Exogenous Semantics Approach to Enriching Logics

Paulo Mateus, Amílcar Sernadas, and Cristina Sernadas

Abstract. The exogenous semantics approach to enriching a logic consists in defining each model in the enrichment as a set of models in the original logic plus some relevant structure. We illustrate the approach by probabilizing classical propositional logic, including a novel global propositional logic. A model of the probability logic is a probability space where the outcomes are classical valuations. A model of the global propositional logic is simply a set of classical valuations. Syntactically, probabilities appear as constructors of new terms from classical formulas. Soundness and completeness results are proved for the calculi of both logics. A simple zero-knowledge protocol is used to illustrate the capabilities of the proposed probability logic.

1 Introduction

The exogenous semantics approach to enriching a logic consists in defining each model in the enrichment as a set of models in the original logic plus some relevant structure. We illustrate the approach by probabilizing classical propositional logic.

Adding probability features to a base logic is a recurrent research topic [2, 11, 23, 24]. Carnap, [4], was one of the first attempting to combine logic and probability. The issue is not an easy one since there is the need to accommodate the continuous nature of probabilities (namely dealing with probability spaces and the real numbers) in the discrete setting of logic.

Three main issues must be dealt. The first is of a syntactic nature related to the introduction of probabilities in the language. The second is the choice of the models and how they are related with the models of the base logic. Finally, a calculus should be defined to work hand in hand with the semantics, that is, it should be sound and complete.

Propositional logic, first-order logics and modal logic have been the main base logics. Also many-sorted first-order logic has been the base logic as in the case of the situation calculus. No work can be found answering the question of how to enrich a general base logic in order to get a probability logic as well as on the general requirements for the base logics.

Language-wise, in many approaches, a probability operator is introduced which allows the construction of new formulas/terms from the base formulas. In the case of the so called qualitative reasoning (usual in the context of modal logic), see for instance [3], there is an operator that when applied to a formula states that the formula should be more probable than the respective negation. Also in this direction see [7]. Lately, the probability operators appear as modal operators. In [21], a binary modal operator $\leq$ was adopted: $\varphi_1 \leq \varphi_2$ states that formula φ_1 is less probable than formula φ_2. Yet another approach has been used in knowledge representation, see [5, 1], where modal operators of the kind $w(\phi) \geq b$ expressing that according to an agent formula ϕ holds with probability at least b. See also [18, 19]. More recently, the qualitative approach was investigated through a new modal operator: that is, the fact that a formula should be more probable than its negation is introduced via a unary modal operator, see [22]. This operator does not distribute with conjunctions which means that it is not a normal modal operator. In the context of probability situation calculus [12] probabilities were added to the reactions.

From a syntactic point of view, our approach is as follows. We start with the base language, define a probability language by taking the base formulas as terms and the probability as a term constructor. The main primitive is the probability formula $((\int \varphi) \leq p)$ indicating that the probability of the formula φ should be less than or equal to p. These atomic probability assertions are then combined using what we call a global propositional logic. A normal modality is introduced as an abbreviation: $(\Box \varphi)$ for $((\int \varphi) = 1)$.

From a semantic point point of view, there are two basic approaches. Either the models of the base logic are modified so that they accommodate the probability component (endogenous approach) or the models are kept as they are and the probabilities are assigned outside the models (exogenous approach). The endogenous approach was adopted in the context of modal logic endowed with Kripke semantics (recall that a Kripke frame is a pair $\langle W, R\rangle$ where W is a non empty set and $R \subseteq W \times W$ is a relation). Most of the approaches introducing probabilities to modal logic consist either in giving probabilities to the elements of W, see [5], or giving probabilities to each $\langle w_1, w_2\rangle \in R$, see [8, 22]. The exogenous approach was adopted for instance in [16, 17].

Semantically, we illustrate twice the exogenous approach (some aspects of the exogenous approach appear in [16, 17]). We start by defining a global propositional logic whose models are sets of propositional valuations (no more structure is needed) and then we define a probability logic whose models are probability spaces where the outcome set is a set of propositional valuations (where more structure is needed to represent probabilities).

