Modal Sequent Calculi Labelled with Truth Values: Completeness, Duality and Analyticity

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Abstract

Labelled sequent calculi are provided for a wide class of normal modal systems using truth values as labels. The rules for formula constructors are common to all modal systems. For each modal system, specific rules for truth values are provided that reflect the envisaged properties of the accessibility relation. Both local and global reasoning are supported. Strong completeness is proved for a natural two-sorted algebraic semantics. As a corollary, strong completeness is also obtained over general Kripke semantics. A duality result is established between the category of sober algebras and the category of general Kripke structures. A simple enrichment of the proposed sequent calculi is proved to be complete over standard Kripke structures. The calculi are shown to be analytic in a useful sense.

Key words: modal logic, sequent calculus, labelled deduction.

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1 Introduction

Labelled deduction has been attracting much attention, namely within modal logic where it is natural to set up deduction systems using worlds as labels. The idea of using worlds as labels is already found in [15]. More recently, the idea was further explored in an attempt to produce modular calculi appropriate for automation. Indeed, while for defining a Hilbert calculus for a specific modal system one has to add just some extra axioms to the core calculus for system K, when setting up a sequent calculus (or a tableaux calculus, or a natural deduction calculus) for a specific modal system one has to start from scratch (or almost). Usually, the rules needed for the modal system at hand are quite and subtly different from those of system K (see, for instance, [12, 20]). Labelled deduction opened a way out of this difficulty. With labelled deduction it is possible to keep the rules of the core calculus, adding for each modal system some extra rules about the labels (worlds), as proposed in, for instance, [10, 13, 4, 8, 19].

The idea of labelled deduction also appeared outside the context of modal logic, namely for finite many-valued logics where it is natural to use the truth values as labels [9, 3, 14].

More recently, labelled deduction has been applied in the context of research on combining logics. Namely, as discussed in [16], combining deduction systems requires a sufficiently general notion of labelled deduction system in order to make possible to combine, for instance, a natural deduction calculus for a modal system with a natural deduction calculus for a finitely many-valued logic. To this end, it is convenient to set up a deduction system with labels extracted from a suitable algebra of truth values.

In this paper, we adopt this novel approach, studying labelled sequent calculi for normal modal systems with labels extracted from an ordered algebra of truth values. The basic assertions are of the form $\theta \leq \varphi$ expressing that truth value θ is less than or equal to the denotation of formula φ. Observe that by a truth value we intend here a modal truth value that, in the context of Kripke semantics, corresponds to a set of worlds.

The sequent calculi proposed in Section 2 share a common basis composed of: (i) structural rules; (ii) rules about the order among truth values; and (iii) rules for the formula constructors. The sequent calculus for each modal system is obtained by adding to this common basis specific rules imposing the underlying properties of the accessibility relation. The calculi support both global and local reasoning (corresponding to entailment over Kripke structures and entailment over worlds, respectively). Section 2 ends with some metatheorems not only interesting in themselves but also essential to the proof of the completeness results.

In Section 3, we start by proposing a new algebraic semantics involving two sorts (a sort for truth values and a sort for denotations of formulae). A quite general completeness result is proved assuming very little about the sequent calculus at hand (therefore, applicable outside the context of modal logic). Afterwards, we show how to move between general Kripke structures and such algebras in order to: (i) establish a characterization result showing that the proposed specific rules do characterize the frames with the intended properties of the accessibility relation; and (ii) obtain a completeness result over general Kripke structures as a corollary of the completeness theorem over algebraic semantics.

Observe that the proposed labelled language is rich enough to express properties of the accessibility relation that are not expressible by modal formulae (such as, irreflexivity, antisymmetry and asymmetry). Furthermore, we show that the proposed specific rules characterize the envisaged properties even among general Kripke structures. This ability to deal with general Kripke semantics is a key advantage of the "truth values as labels" approach proposed in this paper compared with the traditional "worlds as labels" approach. By the way, we also provide a simple enrichment of the language that leads to complete calculi over standard Kripke semantics.

Section 3 ends with a duality between the category of sober algebras and the category of general Kripke structures (with p-morphisms) and with a semantic proof of the analyticity of the proposed calculi (that is, we only need to apply rules to known terms and formulae).

The significance of the approach is further discussed in Section 4 where interesting future developments are also mentioned, namely towards obtaining proof-theoretic results (like cut elimination), exploring the relationship to hybrid logic and moving out of the context of modal logic to tackle other logics.

2 Sequent calculi

2.1 Language

Assume given three sets $\{\xi_i : i \in \mathbb{N}\}$, $\{\tau_i : i \in \mathbb{N}\}$ and $\{\Gamma_i : i \in \mathbb{N}\}$. The elements of these sets are *meta-variables* of different kinds: each ξ_i may be replaced by a (simple) formula, each τ_i by a (truth value) term and each Γ_i by a (finite) bag of assertions as described below.

A *signature* is a tuple $\Sigma = \langle C, O, X, Y, Z\rangle$ where $C = \{C_k : k \in \mathbb{N}\}$ and $O = \{O_k : k \in \mathbb{N}\}$ such that each C_k and O_k is a countable set and $\bot, \top \in O_0$ and X, Y, Z are countable sets. All these sets are assumed to be pair wise disjoint. The elements of each C_k are known as (formula) *constructors* of arity k. Those of each O_k are known as (truth value) *operators* of arity k. Those of X are known as *truth value unbound variables*, while those of Y are known as *truth value bound variables*. And those of Z are known as *formula unbound variables*.

The sets X and Z are necessary because we need truth-value terms and formula terms to range over all possible values. Thus, an assignment will provide a possible value for each $x \in X$ and each $z \in Z$ and the set of assignments covers all possible values. The set Y is required for a different reason. The truth value bound variables in Y are needed in order to be able to universally quantify in the inference rules.

The set $F(\Sigma)$ of *(schema) simple formulae* over Σ is inductively defined as follows: (i) $\xi_i \in F(\Sigma)$ for every $i \in \mathbb{N}$; (ii) $z \in F(\Sigma)$ for every $z \in Z$; (iii) $c(\varphi_1, \dots, \varphi_k) \in F(\Sigma)$ whenever $c \in C_k$ and $\varphi_1, \dots, \varphi_k \in F(\Sigma)$. The set $gF(\Sigma)$ of *ground simple formulae* is composed of the elements in $F(\Sigma)$ without meta-variables. The set $cgF(\Sigma)$ of *closed simple formulae* is composed of the elements in $gF(\Sigma)$ without variables.

The set $T(\Sigma)$ of *(schema) terms* over Σ is inductively defined as follows: (i) $\tau_i \in T(\Sigma)$ for every $i \in \mathbb{N}$; (ii) $x \in T(\Sigma)$ for every $x \in X$; (iii) $y \in T(\Sigma)$ for every $y \in Y$; (iv) $o(\theta_1, \dots, \theta_k) \in T(\Sigma)$ whenever $o \in O_k$ and $\theta_1, \dots, \theta_k \in T(\Sigma)$; (v) $\#\varphi \in T(\Sigma)$ whenever $\varphi \in F(\Sigma)$. The set $gT(\Sigma)$ of *ground terms* is composed of the elements in $T(\Sigma)$ without meta-variables. The set $cgT(\Sigma)$ of *closed terms* is composed of the elements in $gT(\Sigma)$ without variables.

The intended purpose of $\#\varphi$ is to say that we have a truth value term for each formula.

The set $A(\Sigma)$ of *(schema) assertions* over Σ is composed of the expressions of the following six forms: (i) $\Omega\theta$ and $\mho\theta$ (*positive* and *negative truth value indivisibility assertion*, respectively) with $\theta \in T(\Sigma)$; (ii) $\theta \sqsubseteq \theta'$ and $\theta \not\sqsubseteq \theta'$ (*positive* and *negative truth value comparison assertion*, respectively) with $\theta, \theta' \in T(\Sigma)$; (iii) $\theta \leq \varphi$ and $\theta \not\leq \varphi$ (*positive* and *negative labelled formula*, respectively) with

$\theta \in T(\Sigma)$ and $\varphi \in F(\Sigma)$. The set $gA(\Sigma)$ of *ground assertions* is composed of the elements in $A(\Sigma)$ without meta-variables. The set $cgA(\Sigma)$ of *closed assertions* is composed of the elements in $gA(\Sigma)$ without variables.

The notion of *conjugate* $\overline{\delta}$ of an assertion δ is introduced as follows: (i) $\overline{\Omega\theta}$ is $\mho\theta$; (ii) $\overline{\mho\theta}$ is $\Omega\theta$; (iii) $\overline{\theta \sqsubseteq \theta'}$ is $\theta \not\sqsubseteq \theta'$; (iv) $\overline{\theta \not\sqsubseteq \theta'}$ is $\theta \sqsubseteq \theta'$; (v) $\overline{\theta \leq \varphi}$ is $\theta \not\leq \varphi$; (vi) $\overline{\theta \not\leq \varphi}$ is $\theta \leq \varphi$.

The intended meaning of $\Omega\theta$ is to assert that a truth value term is atomic, that is, there is no term strictly smaller than it besides falsum. Clearly, the meaning of the conjugate $\mho\theta$ is to indicate that θ is not atomic. We do not consider conjunctions and disjunctions of assertions because we do not need them in the sequel, but they could easily be introduced. Instead, we work with sequents of assertions.

The set of (schema) labelled formulae over Σ is denoted by $L(\Sigma)$. And the set of ground labelled formulae is denoted by $gL(\Sigma)$.

A *(schema) substitution* over Σ is a map σ such that[1]: (i) $\sigma(\xi_i) \in F(\Sigma)$; (ii) $\sigma(\tau_i) \in T(\Sigma)$; (iii) $\sigma(\Gamma_i) \in \mathcal{B}_{\mathrm{f}}(A(\Sigma) \cup \{\Gamma_i : i \in \mathbb{N}\})$. We denote the set of (schema) substitutions over Σ by $Sbs(\Sigma)$.

A *ground substitution* over Σ is a schema substitution ρ such that: (i) $\rho(\xi_i) \in gF(\Sigma)$; (ii) $\rho(\tau_i) \in gT(\Sigma)$; (iii) $\rho(\Gamma_i) \in \mathcal{B}_{\mathrm{f}}(gA(\Sigma))$. We denote the set of ground substitutions over Σ by $gSbs(\Sigma)$.

In what concern substitutions we also write $\Gamma_i\sigma$ and $\Gamma_i\rho$ for $\sigma(\Gamma_i)$ and $\rho(\Gamma_i)$ respectively. The same applies to single formula and truth value terms.

2.2 Calculi

A *sequent* over a signature Σ is a pair $s = \langle \Delta_1, \Delta_2 \rangle$, written $\Delta_1 \to \Delta_2$, where $\Delta_1, \Delta_2 \in \mathcal{B}_{\mathrm{f}}(A(\Sigma) \cup \{\Gamma_i : i \in \mathbb{N}\})$. A sequent is said to be *ground* if it is written without meta-variables and it is said to be *closed* if furthermore it has no variables.

The sequent calculi are composed by rules. As is standard for both labelled and unlabelled deduction calculi, the application of the rules is subject to constraints. For example, in the Hilbert calculus for first-order logics we have the axiom $(\forall_x(\xi_1 \Rightarrow \xi_2) \Rightarrow (\xi_1 \Rightarrow (\forall_x \xi_2)))$ provided that x does not occur free in ξ_1. The meaning os such a proviso is to say that we only allow (ground) substitutions of the axiom where ξ_1 is mapped to a formula φ where x does not occur free in φ. Hence a proviso can be looked upon as a set of allowed substitutions.

A *(local) proviso* over Σ is a map $\pi : gSbs(\Sigma) \to \{0, 1\}$. The *unit proviso* **up** is as follows: $\mathbf{up}(\rho) = 1$ for every $\rho \in gSbs(\Sigma)$. The *zero proviso* **zp** is as follows: $\mathbf{zp}(\rho) = 0$ for every $\rho \in gSbs(\Sigma)$.

Given two provisos π, π', their *intersection* is the proviso $(\pi \cap \pi')$ such that $(\pi \cap \pi')(\rho) = \pi(\rho) \times \pi'(\rho)$. And we say that $\pi \subseteq \pi'$ when $\pi(\rho) \leq \pi'(\rho)$ for each ground substitution ρ. Therefore, a proviso π is included in a proviso π' if the latter allows more ground substitutions than the former.

[1]Given a set U, we denote by $\mathcal{B}_{\mathrm{f}}U$ the set of all finite bags (multisets) of elements in U.

Given a schema substitution σ and a proviso π, the proviso $(\pi\sigma)$ is as follows: $(\pi\sigma)(\rho) = \pi(\sigma\rho)$ for every $\rho \in gSbs(\Sigma)$.

Observe that, for every proviso π and ground substitution ρ, the proviso $(\pi\rho)$ is either **up** or **zp**.

A *rule* over Σ is a triple $r = \langle\{s_1, \ldots, s_p\}, s, \pi\rangle$, written

$$\frac{s_1 \ \ldots \ s_p}{s} \lhd \pi \ ,$$

where $s_1, \ldots, s_p, s$ are sequents over Σ and π is a proviso over Σ. When π is **up**, the rule may be written

$$\frac{s_1 \ \ldots \ s_p}{s} \ .$$

Given a sequent $s = \langle\Delta_1, \Delta_2\rangle$ and a substitution σ both over Σ, we denote by $s\sigma$ the instance $\langle\Delta_1\sigma, \Delta_2\sigma\rangle$ of s by σ. Given a rule r and a substitution σ both over Σ, we denote by $r\sigma$ the instance

$$\frac{s_1\sigma \ \ldots \ s_p\sigma}{s\sigma} \lhd \pi\sigma$$

of r by σ.

A *(sequent) calculus* is a pair $\mathcal{C} = \langle\Sigma, \mathcal{R}\rangle$ where Σ is a signature and $\mathcal{R}$ is a finite set of rules over Σ.

Within the context of a sequent calculus $\mathcal{C}$, we say that a sequent s' is *derived* from a set S of sequents with proviso π, written $S \vdash_{\mathcal{C}} s' \lhd \pi$, if there is a sequence $\langle d_1, \pi_1\rangle, \ldots, \langle d_n, \pi_n\rangle$ such that:

- d_1 is s' and $\pi \subseteq \pi_1$;
- for every $i = 1, \ldots, n$:
  1. either $d_i \in S$ and π_i is **up**;
  2. or there is an assertion that occurs in both sides of d_i and π_i is **up**;
  3. or there are $r \in \mathcal{R}$, $\sigma \in Sbs(\Sigma)$, $p \in \mathbb{N}$ and $i_1, \ldots, i_p \in \{i+1, \ldots, n\}$ such that
  $$r\sigma = \frac{d_{i_1} \ \ldots \ d_{i_p}}{d_i} \lhd \pi'$$
  and $\pi_i = \pi' \cap \pi_{i_1} \cap \cdots \cap \pi_{i_p}$.

When the proviso π is **up**, we may write $S \vdash_{\mathcal{C}} s'$. And we may write $\vdash_{\mathcal{C}} s$ when the set of premises is empty. Furthermore, when the signature is obvious from the context we may write $\vdash_{\mathcal{R}}$ instead of $\vdash_{\mathcal{C}}$. In derivations we justify 1 by hyp (hypothesis), 2 by ax (axiom) and 3 by $r[\sigma] : i_1, \ldots, i_p$.

Following [6, 1], we choose to display rules from premises to conclusions, and derivations starting from the conclusion since this simplifies their reading, as is illustrated by the example derivations below.

The *derivation sequence* $\langle d_1, \pi_1\rangle, \ldots, \langle d_n, \pi_n\rangle$ is said to be *sober* if, for each $i = 2, \ldots, n$, i appears in the justification of some d_j such that $j < i$. It is straightforward to set up an algorithm to make sober any given derivation sequence. It is also simple to set up an algorithm for extracting the traditional *derivation tree* from any given sober derivation sequence.

When making proofs, we may write $d_1, \ldots, d_n$ for the derivation sequence $\langle d_1, \mathbf{up}\rangle, \ldots, \langle d_n, \mathbf{up}\rangle$.

Derivation establishes a finitary consequence operator for ground sequents thanks to the following result:

Proposition 2.1 For any sequent calculus $\mathcal{C}$:

Projective If $S \vdash_{\mathcal{C}} s' \lhd \pi$ and $\pi' \subseteq \pi$ then $S \vdash_{\mathcal{C}} s' \lhd \pi'$.

Finitary If $S \vdash_{\mathcal{C}} s' \lhd \pi$ then there is a finite $S_1 \subseteq S$ such that $S_1 \vdash_{\mathcal{C}} s' \lhd \pi$.

Extensive $S \vdash_{\mathcal{C}} s \lhd \mathbf{up}$ for each sequent $s \in S$.

Monotonic If $S \subseteq S_1$ and $S \vdash_{\mathcal{C}} s' \lhd \pi$ then $S_1 \vdash_{\mathcal{C}} s' \lhd \pi$.

Idempotent If $S_1 \vdash_{\mathcal{C}} s \lhd \pi_s$ for each s in a finite set S of sequents and $S \vdash_{\mathcal{C}} s' \lhd \pi$ then $S_1 \vdash_{\mathcal{C}} s' \lhd \pi \cap (\bigcap_{s \in S} \pi_s)$.

The following result is also straightforward to prove (by induction on the length of the derivation).

Proposition 2.2 For any sequent calculus $\mathcal{C} = \langle \Sigma, \mathcal{R}\rangle$ and substitution σ over Σ, if $S \vdash_{\mathcal{C}} s' \lhd \pi$ with derivation sequence $\langle d_1, \pi_1\rangle, \ldots, \langle d_n, \pi_n\rangle$ then $S\sigma \vdash_{\mathcal{C}} s'\sigma \lhd \pi\sigma$ with derivation sequence $\langle d_1\sigma, \pi_1\sigma\rangle, \ldots, \langle d_n\sigma, \pi_n\sigma\rangle$.

2.3 Structural rules

A sequent calculus $\mathcal{C} = \langle \Sigma, \mathcal{R}\rangle$ is said to be *structural* if $\mathcal{R}$ contains the following weakening, contraction, conjugation and cut rules:

LwΩ	$\frac{\Gamma_1 \to \Gamma_2}{\Omega\tau_1, \Gamma_1 \to \Gamma_2}$	RwΩ	$\frac{\Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2, \Omega\tau_1}$
LwT	$\frac{\Gamma_1 \to \Gamma_2}{\tau_1 \sqsubseteq \tau_2, \Gamma_1 \to \Gamma_2}$	RwT	$\frac{\Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \tau_2}$
LwF	$\frac{\Gamma_1 \to \Gamma_2}{\tau_1 \leq \xi_1, \Gamma_1 \to \Gamma_2}$	RwF	$\frac{\Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1}$
LcT	$\frac{\tau_1 \sqsubseteq \tau_2, \tau_1 \sqsubseteq \tau_2, \Gamma_1 \to \Gamma_2}{\tau_1 \sqsubseteq \tau_2, \Gamma_1 \to \Gamma_2}$	RcT	$\frac{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \tau_2, \tau_1 \sqsubseteq \tau_2}{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \tau_2}$
LcF	$\frac{\tau_1 \leq \xi_1, \tau_1 \leq \xi_1, \Gamma_1 \to \Gamma_2}{\tau_1 \leq \xi_1, \Gamma_1 \to \Gamma_2}$	RcF	$\frac{\Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1, \tau_1 \leq \xi_1}{\Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1}$
LxiΩ	$\frac{\Gamma_1 \to \Gamma_2, \Omega\tau_1}{\mho\tau_1, \Gamma_1 \to \Gamma_2}$	RxiΩ	$\frac{\Omega\tau_1, \Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2, \mho\tau_1}$
LxiT	$\frac{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \tau_2}{\tau_1 \not\sqsubseteq \tau_2, \Gamma_1 \to \Gamma_2}$	RxiT	$\frac{\tau_1 \sqsubseteq \tau_2, \Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2, \tau_1 \not\sqsubseteq \tau_2}$
LxiF	$\frac{\Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1}{\tau_1 \not\leq \xi_1, \Gamma_1 \to \Gamma_2}$	RxiF	$\frac{\tau_1 \leq \xi_1, \Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2, \tau_1 \not\leq \xi_1}$
LxeΩ	$\frac{\Gamma_1 \to \Gamma_2, \mho\tau_1}{\Omega\tau_1, \Gamma_1 \to \Gamma_2}$	RxeΩ	$\frac{\mho\tau_1, \Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2, \Omega\tau_1}$
LxeT	$\frac{\Gamma_1 \to \Gamma_2, \tau_1 \not\sqsubseteq \tau_2}{\tau_1 \sqsubseteq \tau_2, \Gamma_1 \to \Gamma_2}$	RxeT	$\frac{\tau_1 \not\sqsubseteq \tau_2, \Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \tau_2}$
LxeF	$\frac{\Gamma_1 \to \Gamma_2, \tau_1 \not\leq \xi_1}{\tau_1 \leq \xi_1, \Gamma_1 \to \Gamma_2}$	RxeF	$\frac{\tau_1 \not\leq \xi_1, \Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1}$
cutT	$\frac{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \tau_2 \quad \tau_1 \sqsubseteq \tau_2, \Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2}$	cutF	$\frac{\Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1 \quad \tau_1 \leq \xi_1, \Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2}$

The rules above are known as *structural rules*. These structural rules are the usual ones in sequent calculi plus those needed to deal with conjugates. The introduction of disjunction and conjunction of assertions would lead to the expected left and right rules. Other rules that may be present in the sequent calculus at hand are known as *proper rules*.

2.4 Order rules

Before proceeding, we need to introduce the following provisos:

- $(\tau_k : \mathbf{y})(\rho) = 1$ iff $\rho(\tau_k) \in Y$;
- $(\tau_k \notin \Delta)(\rho) = 1$ iff $\rho(\tau_k)$ does not occur in $\Delta\rho$.

The first proviso states that we only allow ground substitutions where τ_k is replaced by a truth value bound variable. The second proviso indicates that we only allow a substitution ρ if τ_k is replaced by a truth value not occurring in the bag $\Delta\rho$.

A structural sequent calculus $\mathcal{C} = \langle \Sigma, \mathcal{R} \rangle$ is said to be an *order sequent calculus* if it contains the following additional order rules:

$$\text{L\#} \quad \frac{\tau_1 \leq \xi_1, \Gamma_1 \to \Gamma_2}{\tau_1 \sqsubseteq \# \xi_1, \Gamma_1 \to \Gamma_2} \qquad \text{R\#} \quad \frac{\Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1}{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \# \xi_1}$$

$$\bot\text{T} \quad \frac{}{\Gamma_1 \to \Gamma_2, \bot \sqsubseteq \tau_1} \qquad \bot\text{F} \quad \frac{}{\Gamma_1 \to \Gamma_2, \bot \leq \xi_1}$$

$$\Omega\bot \quad \frac{\Gamma_1 \to \Gamma_2, \Omega\tau_1}{\Gamma_1 \to \Gamma_2, \tau_1 \not\sqsubseteq \bot} \qquad \top \quad \frac{}{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \top}$$

$$\Omega\top \quad \frac{\Omega\tau_1, \Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1}{\Gamma_1 \to \Gamma_2, \top \leq \xi_1} \triangleleft\ \tau_1 : \mathbf{y}, \tau_1 \notin \Gamma_1, \Gamma_2$$

$$\Omega \quad \frac{\Gamma_1 \to \Gamma_2, \tau_1 \not\sqsubseteq \bot \quad \Omega\tau_2, \tau_2 \sqsubseteq \tau_1, \Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \tau_2}{\Gamma_1 \to \Gamma_2, \Omega\tau_1} \triangleleft\ \tau_2 : \mathbf{y}, \tau_2 \notin \tau_1, \Gamma_1, \Gamma_2$$

$$\text{cons} \quad \frac{}{\Gamma_1 \to \Gamma_2, \top \not\sqsubseteq \bot} \qquad \text{ref} \quad \frac{}{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \tau_1}$$

$$\text{transT} \quad \frac{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \tau_2 \quad \Gamma_1 \to \Gamma_2, \tau_2 \sqsubseteq \tau_3}{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \tau_3} \qquad \text{transF} \quad \frac{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \tau_2 \quad \Gamma_1 \to \Gamma_2, \tau_2 \leq \xi_1}{\Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1}$$

$$\text{Lasym} \quad \frac{\Omega\tau_1, \Omega\tau_2, \tau_1 \sqsubseteq \tau_2, \Gamma_1 \to \Gamma_2}{\Omega\tau_1, \Omega\tau_2, \tau_2 \sqsubseteq \tau_1, \Gamma_1 \to \Gamma_2} \qquad \text{Rasym} \quad \frac{\Omega\tau_1, \Omega\tau_2, \Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \tau_2}{\Omega\tau_1, \Omega\tau_2, \Gamma_1 \to \Gamma_2, \tau_2 \sqsubseteq \tau_1}$$

$$\text{LgenT} \quad \frac{\Omega\tau_2, \tau_2 \sqsubseteq \tau_3, \Gamma_1 \to \Gamma_2 \quad \Omega\tau_2, \Gamma_1 \to \Gamma_2, \tau_2 \sqsubseteq \tau_1 \quad \tau_1 \sqsubseteq \tau_3, \Gamma_1 \to \Gamma_2, \Omega\tau_2}{\tau_1 \sqsubseteq \tau_3, \Gamma_1 \to \Gamma_2}$$

$$\text{RgenT} \quad \frac{\Omega\tau_2, \tau_2 \sqsubseteq \tau_1, \Gamma_1 \to \Gamma_2, \tau_2 \sqsubseteq \tau_3}{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \tau_3} \triangleleft\ \tau_2 : \mathbf{y}, \tau_2 \notin \tau_1, \tau_3, \Gamma_1, \Gamma_2$$

$$\text{LgenF} \quad \frac{\Omega\tau_2, \tau_2 \leq \xi_1, \Gamma_1 \to \Gamma_2 \quad \Omega\tau_2, \Gamma_1 \to \Gamma_2, \tau_2 \sqsubseteq \tau_1 \quad \tau_1 \leq \xi_1, \Gamma_1 \to \Gamma_2, \Omega\tau_2}{\tau_1 \leq \xi_1, \Gamma_1 \to \Gamma_2}$$

$$\text{RgenF} \quad \frac{\Omega\tau_2, \tau_2 \sqsubseteq \tau_1, \Gamma_1 \to \Gamma_2, \tau_2 \leq \xi_1}{\Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1} \triangleleft\ \tau_2 : \mathbf{y}, \tau_2 \notin \tau_1, \Gamma_1, \Gamma_2$$

These rules impose intended meanings to the basic assertions that should be obvious. But, it is worthwhile to note that if Ωt holds then t is intended to denote an atomic truth value (where a formula either holds or does not hold).

It is also worthwhile to explain the rules RgenF and LgenF. The rule RgenF indicates that if the value of τ_2 is less than or equal to the value of ξ_1 for all atomic τ_2 included in τ_1, then τ_1 is less than or equal to the value of ξ_1. We also impose that τ_2 is fresh so that the universal quantifier does not capture other variables namely those in $\tau_1, \Gamma_1, \Gamma_2$. The rule LgenF can be interpreted as follows: assuming that we have $\tau_1 \leq \xi_1$ and Γ_1 in order to show that we have γ_2 for some $\gamma_2 \in \Gamma_2$ it is enough to show that there is an element τ_2 such that

- τ_2 is atomic (premise $\tau_1 \leq \xi_1, \Gamma_1 \to \Gamma_2, \Omega\tau_2$);
- $\tau_2 \sqsubseteq \tau_1$ (premise $\Omega\tau_2, \Gamma_1 \to \Gamma_2, \tau_2 \sqsubseteq \tau_1$);
- from $\tau_2 \leq \xi_1$ we have γ_2 for some $\gamma_2 \in \Gamma_2$ (premise $\Omega\tau_2, \tau_2 \leq \xi_1, \Gamma_1 \to \Gamma_2$).

Observe also that this set of rules, although convenient, is by no means minimal. For instance, rules L# and R# allow the derivation of the F rules from the corresponding T rules. Moreover, rule $\Omega\top$ is derived from RgenT. Note also that not all of these rules are needed for obtaining later on the completeness result over Kripke semantics (Theorem 3.25). Indeed, rules ref and asym are only needed for establishing the duality between the algebraic semantics and the Kripke semantics (see Subsection 3.4).

Note that the provisos used above do not change value when the context of the rule at hand is enriched with closed assertions. More precisely, a rule is said to be *endowed with a persistent proviso* if its proviso does not change value when the context of the rule is enriched with a closed assertion.

For example, consider the rule RgenF. The context of the rule is Γ_1 and Γ_2, that is the bags of assertions in the conclusion of the rule. Note that if we change the context by adding a closed assertion either to Γ_1 or Γ_2, the resulting proviso will allow precisely the same substitutions.

More generally, any rule with a "fresh bound variable" proviso like $\tau_2 : \mathbf{y} \cap \tau_2 \notin \tau_1, \Gamma_1, \Gamma_2$ is endowed with a persistent proviso. Indeed, for every ground substitution ρ, it holds $(\tau_2 : \mathbf{y} \cap \tau_2 \notin \tau_1, \Gamma_1, \Gamma_2)(\rho) = (\tau_2 : \mathbf{y} \cap \tau_2 \notin \tau_1, \Gamma_1, \Gamma_2, \delta)(\rho)$ as long as δ is closed. Clearly, every rule endowed with the unit proviso **up** (which is omitted) is also endowed with a persistent proviso.

Therefore, according to this definition, the order rules above are endowed with persistent provisos.

2.5 Modal system K specific rules

We now proceed to define the sequent calculus $\mathcal{C}_K = \langle \Sigma_k, \mathcal{R}_K \rangle$ for modal system K. The *modal signature* $\Sigma_K = \langle C, O, X, Y, Z \rangle$ is as follows:

- $C_0 = \{\mathbf{f}, \mathbf{t}\} \cup \{\mathbf{p}_i : i \in \mathbb{N}\}$;
- $C_1 = \{\neg, \Box, \Diamond\}$;
- $C_2 = \{\wedge, \vee, \Rightarrow\}$;
- $C_k = \emptyset$ for $k \geq 3$;
- $O_0 = \{\bot, \top\}$;
- $O_1 = \{\mathbf{I}, \mathbf{N}\}$;

- $O_2 = \{\mathbf{lb}\}$;
- $O_k = \emptyset$ for $k \geq 3$;
- $X = \{\mathbf{x}_i : i \in \mathbb{N}\}$;
- $Y = \{\mathbf{y}_i : i \in \mathbb{N}\}$;
- $Z = \{\mathbf{z}_i : i \in \mathbb{N}\}$.

