\gamma'_1)\sigma', \dots, h(\gamma'_k)\sigma'\} \vdash_{\mathrm{p}\vec{x}} h(\delta')\sigma' \lhd (\Pi'\sigma')$ and by Lemma 3.3(1) and by the Fact proved above we get $\{h(\delta_{i_1}), \dots, h(\delta_{i_k})\} \vdash_{\mathrm{p}\vec{x}} h(\delta) \lhd (\Pi'\sigma)$. The result follows easily from this. The proof of the desired result for derivations is similar. QED

**Lemma 3.14** Full logic systems are closed under colimits in **Log**.

**Proof:** We start by showing that fullness is preserved by coproducts. For simplification, we just prove that the coproduct of two full logic systems is full. The general case is proved analogously.

Let $\mathcal{L} = \langle \Sigma, \mathcal{M}, \mathcal{R}_{\mathrm{d}}, \mathcal{R}_{\mathrm{p}} \rangle$ be the fibring of $\mathcal{L}'$ and $\mathcal{L}''$. Then

$$\begin{aligned}\mathcal{M} = \{M \in Str(\Sigma) : \; & M|_{i'} \in \mathcal{M}' \;\&\; M|_{i''} \in \mathcal{M}'' \;\& \\ & M|_{i'} \in Ap(\mathcal{D}') \text{ implies } M \in Ap(i'(\mathcal{D}')) \;\& \\ & \quad M|_{i''} \in Ap(\mathcal{D}'') \text{ implies } M \in Ap(i''(\mathcal{D}''))\}.\end{aligned}$$

Since $\mathcal{M}' = Ap(\mathcal{D}')$ and $\mathcal{M}'' = Ap(\mathcal{D}'')$, by hypothesis, then

$$\begin{aligned}\mathcal{M} = \{M \in Str(\Sigma) : M|_{i'} \in Ap(\mathcal{D}') \mathbin{\&} M|_{i''} \in Ap(\mathcal{D}'') \mathbin{\&} \\ M|_{i'} \in Ap(\mathcal{D}') \text{ implies } M \in Ap(i'(\mathcal{D}')) \mathbin{\&} \\ M|_{i''} \in Ap(\mathcal{D}'') \text{ implies } M \in Ap(i''(\mathcal{D}''))\}\end{aligned}$$

and so

$$\begin{aligned}\mathcal{M} = \{M \in Str(\Sigma) : M|_{i'} \in Ap(\mathcal{D}') \mathbin{\&} M|_{i''} \in Ap(\mathcal{D}'') \mathbin{\&} \\ M \in Ap(i'(\mathcal{D}')) \mathbin{\&} M \in Ap(i''(\mathcal{D}''))\},\end{aligned}$$

that is, $\mathcal{M} = Ap(\mathcal{D})$, by Lemma 3.3(3).

Finally, we show that the codomain of the coequalizer is full whenever the logic systems in the given diagram are full.

Consider morphisms $h_1, h_2 : \mathcal{L} \to \mathcal{L}'$ where $\mathcal{L}$ and $\mathcal{L}'$ are full and let $h : \mathcal{L}' \to \mathcal{L}''$ be the coequalizer as in the proof of Proposition 3.8. We have to show that $\mathcal{M}'' = Ap(\mathcal{D}'')$. By Proposition 3.10, we already know that $\mathcal{M}'' \subseteq Ap(\mathcal{D}'')$. To show that $Ap(\mathcal{D}'') \subseteq \mathcal{M}''$ take $M'' \in Ap(\mathcal{D}'')$. Consider $M' = M''|_h$. Using item 2 of Lemma 3.3 and the definition of $\mathcal{D}''$ we get that $M' \in Ap(\mathcal{D}')$ and by fullness we have that $M' \in \mathcal{M}'$. It remains to show that $M' \in \mathcal{M}'_0$, which follows easily using the definition of reduct. QED

**Corollary 3.15** Both forms of fibring preserve fullness.

**Theorem 3.16** Let $\mathcal{L}'$ and $\mathcal{L}''$ be full logic systems with deduction systems including *HOL* and with *MTD*. Then, for every sharing constraint $\mathcal{G}$ over $\Sigma'$ and $\Sigma''$ such that all the symbols in $\Sigma_{HOL}$ are shared, their fibring

$$\mathcal{L}' \overset{\mathcal{G}}{\oplus} \mathcal{L}''$$

is full, includes *HOL*, has *MTD* and is, therefore, complete.