G. Sica (ed.) *Essays on the Foundations of Mathematics and Logic*

The interest in probability logic has recently increased due to the growing importance of probability in security [10], and in quantum logic. An essential issue in quantum logic is to accommodate the fourth postulate of quantum mechanics (when a physical quantity is measured using an observable on a system in a given state, the resulting outcomes are ruled by a probability measure). For more details on a quantum logic encompassing the probability aspects of the fourth postulate see [13, 14]. Note the key role of the exogenous approach in the design of that quantum logic: the models are superpositions of the models of the underlying classical logic. Probability reasoning is also relevant in distributed co-ordination and routing, and in fault-tolerance problems [20]. For other motivations in artificial intelligence see [9].

In Section 2, we motivate the exogenous approach and introduce the relevant languages. In Section 3, we use the exogenous approach for setting up the global propositional logic, investigate its finite model and Craig interpolation properties, and clarify the differences to the local, standard propositional logic. A Hilbert calculus is provided and shown to be sound and complete. In Section 4, the probability logic is presented, namely by enriching the Hilbert calculus of global propositional logic, as we well as by adding a probability measure to each global structure. For illustration purposes, two properties for a simple zero-knowledge protocol are proved using the Hilbert calculus. Finally, generalizing the technique in [6], the calculus is proved to be complete.

2 Setting

We start by an intuitive explanation of the exogenous approach to probability logic. Then, we discuss the different languages that are relevant.

2.1 Exogenous approach

In the context of this work, a logic is composed by a signature, a language, a deductive system and a semantic domain. We assume that the deductive system is a Hilbert calculus including axioms and inference rules. A semantic domain is a class of semantic structures.

Assume that we fix a logic, that we call the *base or local logic* BL where the signature is $\{C_k\}_{k\in\mathbb{N}}$ where C_k is a set of connectives of arity k and the language $L(C)$ is the free algebra generated by C.

The objective is to enrich it so that we can reason about probabilities of formulas and make assertions on probabilities. We do so by defining a new logic, called the *global logic* GL, that takes the local logic BL as a parameter. The general idea is to set up a parameterized mechanism for building GL from an arbitrary given BL.

From a syntactic point of view, the probability of a formula will be a term in GL of the form $(\int\varphi)$ where φ is a formula of BL. Atomic formulas in GL are the

G. Sica (ed.) *Essays on the Foundations of Mathematics and Logic*

formulas in BL plus inequalities of the form $(t_1 \leq t_2)$ where t_1 and t_2 are terms in GL. For instance, an atomic formula is $((\int \varphi) \leq 0.1)$ asserting that the probability of φ should be at most 0.1. The atomic formulas can be composed using global propositional connectives. For example, we can write $(\psi \sqcap ((\int \varphi) \leq 0.1))$, where ψ and φ are in BL, with the meaning that we want to have ψ and the probability of φ to be at most 0.1. Hence, $\sqcap$ is a conjunction in GL. The examples indicate that we must have a way to deal with real numbers.

The primitives of the global language were influenced by the exogenous semantic approach that we had in mind. That is: (i) probabilities are assigned to sets of semantic structures; (ii) as a consequence the probability of a formula is the probability of the set of all semantic structures that satisfy the formula (models of the formula). In consequence, the semantic structures for defining satisfaction of formulas in the global logic are sets of semantic structures of the local logic. As an example consider that the base logics is normal modal logic K endowed with Kripke structures. A semantic structure for the global logic is a set M of Kripke structures along with a probability space assigning probabilities to all subsets of M.

From a deductive point of view, things have to be more worked out. Of course, the base logic is endowed with a Hilbert calculus. The Hilbert calculus for the global logic has three different components. Two of them are defined once and for all: the one related to the probability reasoning that somehow must incorporate the facts known about probability measures; and the one related to reasoning with real numbers. The component that has to be defined and depends on the base logic is the one asserting the interplay between local and global reasoning.