Besides the structural and order rules introduced above, $\mathcal{R}_K$ contains the following specific rules:

$$\mathbf{I}\quad \frac{}{\Gamma_1\to\Gamma_2,\mathbf{I}(\tau_1)\sqsubseteq\tau_1} \qquad\qquad \Omega\mathbf{I}\quad \frac{\Gamma_1\to\Gamma_2,\tau_1\not\sqsubseteq\bot}{\Gamma_1\to\Gamma_2,\Omega\mathbf{I}(\tau_1)}$$

$$\mathrm{L}\mathbf{N}\Omega\quad \frac{\tau_3\sqsubseteq\tau_1,\Omega\tau_3,\Omega\tau_2,\Gamma_1\to\Gamma_2,\tau_2\not\sqsubseteq\mathbf{N}(\tau_3)}{\Omega\tau_2,\Gamma_1\to\Gamma_2,\tau_2\not\sqsubseteq\mathbf{N}(\tau_1)} \lhd\ \tau_3 : \mathbf{y}, \tau_3 \notin \tau_1, \tau_2, \Gamma_1, \Gamma_2$$

$$\mathrm{R}\mathbf{N}\Omega\quad \frac{\Omega\tau_2,\Gamma_1\to\Gamma_2,\Omega\tau_3 \quad \Omega\tau_3,\Omega\tau_2,\Gamma_1\to\Gamma_2,\tau_3\sqsubseteq\tau_1 \quad \Omega\tau_3,\Omega\tau_2,\Gamma_1\to\Gamma_2,\tau_2\sqsubseteq\mathbf{N}(\tau_3)}{\Omega\tau_2,\Gamma_1\to\Gamma_2,\tau_2\sqsubseteq\mathbf{N}(\tau_1)}$$

$$\mathbf{lb}1\quad \frac{}{\Gamma_1\to\Gamma_2,\mathbf{lb}(\tau_1,\tau_2)\sqsubseteq\tau_1} \qquad\qquad \mathbf{lb}2\quad \frac{}{\Gamma_1\to\Gamma_2,\mathbf{lb}(\tau_1,\tau_2)\sqsubseteq\tau_2}$$

$$\mathrm{L}\mathbf{f}\quad \frac{\tau_1\sqsubseteq\bot,\Gamma_1\to\Gamma_2}{\tau_1\leq\mathbf{f},\Gamma_1\to\Gamma_2} \qquad\qquad \mathrm{R}\mathbf{f}\quad \frac{\Gamma_1\to\Gamma_2,\tau_1\sqsubseteq\bot}{\Gamma_1\to\Gamma_2,\tau_1\leq\mathbf{f}}$$

$$\mathrm{L}\mathbf{t}\quad \frac{\tau_1\sqsubseteq\top,\Gamma_1\to\Gamma_2}{\tau_1\leq\mathbf{t},\Gamma_1\to\Gamma_2} \qquad\qquad \mathrm{R}\mathbf{t}\quad \frac{\Gamma_1\to\Gamma_2,\tau_1\sqsubseteq\top}{\Gamma_1\to\Gamma_2,\tau_1\leq\mathbf{t}}$$

$$\mathrm{L}\wedge\quad \frac{\tau_1\leq\xi_1,\tau_1\leq\xi_2,\Gamma_1\to\Gamma_2}{\tau_1\leq(\xi_1\wedge\xi_2),\Gamma_1\to\Gamma_2} \qquad\qquad \mathrm{R}\wedge\quad \frac{\Gamma_1\to\Gamma_2,\tau_1\leq\xi_1 \quad \Gamma_1\to\Gamma_2,\tau_1\leq\xi_2}{\Gamma_1\to\Gamma_2,\tau_1\leq(\xi_1\wedge\xi_2)}$$

$$\mathrm{L}\neg\quad \frac{\Omega\tau_1,\Gamma_1\to\Gamma_2,\tau_1\leq\xi_1}{\Omega\tau_1,\tau_1\leq(\neg\,\xi_1),\Gamma_1\to\Gamma_2} \qquad\qquad \mathrm{R}\neg\quad \frac{\Omega\tau_1,\tau_1\leq\xi_1,\Gamma_1\to\Gamma_2}{\Omega\tau_1,\Gamma_1\to\Gamma_2,\tau_1\leq(\neg\,\xi_1)}$$

$$\mathrm{L}\Rightarrow\quad \frac{\Omega\tau_1,\Gamma_1\to\Gamma_2,\tau_1\leq\xi_1 \quad \Omega\tau_1,\tau_1\leq\xi_2,\Gamma_1\to\Gamma_2}{\Omega\tau_1,\tau_1\leq(\xi_1\Rightarrow\xi_2),\Gamma_1\to\Gamma_2} \qquad\qquad \mathrm{R}\Rightarrow\quad \frac{\Omega\tau_1,\tau_1\leq\xi_1,\Gamma_1\to\Gamma_2,\tau_1\leq\xi_2}{\Omega\tau_1,\Gamma_1\to\Gamma_2,\tau_1\leq(\xi_1\Rightarrow\xi_2)}$$

$$\mathrm{L}\vee\quad \frac{\Omega\tau_1,\tau_1\leq\xi_1,\Gamma_1\to\Gamma_2 \quad \Omega\tau_1,\tau_1\leq\xi_2,\Gamma_1\to\Gamma_2}{\Omega\tau_1,\tau_1\leq(\xi_1\vee\xi_2),\Gamma_1\to\Gamma_2} \qquad\qquad \mathrm{R}\vee\quad \frac{\Omega\tau_1,\Gamma_1\to\Gamma_2,\tau_1\leq\xi_1,\tau_1\leq\xi_2}{\Omega\tau_1,\Gamma_1\to\Gamma_2,\tau_1\leq(\xi_1\vee\xi_2)}$$

$$\mathrm{L}\Box\quad \frac{\mathbf{N}(\tau_1)\leq\xi_1,\Gamma_1\to\Gamma_2}{\tau_1\leq(\ \ \xi_1),\Gamma_1\to\Gamma_2} \qquad\qquad \mathrm{R}\Box\quad \frac{\Gamma_1\to\Gamma_2,\mathbf{N}(\tau_1)\leq\xi_1}{\Gamma_1\to\Gamma_2,\tau_1\leq(\ \ \xi_1)}$$

$$\mathrm{L}\Diamond\quad \frac{\Omega\tau_1,\Omega\tau_2,\tau_2\leq\xi_1,\tau_2\sqsubseteq\mathbf{N}(\tau_1),\Gamma_1\to\Gamma_2}{\Omega\tau_1,\tau_1\leq(\ \ \xi_1),\Gamma_1\to\Gamma_2} \lhd\ \tau_2 : \mathbf{y}, \tau_2 \notin \tau_1, \Gamma_1, \Gamma_2$$

$$\mathrm{R}\Diamond\quad \frac{\Omega\tau_1,\Omega\tau_2,\Gamma_1\to\Gamma_2,\tau_2\sqsubseteq\mathbf{N}(\tau_1) \quad \Omega\tau_1,\Omega\tau_2,\Gamma_1\to\Gamma_2,\tau_2\leq\xi_1 \quad \Omega\tau_1,\Gamma_1\to\Gamma_2,\Omega\tau_2}{\Omega\tau_1,\Gamma_1\to\Gamma_2,\tau_1\leq(\ \ \xi_1)}$$

Rules $\mathbf{I}$ and $\Omega\mathbf{I}$ impose that $\mathbf{I}(t)$ is an atomic truth value contained in t, as long as the latter is not bottom. Rules $\mathbf{N}\Omega$ state that the neighborhood of a truth value t is induced by the neighbors of the atomic truth values contained in t. Rules $\mathbf{lb}$ establish that $\mathbf{lb}(t_1, t_2)$ is some lower bound of t_1 and t_2.

The rules about the formula constructors fall into two main classes. The rules about $\mathbf{f}$, $\mathbf{t}$, $\wedge$ and $\Box$ hold for any truth value, while the rules about $\neg$, $\Rightarrow$, $\vee$ and $\Diamond$ hold only for atomic truth values, as expressed by $\Omega\tau_1$.

In the rest of the paper, we shall refer to a *modal sequent calculus* as any enrichment of the system K sequent calculus with additional rules endowed with persistent provisos. Several modal sequent calculi in this sense are considered in Subsection 2.6.

The two examples below of derivations in $\mathcal{C}_K$ illustrate how the calculus can be used for modal reasoning. As we remarked above, we display derivations starting from the conclusion, and writing for each line the step number, the sequent, and how it is justified (i.e. by which rule it is obtained and applied to which sequents).

Example: Derivation of the necessitation rule

1	$\rightarrow \quad \top \leq \; \xi_1$	R : 2
2	$\rightarrow \quad \mathbf{N}(\top) \leq \xi_1$	transF : 3, 4
3	$\rightarrow \quad \mathbf{N}(\top) \sqsubseteq \top$	$\top$
4	$\rightarrow \quad \top \leq \xi_1$	hyp

Example: Derivation of the normality axiom

1	$\rightarrow \quad \top \leq \; (\xi_1 \Rightarrow \xi_2) \Rightarrow (\; \xi_1 \Rightarrow \; \xi_2)$	RgenF : 2
2	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\rightarrow \quad \mathbf{y}_1 \leq \; (\xi_1 \Rightarrow \xi_2) \Rightarrow (\; \xi_1 \Rightarrow \; \xi_2)$	R$\Rightarrow$: 3
3	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\mathbf{y}_1 \leq \; (\xi_1 \Rightarrow \xi_2)$ $\rightarrow \quad \mathbf{y}_1 \leq \; \xi_1 \Rightarrow \; \xi_2$	R$\Rightarrow$: 4
4	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\mathbf{y}_1 \leq \; (\xi_1 \Rightarrow \xi_2)$ $\mathbf{y}_1 \leq \; \xi_1$ $\rightarrow \quad \mathbf{y}_1 \leq \; \xi_2$	R : 5
5	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\mathbf{y}_1 \leq \; (\xi_1 \Rightarrow \xi_2)$ $\mathbf{y}_1 \leq \; \xi_1$ $\rightarrow \quad \mathbf{N}(\mathbf{y}_1) \leq \xi_2$	RgenF : 6
6	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\mathbf{y}_1 \leq \; (\xi_1 \Rightarrow \xi_2)$ $\mathbf{y}_1 \leq \; \xi_1$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$ $\rightarrow \quad \mathbf{y}_2 \leq \xi_2$	L : 7
7	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\mathbf{y}_1 \leq \; (\xi_1 \Rightarrow \xi_2)$ $\mathbf{N}(\mathbf{y}_1) \leq \xi_1$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$ $\rightarrow \quad \mathbf{y}_2 \leq \xi_2$	LgenF : 8, 9, 10
8	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\mathbf{y}_1 \leq \; (\xi_1 \Rightarrow \xi_2)$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$ $\Omega\mathbf{y}_2$ $\rightarrow \quad \mathbf{y}_2 \leq \xi_2$ $\mathbf{y}_2 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$	ax

9	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\mathbf{y}_1 \leq \quad (\xi_1 \Rightarrow \xi_2)$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \leq \xi_1$	$\rightarrow$	$\mathbf{y}_2 \leq \xi_2$	L : 11
10	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\mathbf{y}_1 \leq \quad (\xi_1 \Rightarrow \xi_2)$ $\mathbf{N}(\mathbf{y}_1) \leq \xi_1$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$	$\rightarrow$	$\mathbf{y}_2 \leq \xi_2$ $\Omega\mathbf{y}_2$	ax
11	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\mathbf{N}(\mathbf{y}_1) \leq \xi_1 \Rightarrow \xi_2$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \leq \xi_1$	$\rightarrow$	$\mathbf{y}_2 \leq \xi_2$	LgenF : 12, 13, 14
12	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \leq \xi_1$ $\Omega\mathbf{y}_2$	$\rightarrow$	$\mathbf{y}_2 \leq \xi_2$ $\mathbf{y}_2 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$	ax
13	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \leq \xi_1$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \leq \xi_1 \Rightarrow \xi_2$	$\rightarrow$	$\mathbf{y}_2 \leq \xi_2$	L$\Rightarrow$: 15, 16
14	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\mathbf{N}(\mathbf{y}_1) \leq \xi_1 \Rightarrow \xi_2$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \leq \xi_1$	$\rightarrow$	$\mathbf{y}_2 \leq \xi_2$ $\Omega\mathbf{y}_2$	ax
15	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \leq \xi_1$ $\Omega\mathbf{y}_2$	$\rightarrow$	$\mathbf{y}_2 \leq \xi_2$ $\mathbf{y}_2 \leq \xi_1$	ax
16	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \leq \xi_1$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \leq \xi_2$	$\rightarrow$	$\mathbf{y}_2 \leq \xi_2$	ax

2.6 Rules for other modal systems

Other modal systems defined by properties of the accessibility relation can be easily obtained by adding suitable rules. For instance:

Additional rule for T (reflexive)

$$\text{T}\ \frac{}{\Gamma_1 \rightarrow \Gamma_2, \tau_1 \sqsubseteq \mathbf{N}(\tau_1)}$$

Additional rule for B (symmetric)

$$\text{B}\ \frac{\Omega\tau_1, \Omega\tau_2, \Gamma_1 \rightarrow \Gamma_2, \tau_1 \sqsubseteq \mathbf{N}(\tau_2)}{\Omega\tau_1, \Omega\tau_2, \Gamma_1 \rightarrow \Gamma_2, \tau_2 \sqsubseteq \mathbf{N}(\tau_1)}$$

Additional rule for K4 (transitive)

$$4\ \frac{}{\Gamma_1 \rightarrow \Gamma_2, \mathbf{N}(\mathbf{N}(\tau_1)) \sqsubseteq \mathbf{N}(\tau_1)}$$

Additional rule for D (serial)

$$\text{D}\ \frac{}{\Omega\tau_1, \Gamma_1 \rightarrow \Gamma_2, \mathbf{N}(\tau_1) \not\sqsubseteq \bot}$$

Additional rule for L (right linear)

$$\text{L}\ \frac{\Omega\tau_1, \Omega\tau_2, \Omega\tau_3, \Gamma_1 \rightarrow \Gamma_2, \tau_1 \sqsubseteq \mathbf{N}(\tau_3) \quad \Omega\tau_1, \Omega\tau_2, \Omega\tau_3, \Gamma_1 \rightarrow \Gamma_2, \tau_2 \sqsubseteq \mathbf{N}(\tau_3)}{\Omega\tau_1, \Omega\tau_2, \Gamma_1 \rightarrow \Gamma_2, \tau_2 \sqsubseteq \mathbf{N}(\tau_1), \tau_1 \sqsubseteq \mathbf{N}(\tau_2), \tau_1 \sqsubseteq \tau_2}$$

Additional rule for K5 (Euclidean)

$$5\ \frac{\Omega\tau_1, \Omega\tau_2, \Omega\tau_3, \Gamma_1 \rightarrow \Gamma_2, \tau_1 \sqsubseteq \mathbf{N}(\tau_3) \quad \Omega\tau_1, \Omega\tau_2, \Omega\tau_3, \Gamma_1 \rightarrow \Gamma_2, \tau_2 \sqsubseteq \mathbf{N}(\tau_3) \quad \Omega\tau_1, \Omega\tau_2, \Gamma_1 \rightarrow \Gamma_2, \Omega\tau_3}{\Omega\tau_1, \Omega\tau_2, \Gamma_1 \rightarrow \Gamma_2, \tau_1 \sqsubseteq \mathbf{N}(\tau_2)}$$

Additional rule for C (confluent)

$$\text{C}\ \frac{\Gamma_1 \rightarrow \Gamma_2, \tau_1 \sqsubseteq \mathbf{N}(\tau_3) \quad \Gamma_1 \rightarrow \Gamma_2, \tau_2 \sqsubseteq \mathbf{N}(\tau_3)}{\Gamma_1 \rightarrow \Gamma_2, \mathbf{lb}(\mathbf{N}(\tau_1), \mathbf{N}(\tau_2)) \not\sqsubseteq \bot}$$

Additional rule for W (transitive and well bounded)

$$\text{W}\ \frac{\Omega\tau_1, \Omega\tau_3, \tau_3 \sqsubseteq \mathbf{N}(\tau_1), \Gamma_1 \rightarrow \Gamma_2, \tau_3 \sqsubseteq \tau_2, \mathbf{N}(\tau_3) \not\sqsubseteq \tau_2}{\Omega\tau_1, \Gamma_1 \rightarrow \Gamma_2, \mathbf{N}(\tau_1) \sqsubseteq \tau_2} \lhd \tau_3 : \mathbf{y}, \tau_3 \notin \tau_1, \tau_2, \Gamma_1, \Gamma_2$$

Additional rule for X (irreflexive)

$$\text{X}\ \frac{}{\Omega\tau_1, \Gamma_1 \rightarrow \Gamma_2, \tau_1 \not\sqsubseteq \mathbf{N}(\tau_1)}$$

Additional rule for Y (antisymmetric)

$$\text{Y}\ \frac{\Omega\tau_1, \Omega\tau_2, \Gamma_1 \rightarrow \Gamma_2, \tau_1 \sqsubseteq \mathbf{N}(\tau_2) \quad \Omega\tau_1, \Omega\tau_2, \Gamma_1 \rightarrow \Gamma_2, \tau_2 \sqsubseteq \mathbf{N}(\tau_1)}{\Omega\tau_1, \Omega\tau_2, \Gamma_1 \rightarrow \Gamma_2, \tau_1 \sqsubseteq \tau_2}$$

Additional rule for Z (asymmetric)

$$\text{Z}\ \frac{\Omega\tau_1,\Omega\tau_2,\Gamma_1\rightarrow\Gamma_2,\tau_1\sqsubseteq\mathbf{N}(\tau_2)}{\Omega\tau_1,\Omega\tau_2,\Gamma_1\rightarrow\Gamma_2,\tau_2\not\sqsubseteq\mathbf{N}(\tau_1)}$$

In Subsection 3.3, these rules are shown to characterize precisely the envisaged properties of the accessibility relation (even among general Kripke structures). Given the greater expressiveness of the proposed labelled language, it is not surprising that we can capture more properties of the accessibility relation than those that are axiomatizable in standard modal language (namely, irreflexivity, antisymmetry and asymmmetry). Note *en passant* that these three properties are also not directly expressible in the language of modal logic labelled with worlds (unless one extends the labelling language to a full quantifier calculus; see [19]). However, they are axiomatizable in hybrid logic (see [2]).

It is important to note that, even in the case of an axiomatizable property, it is worthwhile to replace the axiom by a rule about the truth values. Indeed, by doing so, we hope to preserve the good properties of the formula sub-calculus (namely, cut elimination) and concentrate the unavoidable consequences of the new rule on the sub-calculus for truth values. We return to this issue in the concluding remarks.

Observe also that, in the case of an axiomatizable property, we might be tempted to try to show that the proposed rule on truth values is correct by verifying that the axiom and the rule are inter-derivable in $\mathcal{C}_K$. For instance, consider reflexivity. It is straightforward to build a derivation in $\mathcal{C}_K$ of the corresponding modal axiom from rule T above. Indeed:

1		$\rightarrow$	$\top \leq \quad \xi_1 \Rightarrow \xi_1$	RgenF : 2
2	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$	$\rightarrow$	$\mathbf{y}_1 \leq \quad \xi_1 \Rightarrow \xi_1$	R$\Rightarrow$: 3
3	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\mathbf{y}_1 \leq \quad \xi_1$	$\rightarrow$	$\mathbf{y}_1 \leq \xi_1$	L : 4
4	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\mathbf{N}(\mathbf{y}_1) \leq \xi_1$	$\rightarrow$	$\mathbf{y}_1 \leq \xi_1$	transF : 5, 6
5	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\mathbf{N}(\mathbf{y}_1) \leq \xi_1$	$\rightarrow$	$\mathbf{y}_1 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$	T
6	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\mathbf{N}(\mathbf{y}_1) \leq \xi_1$	$\rightarrow$	$\mathbf{N}(\mathbf{y}_1) \leq \xi_1$	ax

On the other hand, it is not possible in $\mathcal{C}_K$ to derive rule T above from the modal axiom for reflexivity. As we shall see in Subsection 3.3, the sequent calculus $\mathcal{C}_K$ is sound with respect to general Kripke structures as we will see in Theorem 3.17 (recall that a general Kripke structure, see for instance [5], is a tuple $\langle W, \rightsquigarrow, \mathcal{B}, V\rangle$ where W is the non-empty set of worlds, $\rightsquigarrow$ is the accessibility relation between worlds, $\mathcal{B} \subseteq \wp W$ is the set of admissible truth values, and the valuation V maps each propositional symbol $\mathbf{p}_i$ to an admissible truth value). So, it is no surprise that rule T is not derivable from $\rightarrow \top \leq \Box\xi_1 \Rightarrow \xi_1$.

Indeed, rule T (as will be shown by Theorem 3.18 in Subsection 3.3) does characterize the general frames with reflexive accessibility relation, while the axiom does so only among the standard frames. Clearly, the axiom is satisfiable by a general Kripke structure with a non reflexive accessibility relation. For instance, consider the general Kripke frame $\langle W, \leadsto, \mathcal{B}\rangle$ where:

- $W = \{w_1, w_2\}$;
- $\leadsto = \{\langle w_1, w_2\rangle, \langle w_2, w_1\rangle\}$;
- $\mathcal{B} = \{\emptyset, W\}$.

It is trivial to verify that every structure over this general frame does satisfy the axiom for reflexivity.

It is worthwhile to point out that the modal axiom for reflexivity and the following mixed rule (about formulae and truth values)

$$\mathrm{T}' \quad \frac{\Gamma_1 \to \Gamma_2, \mathbf{N}(\tau_1) \leq \xi_1}{\Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1}$$

are inter-derivable. However, we prefer rule T to rule T′ in order to preserve, as much as possible, the separation between the formula sub-calculus and the truth value sub-calculus.

A similar analysis could be done about each of the other properties of the accessibility relation that are axiomatizable but there is no need to enter in details.

Nevertheless, it is worthwhile to produce a derivation of the modal axiom for confluence from rule C since it illustrates the use of both **I** and **lb** rules.

1		$\to$	$\top \leq \quad \xi_1 \Rightarrow \quad \xi_1$	RgenF : 2
2	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$	$\to$	$\mathbf{y}_1 \leq \quad \xi_1 \Rightarrow \quad \xi_1$	R⇒ : 3
3	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\mathbf{y}_1 \leq \quad \xi_1$	$\to$	$\mathbf{y}_1 \leq \quad \xi_1$	R : 4
4	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\mathbf{y}_1 \leq \quad \xi_1$	$\to$	$\mathbf{N}(\mathbf{y}_1) \leq \quad \xi_1$	L : 5
5	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$ $\mathbf{y}_2 \leq \quad \xi_1$	$\to$	$\mathbf{N}(\mathbf{y}_1) \leq \quad \xi_1$	L : 6
6	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$ $\mathbf{N}(\mathbf{y}_2) \leq \xi_1$	$\to$	$\mathbf{N}(\mathbf{y}_1) \leq \quad \xi_1$	RgenF : 7
7	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$ $\mathbf{N}(\mathbf{y}_2) \leq \xi_1$ $\Omega\mathbf{y}_3$ $\mathbf{y}_3 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$	$\to$	$\mathbf{y}_3 \leq \quad \xi_1$	R : 8, 11, 16

8	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$ $\mathbf{N}(\mathbf{y}_2) \le \xi_1$ $\Omega\mathbf{y}_3$ $\mathbf{y}_3 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$ $\Omega\mathbf{I}(\mathbf{lb}(\mathbf{N}(\mathbf{y}_2), \mathbf{N}(\mathbf{y}_3)))$	$\rightarrow$	$\mathbf{I}(\mathbf{lb}(\mathbf{N}(\mathbf{y}_2), \mathbf{N}(\mathbf{y}_3))) \sqsubseteq \mathbf{N}(\mathbf{y}_3)$	$\mathrm{transT} : 9, 10$
9	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$ $\mathbf{N}(\mathbf{y}_2) \le \xi_1$ $\Omega\mathbf{y}_3$ $\mathbf{y}_3 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$ $\Omega\mathbf{I}(\mathbf{lb}(\mathbf{N}(\mathbf{y}_2), \mathbf{N}(\mathbf{y}_3)))$	$\rightarrow$	$\mathbf{I}(\mathbf{lb}(\mathbf{N}(\mathbf{y}_2), \mathbf{N}(\mathbf{y}_3)))$ $\sqsubseteq \mathbf{lb}(\mathbf{N}(\mathbf{y}_2), \mathbf{N}(\mathbf{y}_3))$	$\mathbf{I}$
10	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$ $\mathbf{N}(\mathbf{y}_2) \le \xi_1$ $\Omega\mathbf{y}_3$ $\mathbf{y}_3 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$ $\Omega\mathbf{I}(\mathbf{lb}(\mathbf{N}(\mathbf{y}_2), \mathbf{N}(\mathbf{y}_3)))$	$\rightarrow$	$\mathbf{lb}(\mathbf{N}(\mathbf{y}_2), \mathbf{N}(\mathbf{y}_3)) \sqsubseteq \mathbf{N}(\mathbf{y}_3)$	$\mathbf{lb}2$
11	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$ $\mathbf{N}(\mathbf{y}_2) \le \xi_1$ $\Omega\mathbf{y}_3$ $\mathbf{y}_3 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$ $\Omega\mathbf{I}(\mathbf{lb}(\mathbf{N}(\mathbf{y}_2), \mathbf{N}(\mathbf{y}_3)))$	$\rightarrow$	$\mathbf{I}(\mathbf{lb}(\mathbf{N}(\mathbf{y}_2), \mathbf{N}(\mathbf{y}_3))) \le \xi_1$	$\mathrm{transF} : 12, 15$
12	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$ $\mathbf{N}(\mathbf{y}_2) \le \xi_1$ $\Omega\mathbf{y}_3$ $\mathbf{y}_3 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$ $\Omega\mathbf{I}(\mathbf{lb}(\mathbf{N}(\mathbf{y}_2), \mathbf{N}(\mathbf{y}_3)))$	$\rightarrow$	$\mathbf{I}(\mathbf{lb}(\mathbf{N}(\mathbf{y}_2), \mathbf{N}(\mathbf{y}_3))) \sqsubseteq \mathbf{N}(\mathbf{y}_2)$	$\mathrm{transT} : 13, 14$
13	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$ $\mathbf{N}(\mathbf{y}_2) \le \xi_1$ $\Omega\mathbf{y}_3$ $\mathbf{y}_3 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$ $\Omega\mathbf{I}(\mathbf{lb}(\mathbf{N}(\mathbf{y}_2), \mathbf{N}(\mathbf{y}_3)))$	$\rightarrow$	$\mathbf{I}(\mathbf{lb}(\mathbf{N}(\mathbf{y}_2), \mathbf{N}(\mathbf{y}_3))) \sqsubseteq$ $\mathbf{lb}(\mathbf{N}(\mathbf{y}_2), \mathbf{N}(\mathbf{y}_3))$	$\mathbf{I}$
14	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$ $\mathbf{N}(\mathbf{y}_2) \le \xi_1$ $\Omega\mathbf{y}_3$ $\mathbf{y}_3 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$ $\Omega\mathbf{I}(\mathbf{lb}(\mathbf{N}(\mathbf{y}_2), \mathbf{N}(\mathbf{y}_3)))$	$\rightarrow$	$\mathbf{lb}(\mathbf{N}(\mathbf{y}_2), \mathbf{N}(\mathbf{y}_3)) \sqsubseteq \mathbf{N}(\mathbf{y}_2)$	$\mathbf{lb}1$

15	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$ $\mathbf{N}(\mathbf{y}_2) \leq \xi_1$ $\Omega\mathbf{y}_3$ $\mathbf{y}_3 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$ $\Omega\mathbf{I}(\mathbf{lb}(\mathbf{N}(\mathbf{y}_2), \mathbf{N}(\mathbf{y}_3)))$	$\to$	$\mathbf{N}(\mathbf{y}_2) \leq \xi_1$	ax
16	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$ $\mathbf{N}(\mathbf{y}_2) \leq \xi_1$ $\Omega\mathbf{y}_3$ $\mathbf{y}_3 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$	$\to$	$\Omega\mathbf{I}(\mathbf{lb}(\mathbf{N}(\mathbf{y}_2), \mathbf{N}(\mathbf{y}_3)))$	$\Omega\mathbf{I}$: 17
17	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$ $\mathbf{N}(\mathbf{y}_2) \leq \xi_1$ $\Omega\mathbf{y}_3$ $\mathbf{y}_3 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$	$\to$	$\mathbf{I}(\mathbf{lb}(\mathbf{N}(\mathbf{y}_2), \mathbf{N}(\mathbf{y}_3))) \not\sqsubseteq \bot$	C: 18, 19
18	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$ $\mathbf{N}(\mathbf{y}_2) \leq \xi_1$ $\Omega\mathbf{y}_3$ $\mathbf{y}_3 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$	$\to$	$\mathbf{N}(\mathbf{y}_2) \sqsubseteq \mathbf{N}(\mathbf{y}_1)$	ax
19	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$ $\mathbf{N}(\mathbf{y}_2) \leq \xi_1$ $\Omega\mathbf{y}_3$ $\mathbf{y}_3 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$	$\to$	$\mathbf{N}(\mathbf{y}_3) \sqsubseteq \mathbf{N}(\mathbf{y}_1)$	ax

Again, the converse does not hold because (as will be shown by Theorem 3.18) rule C does caracterize the general frames with confluent accessibility relation while the corresponding modal axiom is more relaxed among general frames.

It is also worthwhile to show that the Löb axiom can be derived in $\mathcal{C}_k$ from rule W since the derivation requires the use of the # rules.