**Proof:** Since both $\mathcal{L}'$ and $\mathcal{L}''$ are full and include *HOL* then their $\mathcal{G}$-constrained fibring $\mathcal{L}$ is full, by Lemma 3.14, and obviously it includes *HOL*. By Theorem 2.25, it suffices to show that $\mathcal{L}$ has *MTD*.

Recall that a rule in the fibring can only come from $\mathcal{L}'$ or from $\mathcal{L}''$. Moreover, it is worth noting that a deduction system including *HOL* has *MTD* iff $\{(\xi \Rightarrow \gamma_1), \ldots, (\xi \Rightarrow \gamma_k)\} \vdash_{\mathrm{d}} (\xi \Rightarrow \delta) \lhd \Pi$ for every derivation rule $\langle\{\gamma_1, \ldots, \gamma_k\}, \delta, \Pi\rangle$ and every schema variable $\xi \in \Xi_{\mathbf{\Omega}}$ not occurring in the rule. The proof of this fact is an easy adaptation of [16]. Now assume that $\mathcal{L}'$ and $\mathcal{L}''$ have *MTD*. Assume without loss of generality that a given derivation rule of $\mathcal{L}$ comes from $\mathcal{L}'$ and it is of the form: $\langle\{i'(\gamma'_1), \ldots, i'(\gamma'_k)\}, i'(\delta'), \Pi'\rangle$. Since $\mathcal{D}'$ has *MTD* then, by our remark above, we have $\{(\xi \Rightarrow \gamma'_1), \ldots, (\xi \Rightarrow \gamma'_k)\} \vdash_{\mathrm{d}} (\xi \Rightarrow \delta') \lhd \Pi'$ where $\xi$ does not occur in the rule. By Lemma 3.13 we get $\{(\xi \Rightarrow i'(\gamma'_1)), \ldots, (\xi \Rightarrow i'(\gamma'_k))\} \vdash_{\mathrm{d}} (\xi \Rightarrow i'(\delta')) \lhd \Pi'$. Since $\xi \in \Xi_{\mathbf{\Omega}}$ does not occur in the derivation rule of $\mathcal{L}$ the result follows. QED

As expected, this result has a counterpart for colimits: the idea of course is to require that every logic system in the diagram be full with *MTD* and include *HOL*. But we refrain to spell it out because it is not relevant for the rest of the paper.

Unfortunately, Theorem 3.16, although useful, requires that each of the given deductive systems includes *HOL*. What about weaker logic systems? Given two such complete systems we might consider their enrichments with *HOL* and then try to compare the fibring of the original systems with the fibring of the enriched systems since the latter is complete thanks to the theorem above. To this end, we start by making precise what we mean by enriching a given logic system with *HOL* and then we establish a very useful property of the enrichment: the enrichment is conservative iff the original system is complete (under some weak conditions).

**Definition 3.17** Let $\mathcal{L} = \langle \Sigma, \mathcal{M}, \mathcal{R}_\mathrm{d}, \mathcal{R}_\mathrm{p} \rangle$ be a logic system and let $\mathcal{L} \oplus HOL = \langle \check{\Sigma}, \check{\mathcal{M}}, \check{\mathcal{R}}_\mathrm{d}, \check{\mathcal{R}}_\mathrm{p} \rangle$ be the unconstrained fibring of $\mathcal{L}$ and *HOL* with injections $e : \Sigma \to \check{\Sigma}$ and $i : \Sigma_{HOL} \to \check{\Sigma}$. Denote by $\mathcal{R}_\mathrm{d}^{HOL}$ the set of derivation rules of *HOL*. Then, the *HOL-enrichment* $\mathcal{L}^* = \langle \Sigma^*, \mathcal{M}^*, \mathcal{R}_\mathrm{d}^*, \mathcal{R}_\mathrm{p}^* \rangle$ of $\mathcal{L}$ is defined as follows:

- $\Sigma^* = \check{\Sigma}$;
- $\mathcal{M}^* = \check{\mathcal{M}}$;
- $\mathcal{R}_\mathrm{d}^* = \{\langle \emptyset, (\bigwedge \Gamma) \Rightarrow \delta, \Pi \rangle : \langle \Gamma, \delta, \Pi \rangle \in e(\mathcal{R}_\mathrm{d})\} \cup i(\mathcal{R}_\mathrm{d}^{HOL})$;
- $\mathcal{R}_\mathrm{p}^* = \check{\mathcal{R}}_\mathrm{p}$. △

Observe that $\mathcal{L}^*$ has *MTD* (the proof is similar to that of Theorem 3.16). Moreover, $\Psi \vdash_{\mathrm{o}\vec{x}}^{\mathcal{L}^*} \varphi$ whenever $\Psi \vdash_{\mathrm{o}\vec{x}}^{\mathcal{L} \oplus HOL} \varphi$, and the converse is true iff $\mathcal{L} \oplus HOL$ has *MTD*.