The properties of the global logic that we are interested in are soundness and completeness. Unfortunately, the aim is weak completeness which is expected when reasoning with real numbers is involved. That is, we cannot consider hypotheses. Hence, we have to impose that the Hilbert calculus of base logic should also be sound and (weak) complete with respect to the semantics.

Herein we show how to enrich propositional logic with probabilities. Even in this case, we can give examples showing the interest and the expressiveness of our approach. We use an oracle for reasoning about real numbers in order to avoid details that would not be very relevant to our approach.

2.2 Language

Assume that we fix a base or a local logic with a signature C and a language $L(C)$ and a set $X = \{x_k : k \in \mathbb{N}\}$. The elements in $L(C)$ are the local formulas and the elements of X variables. The set of *probability terms* $T(C)$ is inductively defined as follows:

- $x \in T(C)$ whenever $x \in X$;
- $r \in T(C)$ for every computable real number r;

G. Sica (ed.) *Essays on the Foundations of Mathematics and Logic*

- $(\int \varphi) \in T(C)$ whenever $\varphi \in L(C)$;
- $(t_1 + t_2), (t_1 t_2) \in T(C)$ whenever $t_1, t_2 \in T(C)$.

Observe that the set of computable real numbers is denumerable and, so, even dealing with real numbers, we are able to remain within the discrete setting. The term $(\int \varphi)$ denotes the probability of φ, to be interpreted as the probability of the set of all valuations that satisfy φ.

The set of *probability assertions* $P(C)$ is the set of all inequalities of the form $(t_1 \leq t_2)$ where $t_1, t_2 \in T(C)$. The set of *global formulas* $G(C)$ is inductively defined as follows:

- $\varphi \in G(C)$ whenever $\varphi \in L(C)$;
- $\nu \in G(C)$ whenever $\nu \in P(C)$;
- $(\boxminus \delta_1) \in G(C)$ and $(\delta_1 \sqsupset \delta_2) \in G(C)$ whenever $\delta_1, \delta_2 \in G(C)$.

The global language allow us to talk about probability assertions. The symbols $\boxminus$ and $\sqsupset$ are the global negation and the global implication, respectively. The *atomic global formulas* are the probability assertions and the local formulas.

It is useful to extract from the global language two languages: (i) the *oracle language* $oL(C)$ which is the subset of the global language whose atomic global formulas are only the probability assertions where no probability terms occur; (ii) the *global propositional logic* $gL(C)$ which is the subset of the global language whose atomic global formulas are only the local formulas.

We use the following abbreviations: $(\delta_1 \sqcup \delta_2)$ for $((\boxminus \delta_1) \sqsupset \delta_2)$, $(\delta_1 \sqcap \delta_2)$ for $(\boxminus((\boxminus \delta_1) \sqcup (\boxminus \delta_2)))$, $(\delta_1 \equiv \delta_2)$ for $((\delta_1 \sqsupset \delta_2) \sqcap (\delta_2 \sqsupset \delta_1))$; $(t_1 = t_2)$ for $((t_1 \leq t_2) \sqcap (t_2 \leq t_1))$; and $(t_1 < t_2)$ for $((t_1 \leq t_2) \sqcap (\boxminus(t_1 = t_2)))$.

We present the main ideas in a simple context by adopting propositional logic as the base or the local logic. Hence, C is such that $C_0 = \Pi$ where Π is the set of propositional symbols, $C_1 = \{\neg\}$ and $C_2 = \{\Rightarrow\}$. The set of *local formulas* as $L(C)$ is then inductively defined as follows: $\Pi \subseteq L(C)$; $(\neg \varphi) \in L(C)$ whenever $\varphi \in L(C)$; $(\varphi_1 \Rightarrow \varphi_2) \in L(C)$ whenever $\varphi_1, \varphi_2 \in L(C)$. We say that $\neg, \Rightarrow$ are the local negation and the local implication, respectively.