1		$\to$	$\top \leq \ \ (\ \ \xi_1 \Rightarrow \xi_1) \Rightarrow \ \ \xi_1$	RgenF : 2
2	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$	$\to$	$\mathbf{y}_1 \leq \ \ (\ \ \xi_1 \Rightarrow \xi_1) \Rightarrow \ \ \xi_1$	R$\Rightarrow$: 3
3	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\mathbf{y}_1 \leq \ \ (\ \ \xi_1 \Rightarrow \xi_1)$	$\to$	$\mathbf{y}_1 \leq \ \ \xi_1$	R : 4
4	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\mathbf{y}_1 \leq \ \ (\ \ \xi_1 \Rightarrow \xi_1)$	$\to$	$\mathbf{N}(\mathbf{y}_1) \leq \xi_1$	cutT : 5, 7
5	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\mathbf{y}_1 \leq \ \ (\ \ \xi_1 \Rightarrow \xi_1)$ $\mathbf{N}(\mathbf{y}_1) \sqsubseteq \#\xi_1$	$\to$	$\mathbf{N}(\mathbf{y}_1) \leq \xi_1$	L# : 6

6	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\mathbf{y}_1 \leq \quad (\ \ \xi_1 \Rightarrow \xi_1)$ $\mathbf{N}(\mathbf{y}_1) \leq \xi_1$	$\rightarrow$	$\mathbf{N}(\mathbf{y}_1) \leq \xi_1$	ax
7	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\mathbf{y}_1 \leq \quad (\ \ \xi_1 \Rightarrow \xi_1)$	$\rightarrow$	$\mathbf{N}(\mathbf{y}_1) \leq \xi_1$ $\mathbf{N}(\mathbf{y}_1) \sqsubseteq \#\xi_1$	W: 8
8	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\mathbf{y}_1 \leq \quad (\ \ \xi_1 \Rightarrow \xi_1)$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$	$\rightarrow$	$\mathbf{N}(\mathbf{y}_1) \leq \xi_1$ $\mathbf{y}_2 \sqsubseteq \#\xi_1$ $\mathbf{N}(\mathbf{y}_2) \not\sqsubseteq \#\xi_1$	R# : 9
9	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\mathbf{y}_1 \leq \quad (\ \ \xi_1 \Rightarrow \xi_1)$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$	$\rightarrow$	$\mathbf{N}(\mathbf{y}_1) \leq \xi_1$ $\mathbf{y}_2 \leq \xi_1$ $\mathbf{N}(\mathbf{y}_2) \not\sqsubseteq \#\xi_1$	RxiT : 10
10	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\mathbf{y}_1 \leq \quad (\ \ \xi_1 \Rightarrow \xi_1)$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$ $\mathbf{N}(\mathbf{y}_2) \sqsubseteq \#\xi_1$	$\rightarrow$	$\mathbf{N}(\mathbf{y}_1) \leq \xi_1$ $\mathbf{y}_2 \leq \xi_1$	L# : 11
11	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\mathbf{y}_1 \leq \quad (\ \ \xi_1 \Rightarrow \xi_1)$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$ $\mathbf{N}(\mathbf{y}_2) \leq \xi_1$	$\rightarrow$	$\mathbf{N}(\mathbf{y}_1) \leq \xi_1$ $\mathbf{y}_2 \leq \xi_1$	LxeT : 12
12	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\mathbf{y}_1 \leq \quad (\ \ \xi_1 \Rightarrow \xi_1)$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$	$\rightarrow$	$\mathbf{N}(\mathbf{y}_1) \leq \xi_1$ $\mathbf{y}_2 \leq \xi_1$ $\mathbf{N}(\mathbf{y}_2) \not\leq \xi_1$	L : 13
13	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\mathbf{N}(\mathbf{y}_1) \leq \quad \xi_1 \Rightarrow \xi_1$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$	$\rightarrow$	$\mathbf{N}(\mathbf{y}_1) \leq \xi_1$ $\mathbf{y}_2 \leq \xi_1$ $\mathbf{N}(\mathbf{y}_2) \not\leq \xi_1$	LgenT : 14, 15, 16
14	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$	$\rightarrow$	$\mathbf{N}(\mathbf{y}_1) \leq \xi_1$ $\mathbf{y}_2 \leq \xi_1$ $\mathbf{N}(\mathbf{y}_2) \not\leq \xi_1$ $\Omega\mathbf{y}_2$	ax
15	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$	$\rightarrow$	$\mathbf{N}(\mathbf{y}_1) \leq \xi_1$ $\mathbf{y}_2 \leq \xi_1$ $\mathbf{N}(\mathbf{y}_2) \not\leq \xi_1$ $\mathbf{y}_2 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$	ax
16	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\mathbf{y}_2 \leq \quad \xi_1 \Rightarrow \xi_1$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$	$\rightarrow$	$\mathbf{N}(\mathbf{y}_1) \leq \xi_1$ $\mathbf{y}_2 \leq \xi_1$ $\mathbf{N}(\mathbf{y}_2) \not\leq \xi_1$	L$\Rightarrow$: 17, 18

17	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\mathbf{y}_2 \leq \xi_1$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$	$\rightarrow$	$\mathbf{N}(\mathbf{y}_1) \leq \xi_1$ $\mathbf{y}_2 \leq \xi_1$ $\mathbf{N}(\mathbf{y}_2) \not\leq \xi_1$	ax
18	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$	$\rightarrow$	$\mathbf{N}(\mathbf{y}_1) \leq \xi_1$ $\mathbf{y}_2 \leq \xi_1$ $\mathbf{N}(\mathbf{y}_2) \not\leq \xi_1$ $\mathbf{y}_2 \leq \quad \xi_1$	R : 19
19	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$	$\rightarrow$	$\mathbf{N}(\mathbf{y}_1) \leq \xi_1$ $\mathbf{y}_2 \leq \xi_1$ $\mathbf{N}(\mathbf{y}_2) \not\leq \xi_1$ $\mathbf{N}(\mathbf{y}_2) \leq \xi_1$	RxiF : 20
20	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\Omega\mathbf{y}_2$ $\mathbf{y}_2 \sqsubseteq \mathbf{N}(\mathbf{y}_1)$ $\mathbf{N}(\mathbf{y}_2) \leq \xi_1$	$\rightarrow$	$\mathbf{N}(\mathbf{y}_1) \leq \xi_1$ $\mathbf{y}_2 \leq \xi_1$ $\mathbf{N}(\mathbf{y}_2) \leq \xi_1$	ax

Obviously, the converse does not hold since (as will be shown by Theorem 3.18) rule W does caracterize the general frames with transitive and well bounded accessibility relation while Löb's axiom is more relaxed among general frames.

This provides the basis for giving other rules for modal and other non-classical logics. A detailed discussion of such rules and of the semantic and proof-theoretic properties of the resulting labelled sequent calculi (e.g. a form of correspondence theory [18] or the eliminability of cut) is out of the scope of this paper and we leave it as future work.

2.7 Towards a hybrid version of $\mathcal{C}_K$

The discussion above (about, for instance, the non inter-derivability of rule T and the modal axiom for reflexivity) motivates the following question: is it possible to enrich $\mathcal{C}_K$ in order to recover that inter-derivability? Semantically, as we saw, this will mean moving from general Kripke semantics towards standard Kripke semantics (as we shall further comment at the end of Subsection 3.3).

The answer turns out to be surprisingly simple and possibly useful for other purposes. It is enough: (i) first, to enrich the language with a *coercion operator* @ transforming any term t into a simple formula $@t$; (ii) and, second, add the following order rules:

$$\text{L@} \quad \frac{\tau_1 \sqsubseteq \tau_2, \Gamma_1 \rightarrow \Gamma_2}{\tau_1 \leq @\tau_2, \Gamma_1 \rightarrow \Gamma_2} \qquad\qquad \text{R@} \quad \frac{\Gamma_1 \rightarrow \Gamma_2, \tau_1 \sqsubseteq \tau_2}{\Gamma_1 \rightarrow \Gamma_2, \tau_1 \leq @\tau_2}$$

In this way we established an enrichment of $\mathcal{C}_K$ that we denote by $\mathcal{C}_K^{@}$. In this enriched calculus, it is possible to derive, for instance, rule T from the modal axiom for reflexivity:

1	$\Omega\tau_1$	$\rightarrow$	$\tau_1 \sqsubseteq \mathbf{N}(\tau_1)$	cutF : 2, 6
2	$\Omega\tau_1$	$\rightarrow$	$\tau_1 \sqsubseteq \mathbf{N}(\tau_1)$ $\tau_1 \leq \quad @\mathbf{N}(\tau_1) \Rightarrow @\mathbf{N}(\tau_1)$	transF : 3, 4

3	$\Omega\tau_1$	$\rightarrow$	$\tau_1 \sqsubseteq \mathbf{N}(\tau_1)$ $\tau_1 \sqsubseteq \top$	$\top$
4	$\Omega\tau_1$	$\rightarrow$	$\tau_1 \sqsubseteq \mathbf{N}(\tau_1)$ $\top \leq \quad @\mathbf{N}(\tau_1) \Rightarrow @\mathbf{N}(\tau_1)$	w : 5
5		$\rightarrow$	$\top \leq \quad @\mathbf{N}(\tau_1) \Rightarrow @\mathbf{N}(\tau_1)$	hyp
6	$\Omega\tau_1$ $\tau_1 \leq \quad @\mathbf{N}(\tau_1) \Rightarrow @\mathbf{N}(\tau_1)$	$\rightarrow$	$\tau_1 \sqsubseteq \mathbf{N}(\tau_1)$	L$\Rightarrow$: 7, 9
7	$\Omega\tau_1$ $\tau_1 \leq @\mathbf{N}(\tau_1)$	$\rightarrow$	$\tau_1 \sqsubseteq \mathbf{N}(\tau_1)$	L@ : 8
8	$\Omega\tau_1$ $\tau_1 \sqsubseteq \mathbf{N}(\tau_1)$	$\rightarrow$	$\tau_1 \sqsubseteq \mathbf{N}(\tau_1)$	ax
9	$\Omega\tau_1$	$\rightarrow$	$\tau_1 \sqsubseteq \mathbf{N}(\tau_1)$ $\tau_1 \leq \quad @\mathbf{N}(\tau_1)$	R : 10
10	$\Omega\tau_1$	$\rightarrow$	$\tau_1 \sqsubseteq \mathbf{N}(\tau_1)$ $\mathbf{N}(\tau_1) \leq @\mathbf{N}(\tau_1)$	R@ : 11
11	$\Omega\tau_1$	$\rightarrow$	$\tau_1 \sqsubseteq \mathbf{N}(\tau_1)$ $\mathbf{N}(\tau_1) \sqsubseteq \mathbf{N}(\tau_1)$	ref

The same holds for the other properties of the accessibility relation but we refrain from going into details.

The sequent calculus $\mathcal{C}_K^@$ represents a first step towards a hybrid version of $\mathcal{C}_K$ combining the ideas in this paper and those of hybrid logics [7, 6, 2].

2.8 Local and global reasoning

In the context of a modal sequent calculus, local and global notions of proof-theoretic consequence can be defined as follows:

- $\psi_1, \ldots, \psi_k \vdash^{\mathrm{g}}_{\mathcal{R}} \varphi$ iff $\vdash_{\mathcal{R}} \top \leq \psi_1, \ldots, \top \leq \psi_k \rightarrow \top \leq \varphi$;
- $\psi_1, \ldots, \psi_k \vdash^{\ell}_{\mathcal{R}} \varphi$ iff $\vdash_{\mathcal{R}} \Omega\mathbf{y}_1, \mathbf{y}_1 \leq \psi_1, \ldots, \mathbf{y}_1 \leq \psi_k \rightarrow \mathbf{y}_1 \leq \varphi$.

Thus, φ is globally derived from $\psi_1, \ldots, \psi_k$ provided that φ is true ($\top$) whenever ψ_i is true for all $i = 1, \ldots k$. When working with worlds this means that the denotation φ is W whenever the denotation of ψ_i is W for all $i = 1, \ldots k$.

On the other hand, φ is locally derived from $\psi_1, \ldots, \psi_k$ provided that for every atomic element $\mathbf{y}_1$ the value of φ is greater than or equal to the value of $\mathbf{y}_1$ whenever the value of ψ_i is greater than or equal to the value of $\mathbf{y}_1$ for all $i = 1, \ldots k$. When working with worlds we get the usual definition stating that φ is true at w whenever ψ_i is true at w for all $i = 1, \ldots k$.

Lemma 2.3 Within the context of a modal sequent calculus:

1. $\Omega\mathbf{y}_1, \mathbf{y}_1 \leq \psi_1, \ldots, \mathbf{y}_1 \leq \psi_k \rightarrow \mathbf{y}_1 \leq \varphi \vdash_{\mathcal{R}} \rightarrow \top \leq ((\psi_1 \wedge \ldots \wedge \psi_k) \Rightarrow \varphi)$;
2. $\rightarrow \top \leq ((\psi_1 \wedge \ldots \wedge \psi_k) \Rightarrow \varphi) \vdash_{\mathcal{R}} \Omega\mathbf{y}_1, \mathbf{y}_1 \leq \psi_1, \ldots, \mathbf{y}_1 \leq \psi_k \rightarrow \mathbf{y}_1 \leq \varphi$.

Proof: Without loss of generality consider $k = 2$.
1. Consider the following derivation:

1		$\rightarrow$	$\top \leq ((\psi_1 \wedge \psi_2) \Rightarrow \varphi)$	RgenF : 2
2	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$	$\rightarrow$	$\mathbf{y}_1 \leq ((\psi_1 \wedge \psi_2) \Rightarrow \varphi)$	R$\Rightarrow$: 3
3	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\mathbf{y}_1 \leq (\psi_1 \wedge \psi_2)$	$\rightarrow$	$\mathbf{y}_1 \leq \varphi$	L$\wedge$: 4
4	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \top$ $\mathbf{y}_1 \leq \psi_1$ $\mathbf{y}_1 \leq \psi_2$	$\rightarrow$	$\mathbf{y}_1 \leq \varphi$	Lw : 5
5	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \leq \psi_1$ $\mathbf{y}_1 \leq \psi_2$	$\rightarrow$	$\mathbf{y}_1 \leq \varphi$	hyp

2. Consider the following derivation:

1	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \leq \psi_1$ $\mathbf{y}_1 \leq \psi_2$	$\rightarrow$	$\mathbf{y}_1 \leq \varphi$	cutF : 2, 3
2	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \leq \psi_1$ $\mathbf{y}_1 \leq \psi_2$	$\rightarrow$	$\mathbf{y}_1 \leq \varphi$ $\mathbf{y}_1 \leq ((\psi_1 \wedge \psi_2) \Rightarrow \varphi)$	transF : 4, 9
3	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \leq \psi_1$ $\mathbf{y}_1 \leq \psi_2$ $\mathbf{y}_1 \leq ((\psi_1 \wedge \psi_2) \Rightarrow \varphi)$	$\rightarrow$	$\mathbf{y}_1 \leq \varphi$	L$\Rightarrow$: 5, 6
4	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \leq \psi_1$ $\mathbf{y}_1 \leq \psi_2$	$\rightarrow$	$\mathbf{y}_1 \leq \varphi$ $\mathbf{y}_1 \sqsubseteq \top$	$\top$
5	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \leq \psi_1$ $\mathbf{y}_1 \leq \psi_2$	$\rightarrow$	$\mathbf{y}_1 \leq \varphi$ $\mathbf{y}_1 \leq (\psi_1 \wedge \psi_2)$	R$\wedge$: 7, 8
6	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \leq \psi_1$ $\mathbf{y}_1 \leq \psi_2$ $\mathbf{y}_1 \leq \varphi$	$\rightarrow$	$\mathbf{y}_1 \leq \varphi$	ax
7	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \leq \psi_1$ $\mathbf{y}_1 \leq \psi_2$	$\rightarrow$	$\mathbf{y}_1 \leq \varphi$ $\mathbf{y}_1 \leq \psi_1$	ax
8	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \leq \psi_1$ $\mathbf{y}_1 \leq \psi_2$	$\rightarrow$	$\mathbf{y}_1 \leq \varphi$ $\mathbf{y}_1 \leq \psi_2$	ax
9	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \leq \psi_1$ $\mathbf{y}_1 \leq \psi_2$	$\rightarrow$	$\mathbf{y}_1 \leq \varphi$ $\top \leq ((\psi_1 \wedge \psi_2) \Rightarrow \varphi)$	ws : 10
10		$\rightarrow$	$\top \leq ((\psi_1 \wedge \psi_2) \Rightarrow \varphi)$	hyp

QED

With this lemma it is straightforward to establish the following result relating global and local reasoning.

Proposition 2.4 Within any modal sequent calculus: $\psi_1, \ldots, \psi_k \ \vdash^{\ell}_{\mathcal{R}} \varphi$ iff $\vdash^{\mathrm{g}}_{\mathcal{R}} (\psi_1 \wedge \ldots \wedge \psi_k) \Rightarrow \varphi$.

Proof: Indeed, $\psi_1, \ldots, \psi_k \ \vdash^{\ell}_{\mathcal{R}} \varphi$ iff $\vdash_{\mathcal{R}} \Omega \mathbf{y}_1, \mathbf{y}_1 \leq \psi_1, \ldots, \mathbf{y}_1 \leq \psi_k \to \mathbf{y}_1 \leq \varphi$ iff (using Lemma 2.3 and taking into account idempotence in Proposition 2.1) $\vdash_{\mathcal{R}} \to \top \leq ((\psi_1 \wedge \ldots \wedge \psi_k) \Rightarrow \varphi)$ iff $\vdash^{\mathrm{g}}_{\mathcal{R}} (\psi_1 \wedge \ldots \wedge \psi_k) \Rightarrow \varphi$. QED

2.9 Metatheorems

It is useful to denote by $\overline{\Delta}$ the bag $\{\overline{\delta} : \delta \in \Delta\}$. Then, it is straightforward to prove the following result taking into account that the conjugate of a conjugate of an assertion is the original assertion:

Theorem 2.5 (Metatheorem of conjugation) Let $\mathcal{C}$ be a structural sequent calculus. Then, for every set S of ground sequents and every ground sequent $\Delta' \to \Delta''$:

$$S \vdash_{\mathcal{C}} \Delta' \to \Delta'' \ \text{ iff } \ S \vdash_{\mathcal{C}} \to \Delta'', \overline{\Delta'} \,.$$

The two following metatheorems will also be useful later on when establishing the completeness theorem.

Theorem 2.6 (Metatheorem of contradiction) Let $\mathcal{C}$ be a structural sequent calculus. Then, for every set S of ground sequents and every ground sequent $\to \Delta$, if

$$\begin{cases} S \vdash_{\mathcal{C}} \to \Delta & (*) \\ S \vdash_{\mathcal{C}} \to \overline{\delta} \text{ for every } \delta \in \Delta & (**) \end{cases}$$

then $S \vdash_{\mathcal{C}} \to \upsilon$ for every ground assertion υ.

Proof: Let Δ be δ_1, δ_2 without loss of generality. Then:

1		$\to \upsilon$	cut : 2, 5
2	δ_1	$\to \upsilon$	Rx : 3
3		$\to \upsilon, \overline{\delta_1}$	Rw : 4
4		$\to \overline{\delta_1}$	(**)
5		$\to \upsilon, \delta_1$	cut : 6, 9
6	δ_2	$\to \upsilon, \delta_1$	Rx : 7
7		$\to \upsilon, \delta_1, \overline{\delta_2}$	Rws : 8
8		$\to \overline{\delta_2}$	(**)
9		$\to \upsilon, \delta_1, \delta_2$	Rw : 10
10		$\to \delta_1, \delta_2$	(*)

QED

Theorem 2.7 (Metatheorem of deduction) Let $\mathcal{C}$ be a structural sequent calculus with rules endowed with persistent provisos. Then, for every set S of ground sequents and closed sequent $\delta'_1, \ldots, \delta'_m \to \Delta''$:

$$S \vdash_{\mathcal{C}} \delta'_1, \ldots, \delta'_m \to \Delta'' \ \text{ iff } \ S, \to \delta'_1, \ldots, \to \delta'_m \vdash_{\mathcal{C}} \to \Delta'' \,.$$

Proof:
($\Rightarrow$) Assume $S \vdash_{\mathcal{C}} \delta'_1, \ldots, \delta'_m \to \Delta''$ with derivation sequence D. Then, the following sequence outline establishes $S, \to \delta'_1, \ldots, \to \delta'_m \vdash_{\mathcal{C}} \to \Delta''$:

1		$\to \Delta''$	cut : 2, 3
2		$\to \Delta'', \delta'_1$	Rws : 4
3	δ'_1	$\to \Delta''$	cut : 5, 6
4		$\to \delta'_1$	hyp
5	δ'_1	$\to \Delta'', \delta'_2$	Rws : 7
6	δ'_2, δ'_1	$\to \Delta''$	cut : 9, 10
7	δ'_1	$\to \delta'_2$	Lw : 8
8		$\to \delta'_2$	hyp
		...	
i	$\delta'_1, \ldots, \delta'_m$	$\to \Delta''$	D

($\Leftarrow$) Assume $S, \ \to \delta'_1, \ldots, \ \to \delta'_m \vdash_{\mathcal{C}} \ \to \Delta''$ with the derivation sequence $d_1, \ldots, d_n$. Then we can build a derivation of $S \vdash_{\mathcal{C}} \delta'_1, \ldots, \delta'_m \to \Delta''$ by changing each $d_i = \Theta^i_1 \to \Theta^i_2$ to $d'_i = \delta'_1, \ldots, \delta'_m, \Theta^i_1 \to \Theta^i_2$ replacing the justification hyp on each $d_i = \to \delta'_j$ by ax. Observe that the sequence $d'_1, \ldots, d'_n$ does constitute a derivation because the unchanged justifications still hold thanks to the fact that $\delta'_1, \ldots, \delta'_m$ are closed assertions and therefore any (persistent) proviso that otherwise might be violated is still fulfilled. QED

Therefore, the metatheorem of deduction holds in any modal sequent calculus as defined at the end of Subsection 2.5 (precluding the use of non persistent provisos).

3 Semantics

3.1 Algebraic semantics

Let $\Sigma = \langle C, O, X, Y, Z \rangle$ be a signature. A Σ-*algebra* is a triple $\mathbb{A} = \langle F, T, \cdot_{\mathbb{A}} \rangle$ where:

- F and T are sets;
- $\cdot_{\mathbb{A}}$ is a map such that:
  - $c_{\mathbb{A}} : F^k \to F$ for each $c \in C_k$;
  - $o_{\mathbb{A}} : T^k \to T$ for each $o \in O_k$;
  - $\#_{\mathbb{A}} : F \to T$;
  - $\Omega_{\mathbb{A}} \subseteq T$;
  - $\sqsubseteq_{\mathbb{A}} \subseteq T \times T$;
  - $\leq_{\mathbb{A}} \subseteq T \times F$.

Let $\mathbb{A}$ be a Σ-algebra. An *unbound variable assignment* over $\mathbb{A}$ is a map α that maps each element of X to an element of T and each element of Z to an element of F. A *bound variable assignment* over $\mathbb{A}$ is a map β from Y to T.

The *denotation* at Σ-algebra $\mathbb{A}$ for unbound variable assignment α of ground simple formulae is inductively defined with the following rules:

- $[\![z]\!]_{\mathbb{A}\alpha} = \alpha(z)$;
- $[\![c(\varphi_1, \ldots, \varphi_k)]\!]_{\mathbb{A}\alpha} = c_{\mathbb{A}}([\![\varphi_1]\!]_{\mathbb{A}\alpha}, \ldots, [\![\varphi_k]\!]_{\mathbb{A}\alpha})$.

The *denotation* at $\mathbb{A}$ for assignments α, β over $\mathbb{A}$ of ground terms is inductively defined with the following rules:

- $[\![x]\!]_{\mathbb{A}\alpha\beta} = \alpha(x)$;
- $[\![y]\!]_{\mathbb{A}\alpha\beta} = \beta(y)$;
- $[\![o(\theta_1, \dots, \theta_k)]\!]_{\mathbb{A}\alpha\beta} = o_{\mathbb{A}}([\![\theta_1]\!]_{\mathbb{A}\alpha\beta}, \dots, [\![\theta_k]\!]_{\mathbb{A}\alpha\beta})$;
- $[\![\#\varphi]\!]_{\mathbb{A}\alpha\beta} = \#_{\mathbb{A}}([\![\varphi]\!]_{\mathbb{A}\alpha})$.

The *satisfaction* by $\mathbb{A}$ for α, β of ground assertions and sequents is defined as follows:

- $\mathbb{A}\alpha\beta \Vdash \Omega\theta$ iff $[\![\theta]\!]_{\mathbb{A}\alpha\beta} \in \Omega_{\mathbb{A}}$;
- $\mathbb{A}\alpha\beta \Vdash \mho\theta$ iff $[\![\theta]\!]_{\mathbb{A}\alpha\beta} \notin \Omega_{\mathbb{A}}$;
- $\mathbb{A}\alpha\beta \Vdash \theta \sqsubseteq \theta'$ iff $\langle [\![\theta]\!]_{\mathbb{A}\alpha\beta}, [\![\theta']\!]_{\mathbb{A}\alpha\beta} \rangle \in \sqsubseteq_{\mathbb{A}}$;
- $\mathbb{A}\alpha\beta \Vdash \theta \not\sqsubseteq \theta'$ iff $\langle [\![\theta]\!]_{\mathbb{A}\alpha\beta}, [\![\theta']\!]_{\mathbb{A}\alpha\beta} \rangle \notin \sqsubseteq_{\mathbb{A}}$;
- $\mathbb{A}\alpha\beta \Vdash \theta \leq \varphi$ iff $\langle [\![\theta]\!]_{\mathbb{A}\alpha\beta}, [\![\varphi]\!]_{\mathbb{A}\alpha} \rangle \in \leq_{\mathbb{A}}$;
- $\mathbb{A}\alpha\beta \Vdash \theta \not\leq \varphi$ iff $\langle [\![\theta]\!]_{\mathbb{A}\alpha\beta}, [\![\varphi]\!]_{\mathbb{A}\alpha} \rangle \notin \leq_{\mathbb{A}}$;
- $\mathbb{A}\alpha\beta \Vdash \Delta' \to \Delta''$ iff $\mathbb{A}\alpha\beta \Vdash \delta$ for some $\delta \in \Delta'' \cup \overline{\Delta'}$.

Furthermore, the *satisfaction* by $\mathbb{A}$ for α of ground assertions and sequents is defined as follows:

- $\mathbb{A}\alpha \Vdash \delta$ iff $\mathbb{A}\alpha\beta \Vdash \delta$ for every bound variable assignment β over $\mathbb{A}$;
- $\mathbb{A}\alpha \Vdash \Delta' \to \Delta''$ iff $\mathbb{A}\alpha\beta \Vdash \Delta' \to \Delta''$ for every bound variable assignment β over $\mathbb{A}$.

Observe that, when dealing with closed simple formulae, terms, assertions and sequents, we may drop the reference to the assignments in denotations and satisfactions since they do not depend on them. For instance, if δ is a closed assertion then we may write $\mathbb{A} \Vdash \delta$ since, for any assignments $\alpha, \alpha', \beta, \beta'$, $\mathbb{A}\alpha\beta \Vdash \delta$ iff $\mathbb{A}\alpha'\beta' \Vdash \delta$. A similar principle applies when we deal with terms, assertions and sequents without bound variables in which case we may drop the reference to the bound variable assignment. In the same vein, we may drop the reference to the unbound variable assignment when dealing with terms, assertions and sequents without unbound variables.

Given a class $\mathcal{A}$ of Σ-algebras, a ground sequent s is $\mathcal{A}$-*entailed* by the ground sequents $s_1, \dots, s_p$, written $s_1, \dots, s_p \vDash_{\mathcal{A}} s$, iff, for each $\mathbb{A} \in \mathcal{A}$ and unbound variable assignment α over $\mathbb{A}$, $\mathbb{A}\alpha \Vdash s$ whenever $\mathbb{A}\alpha \Vdash s_i$ for every $i = 1, \dots, p$.

The notion of entailment is easily extended to (schema) sequents possibly with provisos. A sequent s is $\mathcal{A}$-*entailed* by the sequents $s_1, \dots, s_p$ with proviso π, written $s_1, \dots, s_p \vDash_{\mathcal{A}} s \lhd \pi$, iff $s_1\rho, \dots, s_p\rho \vDash_{\mathcal{A}} s\rho$ for every ground substitution ρ over Σ such that $\pi(\rho) = 1$.

The following results are the semantic counterparts of Proposition 2.1 and Proposition 2.2.

Proposition 3.1 Given a class $\mathcal{A}$ of Σ-algebras:

Projective If $S \vDash_{\mathcal{A}} s' \lhd \pi$ and $\pi' \subseteq \pi$ then $S \vDash_{\mathcal{A}} s' \lhd \pi'$.

Extensive $S \vDash_{\mathcal{A}} s \lhd \mathbf{up}$ for each sequent $s \in S$.

Monotonic If $S \subseteq S_1$ and $S \vDash_{\mathcal{A}} s' \lhd \pi$ then $S_1 \vDash_{\mathcal{A}} s' \lhd \pi$.

Idempotent If $S_1 \vDash_{\mathcal{A}} s \lhd \pi_s$ for each s in a finite set S of sequents and $S \vDash_{\mathcal{A}} s' \lhd \pi$ then $S_1 \vDash_{\mathcal{A}} s' \lhd \pi \cap (\bigcap_{s \in S} \pi_s)$.

Proof: Straightforward. We prove only the last property. Assume $S_1 \vDash_{\mathcal{A}} s \lhd \pi_s$ for each sequent $s \in S$ and $S \vDash_{\mathcal{A}} s' \lhd \pi$. So, by the projective property, $S_1 \vDash_{\mathcal{A}} s \lhd \pi \cap (\bigcap_{s \in S} \pi_s)$ for each sequent $s \in S$ and $S \vDash_{\mathcal{A}} s' \lhd \pi \cap (\bigcap_{s \in S} \pi_s)$. Thus, by definition of entailment, for every ρ such that $(\pi \cap (\bigcap_{s \in S} \pi_s))(\rho) = 1$, $S_1\rho \vDash_{\mathcal{A}} s\rho$ for each sequent $s \in S$ and $S\rho \vDash_{\mathcal{A}} s'\rho$. Therefore, for every such ρ, every $\mathbb{A} \in \mathcal{A}$ and unbound variable assignment α over $\mathbb{A}$: (i) if $\mathbb{A}\alpha \Vdash s_1\rho$ for every $s_1 \in S_1$ then $\mathbb{A}\alpha \Vdash s\rho$ for every $s \in S$; and (ii) if $\mathbb{A}\alpha \Vdash s\rho$ for every $s \in S$ then $\mathbb{A}\alpha \Vdash s'\rho$. So, for every such ρ, every $\mathbb{A} \in \mathcal{A}$ and α, if $\mathbb{A}\alpha \Vdash s_1\rho$ for every $s_1 \in S_1$ then $\mathbb{A}\alpha \Vdash s'\rho$. QED

Proposition 3.2 Given a class $\mathcal{A}$ of Σ-algebras, for every substitution σ, if $S \vDash_{\mathcal{A}} s' \lhd \pi$ then $S\sigma \vDash_{\mathcal{A}} s'\sigma \lhd \pi\sigma$.

Proof: We have to show $S\sigma \vDash_{\mathcal{A}} s'\sigma \lhd \pi\sigma$. That is, for an arbitrary ground substitution ρ such that $(\pi\sigma)(\rho) = 1$, we have to show $(S\sigma)\rho \vDash_{\mathcal{A}} (s'\sigma)\rho$. By hypothesis, we know $S \vDash_{\mathcal{A}} s' \lhd \pi$. That is, for every ground substitution ρ' such that $\pi(\rho') = 1$, we know $S\rho' \vDash_{\mathcal{A}} s'\rho'$. Since the substitution $\sigma\rho$ is ground and, furthermore, $\pi(\sigma\rho) = (\pi\sigma)(\rho) = 1$, we know from the hypothesis $S(\sigma\rho) \vDash_{\mathcal{A}} s'(\sigma\rho)$ which establishes the thesis taking into account the following property of substitutions $\delta(\sigma\rho) = (\delta\sigma)\rho$. QED

A class $\mathcal{A}$ of Σ-algebras is said to be *appropriate* for a Σ-rule $\langle\{s_1, \ldots, s_p\}, s, \pi\rangle$ if $s_1, \ldots, s_p \vDash_{\mathcal{A}} s \lhd \pi$. And an algebra $\mathbb{A}$ is said to be appropriate for a rule if so is the class $\{\mathbb{A}\}$.

Theorem 3.3 (Structural soundness) The class of all Σ-algebras is appropriate for every structural rule over Σ.

Proof: It is straighforward to verify the thesis for each of the structural rules. For instance, consider the rule RwT. We have to verify that, for each Σ-algebra $\mathbb{A}$, assignment α and ground substitution ρ, if $\mathbb{A}\alpha \Vdash \Gamma_1\rho \to \Gamma_2\rho$ then $\mathbb{A}\alpha \Vdash \Gamma_1\rho \to \Gamma_2\rho, \tau_1\rho \sqsubseteq \tau_2\rho$. Assume that $\mathbb{A}\alpha \Vdash \Gamma_1\rho \to \Gamma_2\rho$. That is, for every assignment β, $\mathbb{A}\alpha\beta \Vdash \Gamma_1\rho \to \Gamma_2\rho$. Thus, for every assignment β there is $\delta \in \Gamma_2\rho \cup \overline{\Gamma_1\rho}$ such that $\mathbb{A}\alpha\beta \Vdash \delta$. So, for every assignment β there is $\delta \in \Gamma_2\rho \cup \{\tau_1\rho \sqsubseteq \tau_2\rho\} \cup \overline{\Gamma_1\rho}$ such that $\mathbb{A}\alpha\beta \Vdash \delta$. Therefore, for every assignment β, $\mathbb{A}\alpha\beta \Vdash \Gamma_1\rho \to \Gamma_2\rho, \tau_1\rho \sqsubseteq \tau_2\rho$. That is, $\mathbb{A}\alpha \Vdash \Gamma_1\rho \to \Gamma_2\rho, \tau_1\rho \sqsubseteq \tau_2\rho$. QED

A class $\mathcal{A}$ of Σ-algebras is said to be appropriate for a sequent calculus $\langle\Sigma, \mathcal{R}\rangle$ if it is appropriate for each proper rule in $\mathcal{R}$ (and also for the structural rules thanks to the theorem above). And an algebra $\mathbb{A}$ is said to be appropriate for a sequent calculus if so is the class $\{\mathbb{A}\}$.