In the sequel, it is convenient to use $e : \Sigma \to \Sigma^*$, the embedding morphism of $\Sigma$ into $\Sigma^*$. We say that $\mathcal{L}^*$ is a *conservative extension of* $\mathcal{L}$ iff, for finite $\Psi \cup \{\varphi\} \subseteq L(\Sigma)$ and $\mathrm{o} \in \{\mathrm{p}, \mathrm{d}\}$, $\Psi {\vdash_\mathrm{o}}^{\mathcal{L}} \varphi$ whenever $e(\Psi) {\vdash_\mathrm{o}}^{\mathcal{L}^*} e(\varphi)$. Observe that, for $\mathrm{o} \in \{\mathrm{p}, \mathrm{d}\}$, $\Psi {\vdash_\mathrm{o}}^{\mathcal{L}} \varphi$ implies $e(\Psi) {\vdash_\mathrm{o}}^{\mathcal{L}^*} e(\varphi)$, because of the definition of $\mathcal{L}^*$.

It is clear that, as long as the rules at hand are robust (recall Definition 3.1), every structure of $\mathcal{M}_\mathcal{L}$ appears in $\mathcal{M}^*$ with its interpretation map extended to the symbols of $\Sigma_{HOL}$ as in $\mathcal{M}_{HOL}^0$: no model of $\mathcal{L}$ is lost. Hence:

**Lemma 3.18** Let $\Psi \cup \{\varphi\} \subseteq L(\Sigma, \vec{x})$ be a finite set and $\mathrm{o} \in \{\mathrm{p}, \mathrm{d}\}$. Assume that $\mathcal{L}$ is robust. Then $\Psi \vDash_{\mathrm{o}\vec{x}}^{\mathcal{L}} \varphi$ iff $e(\Psi) \vDash_{\mathrm{o}\vec{x}}^{\mathcal{L}^*} e(\varphi)$.

**Proof:** Observe that $[\![\psi]\!]_{\vec{x}}^{M^*|_e} = [\![\hat{e}(\psi)]\!]_{\vec{x}}^{M^*}$ for every $M^* \in \mathit{Str}(\Sigma^*)$ and $\psi \in L(\Sigma)$. Let $\mathcal{M}^*|_e = \{M^*|_e : M^* \in \mathcal{M}^*\}$. Therefore,

$$(\dagger) \qquad e(\Psi) \vDash_{\mathrm{o}\vec{x}}^{\langle \Sigma^*, \mathcal{M}^* \rangle} e(\varphi) \text{ iff } \Psi \vDash_{\mathrm{o}\vec{x}}^{\langle \Sigma, \mathcal{M}^*|_e \rangle} \varphi.$$

Since $\mathcal{M}^*|_e \subseteq \mathcal{M}$, $\Psi \vDash_{\mathrm{o}\vec{x}}^{\langle \Sigma, \mathcal{M} \rangle} \varphi$ implies $\Psi \vDash_{\mathrm{o}\vec{x}}^{\langle \Sigma, \mathcal{M}^*|_e \rangle} \varphi$. Furthermore, the latter implies $e(\Psi) \vDash_{\mathrm{o}\vec{x}}^{\langle \Sigma^*, \mathcal{M}^* \rangle} e(\varphi)$.

Conversely, since $\mathcal{L}$ and *HOL* are assumed to be robust, we have

$$\mathcal{M}^* = \{M^* \in Str(\Sigma^*) \ : M^*|_e \in \mathcal{M} \text{ and } M^*|_i \in Ap(\mathcal{D}_{HOL})\}.$$

Observe that every $M \in \mathcal{M}$ can be extended to a $\Sigma^*$-structure $M^*$ such that $M^*|_e = M$ and $M^*|_i \in \mathcal{M}^0_{HOL}$, thus $\mathcal{M} = \mathcal{M}^*|_e$ and the result follows from (†). QED

**Lemma 3.19** Let $\mathcal{L}$ be a full logic system. Then $\mathcal{L}^*$ is full and complete.

**Proof:** As observed above, $\mathcal{L}^*$ has *MTD*. On the other hand $\mathcal{L}^*$ is full by Lemma 3.14 and the fact that $Ap_{\mathrm{d}}(\mathcal{R}^*_{\mathrm{d}}) = Ap_{\mathrm{d}}(\check{\mathcal{R}}_{\mathrm{d}})$. The result follows from Theorem 2.25. QED

With these two lemmas and the following definition, we are finally ready to establish the envisaged result.