3 Global propositional logic

We start with the global propositional logic (avoiding the probability reasoning) so that we can understand the differences between the local and global connectives e.g. the local/global negations and the local/global implications. For instance, we have to understand, both from a semantic and a deductive point of view, the difference between the local formula $(\varphi_1 \Rightarrow \varphi_2)$, the global formula $(\varphi_1 \Rightarrow \varphi_2)$ and the global formula $(\varphi_1 \sqsupset \varphi_2)$. Also of interest is the relationship between local and global consequences.

G. Sica (ed.) *Essays on the Foundations of Mathematics and Logic*

3.1 Semantics

We assume that the reader is familiar with propositional logics. We only introduce some notation and terminology. A *local valuation* is a map $v : \Pi \to \{0,1\}$. We write $v \Vdash_\ell \varphi$ when v satisfies φ and also $V \Vdash_\ell \varphi$ when V is a set of valuations and $v \Vdash_\ell \varphi$ for every $v \in V$. Let $\text{mod}(\varphi) = \{v \in \mathcal{V} : v \Vdash_\ell \varphi\}$. That is, $\text{mod}(\varphi)$ is the set of all local models of the local formulas.

Global propositional formulas are to be interpreted over non-empty sets of valuations. We denote by $\mathcal{V}$ the set of all such valuations. Each $V \subseteq \mathcal{V}$ is called a *global valuation*. The *satisfaction of a global formula* δ by a global valuation V, denoted by $V \Vdash_g \delta$, is inductively defined as follows:

- $V \Vdash_g \varphi$ iff $V \Vdash_\ell \varphi$;
- $V \Vdash_g (\boxminus \delta_1)$ iff $V \not\Vdash_g \delta_1$;
- $V \Vdash_g (\delta_1 \sqsupset \delta_2)$ iff $V \not\Vdash_g \delta_1$ or $V \Vdash_g \delta_2$.

Let $\text{gmod}(\delta) = \{V \subseteq \mathcal{V} : V \Vdash_g \delta\}$. That is, $\text{gmod}(\delta)$ is the set of all global models of δ. The following result is a straightforward consequence of the definition of global satisfaction.

Proposition 3.1 Let φ be a local formula. Then,

- $V \subseteq \text{mod}(\varphi)$ for all $V \in \text{gmod}(\varphi)$;
- $V \in \text{gmod}(\varphi)$ for all $V \subseteq \text{mod}(\varphi)$.

The following result shows the structural nature of the global models for the global connectives.

Proposition 3.2 Let δ_1, δ_2 be global propositional formulas. Then:

i) $\text{gmod}(\boxminus \delta) = \mathcal{V} \setminus \text{gmod}(\delta)$;

ii) $\text{gmod}(\delta_1 \sqcup \delta_2) = \text{gmod}(\delta_1) \cup \text{gmod}(\delta_2)$;

iii) $\text{gmod}(\delta_1 \sqcap \delta_2) = \text{gmod}(\delta_1) \cap \text{gmod}(\delta_2)$;

iv) $\text{gmod}(\delta_1 \sqsupset \delta_2) = \text{gmod}(\boxminus \delta_1) \cup \text{gmod}(\delta_2)$.

Proof: We only prove the second statement. Using the abbreviation for $(\delta_1 \sqcup \delta_2)$, we have $\text{gmod}(\delta_1 \sqcup \delta_2) = \text{gmod}((\boxminus \delta_1) \sqsupset \delta_2)$. Then, $V \in \text{gmod}((\boxminus \delta_1) \sqsupset \delta_2)$ iff either $V \not\Vdash_g (\boxminus \delta_1)$ or $V \Vdash_g \delta_2$ iff either $V \Vdash_g \delta_1$ or $V \Vdash_g \delta_2$ iff $V \in \text{gmod}(\delta_1) \cup \text{gmod}(\delta_2)$.
QED

We give characterizations of local and global negations as well as investigate the relationship between both when interpreted in the global context.