A *(sequent) logic* is a triple $\mathcal{L} = \langle\Sigma, \mathcal{R}, \mathcal{A}\rangle$ where $\langle\Sigma, \mathcal{R}\rangle$ is a sequent calculus and $\mathcal{A}$ is a class of Σ-algebras. A sequent logic is said to be:

- *sound* if $\mathcal{A}$ is appropriate for $\langle\Sigma, \mathcal{R}\rangle$;
- *full* if $\mathcal{A}$ is the class of all Σ-algebras that are appropriate for $\langle\Sigma, \mathcal{R}\rangle$;
- *complete* if $s_1, \ldots, s_p \vdash_{\mathcal{R}} s$ whenever $s_1, \ldots, s_p \vDash_{\mathcal{A}} s$ for any closed sequents $s, s_1, \ldots, s_p$.

Thus, every full logic is sound. Furthermore:

Theorem 3.4 A sequent logic $\mathcal{L} = \langle\Sigma, \mathcal{R}, \mathcal{A}\rangle$ is sound iff $s_1, \ldots, s_p \vDash_{\mathcal{A}} s \lhd \pi$ whenever $s_1, \ldots, s_p \vdash_{\mathcal{R}} s \lhd \pi$.

Proof:

($\Leftarrow$) Given a rule $r = \frac{s'_1 \ldots s'_q}{s'} \lhd \pi'$ of $\mathcal{R}$, we know how to build a derivation of $s'_1, \ldots, s'_q \vdash_{\mathcal{R}} s' \lhd \pi'$. Thus, by hypothesis, $s'_1, \ldots, s'_q \vDash_{\mathcal{A}} s' \lhd \pi'$ and, so, $\mathcal{A}$ is appropriate for r.

($\Rightarrow$) Assume that $\mathcal{L}$ is sound and $s_1, \ldots, s_p \vdash_{\mathcal{R}} s \lhd \pi$ with derivation sequence D. We prove $s_1, \ldots, s_p \vDash_{\mathcal{A}} s \lhd \pi$ by complete induction on the length of D. Assume the thesis for derivation sequences of length less than n (induction hypothesis). Consider a derivation sequence $D = \langle d_1, \pi_1\rangle, \ldots, \langle d_n, \pi_n\rangle$. We have to show $s_1, \ldots, s_p \vDash_{\mathcal{A}} d_1 \lhd \pi_1$. Looking at the justification of the first element in the derivation we have to consider three cases:

(hypothesis) We have to show $s_1, \ldots, s_p \vDash_{\mathcal{A}} s_i \lhd \pi_1$. Indeed, Proposition 3.1 allows us to obtain successively: $s_1, \ldots, s_p \vDash_{\mathcal{A}} s_i$ (extensive) and $s_1, \ldots, s_p \vDash_{\mathcal{A}} s_i \lhd \pi_1$ (projective).

(axiom) We have to show $s_1, \ldots, s_p \vDash_{\mathcal{A}} \delta, \Gamma_1 \to \Gamma_2, \delta \lhd \pi_1$. Indeed, $\vDash_{\mathcal{A}} \delta \to \delta$ and, therefore, by weakening (Theorem 3.3), $\vDash_{\mathcal{A}} \delta, \Gamma_1 \to \Gamma_2, \delta$. Therefore, using Proposition 3.1 we obtain successively: $s_1, \ldots, s_p \vDash_{\mathcal{A}} \delta, \Gamma_1 \to \Gamma_2, \delta$ (monotonic) and $s_1, \ldots, s_p \vDash_{\mathcal{A}} \delta, \Gamma_1 \to \Gamma_2, \delta \lhd \pi_1$ (projective).

(rule) Assume that rule $r \in \mathcal{R}$ was used with substitution σ for justifying d_1. Let

$$r\sigma = \frac{d_{i_1} \ \ldots \ d_{i_q}}{d_1} \lhd \pi'$$

with $i_1, \ldots, i_q \in \{2, \ldots, n\}$. Thanks to the projective property in Proposition 3.1, it is enough to show $s_1, \ldots, s_p \vDash_{\mathcal{A}} d_1 \lhd \pi' \cap \pi_{i_1} \cap \cdots \cap \pi_{i_q}$. Since $\mathcal{A}$ is appropriate for every rule in $\mathcal{R}$ and using Proposition 3.2, we know $d_{i_1}, \ldots, d_{i_q} \vDash_{\mathcal{A}} d_1 \lhd \pi'$. On the other hand, by induction hypothesis, we know $s_1, \ldots, s_p \vDash_{\mathcal{A}} d_{i_j} \lhd \pi_{i_j}$ for $j = 1, \ldots, q$. Therefore, by Proposition 3.1 (idempotent) we obtain the envisaged result. QED

A sequent logic $\mathcal{L} = \langle \Sigma, \mathcal{R}, \mathcal{A} \rangle$ is said to be *structural/order/modal* if its calculus $\langle \Sigma, \mathcal{R} \rangle$ is structural/order/modal, respectively.

Within a modal logic, besides the local and global notions of proof-theoretic consequence presented before, we can also introduce their model-theoretic counterparts:

- $\psi_1, \ldots, \psi_k \vDash^{\mathrm{g}}_{\mathcal{A}} \varphi$ iff $\vDash_{\mathcal{A}} \top \leq \psi_1, \ldots, \top \leq \psi_k \to \top \leq \varphi$;
- $\psi_1, \ldots, \psi_k \vDash^{\ell}_{\mathcal{A}} \varphi$ iff $\vDash_{\mathcal{A}} \Omega \mathbf{y}_1, \mathbf{y}_1 \leq \psi_1, \ldots, \mathbf{y}_1 \leq \psi_k \to \mathbf{y}_1 \leq \varphi$.

The following result is the semantic counterpart of Proposition 2.4.

Proposition 3.5 Within the context of a sound modal sequent logic $\mathcal{L} = \langle \Sigma, \mathcal{R}, \mathcal{A} \rangle$: $\psi_1, \ldots, \psi_k \vDash^{\ell}_{\mathcal{A}} \varphi$ iff $\vDash^{\mathrm{g}}_{\mathcal{A}} (\psi_1 \wedge \ldots \wedge \psi_k) \Rightarrow \varphi$.

Proof: Taking into account Theorem 3.4, from Lemma 2.3 we obtain:

1. $\Omega \mathbf{y}_1, \mathbf{y}_1 \leq \psi_1, \ldots, \mathbf{y}_1 \leq \psi_k \to \mathbf{y}_1 \leq \varphi \vDash_{\mathcal{A}} \to \top \leq ((\psi_1 \wedge \ldots \wedge \psi_k) \Rightarrow \varphi)$;
2. $\to \top \leq ((\psi_1 \wedge \ldots \wedge \psi_k) \Rightarrow \varphi) \vDash_{\mathcal{A}} \Omega \mathbf{y}_1, \mathbf{y}_1 \leq \psi_1, \ldots, \mathbf{y}_1 \leq \psi_k \to \mathbf{y}_1 \leq \varphi$.

Therefore, $\psi_1, \ldots, \psi_k \vDash^{\ell}_{\mathcal{A}} \varphi$ iff $\vDash_{\mathcal{A}} \Omega \mathbf{y}_1, \mathbf{y}_1 \leq \psi_1, \ldots, \mathbf{y}_1 \leq \psi_k \to \mathbf{y}_1 \leq \varphi$ iff (from 1. and 2. taking into account idempotence in Proposition 3.1) $\vDash_{\mathcal{A}} \to \top \leq ((\psi_1 \wedge \ldots \wedge \psi_k) \Rightarrow \varphi)$ iff $\vDash^{\mathrm{g}}_{\mathcal{A}} (\psi_1 \wedge \ldots \wedge \psi_k) \Rightarrow \varphi$. QED

3.2 Algebraic completeness

A set S of closed sequents is said to be *consistent* if for no closed assertion δ both $\to \delta \in S$ and $\to \overline{\delta} \in S$ hold. And it is said to be *maximal consistent* if for every closed assertion δ either $\to \delta \in S$ or $\to \overline{\delta} \in S$ but not both.

Given a sequent calculus $\mathcal{C} = \langle \Sigma, \mathcal{R} \rangle$ and a maximal consistent set S of closed sequents over Σ, the *syntactic algebra* induced by $\mathcal{C}$ and S is the following Σ-algebra:

$$\mathbb{A}(\mathcal{C}, S) = \langle cgF(\Sigma), cgT(\Sigma), \cdot_{\mathbb{A}(\mathcal{C},S)} \rangle$$

where:

- $c_{\mathbb{A}(\mathcal{C},S)} = \lambda f_1 \ldots f_k.\, c(f_1, \ldots, f_k)$;
- $o_{\mathbb{A}(\mathcal{C},S)} = \lambda t_1 \ldots t_k.\, o(t_1, \ldots, t_k)$;
- $\#_{\mathbb{A}(\mathcal{C},S)} = \lambda f.\, \# f$;
- $\theta \in \Omega_{\mathbb{A}(\mathcal{C},S)}$ iff $S \vdash_{\mathcal{C}} \to \Omega \theta$;
- $\langle \theta, \theta' \rangle \in \sqsubseteq_{\mathbb{A}(\mathcal{C},S)}$ iff $S \vdash_{\mathcal{C}} \to \theta \sqsubseteq \theta'$;
- $\langle \theta, \varphi \rangle \in \leq_{\mathbb{A}(\mathcal{C},S)}$ iff $S \vdash_{\mathcal{C}} \to \theta \leq \varphi$.

Let $\varphi \in gF(\Sigma)$. Given an unbounded variable assignment α over a syntactic algebra $\mathbb{A}(\mathcal{C}, S)$, we denote by $\varphi\alpha$ the closed simple formula obtained from φ by replacing each variable $z \in Z$ by $\alpha(z)$.

Let $\theta \in gT(\Sigma)$. Given an unbounded variable assignment α and a bounded variable assignment β both over a syntactic algebra $\mathbb{A}(\mathcal{C}, S)$, we denote by $\theta\alpha\beta$ the closed term obtained from θ by replacing each variable $x \in X$ by $\alpha(x)$ and each variable $y \in Y$ by $\beta(y)$.

This notation is extended to ground assertions and bags of ground assertions by identifying $\varphi\alpha\beta$ with $\varphi\alpha$.

Lemma 3.6 Let $\mathcal{C}$ be a structural calculus, S a maximal consistent set of closed sequents, α an unbound variable assignment over $\mathbb{A}(\mathcal{C}, S)$, β a bound variable assignment over $\mathbb{A}(\mathcal{C}, S)$, φ a ground simple formula, and θ a ground term. Then:

- $[\![\varphi]\!]_{\mathbb{A}(\mathcal{C},S)\alpha} = \varphi\alpha$;
- $[\![\theta]\!]_{\mathbb{A}(\mathcal{C},S)\alpha\beta} = \theta\alpha\beta$.

Proof: Straightforward induction on the complexity of simple formula φ and term θ, respectively. QED

Lemma 3.7 Let $\mathcal{C}$ be a structural calculus, S a maximal consistent set of closed sequents, α an unbound variable assignment over $\mathbb{A}(\mathcal{C}, S)$, β a bound variable assignment over $\mathbb{A}(\mathcal{C}, S)$, and δ a ground assertion. Then:

$$\mathbb{A}(\mathcal{C}, S)\alpha\beta \Vdash \delta \quad \text{iff} \quad S \vdash_{\mathcal{C}} \to \delta\alpha\beta\,.$$

Proof:

(i) $\mathbb{A}(\mathcal{C}, S)\alpha\beta \Vdash \Omega\theta$ iff $[\![\theta]\!]_{\mathbb{A}(\mathcal{C},S)\alpha\beta} \in \Omega_{\mathbb{A}(\mathcal{C},S)}$ iff (by Lemma 3.6) $\theta\alpha\beta \in \Omega_{\mathbb{A}(\mathcal{C},S)}$ iff $S \vdash_{\mathcal{C}} \to \Omega(\theta\alpha\beta)$ iff $S \vdash_{\mathcal{C}} \to (\Omega\theta)\alpha\beta$.

(ii) Both $\mathbb{A}(\mathcal{C}, S)\alpha\beta \Vdash \theta \sqsubseteq \theta'$ iff $S \vdash_{\mathcal{C}} \to (\theta \sqsubseteq \theta')\alpha\beta$ and $\mathbb{A}(\mathcal{C}, S)\alpha\beta \Vdash \theta \leq \varphi$ iff $S \vdash_{\mathcal{C}} \to (\theta \leq \varphi)\alpha\beta$ are obtained in a similar way.

(iii) $\mathbb{A}(\mathcal{C}, S)\alpha\beta \Vdash \mho\theta$ iff $[\![\theta]\!]_{\mathbb{A}(\mathcal{C},S)\alpha\beta} \notin \Omega_{\mathbb{A}(\mathcal{C},S)}$ iff (by Lemma 3.6) $\theta\alpha\beta \notin \Omega_{\mathbb{A}(\mathcal{C},S)}$ iff $S \not\vdash_{\mathcal{C}} \to \Omega(\theta\alpha\beta)$ iff (since S is maximal consistent) $S \vdash_{\mathcal{C}} \to \mho(\theta\alpha\beta)$ iff $S \vdash_{\mathcal{C}} \to (\mho\theta)\alpha\beta$.

(iv) Again, both $\mathbb{A}(\mathcal{C}, S)\alpha\beta \Vdash \theta \not\sqsubseteq \theta'$ iff $S \vdash_{\mathcal{C}} \to (\theta \not\sqsubseteq \theta')\alpha\beta$ and $\mathbb{A}(\mathcal{C}, S)\alpha\beta \Vdash \theta \not\leq \varphi$ iff $S \vdash_{\mathcal{C}} \to (\theta \not\leq \varphi)\alpha\beta$ are obtained in a similar way. QED

Lemma 3.8 (Lifting) Let $\mathcal{C}$ be a structural calculus, S a maximal consistent set of closed sequents, α an unbound variable assignment over $\mathbb{A}(\mathcal{C}, S)$, β a bound variable assignment over $\mathbb{A}(\mathcal{C}, S)$, and $\Delta' \to \Delta''$ a ground sequent. Then:

$$\mathbb{A}(\mathcal{C}, S)\alpha\beta \Vdash \Delta' \to \Delta'' \;\text{ iff }\; S \vdash_{\mathcal{C}} \Delta'\alpha\beta \to \Delta''\alpha\beta\,.$$

Proof: Indeed, $\mathbb{A}(\mathcal{C},S)\alpha\beta \Vdash \Delta' \rightarrow \Delta''$ iff $\mathbb{A}(\mathcal{C},S)\alpha\beta \Vdash \delta$ for some $\delta \in \Delta'' \cup \overline{\Delta'}$ iff (Lemma 3.7) $S \vdash_{\mathcal{C}} \rightarrow \delta\alpha\beta$ for some $\delta \in \Delta'' \cup \overline{\Delta'}$ iff $S \vdash_{\mathcal{C}} \rightarrow \delta$ for some $\delta \in \Delta''\alpha\beta \cup \overline{\Delta'}\alpha\beta$ iff (see justification below) $S \vdash_{\mathcal{C}} \rightarrow \Delta''\alpha\beta, \overline{\Delta'}\alpha\beta$ iff (Theorem 2.5), $S \vdash_{\mathcal{C}} \Delta'\alpha\beta \rightarrow \Delta''\alpha\beta$. It remains to explain:

(1) If $S \vdash_{\mathcal{C}} \rightarrow \delta$ for some $\delta \in \Delta''\alpha\beta \cup \overline{\Delta'}\alpha\beta$ then $S \vdash_{\mathcal{C}} \rightarrow \Delta''\alpha\beta, \overline{\Delta'}\alpha\beta$. This fact is trivially obtained by applications of right weakening.

(2) If $S \vdash_{\mathcal{C}} \rightarrow \Delta''\alpha\beta, \overline{\Delta'}\alpha\beta$ then $S \vdash_{\mathcal{C}} \rightarrow \delta$ for some $\delta \in \Delta''\alpha\beta \cup \overline{\Delta'}\alpha\beta$. Indeed, otherwise, since S is maximal consistent, $S \vdash_{\mathcal{C}} \rightarrow \overline{\delta}$ for every $\delta \in \Delta''\alpha\beta \cup \overline{\Delta'}\alpha\beta$. Then, using Theorem 2.6, we would be able to show that every closed assertion is derivable from S, therefore contradicting that S is consistent. QED

Observe that, in the conditions of the previous lemma, if, furthermore, the sequent $\Delta' \rightarrow \Delta''$ is closed then we have:

$$\mathbb{A}(\mathcal{C},S) \Vdash \Delta' \rightarrow \Delta'' \;\text{ iff }\; S \vdash_{\mathcal{C}} \Delta' \rightarrow \Delta'' .$$

Lemma 3.9 (Appropriateness) The class of all syntactic algebras induced by a sequent calculus is appropriate for it.

Proof: Let $r = \frac{s_1 \ldots s_p}{s}$ be a ground instance of a (proper) rule of a sequent calculus $\mathcal{C}$. Let α be an arbitrary unbound variable assignment over a syntactic algebra $\mathbb{A}(\mathcal{C},S)$. Assume that $\mathbb{A}(\mathcal{C},S)\alpha \Vdash s_i$ for each $i = 1,\ldots,p$. That is, for every bound variable assignment β, $\mathbb{A}(\mathcal{C},S)\alpha\beta \Vdash s_i$ for each $i = 1,\ldots,p$. So, by Lemma 3.8, for every such β, $S \vdash_{\mathcal{C}} s_i\alpha\beta$, say with derivation sequence $D^{s_i\alpha\beta}$, for each such i. Then, it is straightforward to build, for every pair α, β, a derivation sequence for $S \vdash_{\mathcal{C}} s\alpha\beta$ using rule r and those derivation sequences. Thus, again by Lemma 3.8, for every such β, $\mathbb{A}(\mathcal{C},S)\alpha\beta \Vdash s$. That is, $\mathbb{A}(\mathcal{C},S)\alpha \Vdash s$. QED

Lemma 3.10 (Consistent extension) Let $\mathcal{C}$ be a structural sequent calculus with rules endowed with persistent provisos. If S is a consistent set of closed sequents and $S \not\vdash_{\mathcal{C}} \rightarrow \upsilon_1,\ldots,\upsilon_m$ for closed assertions $\upsilon_1,\ldots,\upsilon_m$ then the set $S \cup \{\rightarrow\overline{\upsilon_1},\ldots,\rightarrow\overline{\upsilon_m}\}$ is still consistent.

Proof: Assume that $S \cup \{\rightarrow\overline{\upsilon_1},\ldots,\rightarrow\overline{\upsilon_m}\}$ is inconsistent. Then, there is a closed assertion δ such that $S, \rightarrow\overline{\upsilon_1},\ldots,\rightarrow\overline{\upsilon_m} \vdash_{\mathcal{C}} \rightarrow \delta$ and $S, \rightarrow\overline{\upsilon_1},\ldots,\rightarrow\overline{\upsilon_m} \vdash_{\mathcal{C}} \rightarrow \overline{\delta}$. So, using the metatheorem of contradiction (Theorem 2.6),

$$S, \rightarrow\overline{\upsilon_1},\ldots,\rightarrow\overline{\upsilon_m} \vdash_{\mathcal{C}} \rightarrow \upsilon_1 .$$

Therefore, using the metatheorem of deduction (Theorem 2.7),

$$S \vdash_{\mathcal{C}} \overline{\upsilon_1},\ldots,\overline{\upsilon_m} \rightarrow \upsilon_1 .$$

Thus, applying the metatheorem of conjugation (Theorem 2.5), we get

$$S \vdash_{\mathcal{C}} \rightarrow \upsilon_1, \upsilon_1, \upsilon_2, \ldots, \upsilon_m$$

and, by right contraction,

$$S \vdash_{\mathcal{C}} \rightarrow \upsilon_1, \ldots, \upsilon_m$$

which contradicts the second hypothesis. QED

Theorem 3.11 (Algebraic completeness) Every full structural sequent logic with rules endowed with persistent provisos is complete.

Proof: Consider the logic $\mathcal{L} = \langle \Sigma, \mathcal{R}, \mathcal{A} \rangle$ and let $\mathcal{C} = \langle \Sigma, \mathcal{R} \rangle$. Assume that $S \nvdash_{\mathcal{R}} \Delta' \rightarrow \Delta''$ with $S \cup \{\Delta' \rightarrow \Delta''\}$ composed of closed sequents.
Given an enumeration υ_n with $n \in \mathbb{N}$ of the set of closed assertions, we start by extending S to a maximal consistent set $S^\bullet$ as follows:

- $S_0 = S \cup \{\rightarrow \delta : \delta \in \overline{\Delta''} \cup \Delta'\}$;
- $S_{n+1} = \begin{cases} S \cup \{\rightarrow \upsilon_n\} & \text{provided that } \; S_n \vdash_{\mathcal{R}} \rightarrow \upsilon_n \\ S \cup \{\rightarrow \overline{\upsilon_n}\} & \text{otherwise} \end{cases}$;
- $S^\bullet = \displaystyle\bigcup_{n \in \mathbb{N}} S_n$.

Observe that $S^\bullet$ is still consistent thanks to Lemma 3.10. Furthermore, by construction, it is maximal consistent. Therefore, $S^\bullet \nvdash_{\mathcal{R}} \Delta' \rightarrow \Delta''$ because otherwise $S^\bullet \vdash_{\mathcal{R}} \rightarrow \delta$ for some $\delta \in \Delta'' \cup \overline{\Delta'}$ (using the same reasoning as in justification (2) in the proof of Lemma 3.8) and, hence, $S^\bullet$ would be inconsistent. Thus, by Lemma 3.8 applied to a closed sequent, $\mathbb{A}(\mathcal{C}, S^\bullet) \nVdash \Delta' \rightarrow \Delta''$.
On the other hand, for every $s \in S$ we know that $S \vdash_{\mathcal{R}} s$ and, thus, again thanks to Lemma 3.8, $\mathbb{A}(\mathcal{C}, S^\bullet) \Vdash s$.
Since the logic is full and taking into account Lemma 3.9, $\mathbb{A}(\mathcal{C}, S^\bullet)$ is in $\mathcal{A}$. Hence, $S \nvDash_{\mathcal{A}} \Delta' \rightarrow \Delta''$. QED

Corollary 3.12 (Modal algebraic completeness) Within the context of a full modal sequent logic $\mathcal{L} = \langle \Sigma_K, \mathcal{R}, \mathcal{A} \rangle$:

1. $\psi_1, \ldots, \psi_k \vdash^{\mathrm{g}}_{\mathcal{R}} \varphi$ iff $\psi_1, \ldots, \psi_k \vDash^{\mathrm{g}}_{\mathcal{A}} \varphi$;
2. $\psi_1, \ldots, \psi_k \vdash^{\ell}_{\mathcal{R}} \varphi$ iff $\psi_1, \ldots, \psi_k \vDash^{\ell}_{\mathcal{A}} \varphi$.

Proof:
1. $\psi_1, \ldots, \psi_k \vdash^{\mathrm{g}}_{\mathcal{R}} \varphi$ iff (by definition) $\vdash_{\mathcal{R}} \top \leq \psi_1, \ldots, \top \leq \psi_k \rightarrow \top \leq \varphi$ iff ($\Rightarrow$: since $\mathcal{L}$ is full and therefore sound; $\Leftarrow$: thanks to the completeness theorem above, since we are dealing with closed sequents and modal logics are assumed to use only persistent provisos) $\vDash_{\mathcal{A}} \top \leq \psi_1, \ldots, \top \leq \psi_k \rightarrow \top \leq \varphi$ iff (by definition) $\psi_1, \ldots, \psi_k \vDash^{\mathrm{g}}_{\mathcal{A}} \varphi$.

2. $\psi_1, \ldots, \psi_k \vdash^{\ell}_{\mathcal{R}} \varphi$ iff (by Proposition 2.4) $\vdash^{\mathrm{g}}_{\mathcal{R}} (\psi_1 \wedge \ldots \wedge \psi_k) \Rightarrow \varphi$ iff (by 1.) $\vDash^{\mathrm{g}}_{\mathcal{A}} (\psi_1 \wedge \ldots \wedge \psi_k) \Rightarrow \varphi$ iff (by Proposition 3.5) $\psi_1, \ldots, \psi_k \vDash^{\ell}_{\mathcal{A}} \varphi$. QED

3.3 Kripke completeness

We now turn our attention to the traditional semantics of modal logic (based on, possibly general, Kripke structures). Namely, it is worthwhile to analyze the class of Kripke structures characterized by a given set of rules. More precisely, we would like to prove that, for instance, rule T does characterize the reflexive frames. It would also be nice to establish soundness and completeness results over general Kripke semantics by capitalizing on the algebraic completeness theorem proved in the previous subsection.

Given a suitable choice function needed for interpreting **I**, it is straightforward to extract from any (possibly general) Kripke structure a Σ_K-algebra while respecting the denotation of ground simple formulae. Recall that $\Sigma_K = \langle C, O, X, Y, Z\rangle$ is the modal signature introduced in Subsection 2.5.

Let $\mathbb{K} = \langle W, \rightsquigarrow, \mathcal{B}, V\rangle$ be a *general Kripke structure* over C where W is the non-empty set of worlds, $\rightsquigarrow$ is the accessibility relation between worlds, $\mathcal{B} \subseteq \wp W$ is the set of admissible truth values, and the valuation V maps each propositional symbol $\mathbf{p}_i$ to an admissible truth value. Let ι be a choice function for W. Then, the Σ_K-algebra $Alg(\mathbb{K}) = \langle F, T, \cdot_{Alg(\mathbb{K})}\rangle$ induced by $\mathbb{K}$ is as follows:

- $F = \mathcal{B}$;
- $T = \wp W$;
- $\#_{Alg(\mathbb{K})} = \lambda b.\ b$;
- $a \in \Omega_{Alg(\mathbb{K})}$ iff a is a singleton;
- $\langle a, a'\rangle \in \sqsubseteq_{Alg(\mathbb{K})}$ iff $a \subseteq a'$;
- $\langle a, b\rangle \in \leq_{Alg(\mathbb{K})}$ iff $a \subseteq b$;
- $\bot_{Alg(\mathbb{K})} = \emptyset$;
- $\top_{Alg(\mathbb{K})} = W$;
- $\mathbf{I}_{Alg(\mathbb{K})} = \lambda a.\ \iota(a)$;
- $\mathbf{N}_{Alg(\mathbb{K})} = \lambda a.\ \{w' \in W :\ \text{exists } w \in a \text{ such that } w \rightsquigarrow w'\}$;
- $\mathbf{lb}_{Alg(\mathbb{K})} = \lambda a a'.\ a \cap a'$;
- $\mathbf{f}_{Alg(\mathbb{K})} = \emptyset$;
- $\mathbf{t}_{Alg(\mathbb{K})} = W$;
- $\mathbf{p}_{i Alg(\mathbb{K})} = V(\mathbf{p}_i)$;
- $\neg_{Alg(\mathbb{K})} = \lambda b.\ W \setminus b$;
- $\Box_{Alg(\mathbb{K})} = \lambda b.\ \{w \in W : \mathbf{N}_{Alg(\mathbb{K})}(\{w\}) \subseteq b\}$;
- $\Diamond_{Alg(\mathbb{K})} = \lambda b.\ \{w \in W : \mathbf{N}_{Alg(\mathbb{K})}(\{w\}) \cap b \neq \emptyset\}$;
- $\wedge_{Alg(\mathbb{K})} = \lambda b b'.\ b \cap b'$;
- $\vee_{Alg(\mathbb{K})} = \lambda b b'.\ b \cup b'$;
- $\Rightarrow_{Alg(\mathbb{K})} = \lambda b b'.\ (W \setminus b) \cup b'$.

It is straightforward to verify the following facts that will allow us later on to concentrate on the semantics of $\mathbf{f}, \mathbf{p}_i, \neg, \Box, \vee$.

- $\mathbf{t}_{Alg(\mathbb{K})} = \neg_{Alg(\mathbb{K})}(\mathbf{f}_{Alg(\mathbb{K})})$;
- $\Diamond_{Alg(\mathbb{K})}(b) = \neg_{Alg(\mathbb{K})}(\Box_{Alg(\mathbb{K})}(\neg_{Alg(\mathbb{K})}(b)))$;
- $\Rightarrow_{Alg(\mathbb{K})}(b, b') = \vee_{Alg(\mathbb{K})}(\neg_{Alg(\mathbb{K})}(b), b')$;
- $\wedge_{Alg(\mathbb{K})}(b, b') = \neg_{Alg(\mathbb{K})}(\vee_{Alg(\mathbb{K})}(\neg_{Alg(\mathbb{K})}(b), \neg_{Alg(\mathbb{K})}(b')))$.

The following results show that the semantics of closed simple formulae is preserved when we move from a general Kripke structure to the corresponding Σ_K-algebra. In the sequel we use $[\![\varphi]\!]_{\mathbb{K}}$ for the denotation of φ over the general Kripke structure $\mathbb{K}$. This denotation is inductively defined over the structure of φ in the usual way.

Lemma 3.13 Let $\mathbb{K}$ be a general Kripke structure. Then, for every closed simple formula φ, $[\![\varphi]\!]_{\mathbb{K}} = [\![\varphi]\!]_{Alg(\mathbb{K})}$.

Proof:

The proof is carried out by induction on the complexity of φ:

(Base) We have to consider only two representative cases:

(i) φ is $\mathbf{f}$. Then:

$$[\![\mathbf{f}]\!]_{\mathbb{K}} = \emptyset = \mathbf{f}_{Alg(\mathbb{K})} = [\![\mathbf{f}]\!]_{Alg(\mathbb{K})} .$$

(ii) φ is $\mathbf{p}_i$. Then:

$$[\![p_i]\!]_{\mathbb{K}} = V(p_i) = p_{i Alg(\mathbb{K})} = [\![p_i]\!]_{Alg(\mathbb{K})} .$$

(Step) We have to consider only three representative cases:

(i) φ is $\neg\,\varphi'$. Then:

$$[\![\neg\,\varphi']\!]_{\mathbb{K}} = W \setminus [\![\varphi']\!]_{\mathbb{K}} = W \setminus [\![\varphi']\!]_{Alg(\mathbb{K})} = \neg_{Alg(\mathbb{K})}([\![\varphi']\!]_{Alg(\mathbb{K})}) = [\![\neg\,\varphi']\!]_{Alg(\mathbb{K})} .$$

(ii) φ is $\Box\varphi'$. Then:

$$[\![\Box\varphi']\!]_{\mathbb{K}} = \{w \in W : w \rightsquigarrow w' \text{ implies } w' \in [\![\varphi']\!]_{\mathbb{K}} \text{ for every } w' \in W\} = \{w \in W : w' \in \mathbf{N}_{Alg(\mathbb{K})}(\{w\}) \text{ implies } w' \in [\![\varphi']\!]_{Alg(\mathbb{K})} \text{ for every } w' \in W\} = \{w \in W : \mathbf{N}_{Alg(\mathbb{K})}(\{w\}) \subseteq [\![\varphi']\!]_{Alg(\mathbb{K})}\} = \Box_{Alg(\mathbb{K})}([\![\varphi']\!]_{Alg(\mathbb{K})}) = [\![\Box\varphi']\!]_{Alg(\mathbb{K})} .$$

(iii) φ is $\varphi' \vee \varphi''$. Then:

$$[\![\varphi' \vee \varphi'']\!]_{\mathbb{K}} = [\![\varphi']\!]_{\mathbb{K}} \cup [\![\varphi'']\!]_{\mathbb{K}} = [\![\varphi']\!]_{Alg(\mathbb{K})} \cup [\![\varphi'']\!]_{Alg(\mathbb{K})} = \vee_{Alg(\mathbb{K})}([\![\varphi']\!]_{Alg(\mathbb{K})}, [\![\varphi'']\!]_{Alg(\mathbb{K})}) = [\![\varphi' \vee \varphi'']\!]_{Alg(\mathbb{K})} .$$

QED

Proposition 3.14 Given a general Kripke structure $\mathbb{K}$ and a closed simple formula φ, $\mathbb{K} \Vdash \varphi$ iff $Alg(\mathbb{K}) \Vdash \top \leq \varphi$.