**Definition 3.20** A logic system $\mathcal{L}$ is said to be *expressive* iff for every context $\vec{x} = x_1 \ldots x_n$ there is a finite set $\Delta_{\vec{x}} \subseteq L(\Sigma)$ such that the set of variables occurring free in $\Delta_{\vec{x}}$ is $\{x_1, \ldots, x_n\}$, and $\vdash^{\mathcal{L}}_{\mathrm{d}\vec{x}} \varphi$ for every $\varphi \in \Delta_{\vec{x}}$. △

This condition is used in the proof of item 2 of Theorem 3.21 in order to obtain canonical contexts in derivations. It should be noted that the most common logics are expressive in this sense.

**Theorem 3.21** Let $\mathcal{L}$ be a full logic system.

1. The logic system $\mathcal{L}$ is complete whenever $\mathcal{L}^*$ is a conservative extension of $\mathcal{L}$.

2. Assume that $\mathcal{L}$ is expressive and robust. Then $\mathcal{L}^*$ is a conservative extension of $\mathcal{L}$ whenever $\mathcal{L}$ is complete.

**Proof:** 1. Assume that $\mathcal{L}^*$ is conservative and $\Psi \vDash^{\mathcal{L}}_{\mathrm{o}} \varphi$. Then by Lemma 3.18 $e(\Psi) \vDash^{\mathcal{L}^*}_{\mathrm{o}} e(\varphi)$. Using Lemma 3.19 we obtain $e(\Psi) {\vdash_{\mathrm{o}}}^{\mathcal{L}^*} e(\varphi)$ and so $\Psi {\vdash_{\mathrm{o}}}^{\mathcal{L}} \varphi$ by conservativeness of $\mathcal{L}^*$.
2. Observe that fullness of $\mathcal{L}$ implies fullness of $\mathcal{L}^*$, by Lemma 3.19. Let $\Psi \cup \{\varphi\} \subseteq L(\Sigma)$ be a finite set such that $e(\Psi) {\vdash_{\mathrm{o}}}^{\mathcal{L}^*} e(\varphi)$. Then there exists a context $\vec{x}$ such that $e(\Psi) \vdash^{\mathcal{L}^*}_{\mathrm{o}\vec{x}} e(\varphi)$ and so $e(\Psi) \vDash^{\mathcal{L}^*}_{\mathrm{o}\vec{x}} e(\varphi)$ using the soundness of $\mathcal{L}^*$. Let $\Delta_{\vec{x}}$ be a set of theorems associated with $\vec{x}$ using the expressiveness of $\mathcal{L}$. Therefore $e(\Psi \cup \Delta_{\vec{x}}) \vDash^{\mathcal{L}^*}_{\mathrm{o}\vec{x}} e(\varphi)$ and so $\Psi \cup \Delta_{\vec{x}} \vDash^{\mathcal{L}}_{\mathrm{o}\vec{x}} \varphi$ by Lemma 3.18, i.e., $\Psi \cup \Delta_{\vec{x}} \vDash^{\mathcal{L}}_{\mathrm{o}} \varphi$. Since $\mathcal{L}$ is complete then $\Psi \cup \Delta_{\vec{x}} {\vdash_{\mathrm{o}}}^{\mathcal{L}} \varphi$, i.e., $\Psi {\vdash_{\mathrm{o}}}^{\mathcal{L}} \varphi$ and so $\mathcal{L}^*$ is a conservative extension of $\mathcal{L}$. QED

Indeed, this result suggests an approach to the problem of improving the Theorem 3.16 towards ensuring the preservation by fibring of completeness without requiring the inclusion of *HOL*. The idea is to identify under which conditions fibring preserves the conservativeness of *HOL*-enrichment. In those conditions plus those of Theorem 3.21, the completeness of the fibring

$$\mathcal{L} = \mathcal{L}' \overset{\mathcal{G}}{\oplus} \mathcal{L}''$$

would result from the conservativeness of $\mathcal{L}^*$.

In short, we are led to the following theorem:

**Theorem 3.22** Let $\mathcal{L}'$ and $\mathcal{L}''$ be full, expressive, robust and complete logic systems. Assume also that the conservativeness of

$$(\mathcal{L}' \overset{\mathcal{G}}{\oplus} \mathcal{L}'')^*$$

follows from the conservativeness of $\mathcal{L}'^*$ and $\mathcal{L}''^*$. Then, their fibring

$$\mathcal{L}' \overset{\mathcal{G}}{\oplus} \mathcal{L}''$$

is also full, expressive, robust and complete.