Proposition 3.3 Let φ be a local propositional formula. Then:

i) $\text{gmod}(\boxminus \varphi) = \{V \subseteq \mathcal{V} : V \cap \text{mod}(\neg \varphi) \neq \emptyset\}$;

ii) $\text{gmod}(\neg \varphi) = \{V \subseteq \mathcal{V} : V \subseteq \text{mod}(\neg \varphi)\}$;

iii) $\text{gmod}(\neg \varphi) \subseteq \text{gmod}(\boxminus \varphi)$;

iv) Not always $\text{gmod}(\boxminus \varphi) \subseteq \text{gmod}(\neg \varphi)$.

Proof: 1. $V \in \text{gmod}(\boxminus \varphi)$ iff $V \not\Vdash_g \varphi$ iff $V \not\Vdash_\ell \varphi$ iff there is $v \in V$ such that $v \not\Vdash_\ell \varphi$ iff there is $v \in V$ such that $v \Vdash_\ell (\neg \varphi)$ iff there is $v \in V$ such that $v \in \text{mod}(\neg \varphi)$ iff $V \cap \text{mod}(\neg \varphi) \neq \emptyset$.
2. $V \Vdash_g (\neg \varphi)$ iff $V \Vdash_\ell (\neg \varphi)$ iff $v \Vdash_\ell (\neg \varphi)$ for every $v \in V$ iff $v \in \text{mod}(\neg \varphi)$ for every $v \in V$ iff $V \subseteq \text{mod}(\neg \varphi)$.
3. Direct consequence of both 1. and 2.
4. It is enough to give a counterexample. Take φ as π and v_1, v_2 valuations such that $v_1(\pi) = 1$ and $v_2(\pi) = 0$. Then, $\{v_1, v_2\} \in \text{gmod}(\boxminus \pi)$, but $\{v_1, v_2\} \notin \text{gmod}(\neg \pi)$. QED

The local and the global negations can be interpreted in a modal context as follows. Assume that we introduce $\Box$ and $\Diamond$ as a unary connectives that can be applied to local formulas. Consider the class of Kripke frames of the form $\langle V, \{\langle v, v\rangle : v \in V\}\rangle$ where V is a non-empty set of valuations. Then, $V \Vdash_g (\Box\varphi)$ iff $V \Vdash_g \varphi$, and $V \Vdash_g (\Diamond\varphi)$ iff there is $v \in V$ such that $v \Vdash_\ell \varphi$. Then, $\boxminus$ and $\neg$, when interpreted in the global logic, can be seen as abbreviations:

$$(\boxminus \varphi) \text{ is } (\Diamond(\neg \varphi)) \text{ and } (\neg \varphi) \text{ is } (\Box(\neg \varphi)).$$

Local implication and global implication can also be characterized as well as related with each other.

Proposition 3.4 Let φ_1, φ_2 be global propositional formulas. Then:

i) $\text{gmod}(\varphi_1 \sqsupset \varphi_2) = \{V \in \mathcal{V} : V \cap \text{mod}(\neg \varphi_1) \neq \emptyset\} \cup \{V \in \mathcal{V} : V \subseteq \text{mod}(\varphi_2)\}$;

ii) $\text{gmod}(\varphi_1 \Rightarrow \varphi_2) = \{V \in \mathcal{V} : V \subseteq \text{mod}(\neg \varphi_1) \cup \text{mod}(\varphi_2)\}$;

iii) $\text{gmod}(\varphi_1 \Rightarrow \varphi_2) \subseteq \text{gmod}(\varphi_1 \sqsupset \varphi_2)$;

iv) Not always $\text{gmod}(\varphi_1 \sqsupset \varphi_2) \subseteq \text{gmod}(\varphi_1 \Rightarrow \varphi_2)$.