Proof:

$\mathbb{K} \Vdash \varphi$ iff $[\![\varphi]\!]_{\mathbb{K}} = W$ iff $W \subseteq [\![\varphi]\!]_{\mathbb{K}}$ iff $\top_{Alg(\mathbb{K})} \subseteq [\![\varphi]\!]_{Alg(\mathbb{K})}$ iff $[\![\top]\!]_{Alg(\mathbb{K})} \subseteq [\![\varphi]\!]_{Alg(\mathbb{K})}$ iff $\langle [\![\top]\!]_{Alg(\mathbb{K})}, [\![\varphi]\!]_{Alg(\mathbb{K})} \rangle \in \leq_{Alg(\mathbb{K})}$ iff $Alg(\mathbb{K}) \Vdash \top \leq \varphi$. QED

Proposition 3.15 Given a general Kripke structure $\mathbb{K}$, an unbound variable assignment β over $Alg(\mathbb{K})$ such that $\beta(\mathbf{y}_1) = \{w\}$ and a ground simple formula φ, $\mathbb{K}w \Vdash \varphi$ iff $Alg(\mathbb{K})\beta \Vdash \mathbf{y}_1 \leq \varphi$.

Proof:
$\mathbb{K}w \Vdash \varphi$ iff $w \in [\![\varphi]\!]_{\mathbb{K}}$ iff $\{w\} \subseteq [\![\varphi]\!]_{\mathbb{K}}$ iff $\beta(\mathbf{y}_1) \subseteq [\![\varphi]\!]_{Alg(\mathbb{K})}$ iff $[\![\mathbf{y}_1]\!]_{Alg(\mathbb{K})\beta} \subseteq [\![\varphi]\!]_{Alg(\mathbb{K})}$ iff $\langle [\![\mathbf{y}_1]\!]_{Alg(\mathbb{K})\beta}, [\![\varphi]\!]_{Alg(\mathbb{K})} \rangle \in \leq_{Alg(\mathbb{K})}$ iff $Alg(\mathbb{K}) \Vdash \mathbf{y}_1 \leq \varphi$. QED

Given a class $\mathcal{K}$ of general Kripke structures, let $Alg(\mathcal{K})$ be the class $\{Alg(\mathbb{K}) : \mathbb{K} \in \mathcal{K}\}$, and $\vDash^{\mathrm{g}}_{\mathcal{K}}$, $\vDash^{\ell}_{\mathcal{K}}$ be the global, local entailment over $\mathcal{K}$, respectively.

Theorem 3.16 Given a class $\mathcal{K}$ of general Kripke structures:

1. $\psi_1, \ldots, \psi_k \vDash^{\mathrm{g}}_{\mathcal{K}} \varphi$ iff $\psi_1, \ldots, \psi_k \vDash^{\mathrm{g}}_{Alg(\mathcal{K})} \varphi$;
2. $\psi_1, \ldots, \psi_k \vDash^{\ell}_{\mathcal{K}} \varphi$ iff $\psi_1, \ldots, \psi_k \vDash^{\ell}_{Alg(\mathcal{K})} \varphi$.

Proof: Without loss of generality consider $k = 2$:

1. $\psi_1, \psi_2 \vDash^{\mathrm{g}}_{\mathcal{K}} \varphi$ iff, for every $\mathbb{K} \in \mathcal{K}$, $\mathbb{K} \Vdash \varphi$ whenever $\mathbb{K} \Vdash \psi_1$ and $\mathbb{K} \Vdash \psi_2$ iff (thanks to Proposition 3.14), for every $\mathbb{A} \in Alg(\mathcal{K})$, $\mathbb{A} \Vdash \top \leq \varphi$ whenever $\mathbb{A} \Vdash \top \leq \psi_1$ and $\mathbb{A} \Vdash \top \leq \psi_2$ iff $\psi_1, \psi_2 \vDash^{\mathrm{g}}_{Alg(\mathcal{K})} \varphi$.

2. $\psi_1, \psi_2 \vDash^{\ell}_{\mathcal{K}} \varphi$ iff, for every $\mathbb{K} \in \mathcal{K}$ and $w \in W$, $\mathbb{K}w \Vdash \varphi$ whenever $\mathbb{K}w \Vdash \psi_1$ and $\mathbb{K}w \Vdash \psi_2$ iff (thanks to Proposition 3.15), for every $\mathbb{A} \in Alg(\mathcal{K})$ and every assignment β such that $\beta(\mathbf{y}_1) = \{w\}$, $\mathbb{A}\beta \Vdash \mathbf{y}_1 \leq \varphi$ whenever $\mathbb{A}\beta \Vdash \mathbf{y}_1 \leq \psi_1$ and $\mathbb{A}\beta \Vdash \mathbf{y}_1 \leq \psi_2$ iff for every $\mathbb{A} \in Alg(\mathcal{K})$ and every assignment β, $\mathbb{A}\beta \Vdash \mathbf{y}_1 \leq \varphi$ whenever $\mathbb{A}\beta \Vdash \mathbf{y}_1 \leq \psi_1$, $\mathbb{A}\beta \Vdash \mathbf{y}_1 \leq \psi_2$ and $\mathbb{A}\beta \Vdash \Omega\mathbf{y}_1$ iff $\psi_1, \psi_2 \vDash^{\mathrm{g}}_{Alg(\mathcal{K})} \varphi$. QED

Theorem 3.17 For every general Kripke structure $\mathbb{K}$, the Σ_K-algebra $Alg(\mathbb{K})$ is appropriate for each rule in $\mathcal{R}_K$.

Proof:

(i) The algebra $Alg(\mathbb{K})$ is appropriate for each structural rule thanks to Theorem 3.3.

(ii) The algebra $Alg(\mathbb{K})$ is appropriate for each order rule in Subsection 2.4 because inclusion does fulfill the properties imposed by those rules.

For instance, consider rule RgenF. Let ρ be a ground substitution such that $(\tau_2 : \mathbf{y})(\rho) = 1$ and $(\tau_2 \notin \tau_1, \Gamma_1, \Gamma_2)(\rho) = 1$. So $\tau_2\rho$ is a variable, say $\mathbf{y}_i$, that is fresh.

Let α be an arbitrary unbound variable assignment over $Alg(\mathbb{K})$. Assume that the pair $Alg(\mathbb{K})\alpha$ satisfies the premise $\Omega\mathbf{y}_i, \mathbf{y}_i \sqsubseteq \tau_1\rho, \Gamma_1\rho \to \Gamma_2\rho, \mathbf{y}_i \leq \xi_1\rho$. We have to prove that the pair $Alg(\mathbb{K})\alpha$ satisfies the conclusion $\Gamma_1\rho \to \Gamma_2\rho, \tau_1\rho \leq \xi_1\rho$ where $\mathbf{y}_i$ does not occur.

Since $Alg(\mathbb{K})\alpha$ satisfies the premise, we know that, for every bound variable assignment β, there is δ_β in $\Gamma_2\rho \cup \{\mathbf{y}_i \leq \xi_1\rho\} \cup \{\mho\mathbf{y}_i, \mathbf{y}_i \not\sqsubseteq \tau_1\rho\} \cup \overline{\Gamma_1\rho}$ such that the triple $Alg(\mathbb{K})\alpha\beta$ satisfies δ_β.

For each β we have to consider the following two cases:

(a) there is δ_β in $\Gamma_2\rho \cup \overline{\Gamma_1\rho}$ such that $Alg(\mathbb{K})\alpha\beta \Vdash \delta_\beta$ which immediately establishes that $Alg(\mathbb{K})\alpha\beta$ satisfies $\Gamma_1\rho \to \Gamma_2\rho$ and, hence, the conclusion of the rule.

(b) Otherwise, we know that there is δ_β in $\{\mathbf{y}_i \leq \xi_1\rho\} \cup \{\mho\mathbf{y}_i, \mathbf{y}_i \not\sqsubseteq \tau_1\rho\}$ such that $Alg(\mathbb{K})\alpha\beta \Vdash \delta_\beta$. Furthermore, we also know that, for every assignment η $\mathbf{y}_i$-equivalent to β, there is δ_η in $\Gamma_2\rho \cup \{\mathbf{y}_i \leq \xi_1\rho\} \cup \{\mho\mathbf{y}_i, \mathbf{y}_i \not\sqsubseteq \tau_1\rho\} \cup \overline{\Gamma_1\rho}$ such that $Alg(\mathbb{K})\alpha\eta \Vdash \delta_\eta$. Moreover, since $\mathbf{y}_i$ does not occur in $\Gamma_2\rho \cup \overline{\Gamma_1\rho}$, $Alg(\mathbb{K})\alpha\eta \nVdash \to \Gamma_2\rho, \overline{\Gamma_1\rho}$. So, for every such η, there is δ_η in $\{\mathbf{y}_i \leq \xi_1\rho\} \cup \{\mho\mathbf{y}_i, \mathbf{y}_i \not\sqsubseteq \tau_1\rho\}$ such that $Alg(\mathbb{K})\alpha\eta \Vdash \delta_\eta$. In particular, for every η $\mathbf{y}_i$-equivalent to β and such that $\eta(\mathbf{y}_i)$ is a singleton and is included in $[\![\tau_1\rho]\!]_{Alg(\mathbb{K})\beta}$, we know that $Alg(\mathbb{K})\alpha\eta$ satisfies $\mathbf{y}_i \leq \xi_1\rho$, that is, we know that $\eta(\mathbf{y}_i)$ is included in $[\![\xi_1\rho]\!]_{Alg(\mathbb{K})\alpha}$. Therefore, $[\![\tau_1\rho]\!]_{Alg(\mathbb{K})\alpha\beta}$ is included in $[\![\xi_1\rho]\!]_{Alg(\mathbb{K})\alpha}$. So, $Alg(\mathbb{K})\alpha\beta$ satisfies $\to \tau_1\rho \leq \xi_1\rho$ and, hence, the conclusion of the rule.

(iii) It remains to check that $Alg(\mathbb{K})$ is appropriate for each specific rule of the modal calculus K given in Subsection 2.5.

For instance, consider rule R$\Diamond$. Let ρ be any ground substitution. Assume that $Alg(\mathbb{K})\alpha$ satisfies the premises of the rule $\Omega\tau_1\rho, \Omega\tau_2\rho, \Gamma_1\rho \to \Gamma_2\rho, \tau_2\rho \sqsubseteq \mathbf{N}(\tau_1\rho)$, $\Omega\tau_1\rho, \Omega\tau_2\rho, \Gamma_1\rho \to \Gamma_2\rho, \tau_2\rho \leq \xi_1\rho$ and $\Omega\tau_1\rho, \Gamma_1\rho \to \Gamma_2\rho, \Omega\tau_2\rho$. We have to prove that $Alg(\mathbb{K})\alpha$ satisfies the conclusion $\Omega\tau_1\rho, \Gamma_1\rho \to \Gamma_2\rho, \tau_1\rho \leq (\Diamond\xi_1\rho)$.

Since $Alg(\mathbb{K})\alpha$ satisfies the premises, we know that, for every bound variable assignment β, there are:

- δ_1 in $\Gamma_2\rho \cup \{\tau_2\rho \sqsubseteq \mathbf{N}(\tau_1\rho)\} \cup \{\mho\tau_1\rho, \mho\tau_2\rho\} \cup \overline{\Gamma_1\rho}$ such that $Alg(\mathbb{K})\alpha\beta \Vdash \delta_1$;
- δ_2 in $\Gamma_2\rho \cup \{\tau_2\rho \leq \xi_1\rho\} \cup \{\mho\tau_1\rho, \mho\tau_2\rho\} \cup \overline{\Gamma_1\rho}$ such that $Alg(\mathbb{K})\alpha\beta \Vdash \delta_2$;
- and δ_3 in $\Gamma_2\rho \cup \{\Omega\tau_2\rho\} \cup \{\mho\tau_1\rho\} \cup \overline{\Gamma_1\rho}$ such that $Alg(\mathbb{K})\alpha\beta \Vdash \delta_3$.

For each β, we have to consider two cases:

(a) $Alg(\mathbb{K})\alpha\beta \Vdash \delta_i$ with δ_i in $\Gamma_2\rho \cup \{\mho\tau_1\rho\} \cup \overline{\Gamma_1\rho}$ for some $i = 1, \ldots, 3$ which immediately establishes the conclusion of the rule.

(b) Otherwise, we know that:

- $Alg(\mathbb{K})\alpha\beta \Vdash \Omega\tau_1\rho$;
- $Alg(\mathbb{K})\alpha\beta \Vdash \delta_1$ for some δ_1 in $\{\tau_2\rho \sqsubseteq \mathbf{N}(\tau_1\rho), \mho\tau_2\rho\}$;
- $Alg(\mathbb{K})\alpha\beta \Vdash \delta_2$ for some δ_2 in $\{\tau_2\rho \leq \xi_1\rho, \mho\tau_2\rho\}$;
- $Alg(\mathbb{K})\alpha\beta \Vdash \delta_3$ for some δ_3 in $\{\Omega\tau_2\rho\}$, that is, $Alg(\mathbb{K})\beta \Vdash \Omega\tau_2\rho$.

That is:

- $Alg(\mathbb{K})\alpha\beta \Vdash \Omega\tau_1\rho$;
- $Alg(\mathbb{K})\alpha\beta \Vdash \tau_2\rho \sqsubseteq \mathbf{N}(\tau_1\rho)$;
- $Alg(\mathbb{K})\alpha\beta \Vdash \tau_2\rho \leq \xi_1\rho$;

- $Alg(\mathbb{K})\alpha\beta \Vdash \Omega\tau_2\rho$.

Let w_1, w_2 be such that $[\![\tau_1\rho]\!]_{Alg(\mathbb{K})\alpha\beta} = \{w_1\}$ and $[\![\tau_2\rho]\!]_{Alg(\mathbb{K})\alpha\beta} = \{w_2\}$. Thus:

- $\{w_2\} \subseteq \mathbf{N}_{Alg(\mathbb{K})}(\{w_1\})$;
- $\{w_2\} \subseteq [\![\xi_1\rho]\!]_{Alg(\mathbb{K})\alpha}$.

Hence, $\mathbf{N}_{Alg(\mathbb{K})}(\{w_1\}) \cap [\![\xi_1\rho]\!]_{Alg(\mathbb{K})\alpha} \neq \emptyset$. So, $\{w_1\} \subseteq \{w \in W : \mathbf{N}_{Alg(\mathbb{K})}(\{w\}) \cap [\![\xi_1\rho]\!]_{Alg(\mathbb{K})\alpha} \neq \emptyset\}$. That is, $[\![\tau_1\rho]\!]_{Alg(\mathbb{K})\alpha} \subseteq \Diamond_{Alg(\mathbb{K})}([\![\xi_1\rho]\!]_{Alg(\mathbb{K})\alpha})$ which establishes that $Alg(\mathbb{K})\alpha\beta \Vdash \tau_1\rho \leq (\Diamond\xi_1\rho)$ and, so, the conclusion of the rule. QED

In order to state and prove the envisaged characterization results, we need some notation:

- P denotes a property of the accessibility relation among those considered in Subsection 2.6;
- r_P denotes the corresponding sequent rule as indicated in Subsection 2.6;
- $\mathcal{K}_P$ denotes the class of all general Kripke structures fulfilling P;
- $\mathcal{C}_P$ denotes the sequent modal calculus with the extra rule r_P;
- $\text{app}(\mathcal{C}_P)$ denotes the class of algebras appropriate for $\mathcal{C}_P$.

We also extend this notation to any finite set $\mathcal{P}$ of such properties in the obvious way.

Theorem 3.18 (Characterization) For each property P of the accessibility relation and each general Kripke structure $\mathbb{K}$:

$$\mathbb{K} \in \mathcal{K}_P \quad \text{iff} \quad Alg(\mathbb{K}) \in \text{app}(\mathcal{C}_P)\,.$$

Proof: The proof is straightforward for each of the properties in Subsection 2.6. We provide the details only for two cases:

(rule X) Let P state that the relation is *irreflexive*. Then: $\mathbb{K} \in \mathcal{K}_P$ iff $w \not\rightarrow w$ for every $w \in W$ iff $\{w\} \not\subseteq \mathbf{N}_{Alg(\mathbb{K})}(w)$ for every $w \in W$. We have to show that the latter holds iff, for every $\Gamma_1, \Gamma_2, \rho, \alpha, \beta$, $Alg(\mathbb{K})\alpha\beta \Vdash \Omega\tau_1\rho, \Gamma_1\rho \rightarrow \Gamma_2\rho, \tau_1\rho \not\sqsubseteq \mathbf{N}(\tau_1\rho)$.

($\Rightarrow$) Assume that, for every $w \in W$, $\{w\} \not\subseteq \mathbf{N}_{Alg(\mathbb{K})}(w)$. It is sufficient to show that $Alg(\mathbb{K})\alpha\beta \Vdash \Omega\tau_1\rho \rightarrow \tau_1\rho \not\sqsubseteq \mathbf{N}(\tau_1\rho)$, that is, $Alg(\mathbb{K})\alpha\beta \Vdash \rightarrow \tau_1\rho \not\sqsubseteq \mathbf{N}(\tau_1\rho), \mho\tau_1\rho$. We have to consider two cases:
(a) $Alg(\mathbb{K})\alpha\beta \Vdash \mho\tau_1\rho$ which immediately establishes the result.
(b) Otherwise, we know $Alg(\mathbb{K})\alpha\beta \Vdash \Omega\tau_1\rho$ and, hence, that $[\![\tau_1\rho]\!]_{Alg(\mathbb{K})\alpha\beta}$ is a singleton. So, from the hypothesis, we get $[\![\tau_1\rho]\!]_{Alg(\mathbb{K})\alpha\beta} \not\subseteq \mathbf{N}_{Alg(\mathbb{K})}([\![\tau_1\rho]\!]_{Alg(\mathbb{K})\alpha\beta})$. Hence, $Alg(\mathbb{K})\alpha\beta \Vdash \tau_1\rho \not\sqsubseteq \mathbf{N}(\tau_1\rho)$.

($\Leftarrow$) Assume that, for every $\Gamma_1, \Gamma_2, \rho, \alpha, \beta$, $Alg(\mathbb{K})\alpha\beta \Vdash \Omega\tau_1\rho, \Gamma_1\rho \rightarrow \Gamma_2\rho, \tau_1\rho \not\sqsubseteq \mathbf{N}(\tau_1\rho)$. Then, for each $w \in W$, choose $\Gamma_1 = \Gamma_2 = \emptyset$, ρ such that $\rho(\tau_1) = \mathbf{x}_1$, and α such that $\alpha(\mathbf{x}_1) = \{w\}$. Therefore, $Alg(\mathbb{K})\alpha \Vdash \tau_1\rho \not\sqsubseteq \mathbf{N}(\tau_1\rho)$ and, so, $\{w\} \not\subseteq \mathbf{N}_{Alg(\mathbb{K})}(w)$.

(rule W) Let P state that the relation is *transitive* and *well bounded*.

($\Rightarrow$) Assume that the relation is transitive and well bounded and that rule W does not hold. Then, there is ρ fulfilling the proviso and an assignment α such that $Alg(\mathbb{K})\alpha\beta \Vdash \mho\tau_1\rho, \mho\tau_3\rho, \tau_3\rho \sqsubseteq \mathbf{N}(\tau_1\rho), \Gamma_1\rho \to \Gamma_2\rho, \tau_1\rho \sqsubseteq \tau_2\rho, \mathbf{N}(\tau_3\rho) \not\sqsubseteq \tau_2\rho$ for all assignments β and $Alg(\mathbb{K})\alpha \not\Vdash \mho\tau_1\rho, \Gamma_1\rho \to \Gamma_2\rho, \mathbf{N}(\tau_1\rho) \sqsubseteq \tau_2\rho$. So, $Alg(\mathbb{K})\alpha \Vdash \tau_1\rho$, $Alg(\mathbb{K})\alpha \Vdash \Gamma_1\rho$, $Alg(\mathbb{K})\alpha \not\Vdash \gamma_2\rho$ for every $\gamma_2\rho \in \Gamma_2\rho$ and $Alg(\mathbb{K})\alpha \not\Vdash \mathbf{N}(\tau_1\rho) \sqsubseteq \tau_2$. So, there is w such that $\{w\} \subseteq \mathbf{N}_{Alg(\mathbb{K})}([\![\tau_1\rho]\!]_{Alg(\mathbb{K})\alpha})$ and $\{w\} \not\subseteq [\![\tau_2\rho]\!]_{Alg(\mathbb{K})\alpha}$. Since the relation is transitive and well bounded, we can choose a maximal w fulfilling the previous conditions, that is, if $\{w'\} \subseteq \mathbf{N}_{Alg(\mathbb{K})}(\{w\})$ then either $\{w'\} \not\subseteq \mathbf{N}_{Alg(\mathbb{K})}([\![\tau_1\rho]\!]_{Alg(\mathbb{K})\alpha})$ or $\{w'\} \subseteq [\![\tau_2\rho]\!]_{Alg(\mathbb{K})\alpha}$. Consider β such that $[\![\tau_3\rho]\!]_{Alg(\mathbb{K})\alpha\beta} = \{w\}$, then $Alg(\mathbb{K})\alpha\beta \Vdash \tau_3\rho \sqsubseteq \mathbf{N}(\tau_1\rho)$, $Alg(\mathbb{K})\alpha\beta \Vdash \mho\tau_1\rho$, $Alg(\mathbb{K})\alpha\beta \Vdash \mho\tau_3\rho$ and $Alg(\mathbb{K})\alpha\beta \Vdash \Gamma_1\rho$. On the other hand, $Alg(\mathbb{K})\alpha\beta \not\Vdash \gamma_2\rho$ for all $\gamma_2\rho \in \Gamma_2\rho$ and $Alg(\mathbb{K})\alpha\beta \not\Vdash \tau_3\rho \sqsubseteq \tau_2\rho$. So it remains to show that $Alg(\mathbb{K})\alpha\beta \not\Vdash \mathbf{N}(\tau_3\rho) \not\sqsubseteq \tau_2\rho$. Assume that $Alg(\mathbb{K})\alpha\beta \Vdash \mathbf{N}(\tau_3\rho) \not\sqsubseteq \tau_2\rho$, then there is a $\{w'\} \subseteq \mathbf{N}_{Alg(\mathbb{K})}(\{w\})$ such that $\{w'\} \not\subseteq [\![\tau_2\rho]\!]_{Alg(\mathbb{K})\alpha}$. By transitivity $\{w'\} \subseteq \mathbf{N}_{Alg(\mathbb{K})}([\![\tau_1\rho]\!]_{Alg(\mathbb{K})\alpha})$ and so, since $\{w'\} \not\subseteq [\![\tau_2\rho]\!]_{Alg(\mathbb{K})\alpha}$, w would not be maximal. Hence, $Alg(\mathbb{K})\alpha\beta \not\Vdash \mathbf{N}(\tau_3\rho) \not\sqsubseteq \tau_2\rho$. Therefore there is β such that the premise of W is not fulfilled, which contradicts our hypothesis.

($\Leftarrow$) Assume that rule W holds.

(a) Assume that the relation is not transitive. Then, there are w_1, w_2, w_3 such that $\{w_2\} \subseteq \mathbf{N}_{Alg(\mathbb{K})}(w_1)$, $\{w_3\} \subseteq \mathbf{N}_{Alg(\mathbb{K})}(w_2)$ and $\{w_3\} \not\subseteq \mathbf{N}_{Alg(\mathbb{K})}(w_1)$. Consider the ground substitution ρ such that $\Gamma_1\rho = \Gamma_2\rho = \emptyset$, $\tau_1\rho = \mathbf{x}_1$, $\tau_2\rho = \mathbf{x}_2$ and $\tau_3\rho = \mathbf{y}_1$. Let α be an assignment such that $\alpha(\mathbf{x}_1) = \{w_1\}$ and $\alpha(\mathbf{x}_2) = W \setminus \{w_1, w_2, w_3\}$. Then $Alg(\mathbb{K})\alpha \not\Vdash \mho\mathbf{x}_1$ and $Alg(\mathbb{K})\alpha \not\Vdash \mathbf{N}(\mathbf{x}_1) \sqsubseteq \mathbf{x}_2$ and so $Alg(\mathbb{K})\alpha \not\Vdash \mho\mathbf{x}_1 \to \mathbf{N}(\mathbf{x}_1) \sqsubseteq \mathbf{x}_2$. Next we show that $Alg(\mathbb{K})\alpha \Vdash \mho\mathbf{x}_1, \mho\mathbf{y}_1, \mathbf{y}_1 \sqsubseteq \mathbf{N}(\mathbf{x}_1) \to \mathbf{y}_1 \sqsubseteq \mathbf{x}_2, \mathbf{N}(\mathbf{y}_1) \not\sqsubseteq \mathbf{x}_2$ by considering all possibilities for $\beta(\mathbf{y}_1)$: (i) if $\beta(\mathbf{y}_1)$ is not a singleton then the result follows straightforwardly; (ii) if $\beta(\mathbf{y}_1) = \{w_2\}$ then $Alg(\mathbb{K})\alpha\beta \Vdash \mathbf{N}(\mathbf{y}_1) \not\sqsubseteq \mathbf{x}_2$; (iii) if $\beta(\mathbf{y}_1) = \{w_1\}$ or $\beta(\mathbf{y}_1) = \{w_3\}$ then $Alg(\mathbb{K})\alpha\beta \not\Vdash \mathbf{y}_1 \sqsubseteq \mathbf{N}(\mathbf{x}_1)$; (iv) if $\beta(\mathbf{y}_1) \in W \setminus \{w_1, w_2, w_3\}$ then $Alg(\mathbb{K})\alpha\beta \Vdash \mathbf{y}_1 \sqsubseteq \mathbf{x}_2$). Therefore the rule does not hold, which is a contradiction with the hypothesis.

(b) Assume that the relation is transitive but not well bounded. Then, there is a sequence $\{w_i\}_{i\in\mathbb{N}}$ such that $\{w_{i+1}, w_{i+2}, \dots\} \subseteq \mathbf{N}_{Alg(\mathbb{K})}(w_i)$ for all $i \in \mathbb{N}$. Consider the ground substitution ρ such that $\Gamma_1\rho = \Gamma_2\rho = \emptyset$, $\tau_1\rho = \mathbf{x}_1$, $\tau_2\rho = \mathbf{x}_2$ and $\tau_3\rho = \mathbf{y}_1$. Let α be an assignment such that $\alpha(\mathbf{x}_1) = \{w_0\}$ and $\alpha(\mathbf{x}_2) = W \setminus \{w_0, w_1, \dots\}$. Then $Alg(\mathbb{K})\alpha \not\Vdash \mho\mathbf{x}_1$ and $Alg(\mathbb{K})\alpha \not\Vdash \mathbf{N}(\mathbf{x}_1) \sqsubseteq \mathbf{x}_2$ and so $Alg(\mathbb{K})\alpha \not\Vdash \mho\mathbf{x}_1 \to \mathbf{N}(\mathbf{x}_1) \sqsubseteq \mathbf{x}_2$. Next we show that $Alg(\mathbb{K})\alpha \Vdash \mho\mathbf{x}_1, \mho\mathbf{y}_1 \to \mathbf{y}_1 \sqsubseteq \mathbf{x}_2, \mathbf{N}(\mathbf{y}_1) \not\sqsubseteq \mathbf{x}_2$ by considering all possibilities for $\beta(\mathbf{y}_1)$: (i) if $\beta(\mathbf{y}_1)$ is not a singleton then the result follows straightforwardly; (ii) assume that $\beta(\mathbf{y}_1) = \{w_i\}$, then $Alg(\mathbb{K})\alpha\beta \Vdash \mathbf{N}(\mathbf{y}_1) \not\sqsubseteq \mathbf{x}_2$; (iii) assume that $\beta(y_1) \in W \setminus \{w_0, w_1, \dots\}$, then $Alg(\mathbb{K})\alpha\beta \Vdash \mathbf{y}_1 \sqsubseteq \mathbf{x}_2$. Therefore the rule does not hold, which is a contradiction with the hypothesis. QED

Observe that the characterization theorem above shows that the rules proposed in Subsection 2.6 characterize the envisaged properties of the accessibility relation even among general Kripke structures.

We now turn our attention to soundness and completeness of the modal sequent calculi over the general Kripke semantics. Soundness is easy to obtain, but, in order to establish completeness, we have to start by showing how to extract a general Kripke structure from a Σ_K-algebra appropriate for the sequent rules of a modal system. Recall that $\Sigma_K = \langle C, O, X, Y, Z\rangle$ is the modal signature introduced in Subsection 2.5.

Let $\mathbb{A} = \langle F, T, \cdot_\mathbb{A}\rangle$ be a Σ_K-algebra. Consider $Kpk(\mathbb{A}) = \langle W, \rightsquigarrow, \mathcal{B}, V\rangle$ where:

- $W = \Omega_\mathbb{A}$
- $t \rightsquigarrow t'$ iff $t' \sqsubseteq_\mathbb{A} \mathbf{N}_\mathbb{A}(t)$;
- $\mathcal{B} = \{\langle f\rangle_\mathbb{A} : f \in F\}$ where $\langle f\rangle_\mathbb{A}$ denotes the set $\{t \in \Omega_\mathbb{A} : t \leq_\mathbb{A} f\}$;
- $V(\mathbf{p}_i) = \langle \mathbf{p}_{i\mathbb{A}}\rangle_\mathbb{A}$.

In the sequel, we may write $\langle f\rangle$ for $\langle f\rangle_\mathbb{A}$ when the underlying algebra is clear from the context.

Proposition 3.19 Given a Σ_K-algebra $\mathbb{A}$ appropriate for $\mathcal{C}_K$, the tuple $Kpk(\mathbb{A})$ is a general Kripke structure over C.

Proof:

(1) W is non empty. Indeed, by absurd, assume that $W = \Omega_\mathbb{A} = \emptyset$. Then, since $\mathbb{A}$ is appropriate for rules cons and Ω, we would conclude that $\top \in \Omega_\mathbb{A}$.

(2) $W \in \mathcal{B}$. Indeed, $W = \langle [\![\mathbf{t}]\!]_\mathbb{A}\rangle$ since $\mathbb{A}$ is appropriate for rule R$\mathbf{t}$.

(3) $\mathcal{B}$ is closed for complements. More precisely, we have to show that if $\langle f\rangle \in \mathcal{B}$ then $(W \setminus \langle f\rangle) \in \mathcal{B}$. Observe that $W \setminus \langle f\rangle = \{t \in W : t \not\leq_\mathbb{A} f\}$. We show below that $\{t \in W : t \not\leq_\mathbb{A} f\} = \{t \in W : t \leq_\mathbb{A} \neg_\mathbb{A}(f)\}$. Thus, the result follows since the latter set is $\langle \neg_\mathbb{A}(f)\rangle$ which is in $\mathcal{B}$.