If we know under which conditions we can infer the conservativeness of

$$(\mathcal{L}' \overset{\mathcal{G}}{\oplus} \mathcal{L}'')^*$$

from the conservativeness of $\mathcal{L}'^*$ and $\mathcal{L}''^*$, this result can become quite useful. Unfortunately, it is an open problem to find those conditions. It seems that those conditions will include, at least, the following requirements on each of the given logic systems $\mathcal{L}'$ and $\mathcal{L}''$: (i) adding new constants (of any sort) should produce a conservative extension and (ii) should not destroy the conservativeness of the *HOL*-enrichment. Note that the logic system $\mathcal{L}'$ in the counter-example at the beginning of this subsection does not fulfill requirement (ii).

## Acknowledgments

The idea for this paper came up during a visit of John Bell to Lisbon and the second and third authors are grateful for his encouragement. The authors wish also to express their gratitude to Claudio Hermida for many useful pointers into categorical logic and for correcting a couple of our misunderstandings at the early stages of the work, and to Alberto Zanardo for many useful discussions and for an important hint concerning the relationship between completeness and conservativeness of *HOL*-enrichment. The second and third authors are also grateful to their Coimbra colleagues in the ACL Project for introducing them to the joys of topoi some years ago. Finally, the authors are indebted to the anonymous referees that made significant suggestions towards improving the readability and effectiveness of the paper to the intended audience. Furthermore, one referee pointed out a technical bug and suggested the correction.

This work was partially supported by *Fundação para a Ciência e a Tecnologia* (FCT, Portugal), namely via the FEDER Project FibLog POCTI/MAT/372 39/2001. Most of the work was carried out during a long term visit by the first author to Lisbon with the postdoctoral grant 01/1045-0 of *Fundação de Amparo à Pesquisa do Estado de São Paulo* (FAPESP, Brazil).

## References

[1] J. L. Bell. *Toposes and Local Set Theories*. Oxford University Press, 1988.

[2] P. Blackburn and M. de Rijke. Why combine logics? *Studia Logica*, 59(1):5–27, 1997.

[3] M. E. Coniglio. Categorical logic with partial elements. In W. A. Carnielli and I. M. L. D'Ottaviano, editors, *Advances in Contemporary Logic and Computer Science - Proceedings of the XI Brazilian Conference on Mathematical Logic*, volume 235 of *Contemporary Mathematics*, pages 63–82. American Mathematical Society, 1999.

[4] M. E. Coniglio and C. Sernadas. A complete axiomatization of higher-order intuitionistic logic. *CLE e-Prints*, 2(6), 2002. CLE-UNICAMP, Campinas, Brazil. Available at http://www.cle.unicamp.br/e-prints/abstract_15.html.

[5] D. Gabbay. Fibred semantics and the weaving of logics: part 1. *Journal of Symbolic Logic*, 61(4):1057–1120, 1996.

[6] D. Gabbay. *Fibring logics*. Oxford University Press, 1999.

[7] J. Lambek and P. J. Scott. *Introduction to Higher-Order Categorical Logic*. Cambridge University Press, 1986.

[8] S. Mac Lane and I. Moerdijk. *Sheaves in Geometry and Logic*. Springer-Verlag, New York, 1994.

[9] A. M. Pitts. Categorical logic. In S. Abramski, D. M. Gabbay, and T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science*, volume 5, pages 39–128. Oxford University Press, 2001.

[10] A. Sernadas, C. Sernadas, and C. Caleiro. Fibring of logics as a categorial construction. *Journal of Logic and Computation*, 9(2):149–179, 1999.

[11] A. Sernadas, C. Sernadas, C. Caleiro, and T. Mossakowski. Categorial fibring of logics with terms and binding operators. In D. Gabbay and M. de Rijke, editors, *Frontiers of Combining Systems 2*, pages 295–316. Research Studies Press, 2000.

[12] A. Sernadas, C. Sernadas, and A. Zanardo. Fibring modal first-order logics: Completeness preservation. *Logic Journal of the IGPL*, 10(4):413–451, 2002.

[13] C. Sernadas, J. Rasga, and W. A. Carnielli. Modulated fibring and the collapsing problem. *Journal of Symbolic Logic*, in print.

[14] A. Tarlecki, R. Burstall, and J. Goguen. Some fundamental algebraic tools for the semantics of computation. III: Indexed categories. *Theoretical Computer Science*, 91(2):239–264, 1991.

[15] O. Wyler. *Lecture Notes on Topoi and Quasitopoi*. World Scientific, 1991.

[16] A. Zanardo, A. Sernadas, and C. Sernadas. Fibring: Completeness preservation. *Journal of Symbolic Logic*, 66(1):414–439, 2001.