Proof: We prove the third statement and give a counterexample for the fourth.
3. Assume that $V \in \text{gmod}(\varphi_1 \Rightarrow \varphi_2)$. We have two cases to consider. (i) $V \cap \text{mod}(\neg \varphi_1) \neq \emptyset$ and, so, $V \in \text{gmod}(\varphi_1 \sqsupset \varphi_2)$. (ii) $V \cap \text{mod}(\neg \varphi_1) = \emptyset$. Then, $V \subseteq \text{mod}(\varphi_1)$, hence, $v \in \text{mod}(\varphi_1)$ for every $v \in V$, therefore $v \in \text{mod}(\varphi_2)$ for every $v \in V$, so $V \subseteq \text{mod}(\varphi_2)$ and, finally, $V \in \text{gmod}(\varphi_1 \sqsupset \varphi_2)$.
4. Assume that φ_1 is π_1, φ_2 is π_2, $v_1(\pi_1) = 0$, $v_1(\pi_2) = 0$, $v_2(\pi_1) = 1$ and $v_2(\pi_2) = 1$. Then, $\{v_1, v_2\} \in \text{gmod}(\pi_1 \Rightarrow \pi_2)$, but $\{v_1, v_2\} \notin \text{mod}(\pi_1 \sqsupset \pi_2)$. QED

The following result asserts the relationship between the local and global counterparts for the remaining connectives.

Proposition 3.5 Let φ_1, φ_2 be global propositional formulas. Then:

i) $\text{gmod}(\varphi_1 \sqcup \varphi_2) \subseteq \text{gmod}(\varphi_1 \vee \varphi_2)$;

ii) Not always $\text{gmod}(\varphi_1 \vee \varphi_2) \subseteq \text{gmod}(\varphi_1 \sqcup \varphi_2)$;

iii) $\text{gmod}(\varphi_1 \sqcap \varphi_2) = \text{gmod}(\varphi_1 \wedge \varphi_2)$.

Proof: 1. Assume that $V \in \text{gmod}(\varphi_1 \sqcup \varphi_2)$ and that there is $v \in V$ such that $v \not\Vdash_g \varphi_1$. Then, $V \not\Vdash_\ell \varphi_1$ and so $V \not\Vdash_g \varphi_1$. Hence, $V \Vdash_g \varphi_2$, therefore $V \Vdash_\ell \varphi_2$, so $V \Vdash_\ell (\varphi_2 \vee \varphi_2)$ and $V \in \text{gmod}(\varphi_1 \vee \varphi_2)$.
2. Assume that φ_1 is π_1, φ_2 is π_2, $v_1(\pi_1) = 0$, $v_1(\pi_2) = 0$, $v_2(\pi_1) = 1$ and $v_2(\pi_2) = 1$. Then, $\{v_1, v_2\} \in \text{gmod}(\pi_1 \vee \pi_2)$, but $\{v_1, v_2\} \notin \text{mod}(\pi_1 \sqcup \pi_2)$. QED

We now investigate whether some well known properties of local propositional logic also hold in the global propositional logic. Let $\text{gvar}(\delta)$, the global set of variables, be inductively defined as follows: $\text{gvar}(\varphi) = \{\varphi\}$; $\text{gvar}(\boxminus \delta_1) = \text{gvar}(\delta_1)$; and $\text{gvar}(\delta_1 \sqsupset \delta_2) = \text{gvar}(\delta_1) \cup \text{gvar}(\delta_2)$.

Lemma 3.6 Let $V_1, V_2 \in \wp\mathcal{V}$ and assume $V_1 \subseteq \text{mod}(\varphi)$ iff $V_2 \subseteq \text{mod}(\varphi)$, for each $\varphi \subseteq \text{gvar}(\eta)$. Then, $V_1 \in \text{gmod}(\eta)$ iff $V_2 \in \text{gmod}(\eta)$.