(i) $\{t \in W : t \not\leq_\mathbb{A} f\} \subseteq \{t \in W : t \leq_\mathbb{A} \neg_\mathbb{A}(f)\}$. Indeed, assume that $t \not\leq_\mathbb{A} f$ whenever $t \in \Omega_\mathbb{A}$. So, by definition of satisfaction, choosing α such that $\alpha(\mathbf{x}_1) = t$ and $\alpha(\mathbf{z}_1) = f$, we have $\mathbb{A}\alpha \Vdash \Omega\mathbf{x}_1 \to \mathbf{x}_1 \not\leq \mathbf{z}_1$. Observe that $\Omega\mathbf{x}_1 \to \mathbf{x}_1 \not\leq \mathbf{z}_1 \vdash_{\mathcal{C}_K} \Omega\mathbf{x}_1 \to \mathbf{x}_1 \leq \neg\,\mathbf{z}_1$:

1	$\Omega\mathbf{x}_1$	$\to$	$\mathbf{x}_1 \leq \neg\,\mathbf{z}_1$	R$\neg$: 2
2	$\mathbf{x}_1 \leq \mathbf{z}_1, \Omega\mathbf{x}_1$	$\to$		LxeF : 3
3	$\Omega\mathbf{x}_1$	$\to$	$\mathbf{x}_1 \not\leq \mathbf{z}_1$	hyp

Therefore, by Theorem 3.4, $\Omega\mathbf{x}_1 \to \mathbf{x}_1 \not\leq \mathbf{z}_1 \vDash_{\mathrm{app}(\mathcal{C}_K)} \Omega\mathbf{x}_1 \to \mathbf{x}_1 \leq \neg\,\mathbf{z}_1$. Hence, $\mathbb{A}\alpha \Vdash \Omega\mathbf{x}_1 \to \mathbf{x}_1 \leq \neg\,\mathbf{z}_1$. Again by definition of satisfaction and taking into account the choice of α, we get $t \leq_\mathbb{A} \neg_\mathbb{A}(f)$ whenever $t \in \Omega_\mathbb{A}$.

(ii) $\{t \in W : t \not\leq_\mathbb{A} f\} \supseteq \{t \in W : t \leq_\mathbb{A} \neg_\mathbb{A}(f)\}$. The proof is similar taking into account $\Omega\mathbf{x}_1 \to \mathbf{x}_1 \leq \neg\,\mathbf{z}_1 \vdash_{\mathcal{C}_K} \Omega\mathbf{x}_1 \to \mathbf{x}_1 \not\leq \mathbf{z}_1$:

1	$\Omega\mathbf{x}_1$	$\to$	$\mathbf{x}_1 \not\leq \mathbf{z}_1$	cutF : 2, 3
2	$\Omega\mathbf{x}_1$	$\to$	$\mathbf{x}_1 \not\leq \mathbf{z}_1, \mathbf{x}_1 \leq \neg\,\mathbf{z}_1$	RwF : 4
3	$\mathbf{x}_1 \leq \neg\,\mathbf{z}_1, \Omega\mathbf{x}_1$	$\to$	$\mathbf{x}_1 \not\leq \mathbf{z}_1$	L $\neg$: 5
4	$\Omega\mathbf{x}_1$	$\to$	$\mathbf{x}_1 \leq \neg\,\mathbf{z}_1$	hyp
5	$\Omega\mathbf{x}_1$	$\to$	$\mathbf{x}_1 \not\leq \mathbf{z}_1, \mathbf{x}_1 \leq \mathbf{z}_1$	RxiF : 6
6	$\mathbf{x}_1 \leq \mathbf{z}_1, \Omega\mathbf{x}_1$	$\to$	$\mathbf{x}_1 \leq \mathbf{z}_1$	ax

(4) $\mathcal{B}$ is closed for unions. More precisely, we have to show that if $\langle f\rangle, \langle g\rangle \in \mathcal{B}$ then $(\langle f\rangle \cup \langle g\rangle) \in \mathcal{B}$. Observe that $\langle f\rangle \cup \langle g\rangle = \{t \in W : t \leq_{\mathbb{A}} f \text{ or } t \leq_{\mathbb{A}} g\}$. We show below that $\{t \in W : t \leq_{\mathbb{A}} f \text{ or } t \leq_{\mathbb{A}} g\} = \{t \in W : t \leq_{\mathbb{A}} \vee_{\mathbb{A}}(f, g)\}$. Thus, the result follows since the latter set is $\langle \vee_{\mathbb{A}}(f, g)\rangle$ which is in $\mathcal{B}$.

(i) $\{t \in W : t \leq_{\mathbb{A}} f \text{ or } t \leq_{\mathbb{A}} g\} \subseteq \{t \in W : t \leq_{\mathbb{A}} \vee_{\mathbb{A}}(f, g)\}$. Indeed, assume that $t \leq_{\mathbb{A}} f$ or $t \leq_{\mathbb{A}} g$ whenever $t \in \Omega_{\mathbb{A}}$. So, by definition of satisfaction, choosing α such that $\alpha(\mathbf{x}_1) = t$, $\alpha(\mathbf{z}_1) = f$ and $\alpha(\mathbf{z}_2) = g$ we have $\mathbb{A}\alpha \Vdash \Omega\mathbf{x}_1 \to \mathbf{x}_1 \leq \mathbf{z}_1, \mathbf{x}_1 \leq \mathbf{z}_2$. Observe that $\Omega\mathbf{x}_1 \to \mathbf{x}_1 \leq \mathbf{z}_1, \mathbf{x}_1 \leq \mathbf{z}_2 \vdash_{\mathcal{C}_K} \Omega\mathbf{x}_1 \to \mathbf{x}_1 \leq (\mathbf{z}_1 \vee \mathbf{z}_2)$ by rule R$\vee$. Therefore, by Theorem 3.4, $\Omega\mathbf{x}_1 \to \mathbf{x}_1 \leq \mathbf{z}_1, \mathbf{x}_1 \leq \mathbf{z}_2 \vDash_{\mathrm{app}(\mathcal{C}_K)} \Omega\mathbf{x}_1 \to \mathbf{x}_1 \leq (\mathbf{z}_1 \vee \mathbf{z}_2)$. Hence, $\mathbb{A}\alpha \Vdash \Omega\mathbf{x}_1 \to \mathbf{x}_1 \leq (\mathbf{z}_1 \vee \mathbf{z}_2)$. Again by definition of satisfaction and taking into account the choice of α, we get $t \leq_{\mathbb{A}} \vee_{\mathbb{A}}(f, g)$ whenever $t \in \Omega_{\mathbb{A}}$.

(ii) $\{t \in W : t \leq_{\mathbb{A}} f \text{ or } t \leq_{\mathbb{A}} g\} \supseteq \{t \in W : t \leq_{\mathbb{A}} \vee_{\mathbb{A}}(f, g)\}$. The proof is similar taking into account $\Omega\mathbf{x}_1 \to \mathbf{x}_1 \leq (\mathbf{z}_1 \vee \mathbf{z}_2) \vdash_{\mathcal{C}_K} \Omega\mathbf{x}_1 \to \mathbf{x}_1 \leq \mathbf{z}_1, \mathbf{x}_1 \leq \mathbf{z}_2$:

1	$\Omega\mathbf{x}_1$	$\to$	$\mathbf{x}_1 \leq \mathbf{z}_1$ $\mathbf{x}_1 \leq \mathbf{z}_2$	cutF : 2, 3
2	$\Omega\mathbf{x}_1$	$\to$	$\mathbf{x}_1 \leq (\mathbf{z}_1 \vee \mathbf{z}_2)$ $\mathbf{x}_1 \leq \mathbf{z}_1$ $\mathbf{x}_1 \leq \mathbf{z}_2$	RwF : 4
3	$\mathbf{x}_1 \leq (\mathbf{z}_1 \vee \mathbf{z}_2)$ $\Omega\mathbf{x}_1$	$\to$	$\mathbf{x}_1 \leq \mathbf{z}_1$ $\mathbf{x}_1 \leq \mathbf{z}_2$	L$\vee$: 5, 6
4	$\Omega\mathbf{x}_1$	$\to$	$\mathbf{x}_1 \leq (\mathbf{z}_1 \vee \mathbf{z}_2)$	hyp
5	$\mathbf{x}_1 \leq \mathbf{z}_1$ $\Omega\mathbf{x}_1$	$\to$	$\mathbf{x}_1 \leq \mathbf{z}_1$ $\mathbf{x}_1 \leq \mathbf{z}_2$	ax
6	$\mathbf{x}_1 \leq \mathbf{z}_2$ $\Omega\mathbf{x}_1$	$\to$	$\mathbf{x}_1 \leq \mathbf{z}_1$ $\mathbf{x}_1 \leq \mathbf{z}_2$	ax

(5) $\mathcal{B}$ is closed for necessitations. More precisely, denoting by $L(\langle f\rangle)$ the set $\{t \in W : t \rightsquigarrow t' \text{ implies } t' \in \langle f\rangle \text{ for every } t' \in W\}$, we have to show that if $\langle f\rangle \in \mathcal{B}$ then $L(\langle f\rangle) \in \mathcal{B}$. Observe that $L(\langle f\rangle) = \{t \in W : t' \sqsubseteq_{\mathbb{A}} \mathbf{N}_{\mathbb{A}}(t) \text{ implies } t' \in \langle f\rangle \text{ for every } t' \in W\} = \{t \in W : t' \sqsubseteq_{\mathbb{A}} \mathbf{N}_{\mathbb{A}}(t) \text{ implies } t' \leq_{\mathbb{A}} f \text{ for every } t' \in W\}$. We show below the latter set is equal to $\{t \in W : t \leq_{\mathbb{A}} \Box_{\mathbb{A}}(f)\}$. Thus, the result follows since the latter set is $\langle \Box_{\mathbb{A}}(f)\rangle$ which is in $\mathcal{B}$.

(i) $L(\langle f\rangle) \subseteq \{t \in W : t \leq_{\mathbb{A}} \Box_{\mathbb{A}}(f)\}$. Indeed, assume that if $t \in \Omega_{\mathbb{A}}$ then for every $t' \in \Omega_{\mathbb{A}}$ we have $t' \leq_{\mathbb{A}} f$ whenever $t' \sqsubseteq_{\mathbb{A}} \mathbf{N}_{\mathbb{A}}(t)$. So, by definition of satisfaction, choosing α such that $\alpha(\mathbf{x}_1) = t$ and $\alpha(\mathbf{z}_1) = f$, we have

$$\mathbb{A}\alpha \Vdash \Omega\mathbf{x}_1, \Omega\mathbf{y}_1, \mathbf{y}_1 \sqsubseteq \mathbf{N}(\mathbf{x}_1) \to \mathbf{y}_1 \leq \mathbf{z}_1 .$$

Observe that $\Omega\mathbf{x}_1, \Omega\mathbf{y}_1, \mathbf{y}_1 \sqsubseteq \mathbf{N}(\mathbf{x}_1) \to \mathbf{y}_1 \leq \mathbf{z}_1 \vdash_{\mathcal{C}_K} \Omega\mathbf{x}_1 \to \mathbf{x}_1 \leq \Box\mathbf{z}_1$ by rules R$\Box$ and Rgen. Therefore, by Theorem 3.4 $\Omega\mathbf{x}_1, \Omega\mathbf{y}_1, \mathbf{y}_1 \sqsubseteq \mathbf{N}(\mathbf{x}_1) \to$

$\mathbf{y}_1 \leq \mathbf{z}_1 \vDash_{\text{app}(\mathcal{C}_K)} \Omega\mathbf{x}_1 \to \mathbf{x}_1 \leq \Box\mathbf{z}_1$. Hence, $\mathbb{A}\alpha \Vdash \Omega\mathbf{x}_1 \to \mathbf{x}_1 \leq \Box\mathbf{z}_1$. Again by definition of satisfaction and taking into account the choice of α, we get $t \leq_{\mathbb{A}} \Box_{\mathbb{A}}(f)$ whenever $t \in \Omega_{\mathbb{A}}$.

(ii) $L(\langle f \rangle) \supseteq \{t \in W : t \leq_{\mathbb{A}} \Box_{\mathbb{A}}(f)\}$. The proof is similar taking into account $\Omega\mathbf{x}_1 \to \mathbf{x}_1 \leq \Box\mathbf{z}_1 \vdash_{\mathcal{C}_K} \Omega\mathbf{x}_1, \Omega\mathbf{y}_1, \mathbf{y}_1 \sqsubseteq \mathbf{N}(\mathbf{x}_1) \to \mathbf{y}_1 \leq \mathbf{z}_1$:

1	$\Omega\mathbf{x}_1$ $\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \mathbf{N}(\mathbf{x}_1)$	$\to$	$\mathbf{y}_1 \leq \mathbf{z}_1$	cutF : 2, 3
2	$\Omega\mathbf{x}_1$ $\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \mathbf{N}(\mathbf{x}_1)$	$\to$	$\mathbf{x}_1 \leq \ \ \mathbf{z}_1$ $\mathbf{y}_1 \leq \mathbf{z}_1$	w : 4
3	$\mathbf{x}_1 \leq \ \ \mathbf{z}_1$ $\Omega\mathbf{x}_1$ $\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \mathbf{N}(\mathbf{x}_1)$	$\to$	$\mathbf{y}_1 \leq \mathbf{z}_1$	L : 5, 6, 7
4	$\Omega\mathbf{x}_1$	$\to$	$\mathbf{x}_1 \leq \ \ \mathbf{z}_1$	hyp
5	$\Omega\mathbf{y}_1$ $\mathbf{x}_1 \leq \ \ \mathbf{z}_1$ $\Omega\mathbf{x}_1$ $\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \mathbf{N}(\mathbf{x}_1)$	$\to$	$\mathbf{y}_1 \sqsubseteq \mathbf{N}(\mathbf{x}_1)$ $\mathbf{y}_1 \leq \mathbf{z}_1$	ax
6	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \leq \mathbf{z}_1$ $\mathbf{x}_1 \leq \ \ \mathbf{z}_1$ $\Omega\mathbf{x}_1$ $\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \mathbf{N}(\mathbf{x}_1)$	$\to$	$\mathbf{y}_1 \leq \mathbf{z}_1$	ax
7	$\mathbf{x}_1 \leq \ \ \mathbf{z}_1$ $\Omega\mathbf{x}_1$ $\Omega\mathbf{y}_1$ $\mathbf{y}_1 \sqsubseteq \mathbf{N}(\mathbf{x}_1)$	$\to$	$\Omega\mathbf{y}_1$ $\mathbf{y}_1 \leq \mathbf{z}_1$	ax

QED

The following results show that the semantics of ground simple formulae is preserved when we move from a Σ_K-algebra appropriate for $\mathcal{C}_K$ to the corresponding general Kripke structure.

Lemma 3.20 Let $\mathbb{A}$ be a Σ_K-algebra appropriate for $\mathcal{C}_K$. Then, for every ground simple formula φ, $\langle [\![\varphi]\!]_{\mathbb{A}} \rangle = [\![\varphi]\!]_{Kpk(\mathbb{A})}$.

Proof:

The proof is carried out by induction on the complexity of φ:

(Base) We have to consider only two representative cases:

(i) φ is $\mathbf{f}$. Then: $\langle [\![\mathbf{f}]\!]_{\mathbb{A}} \rangle = \langle \mathbf{f}_{\mathbb{A}} \rangle = \{u \in \Omega_{\mathbb{A}} : u \leq_{\mathbb{A}} \mathbf{f}_{\mathbb{A}}\}$ which coincides with $\{u \in \Omega_{\mathbb{A}} : u \sqsubseteq_{\mathbb{A}} \bot_{\mathbb{A}}\}$ since $\mathbb{A}$ is appropriate for rule R$\mathbf{f}$. Moreover, $\{u \in \Omega_{\mathbb{A}} : u \sqsubseteq_{\mathbb{A}} \bot_{\mathbb{A}}\} = \emptyset$ since $\mathbb{A}$ is appropriate for rule $\Omega\bot$. Therefore, $\langle [\![\mathbf{f}]\!]_{\mathbb{A}} \rangle = \mathbf{f}_{Kpk(\mathbb{A})} = [\![\mathbf{f}]\!]_{Kpk(\mathbb{A})}$.

(ii) φ is $\mathbf{p}_i$. Then: $\langle [\![p_i]\!]_{\mathbb{A}} \rangle = \langle p_{i\mathbb{A}} \rangle = V(p_i) = [\![p_i]\!]_{Kpk(\mathbb{A})}$.

(Step) We have to consider only three representative cases:

(i) φ is $\neg\,\varphi'$. Then: $\langle [\![\neg\,\varphi']\!]_{\mathbb{A}} \rangle = \langle \neg_{\mathbb{A}}([\![\varphi']\!]_{\mathbb{A}}) \rangle = \{u \in \Omega_{\mathbb{A}} : u \leq_{\mathbb{A}} \neg_{\mathbb{A}}([\![\varphi']\!]_{\mathbb{A}})\}$

which coincides with $\{u \in \Omega_{\mathbb{A}} : u \not\leq_{\mathbb{A}} [\![\varphi']\!]_{\mathbb{A}}\}$ as seen in part (2) of the proof of Proposition 3.19. Therefore, $\langle [\![\neg\, \varphi']\!]_{\mathbb{A}} \rangle = \Omega_{\mathbb{A}} \setminus \{u \in \Omega_{\mathbb{A}} : u \leq_{\mathbb{A}} [\![\varphi']\!]_{\mathbb{A}}\} = \Omega_{\mathbb{A}} \setminus \langle [\![\varphi']\!]_{\mathbb{A}} \rangle$ which, by the induction hypothesis, is equal to $\Omega_{\mathbb{A}} \setminus [\![\varphi']\!]_{Kpk(\mathbb{A})}$ and, so, identical to $[\![\neg\, \varphi']\!]_{Kpk(\mathbb{A})}$.

(ii) φ is $\Box\varphi'$. Then: $\langle [\![\Box\varphi']\!]_{\mathbb{A}} \rangle = \langle \Box_{\mathbb{A}}([\![\varphi']\!]_{\mathbb{A}}) \rangle = \{u \in \Omega_{\mathbb{A}} : u \leq_{\mathbb{A}} \Box_{\mathbb{A}}([\![\varphi']\!]_{\mathbb{A}})\}$ which coincides with $L(\langle [\![\varphi']\!]_{\mathbb{A}} \rangle)$ as seen in part (4) of the proof of Proposition 3.19. Therefore, by the induction hypothesis, $\langle [\![\Box\varphi']\!]_{\mathbb{A}} \rangle = L([\![\varphi']\!]_{Kpk(\mathbb{A})})$ and, so, identical to $[\![\Box\varphi']\!]_{Kpk(\mathbb{A})}$.

(iii) φ is $\varphi' \vee \varphi''$. Then: $\langle [\![\varphi' \vee \varphi'']\!]_{\mathbb{A}} \rangle = \langle \vee_{\mathbb{A}}([\![\varphi']\!]_{\mathbb{A}}, [\![\varphi'']\!]_{\mathbb{A}}) \rangle$ which coincides with $\{u \in \Omega_{\mathbb{A}} : u \leq_{\mathbb{A}} [\![\varphi']\!]_{\mathbb{A}} \text{ or } u \leq_{\mathbb{A}} [\![\varphi'']\!]_{\mathbb{A}}\}$ as seen in part (3) of the proof of Proposition 3.19. Therefore, $\langle [\![\varphi' \vee \varphi'']\!]_{\mathbb{A}} \rangle = \{u \in \Omega_{\mathbb{A}} : u \leq_{\mathbb{A}} [\![\varphi']\!]_{\mathbb{A}}\} \cup \{u \in \Omega_{\mathbb{A}} : u \leq_{\mathbb{A}} [\![\varphi'']\!]_{\mathbb{A}}\}$ which, by the induction hypothesis, is equal to $[\![\varphi']\!]_{Kpk(\mathbb{A})} \cup [\![\varphi'']\!]_{Kpk(\mathbb{A})}$ and, so, identical to $[\![\varphi' \vee \varphi'']\!]_{Kpk(\mathbb{A})}$. QED

Proposition 3.21 Given a Σ_K-algebra $\mathbb{A}$ appropriate for $\mathcal{C}_K$ and a ground simple formula φ, $\mathbb{A} \Vdash \top \leq \varphi$ iff $Kpk(\mathbb{A}) \Vdash \varphi$.

Proof:

($\Rightarrow$) Assume $\mathbb{A} \Vdash \top \leq \varphi$. Then, $\top_{\mathbb{A}} \leq_{\mathbb{A}} [\![\varphi]\!]_{\mathbb{A}}$. Since $\mathbb{A}$ is appropriate for rules $\top$ and transF, for every $t \in T$, $t \leq_{\mathbb{A}} [\![\varphi]\!]_{\mathbb{A}}$. In particular, for every $u \in \Omega_{\mathbb{A}}$, $u \leq_{\mathbb{A}} [\![\varphi]\!]_{\mathbb{A}}$. Therefore, $\{u \in \Omega_{\mathbb{A}} : u \leq_{\mathbb{A}} [\![\varphi]\!]_{\mathbb{A}}\} = \Omega_{\mathbb{A}}$. Thus, $\langle [\![\varphi]\!]_{\mathbb{A}} \rangle = \Omega_{\mathbb{A}}$ and, so, $[\![\varphi]\!]_{Kpk(\mathbb{A})} = W$ which means $Kpk(\mathbb{A}) \Vdash \varphi$.

($\Leftarrow$) Assume $Kpk(\mathbb{A}) \Vdash \varphi$. Then, $\{u \in \Omega_{\mathbb{A}} : u \leq_{\mathbb{A}} [\![\varphi]\!]_{\mathbb{A}}\} = \Omega_{\mathbb{A}}$. Thus, since $\mathbb{A}$ is appropriate for rule $\Omega\top$, $\top_{\mathbb{A}} \leq_{\mathbb{A}} [\![\varphi]\!]_{\mathbb{A}}$, and, so, $\mathbb{A} \Vdash \top \leq \varphi$. QED

Proposition 3.22 Given a Σ_K-algebra $\mathbb{A}$ appropriate for $\mathcal{C}_K$, an assignment β over $\mathbb{A}$ such that $\beta(\mathbf{y}_1) = u \in \Omega_{\mathbb{A}}$ and a ground simple formula φ, $\mathbb{A}\alpha\beta \Vdash \mathbf{y}_1 \leq \varphi$ iff $Kpk(\mathbb{A})u \Vdash \varphi$.

Proof: $\mathbb{A}\alpha\beta \Vdash \mathbf{y}_1 \leq \varphi$ iff $\beta(\mathbf{y}_1) \leq_{\mathbb{A}} [\![\varphi]\!]_{\mathbb{A}}$ iff $u \in \langle [\![\varphi]\!]_{\mathbb{A}} \rangle_{\mathbb{A}}$ iff $u \in [\![\varphi]\!]_{Kpk(\mathbb{A})}$ iff $Kpk(\mathbb{A})u \Vdash \varphi$. QED

Given a class $\mathcal{A}$ of Σ_K-algebras appropriate for the sequent rules of modal system K, let $Kpk(\mathcal{A})$ be the class $\{Kpk(\mathbb{A}) : \mathbb{A} \in \mathcal{A}\}$.

Theorem 3.23 Given a class $\mathcal{A}$ of Σ_K-algebras appropriate for $\mathcal{C}_K$:

- $\psi_1, \ldots, \psi_k \vDash^{\mathrm{g}}_{\mathcal{A}} \varphi$ iff $\psi_1, \ldots, \psi_k \vDash^{\mathrm{g}}_{Kpk(\mathcal{A})} \varphi$;
- $\psi_1, \ldots, \psi_k \vDash^{\ell}_{\mathcal{A}} \varphi$ iff $\psi_1, \ldots, \psi_k \vDash^{\ell}_{Kpk(\mathcal{A})} \varphi$.

Proof: Immediate using the two propositions above. QED

We are now ready to establish the counterpart of Theorem 3.18.

Theorem 3.24 (Cocharacterization) For each property P of the accessibility relation and each Σ_K-algebra $\mathbb{A}$ appropriate for $\mathcal{C}_K$:

$$Kpk(\mathbb{A}) \in \mathcal{K}_P \quad \text{iff} \quad \mathbb{A} \in \mathrm{app}(\mathcal{C}_P)\,.$$

Proof: Straightforward for each of the properties in Subsection 2.6. For instance, let P state that the relation is *irreflexive*. Then: $Kpk(\mathbb{A}) \in \mathcal{K}_P$ iff $u \not\rightarrow u$ for every $u \in W = \Omega_\mathbb{A}$ iff $u \not\sqsubseteq_\mathbb{A} \mathbf{N}_\mathbb{A}(u)$ for every $u \in W$. So, we have to show that the latter holds iff, for every $\Gamma_1, \Gamma_2, \rho, \alpha, \beta$, $\mathbb{A}\alpha\beta \Vdash \Omega\tau_1\rho, \Gamma_1\rho \to \Gamma_2\rho, \tau_1\rho \not\sqsubseteq \mathbf{N}(\tau_1\rho)$. QED

And, finally, we are now able to prove the envisaged completeness result over general Kripke semantics.

Theorem 3.25 (Modal Kripke soundness and completeness) For each finite set $\mathcal{P}$ of properties of the accessibility relation:

- $\psi_1, \ldots, \psi_k \vDash^{\mathrm{g}}_{\mathcal{K}_\mathcal{P}} \varphi$ iff $\psi_1, \ldots, \psi_k \vdash^{\mathrm{g}}_{\mathcal{C}_\mathcal{P}} \varphi$;
- $\psi_1, \ldots, \psi_k \vDash^{\ell}_{\mathcal{K}_\mathcal{P}} \varphi$ iff $\psi_1, \ldots, \psi_k \vdash^{\ell}_{\mathcal{C}_\mathcal{P}} \varphi$.

Proof:

(Soundness) Immediate by induction on the length of the derivation using ($\Rightarrow$) of Theorem 3.18, Theorem 3.17 and Theorem 3.16.

(Global completeness) Assume that $\psi_1, \ldots, \psi_k \nvdash^{\mathrm{g}}_{\mathcal{C}_\mathcal{P}} \varphi$. Then, by Corollary 3.12, $\psi_1, \ldots, \psi_k \nvDash^{\mathrm{g}}_{\mathrm{app}(\mathcal{C}_\mathcal{P})} \varphi$. That is, there is $\mathbb{A} \in \mathrm{app}(\mathcal{C}_\mathcal{P})$ such that $\mathbb{A} \Vdash \top \leq \psi_i$ for $i = 1, \ldots, k$ and $\mathbb{A} \nVdash \top \leq \varphi$. Let $\mathbb{A}$ be such an algebra and $\mathbb{K} = Kpk(\mathbb{A})$. Then, by Theorem 3.24, we know that $\mathbb{K} \in \mathcal{K}_\mathcal{P}$. Furthermore, by Proposition 3.21, $\mathbb{K} \Vdash \psi_i$ for $i = 1, \ldots, k$ and $\mathbb{K} \nVdash \varphi$. Therefore, $\psi_1, \ldots, \psi_k \nvDash^{\mathrm{g}}_{\mathcal{K}_\mathcal{P}} \varphi$.

(Local completeness) Assume that $\psi_1, \ldots, \psi_k \nvdash^{\ell}_{\mathcal{C}_\mathcal{P}} \varphi$. Then, by Proposition 2.4, $\nvdash^{\mathrm{g}}_{\mathcal{C}_\mathcal{P}} (\psi_1 \wedge \ldots \wedge \psi_k) \Rightarrow \varphi$. So, by global completeness as proved above, $\nvDash^{\mathrm{g}}_{\mathcal{K}_\mathcal{P}} (\psi_1 \wedge \ldots \wedge \psi_k) \Rightarrow \varphi$. Therefore, by Proposition 3.5, $\psi_1, \ldots, \psi_k \nvDash^{\ell}_{\mathcal{K}_\mathcal{P}} \varphi$. QED

It is straightforward to adapt the results in this subsection (about $\mathcal{C}_K$ and its extensions) to the hybrid calculi introduced in Subsection 2.7.

We refrain to develop the details on this side issue, but it is worthwhile to comment on the nature of $Kpk(\mathbb{A})$ when $\mathbb{A}$ is appropriate for $\mathcal{C}^{@}_K$. Since, $F_\mathbb{A}$ contains elements representing all elements of $\wp W$, $\mathcal{B}$ collapses into $\wp W$ in the resulting Kripke structure. Therefore, denoting by $s\mathcal{K}_\mathcal{P}$ the class of all standard Kripke structures with set $\mathcal{P}$ of properties imposed on the accessibility relation, the theorem above becomes:

Theorem 3.26 For each finite set $\mathcal{P}$ of properties of the accessibility relation:

- $\psi_1, \ldots, \psi_k \vDash^{\mathrm{g}}_{s\mathcal{K}_\mathcal{P}} \varphi$ iff $\psi_1, \ldots, \psi_k \vdash^{\mathrm{g}}_{\mathcal{C}^{@}_\mathcal{P}} \varphi$;
- $\psi_1, \ldots, \psi_k \vDash^{\ell}_{s\mathcal{K}_\mathcal{P}} \varphi$ iff $\psi_1, \ldots, \psi_k \vdash^{\ell}_{\mathcal{C}^{@}_\mathcal{P}} \varphi$.

3.4 Duality between sober algebras and Kripke structures

In the previous subsection, we saw how to extract a Σ_K-algebra appropriate for $\mathcal{C}_K$ from a general Kripke structure and vice versa. It is natural to ask what is the relationship, if any, between $\mathbb{A}$ and $Alg(Kpk(\mathbb{A}))$ and also between $\mathbb{K}$ and $Kpk(Alg(\mathbb{K}))$. In order to answer these questions, first we need to set up the category of Σ_k-algebras and the category of general Kripke structures.

Recall that a *p-morphism* $h : \langle W, \rightsquigarrow, \mathcal{B}, V\rangle \to \langle W', \rightsquigarrow', \mathcal{B}', V'\rangle$ is a map $h : W \to W'$ such that:

a) if $w_1 \rightsquigarrow w_2$ then $h(w_1) \rightsquigarrow' h(w_2)$;

b) if $h(w_1) \rightsquigarrow' w_2'$ then there is w_2 such that $h(w_2) = w_2'$ and $w_1 \rightsquigarrow w_2$;

c) $h^{-1}(B') \in \mathcal{B}$ for every $B' \in \mathcal{B}'$;

d) $w \in V(\mathbf{p}_i)$ iff $h(w) \in V'(\mathbf{p}_i)$.

General Kripke structures together with p-morphisms constitute the category **K**. Observe that condition d) is equivalent to

d′) $V(\mathbf{p}_i) = h^{-1}(V'(\mathbf{p}_i))$.

In order to set up the envisaged category of Σ_K-algebras appropriate for $\mathcal{C}_K$, we propose the following notion of morphism between such algebras. To this end, we need the following notation:

- $f_1 \sim_{\mathbb{A}} f_2$ iff $\langle f_1\rangle_{\mathbb{A}} = \langle f_2\rangle_{\mathbb{A}}$, that is, iff $\{u \in \Omega_{\mathbb{A}} : u \leq_{\mathbb{A}} f_1\} = \{u \in \Omega_{\mathbb{A}} : u \leq_{\mathbb{A}} f_2\}$;
- $\lceil t\rceil_{\mathbb{A}} = \{u \in \Omega_{\mathbb{A}} : u \sqsubseteq_{\mathbb{A}} t\}$;
- $t_1 \approx_{\mathbb{A}} t_2$ iff $t_1 \sqsubseteq_{\mathbb{A}} t_2$ and $t_2 \sqsubseteq_{\mathbb{A}} t_1$, that is, iff $\lceil t_1\rceil_{\mathbb{A}} = \lceil t_2\rceil_{\mathbb{A}}$.