Proof: Assume that $V_1 \subseteq \text{mod}(\varphi)$ iff $V_2 \subseteq \text{mod}(\varphi)$, for each $\varphi \in \text{gvar}(\eta)$. We prove the result by induction on the structure of η. Base. Assume that η is the local formula ψ. Then, $V_1 \Vdash_\ell \psi$ iff $V_2 \Vdash_\ell \psi$ and so $V_1 \Vdash_g \psi$ iff $V_2 \Vdash_g \psi$. Step. We have $V_1 \in \text{gmod}(\boxminus \delta)$ iff $V_1 \not\Vdash_g \delta$ iff $V_2 \not\Vdash_g \delta$ iff $V_2 \in \text{gmod}(\boxminus \delta)$. Similarly for the implication. QED

This is the lemma of the omitting symbols generalized to the global logic. The following result states that for a particular global formula we only have to consider a finite number of structures.

Proposition 3.7 Let η be a global formula. Define the following equivalence relation:

$$V_1 \approx_\eta V_2$$

iff ($V_1 \subseteq \text{mod}(\varphi)$ iff $V_2 \subseteq \text{mod}(\varphi)$) for every $\varphi \in \text{gvar}(\eta)$. Then, the set $\{[V]_{\approx_\eta} : V \in \wp\mathcal{V}\}$ is finite.

Proof: Assume that $\text{gvar}(\eta) = \{\varphi_1, \ldots, \varphi_n\}$ and that $V_1 \subseteq \text{mod}(\varphi_i)$ iff $V_2 \subseteq \text{mod}(\varphi_i)$ for every $i = 1, \ldots, n$. Then, the number of equivalence classes $[V]_{\approx_\eta}$ is at most 2^n since each class corresponds to a possible combination of $V \subseteq \text{mod}(\varphi_i)$ or $V \not\subseteq \text{mod}(\varphi_i)$ if that is possible. QED

G. Sica (ed.) *Essays on the Foundations of Mathematics and Logic*

Proposition 3.8 Let η be a global formula. Then,

$$V \in \text{gmod}(\eta) \text{ iff } [V]_{\approx_\eta} \subseteq \text{gmod}(\eta).$$

Proof: Direct consequence of Lemma 3.6. QED

As a consequence for evaluating a global formula η we just consider the the elements in $\{[V]_{\approx_\eta} : V \in \wp\mathcal{V}\}$. We use V instead of $[V]_{\approx_\eta}$.

We can look at the transference of a (local) propositional formula to a global formula as if we have a unary modality $\Box$ such that $\varphi \in L(C)$ implies that $(\Box\varphi) \in gL(C)$. The following properties hold assuming that a structure V induces a Kripke frame $\langle V, \{\langle v, v\rangle : v \in V\}\rangle$:

- $V \Vdash_\ell \varphi$ iff $V \Vdash_g (\Box\varphi)$;
- $V \Vdash_g ((\Box\varphi_1) \sqcap (\Box\varphi_2))$ iff $V \Vdash_g (\Box(\varphi_1 \wedge \varphi_2))$;
- $V \Vdash_g ((\Box(\varphi_1 \Rightarrow \varphi_2)) \sqsupset ((\Box\varphi_1) \sqsupset (\Box\varphi_2)))$.

The properties above state that $\Box$ is a normal modal necessitation: the left to right implication of the first property corresponds to necessitation and the second and third properties are equivalent ways to define normalization.

Now we turn our attention to semantic consequence. We say that φ is a *local semantic consequence* of Γ, written $\Gamma \vDash_\ell \varphi$, if, for every local valuation v, $v \Vdash \varphi$ whenever $v \Vdash \gamma$ for every $\gamma \in \Gamma$. Furthermore, we say that η is a *global semantic consequence* of Δ, written $\Delta \vDash_g \eta$, if, for global valuation V, $V \Vdash \eta$ whenever $V \Vdash \delta$ for every $\delta \in \Delta$. A global formula δ is *valid* if $\vDash_g \delta$. The following results express semantic consequence in terms of models.