A *morphism* $h : \langle F, T, \cdot_{\mathbb{A}}\rangle \to \langle F', T', \cdot_{\mathbb{A}'}\rangle$ is a pair $\langle \overrightarrow{h}, \overleftarrow{h}\rangle$ where:

- $\overrightarrow{h} : F \to F'$;
- $\overleftarrow{h} : T' \to T$;

such that

1) $\overrightarrow{h}(c_{\mathbb{A}}(f_1, \dots, f_k)) \sim_{\mathbb{A}'} c_{\mathbb{A}'}(\overrightarrow{h}(f_1), \dots, \overrightarrow{h}(f_k))$;

2) $\lceil \overleftarrow{h}(t')\rceil_{\mathbb{A}} \subseteq \overleftarrow{h}(\lceil t'\rceil_{\mathbb{A}'})$;

3) $\overleftarrow{h}(\perp_{\mathbb{A}'}) \sqsubseteq_{\mathbb{A}} \perp_{\mathbb{A}}$;

4) $\mathbf{N}_{\mathbb{A}}(\overleftarrow{h}(t')) \approx_{\mathbb{A}} \overleftarrow{h}(\mathbf{N}_{\mathbb{A}'}(t'))$;

5) if $t' \in \Omega_{\mathbb{A}'}$ then $\overleftarrow{h}(t') \in \Omega_{\mathbb{A}}$;

6) if $t_1' \sqsubseteq_{\mathbb{A}'} t_2'$ then $\overleftarrow{h}(t_1') \sqsubseteq_{\mathbb{A}} \overleftarrow{h}(t_2')$;

7) $\overleftarrow{h}(t') \leq_{\mathbb{A}} f$ iff $t' \leq_{\mathbb{A}'} \overrightarrow{h}(f)$ whenever $t' \in \Omega_{\mathbb{A}'}$.

It is straightforward to verify that the Σ_K-algebras appropriate for $\mathcal{C}_K$ together with these morphisms constitute a category that we denote by **A**.

This notion of morphism is sufficiently strong to ensure the preservation and reflection of satisfaction of labelled formula. To this end, it is convenient to make explicit the structure of an unbound variable assignment α over a Σ-algebra by identifying each such α with the pair $\langle \alpha|_X, \alpha|_Z \rangle$.

Theorem 3.27 (Satisfaction condition) Let $h : \mathbb{A} \to \mathbb{A}'$ be a morphism, $\theta \in gT(\Sigma_K)$ without bounded variables, $\varphi \in gF(\Sigma_K)$, α an unbounded variable assignment over $\mathbb{A}$, and α' an unbounded variable assignment over $\mathbb{A}'$. Then:

$$\mathbb{A}\langle \overleftarrow{h} \circ \alpha'|_X, \alpha|_Z \rangle \Vdash \theta \leq \varphi \;\text{ iff }\; \mathbb{A}'\langle \alpha'|_X, \overrightarrow{h} \circ \alpha|_Z \rangle \Vdash \theta \leq \varphi\,.$$

Proof: The following two equalities can be proved by induction on the complexity of θ and φ, respectively:

1. $[\![\theta]\!]_{\mathbb{A}\overleftarrow{h}\circ\alpha'|_X} = \overleftarrow{h}([\![\theta]\!]_{\mathbb{A}'\alpha'|_X})$;
2. $\overrightarrow{h}([\![\varphi]\!]_{\mathbb{A}\alpha|_Z}) = [\![\varphi]\!]_{\mathbb{A}'\overrightarrow{h}\circ\alpha|_Z}$.

Now we have: $\mathbb{A}\langle \overleftarrow{h} \circ \alpha'|_X, \alpha|_Z \rangle \Vdash \theta \leq \varphi$ iff $[\![\theta]\!]_{\mathbb{A}\overleftarrow{h}\circ\alpha'|_X} \leq_{\mathbb{A}} [\![\varphi]\!]_{\mathbb{A}\alpha|_Z}$ iff (by 1.) $\overleftarrow{h}([\![\theta]\!]_{\mathbb{A}'\alpha'|_X}) \leq_{\mathbb{A}} [\![\varphi]\!]_{\mathbb{A}\alpha|_Z}$ iff (by property 7 of h) $[\![\theta]\!]_{\mathbb{A}'\alpha'|_X} \leq_{\mathbb{A}'} \overrightarrow{h}([\![\varphi]\!]_{\mathbb{A}\alpha|_Z})$ iff (by 2.) $[\![\theta]\!]_{\mathbb{A}'\alpha'|_X} \leq_{\mathbb{A}'} [\![\varphi]\!]_{\mathbb{A}'\overrightarrow{h}\circ\alpha|_Z}$ iff $\mathbb{A}'\langle \alpha'|_X, \overrightarrow{h} \circ \alpha|_Z \rangle \Vdash \theta \leq \varphi$. QED

Corollary 3.28 Let $h : \mathbb{A} \to \mathbb{A}'$ be a morphism, $\theta \in cgT(\Sigma_K)$ and $\varphi \in cgF(\Sigma_K)$. Then:

$$\mathbb{A} \Vdash \theta \leq \varphi \;\text{ iff }\; \mathbb{A}' \Vdash \theta \leq \varphi\,.$$

We now turn our attention towards extending the maps *Alg* and *Kpk* to functors.

Proposition 3.29 Given a p-morphism $h : \mathbb{K} \to \mathbb{K}'$, the pair $\langle h^{-1}, h \rangle$ is a morphism from $Alg(\mathbb{K}') \to Alg(\mathbb{K})$.

Proof:

1) $h^{-1}(c_{Alg(\mathbb{K}')}(B'_1, \ldots, B'_k)) = c_{Alg(\mathbb{K})}(h^{-1}(B'_1), \ldots, h^{-1}(B'_k))$ is established for each representative modal constructor:
(**f**) $h^{-1}(\mathbf{f}_{Alg(\mathbb{K}')}) = h^{-1}(\emptyset) = \emptyset = \mathbf{f}_{Alg(\mathbb{K})}$.
($\mathbf{p}_i$) $h^{-1}(\mathbf{p}_{i Alg(\mathbb{K}')}) = h^{-1}(V'(\mathbf{p}_i))$ which by condition d′) is identical to $V(\mathbf{p}_i) = \mathbf{p}_{i Alg(\mathbb{K})}$.
($\neg$) $h^{-1}(\neg_{Alg(\mathbb{K}')}(B')) = h^{-1}(W' \setminus B') = W \setminus h^{-1}(B') = \neg_{Alg(\mathbb{K})}(h^{-1}(B'))$.
($\Box$) $h^{-1}(\Box_{Alg(\mathbb{K}')}(B')) = h^{-1}(\{w' : \mathbf{N}_{Alg(\mathbb{K}')}(\{w'\}) \subseteq B'\})$ which is identical to $\{w : \mathbf{N}_{Alg(\mathbb{K}')}(\{h(w)\}) \subseteq B'\}$ which thanks to 4) below coincides with $\{w : h(\mathbf{N}_{Alg(\mathbb{K})}(\{w\})) \subseteq B'\} = \{w : \mathbf{N}_{Alg(\mathbb{K})}(\{w\}) \subseteq h^{-1}(B')\} = \Box_{Alg(\mathbb{K})}(h^{-1}(B'))$.

($\vee$) $h^{-1}(\vee_{Alg(\mathbb{K}')}(B'_1, B'_2)) = h^{-1}(B'_1 \cup B'_2)$ which coincides with $h^{-1}(B'_1) \cup h^{-1}(B'_2) = \vee_{Alg(\mathbb{K})}(h^{-1}(B'_1), h^{-1}(B'_2))$.

2) $\lceil h(U) \rceil_{Alg(\mathbb{K}')} = h(\lceil U \rceil_{Alg(\mathbb{K})})$. Indeed, $\lceil h(U) \rceil_{Alg(\mathbb{K}')} = \{\{u'\} : u' \in h(U)\} = \{\{h(u)\} : u \in U\} = h(\{\{u\} : u \in U\}) = h(\lceil U \rceil_{Alg(\mathbb{K})})$.

3) $h(\emptyset) \sqsubseteq_{Alg(\mathbb{K}')} \emptyset$. Indeed: $h(\emptyset) = \emptyset$.

4) $\mathbf{N}_{Alg(\mathbb{K}')}(h(U)) = h(\mathbf{N}_{Alg(\mathbb{K})}(U))$. Indeed:

($\subseteq$) Consider any $w' \in \mathbf{N}_{Alg(\mathbb{K}')}(h(U))$. Then, by definition of $Alg(\mathbb{K}')$, there is $u \in U$ such that $h(u) \rightsquigarrow' w'$. So, by property b) of the p-morphism, there is $w \in W$ such that $h(w) = w'$ and $u \rightsquigarrow w$. That is, thanks to the definition of $Alg(\mathbb{K})$, $w \in \mathbf{N}_{Alg(\mathbb{K})}(\{u\})$ and, therefore, $h(w) = w'$ is in $h(\mathbf{N}_{Alg(\mathbb{K})}(U))$.

($\supseteq$) Consider any $w \in \mathbf{N}_{Alg(\mathbb{K})}(U)$. Then, there is $u \in U$ such that $u \rightsquigarrow w$. So, by property a) of the p-morphism, $h(u) \rightsquigarrow' h(w)$. Hence, $h(w) \in \mathbf{N}_{Alg(\mathbb{K}')}(h(U))$.

Since $Alg(\mathbb{K}')$ is appropriate for rule ref, we obtain $\mathbf{N}_{Alg(\mathbb{K}')}(h(U)) \sqsubseteq_{Alg(\mathbb{K}')} h(\mathbf{N}_{Alg(\mathbb{K})}(U))$ and $h(\mathbf{N}_{Alg(\mathbb{K})}(U)) \sqsubseteq_{Alg(\mathbb{K}')} \mathbf{N}_{Alg(\mathbb{K}')}(h(U))$.

5) $h(\{w\}) \in \Omega_{Alg(\mathbb{K}')}$ for every $\{w\} \in \Omega_{Alg(\mathbb{K})}$. Immediate: $h(\{w\}) = \{h(w)\}$.

6) $h(U_1) \sqsubseteq_{Alg(\mathbb{K}')} h(U_2)$ for every $U_1 \sqsubseteq_{Alg(\mathbb{K})} U_2$. Immediate: $h(U_1) \subseteq h(U_2)$ whenever $U_1 \subseteq U_2$.

7) $h(\{w\}) \leq_{Alg(\mathbb{K}')} B'$ iff $\{w\} \leq_{Alg(\mathbb{K})} h^{-1}(B')$ for every $\{w\} \in \Omega_{Alg(\mathbb{K})}$. Immediate: $h(w) \in B'$ iff $w \in h^{-1}(B')$. QED

It is now straightforward to verify that the map Alg can be extended to a functor from $\mathbf{K}$ to $\mathbf{A}^{\mathrm{op}}$. Furthermore, the map Kpk can also be extended to a functor from $\mathbf{A}^{\mathrm{op}}$ to $\mathbf{K}$ thanks to the following result.

Proposition 3.30 Given a morphism $h : \mathbb{A}' \to \mathbb{A}$, the map $\overleftarrow{h}|_{\Omega_{\mathbb{A}}}$ is a p-morphism from $Kpk(\mathbb{A}) \to Kpk(\mathbb{A}')$.

Proof:

a) For every $t_1, t_2 \in W$, if $t_1 \rightsquigarrow t_2$ then $\overleftarrow{h}|_{\Omega_{\mathbb{A}}}(t_1) \rightsquigarrow' \overleftarrow{h}|_{\Omega_{\mathbb{A}}}(t_2)$. Indeed: Assume that $t_1 \rightsquigarrow t_2$. Then, by definition of $Kpk(\mathbb{A})$, $t_2 \sqsubseteq_{\mathbb{A}} \mathbf{N}_{\mathbb{A}}(t_1)$. So, by property 6) of the algebra morphism, $\overleftarrow{h}|_{\Omega_{\mathbb{A}}}(t_2) \sqsubseteq_{\mathbb{A}'} \overleftarrow{h}|_{\Omega_{\mathbb{A}}}(\mathbf{N}_{\mathbb{A}}(t_1))$. Hence, by property 4) of the algebra morphism and since $\mathbb{A}'$ is appropriate for rule transT, $\overleftarrow{h}|_{\Omega_{\mathbb{A}}}(t_2) \sqsubseteq_{\mathbb{A}'} \mathbf{N}_{\mathbb{A}'}(\overleftarrow{h}|_{\Omega_{\mathbb{A}}}(t_1))$. That is, by definition of $Kpk(\mathbb{A}')$, $\overleftarrow{h}|_{\Omega_{\mathbb{A}}}(t_1) \rightsquigarrow' \overleftarrow{h}|_{\Omega_{\mathbb{A}}}(t_2)$.

b) For every $t_1 \in W$ and $t'_2 \in W'$, if $\overleftarrow{h}|_{\Omega_{\mathbb{A}}}(t_1) \rightsquigarrow' t'_2$ then there is $t_2 \in W$ such that $\overleftarrow{h}|_{\Omega_{\mathbb{A}}}(t_2) = t'_2$ and $t_1 \rightsquigarrow t_2$. Indeed: Assume $\overleftarrow{h}|_{\Omega_{\mathbb{A}}}(t_1) \rightsquigarrow' t'_2$. Then, by construction of $Kpk(\mathbb{A}')$, $t'_2 \sqsubseteq_{\mathbb{A}'} \mathbf{N}_{\mathbb{A}'}(\overleftarrow{h}|_{\Omega_{\mathbb{A}}}(t_1))$. So, by property 4) of the algebra morphism and since $\mathbb{A}'$ is appropriate for rule transT, $t'_2 \in \lceil \overleftarrow{h}|_{\Omega_{\mathbb{A}}}(\mathbf{N}_{\mathbb{A}}(t_1)) \rceil_{\mathbb{A}'}$ and, therefore by property 2) of the algebra morphism, $t'_2 \in \overleftarrow{h}|_{\Omega_{\mathbb{A}}}(\lceil \mathbf{N}_{\mathbb{A}}(t_1) \rceil_{\mathbb{A}})$. Thus, there is $t_2 \in \lceil \mathbf{N}_{\mathbb{A}}(t_1) \rceil_{\mathbb{A}}$ such that $\overleftarrow{h}|_{\Omega_{\mathbb{A}}}(t_2) = t'_2$. That is, there is $t_2 \sqsubseteq_{\mathbb{A}} \mathbf{N}_{\mathbb{A}}(t_1)$ such that $\overleftarrow{h}|_{\Omega_{\mathbb{A}}}(t_2) = t'_2$. Or, equivalently, there is $t_2 \in W$ such that $t_1 \rightsquigarrow t_2$ and $\overleftarrow{h}|_{\Omega_{\mathbb{A}}}(t_2) = t'_2$.

c) For each $f' \in F'$, $(\overleftarrow{h}|_{\Omega_{\mathbb{A}}})^{-1}(\langle f' \rangle) \in \mathcal{B}$. Indeed, $(\overleftarrow{h}|_{\Omega_{\mathbb{A}}})^{-1}(\langle f' \rangle)$ is the set $\{u \in \Omega_{\mathbb{A}} : \overleftarrow{h}|_{\Omega_{\mathbb{A}}}(u) \leq_{\mathbb{A}'} f'\}$ which coincides, thanks to property 7) of the algebra morphism, with $\{u \in \Omega_{\mathbb{A}} : u \leq_{\mathbb{A}} \overrightarrow{h}(f')\}$. The latter set is by definition $\langle \overrightarrow{h}(f') \rangle$ which belongs to $\mathcal{B}$.

d′) $V(\mathbf{p}_i) = (\overleftarrow{h}|_{\Omega_{\mathbb{A}}})^{-1}(V'(\mathbf{p}_i))$. Indeed, $(\overleftarrow{h}|_{\Omega_{\mathbb{A}}})^{-1}(V'(\mathbf{p}_i)) = (\overleftarrow{h}|_{\Omega_{\mathbb{A}}})^{-1}(\langle \mathbf{p}_{i\mathbb{A}'} \rangle)$ which, following the same reasoning as in c) above, is $\langle \overrightarrow{h}(\mathbf{p}_{i\mathbb{A}'}) \rangle$ and, so, by property 1) of the algebra morphism, is identical to $\langle \mathbf{p}_{i\mathbb{A}} \rangle$ which coincides with $V(\mathbf{p}_i)$. QED

We are now ready to address the questions stated at the beginning of this subsection. The relationship between $\mathbb{K}$ and $Kpk(Alg(\mathbb{K}))$ is easy to establish.

Proposition 3.31 Let $\mathbb{K}$ be a general Kripke structure. Then, $\mathbb{K}$ is isomorphic in $\mathbf{K}$ to $Kpk(Alg(\mathbb{K}))$.

Proof: Consider the map $h : W_{\mathbb{K}} \to W_{Kpk(Alg(\mathbb{K}))}$ such that $h(w) = \{w\}$. It is straightforward to show that it is a p-morphism. Furthermore, h is a bijective map and its inverse is also a p-morphism. QED

On the other hand, the relationship between $\mathbb{A}$ and $Alg(Kpk(\mathbb{A}))$ is not so simple. Indeed, the best that we can do is to establish an adjunction between $\mathbf{K}$ and the dual of the subcategory of $\mathbf{A}$ containing the Σ_K-algebras appropriate for $\mathcal{C}_K$ where $\sim$ / $\approx$ is the diagonal relation in F/T, respectively. More precisely, a Σ_K-algebra is said to be *sober* iff:

- $f_1 \sim_{\mathbb{A}} f_2$ iff $f_1 = f_2$;
- $t_1 \approx_{\mathbb{A}} t_2$ iff $t_1 = t_2$.

The full subcategory of $\mathbf{A}$ with all sober algebras is denoted by $\mathbf{sA}$. It is straightforward to establish the inclusion functor $I : \mathbf{sA} \to \mathbf{A}$. In the opposite direction, the quotient functor $Q : \mathbf{A} \to \mathbf{sA}$ can be obtained by the composition $Q_2 \circ Q_1$ where:

- $Q_1 : \mathbf{A} \to \tilde{\mathbf{s}}\mathbf{A}$, denoting by $\tilde{\mathbf{s}}\mathbf{A}$ the full subcategory of $\mathbf{A}$ with all algebras that are sober with respect to $\sim$ (but not necessarily with respect to $\approx$), defined as follows:
  - $Q_1(\mathbb{A}) = \langle F/\sim_{\mathbb{A}}, T, \cdot_{Q_1(\mathbb{A})} \rangle$ where:
    - $c_{Q_1(\mathbb{A})}([f_1]_{\sim_{\mathbb{A}}}, \dots, [f_k]_{\sim_{\mathbb{A}}}) = [c_{\mathbb{A}}(f_1, \dots, f_k)]_{\sim_{\mathbb{A}}}$;
    - $o_{Q_1(\mathbb{A})} = o_{\mathbb{A}}$;
    - $\Omega_{Q_1(\mathbb{A})} = \Omega_{\mathbb{A}}$;
    - $\sqsubseteq_{Q_1(\mathbb{A})} = \sqsubseteq_{\mathbb{A}}$;
    - $\leq_{Q_1(\mathbb{A})} = \{\langle t, [f]_{\sim_{\mathbb{A}}} \rangle : \langle t, f \rangle \in \leq_{\mathbb{A}}\}$.

Note that $Q_1(\mathbb{A})$ is well defined since $\sim_{\mathbb{A}}$ is a congruence relation.

- $Q_1(h : \mathbb{A} \to \mathbb{A}') : Q_1(\mathbb{A}) \to Q_1(\mathbb{A}')$ is the pair $\langle \overrightarrow{Q_1(h)}, \overleftarrow{h} \rangle$ where:
  - $\overrightarrow{Q_1(h)}([f]_{\sim_{\mathbb{A}}}) = [\overrightarrow{h}(f)]_{\sim_{\mathbb{A}'}}$.

  Observe that $Q_1(h)$ is well defined thanks to property 7) of h.

- $Q_2 : \tilde{\mathbf{s}}\mathbf{A} \to \mathbf{sA}$ defined as follows:
  - $Q_2(\mathbb{A}) = \langle F, T/\approx_{\mathbb{A}}, \cdot_{Q_2(\mathbb{A})} \rangle$ where:
    - $c_{Q_2(\mathbb{A})} = c_{\mathbb{A}}$;
    - $o_{Q_2(\mathbb{A})}([t_1]_{\approx_{\mathbb{A}}}, \ldots, [t_k]_{\approx_{\mathbb{A}}}) = [o_{\mathbb{A}}(t_1, \ldots, t_k)]_{\approx_{\mathbb{A}}}$;
    - $\Omega_{Q_2(\mathbb{A})} = \{[u]_{\approx_{\mathbb{A}}} : u \in \Omega_{\mathbb{A}}\}$;
    - $\sqsubseteq_{Q_2(\mathbb{A})} = \{\langle [t_1]_{\approx_{\mathbb{A}}}, [t_2]_{\approx_{\mathbb{A}}} \rangle : \langle t_1, t_2 \rangle \in \sqsubseteq_{\mathbb{A}}\}$;
    - $\leq_{Q_2(\mathbb{A})} = \{\langle [t]_{\approx_{\mathbb{A}}}, f \rangle : \langle t, f \rangle \in \leq_{\mathbb{A}}\}$.

    Note that $Q_2(\mathbb{A})$ is well defined since $\approx_{\mathbb{A}}$ is a congruence relation.
  - $Q_2(h : \mathbb{A} \to \mathbb{A}') : Q_2(\mathbb{A}) \to Q_2(\mathbb{A}')$ is the pair $\langle \overrightarrow{h}, \overleftarrow{Q_2(h)} \rangle$ where:
    - $\overleftarrow{Q_2(h)}([t']_{\approx_{\mathbb{A}'}}) = [\overleftarrow{h}(t')]_{\approx_{\mathbb{A}}}$.

    Observe that $Q_2(h)$ is well defined thanks to property 6) of h.

Let $I_1 : \tilde{\mathbf{s}}\mathbf{A} \to \mathbf{A}$ and $I_2 : \mathbf{sA} \to \tilde{\mathbf{s}}\mathbf{A}$ be the inclusion functors. Then, it is straightforward to show that Q_1 is left adjoint for I_1 and Q_2 is right adjoint for I_2. Hence, $\mathbf{sA}$ is a coreflective subcategory of $\tilde{\mathbf{s}}\mathbf{A}$ and $\tilde{\mathbf{s}}\mathbf{A}$ is a reflective category of $\mathbf{A}$. So, Q corresponds to a canonical construction, in this case by composition of two universal constructions.

In fact, for the purposes of this paper, we could work within $\mathbf{sA}$ instead of working in $\mathbf{A}$ since the following result holds:

Proposition 3.32

$$\mathbb{A}\alpha\beta \Vdash \delta \text{ iff } Q(\mathbb{A})([\ ] \circ \alpha)([\ ] \circ \beta) \Vdash \delta\,.$$

Proof: The result follows straightforwardly, by noting that the two equalities hold: $[[\![\theta]\!]_{\mathbb{A}\alpha\beta}]_{\sim_{\mathbb{A}}} = [\![\theta]\!]_{Q(\mathbb{A})([\ ]\circ\alpha)([\ ]\circ\beta)}$ and $[[\![\varphi]\!]_{\mathbb{A}\alpha}]_{\approx_{\mathbb{A}}} = [\![\varphi]\!]_{Q(\mathbb{A})([\ ]\circ\alpha)}$. QED

Furthermore, observe that $Alg(\mathbb{K})$ is in $\mathbf{sA}$. So, the functor $Alg : \mathbf{K} \to \mathbf{A}$ induces a functor from $\mathbf{K}$ to $\mathbf{sA}$ that we denote by *sAlg*. Letting *sKpk* be the restriction of functor *Kpk* to $\mathbf{sA}$, we now establish the envisaged duality between sober algebras and general Kripke structures.

We start by giving the "road map" of the proof of the theorem:

- we introduce a candidate morphism $\eta_{\mathbb{A}} : sAlgsKpk(\mathbb{A}) \to \mathbb{A}$ for each $\mathbb{A}$ (the unit of the adjunction) namely proposing the definition of the truth value map $\overleftarrow{\eta_{\mathbb{A}}}$ and the formula map $\overrightarrow{\eta_{\mathbb{A}}}$;
- we show that $\eta_{\mathbb{A}}$ is a morphism in $\mathbf{sA}^{\mathrm{op}}$ (done in step (i)). According to the definition this requires checking properties:

- $\overrightarrow{\eta_{\mathbb{A}}}(c_{sAlg(sKpk(\mathbb{A}))}(\langle f_1\rangle_{\mathbb{A}},\dots,\langle f_k\rangle_{\mathbb{A}})) \sim_{\mathbb{A}} c_{\mathbb{A}}(f_1,\dots,f_k)$
  (done in substep (1) for each constructor);
- $\lceil \overleftarrow{\eta_{\mathbb{A}}}(t)\rceil_{sAlg(sKpk(\mathbb{A}))} \subseteq \overleftarrow{\eta_{\mathbb{A}}}(\lceil t\rceil_{\mathbb{A}})$
  (done in substep (2));
- $\overleftarrow{\eta_{\mathbb{A}}}(\bot_{\mathbb{A}}) \sqsubseteq_{sAlg(sKpk(\mathbb{A}))} \bot_{sAlg(sKpk(\mathbb{A}))}$
  (done in substep (3));
- $\mathbf{N}_{sAlg(sKpk(\mathbb{A}))}(\overleftarrow{\eta_{\mathbb{A}}}(t)) \approx_{sAlg(sKpk(\mathbb{A}))} \overleftarrow{\eta_{\mathbb{A}}}(\mathbf{N}_{\mathbb{A}}(t))$
  (done in substep (4));
- if $t \in \Omega_{\mathbb{A}}$ then $\overleftarrow{\eta_{\mathbb{A}}}(t) \in \Omega_{sAlg(sKpk(\mathbb{A}))}$
  (done in substep (5));
- if $t_1 \sqsubseteq_{\mathbb{A}} t_2$ then $\overleftarrow{\eta_{\mathbb{A}}}(t_1) \sqsubseteq_{sAlg(sKpk(\mathbb{A}))} \overleftarrow{\eta_{\mathbb{A}}}(t_2)$
  (done in substep (6));
- $\overleftarrow{\eta_{\mathbb{A}}}(t) \leq_{sAlg(sKpk(\mathbb{A}))} \langle f\rangle_{\mathbb{A}}$ iff $t \leq_{\mathbb{A}} \overrightarrow{\eta_{\mathbb{A}}}(\langle f\rangle_{\mathbb{A}})$ whenever $t \in \Omega_{\mathbb{A}}$
  (done in substep (7));

- we show the universal property: given any $\mathbb{K}'$ in $\mathbf{K}$ and any morphism $h : \mathbb{A} \to sAlg(\mathbb{K}')$ in $\mathbf{sA}^{\mathrm{op}}$, there is a unique morphism $h' : sKpk(\mathbb{A}) \to \mathbb{K}'$ in $\mathbf{K}$ such that $sAlg(h') \circ \eta_{\mathbb{A}} = h$ (done in step (ii));
  - definition of candidate morphism h' (substep existence);
  - we show that h' is a a morphism in $\mathbf{K}$:
    * if $w_1 \rightsquigarrow_{sKpk(\mathbb{A})} w_2$ then $h'(w_1) \rightsquigarrow_{\mathbb{K}'} h'(w_2)$
      (done in substep (a));
    * if $h'(w_1) \rightsquigarrow_{\mathbb{K}'} w_2'$ then there is w_2 such that $h'(w_2) = w_2'$ and $w_1 \rightsquigarrow_{sKpk(\mathbb{A})} w_2$
      (done in substep (b));
    * $h'^{-1}(B') \in \mathcal{B}_{sKpk(\mathbb{A})}$ for every $B' \in \mathcal{B}_{\mathbb{K}'}$
      (done in substep (c));
    * $V_{sKpk(\mathbb{A})}(\mathbf{p}_i) = h'^{-1}(V_{\mathbb{K}'}(\mathbf{p}_i))$
      (done in substep (d'));
  - we show the commutativity of the truth value maps;
    (done in substep (e));
  - we show the commutativity of the formula maps;
    (done in substep (f));
  - finally, we show the unicity of h'
    (done in substep (unicity)).

Theorem 3.33 (Duality) Functor $sKpk : \mathbf{sA}^{\mathrm{op}} \to \mathbf{K}$ is left adjoint for functor $sAlg : \mathbf{K} \to \mathbf{sA}^{\mathrm{op}}$.

Proof:

Let $\mathbb{A} = \langle F, T, \cdot_{\mathbb{A}} \rangle$ be a sober Σ_K-algebra. Denote the set $\{\langle f \rangle_{\mathbb{A}} : f \in F\}$ by $\langle F \rangle_{\mathbb{A}}$. Consider the maps:

- $\overrightarrow{\eta_{\mathbb{A}}} : \langle F \rangle_{\mathbb{A}} \to F$ defined as follows: $\overrightarrow{\eta_{\mathbb{A}}}(\langle f \rangle_{\mathbb{A}}) = f$;
- $\overleftarrow{\eta_{\mathbb{A}}} : T \to \wp\Omega_{\mathbb{A}}$ defined as follows: $\overleftarrow{\eta_{\mathbb{A}}}(t) = \lceil t \rceil_{\mathbb{A}}$.

(i) We first show that the pair $\eta_{\mathbb{A}} = \langle \overrightarrow{\eta_{\mathbb{A}}}, \overleftarrow{\eta_{\mathbb{A}}} \rangle$ is a morphism in $\mathbf{sA}^{\mathrm{op}}$ from $\mathbb{A}$ to $sAlg(sKpk(\mathbb{A}))$. That is, $\eta_{\mathbb{A}}$ is a morphism in $\mathbf{sA}$ from $sAlg(sKpk(\mathbb{A}))$ to $\mathbb{A}$. Indeed:

(1) $\overrightarrow{\eta_{\mathbb{A}}}(c_{sAlg(sKpk(\mathbb{A}))}(\langle f_1 \rangle_{\mathbb{A}}, \ldots, \langle f_k \rangle_{\mathbb{A}})) \sim_{\mathbb{A}} c_{\mathbb{A}}(f_1, \ldots, f_k)$ holds for each representative constructor:

($\mathbf{f}$) Observe that $\mathbf{f}_{sAlg(sKpk(\mathbb{A}))} = \emptyset$ by definition of *sAlg*. And so is $\langle \mathbf{f}_{\mathbb{A}} \rangle_{\mathbb{A}}$ because $\mathbb{A}$ is appropriate for rules R$\mathbf{f}$ and $\Omega\bot$. Therefore: $\langle \overrightarrow{\eta_{\mathbb{A}}}(\mathbf{f}_{sAlg(sKpk(\mathbb{A}))}) \rangle_{\mathbb{A}} = \langle \overrightarrow{\eta_{\mathbb{A}}}(\emptyset) \rangle_{\mathbb{A}} = \langle \overrightarrow{\eta_{\mathbb{A}}}(\langle \mathbf{f}_{\mathbb{A}} \rangle_{\mathbb{A}}) \rangle_{\mathbb{A}} = \langle \mathbf{f}_{\mathbb{A}} \rangle_{\mathbb{A}}$.

($\mathbf{p}_i$) Note that $\mathbf{p}_{i sAlg(sKpk(\mathbb{A}))} = V_{sKpk(\mathbb{A})}(\mathbf{p}_i)$ by definition of *sAlg*. And the latter is equal to $\langle \mathbf{p}_{i\mathbb{A}} \rangle_{\mathbb{A}}$ by definition of *sKpk*. Thus: $\langle \overrightarrow{\eta_{\mathbb{A}}}(\mathbf{p}_{i sAlg(sKpk(\mathbb{A}))}) \rangle_{\mathbb{A}} = \langle \overrightarrow{\eta_{\mathbb{A}}}(V_{sKpk(\mathbb{A})}(\mathbf{p}_i)) \rangle_{\mathbb{A}} = \langle \overrightarrow{\eta_{\mathbb{A}}}(\langle \mathbf{p}_{i\mathbb{A}} \rangle_{\mathbb{A}}) \rangle_{\mathbb{A}} = \langle \mathbf{p}_{i\mathbb{A}} \rangle_{\mathbb{A}}$.

($\neg$) $\overrightarrow{\eta_{\mathbb{A}}}(\neg_{sAlg(sKpk(\mathbb{A}))}(\langle f \rangle_{\mathbb{A}})) = \overrightarrow{\eta_{\mathbb{A}}}(\Omega_{\mathbb{A}} \setminus \langle f \rangle_{\mathbb{A}}) = \overrightarrow{\eta_{\mathbb{A}}}(\{u \in \Omega_{\mathbb{A}} : u \not\leq_{\mathbb{A}} f\})$ which, as seen in part (3) of Proposition 3.19, is $\overrightarrow{\eta_{\mathbb{A}}}(\{u \in \Omega_{\mathbb{A}} : u \leq_{\mathbb{A}} \neg_{\mathbb{A}}(f)\}) = \overrightarrow{\eta_{\mathbb{A}}}(\langle \neg_{\mathbb{A}}(f) \rangle_{\mathbb{A}}) = \neg_{\mathbb{A}}(f) = \neg_{\mathbb{A}}(\overrightarrow{\eta_{\mathbb{A}}}(\langle f \rangle_{\mathbb{A}}))$.

($\Box$) $\overrightarrow{\eta_{\mathbb{A}}}(\Box_{sAlg(sKpk(\mathbb{A}))}(\langle f \rangle_{\mathbb{A}})) = \overrightarrow{\eta_{\mathbb{A}}}(\{u \in \Omega_{\mathbb{A}} : \mathbf{N}_{sAlg(sKpk(\mathbb{A}))}(\{u\}) \subseteq \langle f \rangle_{\mathbb{A}}\}) = \overrightarrow{\eta_{\mathbb{A}}}(\{v \in \Omega_{\mathbb{A}} : u \rightsquigarrow_{sKpk(\mathbb{A})} v \text{ implies } v \in \langle f \rangle_{\mathbb{A}}\})$ which, as seen in part (5) of Proposition 3.19, coincides with $\overrightarrow{\eta_{\mathbb{A}}}(\{u \in \Omega_{\mathbb{A}} : u \leq_{\mathbb{A}} \Box_{\mathbb{A}}(f)\})$ and, therefore, is equal to $\overrightarrow{\eta_{\mathbb{A}}}(\langle \Box_{\mathbb{A}}(f) \rangle_{\mathbb{A}}) = \Box_{\mathbb{A}}(f) = \Box_{\mathbb{A}}(\overrightarrow{\eta_{\mathbb{A}}}(\langle f \rangle_{\mathbb{A}}))$.

($\vee$) $\overrightarrow{\eta_{\mathbb{A}}}(\vee_{sAlg(sKpk(\mathbb{A}))}(\langle f \rangle_{\mathbb{A}}, \langle g \rangle_{\mathbb{A}})) = \overrightarrow{\eta_{\mathbb{A}}}(\{u \in \Omega_{\mathbb{A}} : u \leq_{\mathbb{A}} f \text{ or } u \leq_{\mathbb{A}} g\})$ and, so, as in part (4) of Proposition 3.19, equal to $\overrightarrow{\eta_{\mathbb{A}}}(\{u \in \Omega_{\mathbb{A}} : u \leq_{\mathbb{A}} \vee_{\mathbb{A}}(f, g)\}) = \overrightarrow{\eta_{\mathbb{A}}}(\langle \vee_{\mathbb{A}}(f, g) \rangle_{\mathbb{A}}) = \vee_{\mathbb{A}}(f, g) = \vee_{\mathbb{A}}(\overrightarrow{\eta_{\mathbb{A}}}(\langle f \rangle_{\mathbb{A}}), \overrightarrow{\eta_{\mathbb{A}}}(\langle g \rangle_{\mathbb{A}}))$.

(2) $\lceil \overleftarrow{\eta_{\mathbb{A}}}(t) \rceil_{sAlg(sKpk(\mathbb{A}))} \subseteq \overleftarrow{\eta_{\mathbb{A}}}(\lceil t \rceil_{\mathbb{A}})$. Indeed, assume $\{u\} \in \lceil \overleftarrow{\eta_{\mathbb{A}}}(t) \rceil_{sAlg(sKpk(\mathbb{A}))}$. Therefore, $u \in \Omega_{\mathbb{A}}$. Furthermore, $\{u\} \subseteq \overleftarrow{\eta_{\mathbb{A}}}(t)$. So, $\{u\} \subseteq \lceil t \rceil_{\mathbb{A}}$. That is, $u \in \lceil t \rceil_{\mathbb{A}}$ and, so, $u \sqsubseteq_{\mathbb{A}} t$. On the other hand, $\overleftarrow{\eta_{\mathbb{A}}}(\lceil t \rceil_{\mathbb{A}}) = \{\lceil v \rceil_{\mathbb{A}} : v \in \Omega_{\mathbb{A}} \text{ and } v \sqsubseteq_{\mathbb{A}} t\}$ which, since $\mathbb{A}$ is sober and appropriate for rule Rasym, is equal to $\{\{v\} : v \in \Omega_{\mathbb{A}} \text{ and } v \sqsubseteq_{\mathbb{A}} t\}$ which obviously contains $\{u\}$.

(3) $\overleftarrow{\eta_{\mathbb{A}}}(\bot_{\mathbb{A}}) \sqsubseteq_{sAlg(sKpk(\mathbb{A}))} \bot_{sAlg(sKpk(\mathbb{A}))}$. Indeed, thanks to the definition of *sAlg*, this assertion is equivalent to $\overleftarrow{\eta_{\mathbb{A}}}(\bot_{\mathbb{A}}) \subseteq \emptyset$. Moreover, $\overleftarrow{\eta_{\mathbb{A}}}(\bot_{\mathbb{A}}) = \{u \in \Omega_{\mathbb{A}} : u \sqsubseteq_{\mathbb{A}} \bot_{\mathbb{A}}\}$ which is the empty set since $\mathbb{A}$ is appropriate for rule $\Omega\bot$.

(4) $\mathbf{N}_{sAlg(sKpk(\mathbb{A}))}(\overleftarrow{\eta_{\mathbb{A}}}(t)) \approx_{sAlg(sKpk(\mathbb{A}))} \overleftarrow{\eta_{\mathbb{A}}}(\mathbf{N}_{\mathbb{A}}(t))$. Indeed, the former set is equal to $\mathbf{N}_{sAlg(sKpk(\mathbb{A}))}(\{u \in \Omega_{\mathbb{A}} : u \sqsubseteq_{\mathbb{A}} t\})$ which, by the definition of the two functors, coincides with $\{v \in \Omega_{\mathbb{A}} : \text{there is } u \in \Omega_{\mathbb{A}} \text{ such that } u \sqsubseteq_{\mathbb{A}} t \text{ and } v \sqsubseteq_{\mathbb{A}} \mathbf{N}_{\mathbb{A}}(u)\}$. Finally, this set is equal to $\{v \in \Omega_{\mathbb{A}} : v \sqsubseteq_{\mathbb{A}} \mathbf{N}_{\mathbb{A}}(t)\}$ because $\mathbb{A}$ is appropriate for rules R$\mathbf{N}\Omega$ and L$\mathbf{N}\Omega$.

(5) If $t \in \Omega_{\mathbb{A}}$ then $\overleftarrow{\eta_{\mathbb{A}}}(t) \in \Omega_{sAlg(sKpk(\mathbb{A}))}$. Indeed, assume $t \in \Omega_{\mathbb{A}}$. Then,

$\overleftarrow{\eta_{\mathbb{A}}}(t) = \{u \in \Omega_{\mathbb{A}} : u \sqsubseteq_{\mathbb{A}} t\}$ which is the singleton set $\{t\}$ since t is atomic and $\mathbb{A}$ is sober and appropriate for rule Rasym. Therefore, $\overleftarrow{\eta_{\mathbb{A}}}(t)$ is atomic in $sAlg(sKpk(\mathbb{A}))$.

(6) If $t_1 \sqsubseteq_{\mathbb{A}} t_2$ then $\overleftarrow{\eta_{\mathbb{A}}}(t_1) \sqsubseteq_{sAlg(sKpk(\mathbb{A}))} \overleftarrow{\eta_{\mathbb{A}}}(t_2)$. Indeed, the latter assertion is equivalent to $\{u \in \Omega_{\mathbb{A}} : u \sqsubseteq_b At_1\} \subseteq \{u \in \Omega_{\mathbb{A}} : u \sqsubseteq_b At_2\}$. Assuming that $t_1 \sqsubseteq_{\mathbb{A}} t_2$, the assertion holds thanks to the fact that $\mathbb{A}$ is appropriate for rule transT.

(7) $\overleftarrow{\eta_{\mathbb{A}}}(t) \leq_{sAlg(sKpk(\mathbb{A}))} \langle f \rangle_{\mathbb{A}}$ iff $t \leq_{\mathbb{A}} \overrightarrow{\eta_{\mathbb{A}}}(\langle f \rangle_{\mathbb{A}})$ whenever $t \in \Omega_{\mathbb{A}}$. Indeed, $\overleftarrow{\eta_{\mathbb{A}}}(t) = \{v \in \Omega_{\mathbb{A}} : v \sqsubseteq_{\mathbb{A}} t\} = \{t\}$ as seen in (5) above. So, $\overleftarrow{\eta_{\mathbb{A}}}(t) \leq_{sAlg(sKpk(\mathbb{A}))} \langle f \rangle_{\mathbb{A}}$ iff $\{t\} \subseteq \langle f \rangle_{\mathbb{A}}$ iff $\{t\} \subseteq \{v \in \Omega_{\mathbb{A}} : v \leq_{\mathbb{A}} f\}$ iff $t \leq_{\mathbb{A}} f$ iff $t \leq_{\mathbb{A}} \overrightarrow{\eta_{\mathbb{A}}}(\langle f \rangle_{\mathbb{A}})$.

(ii) We now show that the family η, where each $\eta_{\mathbb{A}}$ is the morphism in $\mathbf{sA}^{\mathrm{op}}$ from $\mathbb{A}$ to $sAlg(sKpk(\mathbb{A}))$ defined above, is the unit of the envisaged adjunction. More precisely, we show that, given any $\mathbb{K}'$ in $\mathbf{K}$ and any morphism $h : \mathbb{A} \to sAlg(\mathbb{K}')$ in $\mathbf{sA}^{\mathrm{op}}$, there is a unique morphism $h' : sKpk(\mathbb{A}) \to \mathbb{K}'$ in $\mathbf{K}$ such that $sAlg(h') \circ \eta_{\mathbb{A}} = h$.

(existence) Consider h' defined as follows: for each world $w \in W_{sKpk(\mathbb{A})} = \Omega_{\mathbb{A}}$, $h'(w) = \iota(\{\overleftarrow{h}(w)\}) = \overleftarrow{h}(w)$. Observe that h' is well defined because of property 5) of h and the fact that the atomic elements in $sAlg(\mathbb{K}')$ are the singleton subsets of $W_{\mathbb{K}'}$. First, we show that the map h' is a morphism in $\mathbf{K}$ from $sKpk(\mathbb{A})$ to $\mathbb{K}'$:

(a) If $w_1 \rightsquigarrow_{sKpk(\mathbb{A})} w_2$ then $h'(w_1) \rightsquigarrow_{\mathbb{K}'} h'(w_2)$. Indeed, assume $w_1 \rightsquigarrow_{sKpk(\mathbb{A})} w_2$. Then, $w_2 \sqsubseteq_{\mathbb{A}} \mathbf{N}_{\mathbb{A}}(w_1)$. So, by property 6) of morphism h, $\overleftarrow{h}(w_2) \sqsubseteq_{sAlg(\mathbb{K}')} \overleftarrow{h}(\mathbf{N}_{\mathbb{A}}(w_1))$. Hence, by property 4) of h, $\overleftarrow{h}(w_2) \sqsubseteq_{sAlg(\mathbb{K}')} \mathbf{N}_{sAlg(\mathbb{K}')}(\overleftarrow{h}(w_1))$. Hence, $h'(w_1) \rightsquigarrow_{\mathbb{K}'} h'(w_2)$.

(b) If $h'(w_1) \rightsquigarrow_{\mathbb{K}'} w_2'$ then there is w_2 such that $h'(w_2) = w_2'$ and $w_1 \rightsquigarrow_{sKpk(\mathbb{A})} w_2$. Indeed, from $h'(w_1) \rightsquigarrow_{\mathbb{K}'} w_2'$ we conclude $\{w_2'\} \subseteq \mathbf{N}_{sAlg(\mathbb{K}')}(\{h'(w_1)\})$, that is, $\{w_2'\} \subseteq \mathbf{N}_{sAlg(\mathbb{K}')}(\overleftarrow{h}(w_1))$. Thus, by property 4) of h, $\{w_2'\} \subseteq \overleftarrow{h}(\mathbf{N}_{\mathbb{A}}(w_1))$ and, so, $\{w_2'\} \in \lceil \overleftarrow{h}(\mathbf{N}_{\mathbb{A}}(w_1)) \rceil_{sAlg(\mathbb{K}')}$. Hence, thanks to property 2) of h, $\{w_2'\} \in \overleftarrow{h}(\lceil \mathbf{N}_{\mathbb{A}}(w_1) \rceil_{\mathbb{A}})$. Therefore, there is $\overleftarrow{h}(w_2) = w_2'$ such that $w_2 \in \lceil \mathbf{N}_{\mathbb{A}}(w_1) \rceil_{\mathbb{A}}$. That is, there is $\overleftarrow{h}(w_2) = w_2'$ such that $w_1 \rightsquigarrow_{sKpk(\mathbb{A})} w_2$.

(c) $h'^{-1}(B') \in \mathcal{B}_{sKpk(\mathbb{A})}$ for every $B' \in \mathcal{B}_{\mathbb{K}'}$. Indeed, observe that $h'^{-1}(B') = \{w \in \Omega_{\mathbb{A}} : h'(w) \in B'\} = \{w \in \Omega_{\mathbb{A}} : \overleftarrow{h}(w) \subseteq B'\}$. Using property 7) of h, the latter set coincides with $\{w \in \Omega_{\mathbb{A}} : w \sqsubseteq_{\mathbb{A}} \overrightarrow{h}(B')\} = \langle \overrightarrow{h}(B') \rangle_{\mathbb{A}}$ which is in $\langle F_{\mathbb{A}} \rangle = \mathcal{B}_{sKpk(\mathbb{A})}$.

(d′) $V_{sKpk(\mathbb{A})}(\mathbf{p}_i) = h'^{-1}(V_{\mathbb{K}'}(\mathbf{p}_i))$. Indeed, as seen in (c) above, $h'^{-1}(V_{\mathbb{K}'}(\mathbf{p}_i)) = \langle \overrightarrow{h}(V_{\mathbb{K}'}(\mathbf{p}_i)) \rangle_{\mathbb{A}}$. The latter set, by definition of $sAlg(\mathbb{K}')$, is $\langle \overrightarrow{h}(\mathbf{p}_{i sAlg(\mathbb{K}')}) \rangle_{\mathbb{A}}$ which, thanks to property 1) of h, coincides with $\langle \mathbf{p}_{i\mathbb{A}} \rangle_{\mathbb{A}}$ and, so, with $V_{sKpk(\mathbb{A})}(\mathbf{p}_i)$.

We now verify that $sAlg(h') \circ \eta_{\mathbb{A}} = h$:

(e) $\overleftarrow{sAlg(h')} \circ \overleftarrow{\eta_{\mathbb{A}}} = \overleftarrow{h}$. Indeed, for each $t \in T_{\mathbb{A}}$, $\overleftarrow{sAlg(h')}(\overleftarrow{\eta_{\mathbb{A}}}(t)) = \overleftarrow{sAlg(h')}(\lceil t \rceil_{\mathbb{A}})$.

Furthermore, by definition of $sAlg(h')$, the latter is $h'(\lceil t \rceil_{\mathbb{A}})$. We show that, for every $w' \in W_{\mathbb{K}'}$, $w' \in h'(\lceil t \rceil_{\mathbb{A}})$ iff $w' \in \overleftarrow{h}(t)$:

($\Rightarrow$) Assume $w' \in h'(\lceil t \rceil_{\mathbb{A}})$. Then, there is $u \in \lceil t \rceil_{\mathbb{A}}$ such that $h'(u) = w'$. That is, there is $u \in \Omega_{\mathbb{A}}$ such that $u \sqsubseteq_{\mathbb{A}} t$ and $\overleftarrow{h}(u) = \{w'\}$. Thus, by property 6) of h, there is $u \in \Omega_{\mathbb{A}}$ such that $\overleftarrow{h}(u) \subseteq \overleftarrow{h}(t)$ and $\overleftarrow{h}(u) = \{w'\}$. Therefore, $w' \in \overleftarrow{h}(t)$.

($\Leftarrow$) Assume $w' \in \overleftarrow{h}(t)$. That is, $\{w'\} \subseteq \overleftarrow{h}(t)$. Hence, $\{w'\} \in \lceil \overleftarrow{h}(t) \rceil_{sAlg(\mathbb{K}')}$. Thus, by property 3) of h, $\{w'\} \in \overleftarrow{h}(\lceil t \rceil_{\mathbb{A}})$ and, so, $w' \in h'(\lceil t \rceil_{\mathbb{A}})$.

(f) $\overrightarrow{\eta_{\mathbb{A}}} \circ \overrightarrow{sAlg(h')} = \overrightarrow{h}$. Indeed, for each $B' \in F_{sAlg(\mathbb{K}')} = \mathcal{B}_{\mathbb{K}'}$, $\overrightarrow{\eta_{\mathbb{A}}}(\overrightarrow{sAlg(h')}(B')) = \overrightarrow{\eta_{\mathbb{A}}}(h'^{-1}(B')) = \overrightarrow{\eta_{\mathbb{A}}}(\langle \overrightarrow{h}(B') \rangle_{\mathbb{A}}) = \overrightarrow{h}(B')$.

(unicity) Observe that there is a unique choice of h' that ensures the diagram comutes. Namely, h' should be chosen in order to guarantee that $\overleftarrow{sAlg(h')} \circ \overleftarrow{\eta_{\mathbb{A}}} = \overleftarrow{h}$. So, in particular, for each $w \in W_{sKpk(\mathbb{A})} = \Omega_{\mathbb{A}}$, h' should be chosen in order to guarantee that $h'(\lceil w \rceil_{\mathbb{A}}) = \overrightarrow{h}(w)$. That is, h' should be chosen in order to ensure $h'(\{w\}) = \overrightarrow{h}(w)$, that is, $\{h'(w)\} = \overrightarrow{h}(w)$. Therefore, the choice of h' is unique. QED

3.5 Analyticity of the modal sequent calculi

A rule $r = \langle \{s_1, \ldots, s_p\}, s, \pi \rangle$ is said to be a *creative rule* if there are meta-variables in $\{s_1, \ldots, s_p\}$ that do not occur in s. Most of the structural, order and modal rules introduced in Section 2 are non creative. The obvious exceptions are:

- structural: cutT and cutF;
- order: $\Omega\top$, Ω, transT, transF, LgenT, RgenT, LgenF and RgenF;
- system K: L**N**Ω, R**N**Ω, L$\Diamond$ and R$\Diamond$;
- other modal systems: 5 and C.

We distinguish two kinds of creative rules. A creative rule is said to be a *blindly creative rule* if when applying the rule in a derivation, the instance of the additional meta-variable in the premises is chosen as a fresh variable (in our case a bounded truth value variable $\mathbf{y}$). An example of a blindly creative rule is $\Omega\top$.

On the other hand, a creative rule is said to be a *strongly creative rule* if when applying the rule in a derivation, the instance of the additional meta-variable in the premises is not necessarily a fresh variable. In the case of the sequent calculi at hand, we have the following strongly creative rules:

- structural: cutT and cutF;
- order: transT, transF, LgenT and LgenF;

- system K: R$\mathbf{N}\Omega$ and R$\Diamond$;
- other modal systems: 5.

A derivation $\langle d_1, \pi_1\rangle, \dots, \langle d_n, \pi_n\rangle$ of s' from S is said to be *analytical* if the instance of each additional meta-variable when applying strongly creative rules at step i has to occur either in $d_1, \dots, d_{i-1}$ or in S. A sequent calculus is said to be *analytical* if all derivations are analytical. Clearly, our sequent calculus for modal logic is not analytical since it allows derivations that are not analytical.

The importance of this concept comes from the fact that derivations can be automated. We can make our calculus analytic by keeping all non strongly creative rule and introducing restrictive versions of strongly creative rules as follows:

$$\text{cutT}^\bullet \quad \frac{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \tau_2 \quad \tau_1 \sqsubseteq \tau_2, \Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2} \lhd \tau_1, \tau_2 \in \Gamma_1, \Gamma_2$$

$$\text{cutF}^\bullet \quad \frac{\Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1 \quad \tau_1 \leq \xi_1, \Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2} \lhd \tau_1, \xi_1 \in \Gamma_1, \Gamma_2$$

$$\text{transT}^\bullet \quad \frac{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \tau_2 \quad \Gamma_1 \to \Gamma_2, \tau_2 \sqsubseteq \tau_3}{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \tau_3} \lhd \tau_2 \in \tau_1, \tau_3, \Gamma_1, \Gamma_2$$

$$\text{transF}^\bullet \quad \frac{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \tau_2 \quad \Gamma_1 \to \Gamma_2, \tau_2 \leq \xi_1}{\Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1} \lhd \tau_2 \in \tau_1, \Gamma_1, \Gamma_2$$

$$\text{LgenT}^\bullet \quad \frac{\Omega\tau_2, \tau_2 \sqsubseteq \tau_3, \Gamma_1 \to \Gamma_2 \quad \Omega\tau_2, \Gamma_1 \to \Gamma_2, \tau_2 \sqsubseteq \tau_1 \quad \tau_1 \sqsubseteq \tau_3, \Gamma_1 \to \Gamma_2, \Omega\tau_2}{\tau_1 \sqsubseteq \tau_3, \Gamma_1 \to \Gamma_2} \lhd \tau_2 \in \tau_1, \tau_3, \Gamma_1, \Gamma_2$$

$$\text{LgenF}^\bullet \quad \frac{\Omega\tau_2, \tau_2 \leq \xi_1, \Gamma_1 \to \Gamma_2 \quad \Omega\tau_2, \Gamma_1 \to \Gamma_2, \tau_2 \sqsubseteq \tau_1 \quad \tau_1 \leq \xi_1, \Gamma_1 \to \Gamma_2, \Omega\tau_2}{\tau_1 \leq \xi_1, \Gamma_1 \to \Gamma_2} \lhd \tau_2 \in \tau_1, \Gamma_1, \Gamma_2$$

$$\text{R}\mathbf{N}\Omega^\bullet \quad \frac{\Omega\tau_2, \Gamma_1 \to \Gamma_2, \Omega\tau_3 \quad \Omega\tau_3, \Omega\tau_2, \Gamma_1 \to \Gamma_2, \tau_3 \sqsubseteq \tau_1 \quad \Omega\tau_3, \Omega\tau_2, \Gamma_1 \to \Gamma_2, \tau_2 \sqsubseteq \mathbf{N}(\tau_3)}{\Omega\tau_2, \Gamma_1 \to \Gamma_2, \tau_2 \sqsubseteq \mathbf{N}(\tau_1)} \lhd \tau_3 \in \tau_1, \tau_2, \Gamma_1, \Gamma_2$$

$$\text{R}\Diamond^\bullet \quad \frac{\Omega\tau_1, \Omega\tau_2, \Gamma_1 \to \Gamma_2, \tau_2 \sqsubseteq \mathbf{N}(\tau_1) \quad \Omega\tau_1, \Omega\tau_2, \Gamma_1 \to \Gamma_2, \tau_2 \leq \xi_1 \quad \Omega\tau_1, \Gamma_1 \to \Gamma_2, \Omega\tau_2}{\Omega\tau_1, \Gamma_1 \to \Gamma_2, \tau_1 \leq (\Diamond\xi_1)} \lhd \tau_2 \in \tau_1, \Gamma_1, \Gamma_2$$

$$5^\bullet \quad \frac{\Omega\tau_1, \Omega\tau_2, \Omega\tau_3, \Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \mathbf{N}(\tau_3) \quad \Omega\tau_1, \Omega\tau_2, \Omega\tau_3, \Gamma_1 \to \Gamma_2, \tau_2 \sqsubseteq \mathbf{N}(\tau_3) \quad \Omega\tau_1, \Omega\tau_2, \Gamma_1 \to \Gamma_2, \Omega\tau_3}{\Omega\tau_1, \Omega\tau_2, \Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \mathbf{N}(\tau_2)} \lhd \tau_3 \in \tau_1, \tau_2, \Gamma_1, \Gamma_2$$

where, for instance, the proviso $(\tau_2 \in \tau_1, \Gamma_1, \Gamma_2)(\rho) = 1$ iff $\rho(\tau_2)$ occurs in $\{\rho(\tau_1)\} \cup \rho(\Gamma_1) \cup \rho(\Gamma_2)$ or in the hypotheses. Clearly, the other provisos have a similar meaning.

For each finite set $\mathcal{P}$ of properties of the accessibility relation considered in Subsection 2.6, let also $\mathcal{C}^\bullet_\mathcal{P}$ denote the restricted version of $\mathcal{C}_\mathcal{P}$.

Can we guarantee that by restricting ourselves only to analytical applications of these strongly creative rules nothing is lost? That is, the sequent calculi remain complete if we restrict the use of these rules?

Fortunately, the answer is positive. Indeed, careful examination of the proofs of the relevant metatheorems in Subsection 2.9 and of the completeness result in Subsection 3.2, shows that appropriateness for each of these rules was invoked only in its restricted form. As a consequence, we establish the following extension of Theorem 3.11.

Theorem 3.34 (Algebraic completeness) Every full structural sequent calculus endowed with rules using only persistent provisos and restrictive versions of cutT$^\bullet$ and cutF$^\bullet$ is complete.

Proof: Since this completeness result was established using strongly creative rules in their restricted form, the proof is the same as in Theorem 3.11. QED

This result can be further extended to sequent calculi for modal logic since the proofs in Subsection 3.3 only use strongly creative rules in their restrictive form. As a consequence, we establish an extension of Theorem 3.25.

Theorem 3.35 (Modal Kripke soundness and completeness) For each finite set $\mathcal{P}$ of properties of the accessibility relation:

- $\psi_1, \ldots, \psi_k \vDash^{\mathrm{g}}_{\mathcal{K}_{\mathcal{P}}} \varphi$ iff $\psi_1, \ldots, \psi_k \vdash^{\mathrm{g}}_{\mathcal{C}^{\bullet}_{\mathcal{P}}} \varphi$;
- $\psi_1, \ldots, \psi_k \vDash^{\ell}_{\mathcal{K}_{\mathcal{P}}} \varphi$ iff $\psi_1, \ldots, \psi_k \vdash^{\ell}_{\mathcal{C}^{\bullet}_{\mathcal{P}}} \varphi$.

Proof: We only show the global case since the local case is similar.

Assume that $\psi_1, \ldots, \psi_k \vDash^{\mathrm{g}}_{\mathcal{K}_{\mathcal{P}}} \varphi$, then by Theorem 3.25 we have $\psi_1, \ldots, \psi_k \vdash^{\mathrm{g}}_{\mathcal{C}_{\mathcal{P}}} \varphi$. Since to establish the latter theorem we only used strongly creative rules in their restrictive form, the same derivation can be used to establish $\psi_1, \ldots, \psi_k \vdash^{\mathrm{g}}_{\mathcal{C}^{\bullet}_{\mathcal{P}}} \varphi$. The converse is straightforward. QED

4 Concluding remarks

By using truth values as labels, we were able to provide analytical labelled sequent calculi for a wide class of normal modal systems sharing a common core of rules (structural rules, order rules, and rules for the formula constructors). The calculus for each modal system is obtained by adding to this common core rules on the truth values imposing the envisaged properties of the accessibility relation. We also managed to keep an effective separation between the sub-calculus on the formulae and the sub-calculus on the truth values. Therefore, the way is open for proving desirable proof-theoretic properties of, at least, the formula sub-calculus like, for instance, the elimination of rule cutF (cf the results in [19] where a similar separation was explored in the case of a natural deduction calculus labelled with worlds for obtaining normalization results).

These calculi were shown to be strongly complete (for both global and local reasoning) with respect to the novel two-sorted algebraic semantics, and, as a corollary, also with respect to the general Kripke semantics. For this purpose, we had to indicate how to move back and forth between such algebras and general Kripke structures. This led to a duality between the category of sober algebras and the category of general Kripke structures (with p-morphisms). A simple enrichment of the language (by adding the coercion operator @) was shown to allow reasoning complete with respect to standard Kripke structures.

It is worthwhile to comment on the nature of the two-sorted algebras. In a sense they are halfway between general Kripke structures and modal algebras. We may look at them as general Kripke structures without points. The main results of the paper clearly show that the loss of the points was possible while preserving the ability to spell out the envisaged properties of the accessibility relation. One might argue that we do keep points through the Ω predicate. To some extent this is a fair observation, but we should stress that the calculi impose very little about the atomic truth values, for sure much less than what is implied by set theory about singletons.

The proposed sequent calculi labelled with truth values are relevant to automation and to the theory and applications of the combination of deduction systems. Furthermore, the two-sorted algebraic semantics introduced herein seems to achieve the right balance between modal algebras and Kripke structures by sharing the advantages of both. Indeed, while algebraic in flavor, this new semantics still allows the specification of modal systems with rules reflecting the envisaged properties of the accessibility relation, fully characterizing those systems even among general Kripke structures.

Capitalizing on the results of this paper, several lines of research are evident at this stage. First, taking advantage of the separation between the formula sub-calculus and the truth value sub-calculus, there is hope of obtaining useful proof-theoretic results about the proposed modal calculi (for instance, elimination of rule cutF and control of the application of other rules such as contractions), towards making the calculi even more interesting from an automation point of view. Second, extrapolating from the preliminary ideas in Subsection 2.7, the relationship to hybrid logic seems to be a fruitful line of research, for instance towards a hybrid logic over truth values (instead of worlds). Third, given the generality of the approach (well illustrated by the algebraic completeness theorem that requires very little from the calculus at hand), it is feasible to set up sequent calculi labelled with truth values for other types of logic (like intuitionistic, relevance, many-valued logics) and develop a theory of fibring such calculi (where the general algebraic completeness result obtained in this paper can be used in the style of [21, 17] for establishing the preservation of completeness by fibring).

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