Proposition 3.9 Let $\Gamma \cup \{\varphi\}$ be a set of local formulas and $\Delta \cup \{\eta\}$ be a set of global formulas. Then:

- $\bigcap_{\gamma\in\Gamma} \text{mod}(\gamma) \subseteq \text{mod}(\varphi)$ iff $\Gamma \vDash_l \varphi$;
- $\bigcap_{\delta\in\Delta} \text{gmod}(\delta) \subseteq \text{gmod}(\eta)$ iff $\Delta \vDash_g \eta$.

The following result shows that the global entailment is conservative for the local formulas.

Proposition 3.10 Let $\Gamma \cup \{\varphi\}$ be a set of local formulas. Then:

$$\Gamma \vDash_\ell \varphi \text{ iff } \Gamma \vDash_g \varphi.$$

Proof: Assume that $\Gamma \vDash_\ell \varphi$ and that $V \in \bigcap_{\gamma\in\Gamma} \operatorname{gmod}(\gamma)$. Then,

$$V \subseteq \bigcap_{\gamma\in\Gamma} \operatorname{mod}(\gamma)$$

by Proposition 3.1 and so, by hypothesis, $V \subseteq \operatorname{mod}(\varphi)$. Using the same proposition, we have $V \in \operatorname{gmod}(\varphi)$ and so $\Gamma \vDash_g \varphi$. Assume that $V \in \bigcap_{\gamma\in\Gamma} \operatorname{gmod}(\gamma)$. Then, $V \subseteq \bigcap_{\gamma\in\Gamma} \operatorname{mod}(\gamma)$ for every $\gamma \in \Gamma$, hence $V \subseteq \operatorname{mod}(\varphi)$ and so $V \in \operatorname{gmod}(\varphi)$. QED

As already mentioned, our approach has the advantage that every valid formula in the base logic is also a valid formula in the global logic. The following is a direct consequence of the result above observing that a local tautology is a formula φ such that $\operatorname{mod}(\varphi) = \mathcal{V}$.

Corollary 3.11 A local tautology is a valid global formula.

Moreover, we do not get new tautologies in the global propositional logic when using the connectives in the local logic meaning that the construction is conservative. The semantic version of the metatheorem of deduction also holds for the global propositional logic.

Proposition 3.12 The metatheorem of deduction holds in $gL(C)$. That is:

$$\Delta, \eta_1 \vDash_g \eta_2 \ \text{ implies } \ \Delta \vDash_g (\eta_1 \sqsupset \eta_2).$$

Proof: Assume that $\Delta, \eta_1 \vDash_g \eta_2$ and that $V \in \bigcap_{\delta\in\Delta} \operatorname{gmod}(\delta)$. Suppose that $V \not\Vdash_g \eta_1$. Then, $V \Vdash_g (\boxminus \eta_1)$ and so $V \in \operatorname{gmod}(\eta_1 \sqsupset \eta_2)$. Suppose now that $V \Vdash_g \eta_1$. Then, $V \Vdash_g \eta_2$ and so $V \in \operatorname{gmod}(\eta_1 \sqsupset \eta_2)$. QED

The following semantic consequences and valid formulas hold.

Proposition 3.13 Let $\varphi, \varphi_1, \varphi_2$ be local formulas. Then,

- $\{(\neg \varphi)\} \vDash_g (\boxminus \varphi)$;
- $((\neg \varphi) \sqsupset (\boxminus \varphi))$ is a valid global formula;
- $\{(\boxminus \varphi)\} \nvDash_g (\neg \varphi)$;
- $\{(\varphi_1 \Rightarrow \varphi_2)\} \vDash_g (\varphi_1 \sqsupset \varphi_2)$;
- $((\varphi_1 \Rightarrow \varphi_2) \sqsupset (\varphi_1 \sqsupset \varphi_2))$ is a valid global formula;
- $\{(\varphi_1 \sqsupset \varphi_2)\} \nvDash_g (\varphi_1 \Rightarrow \varphi_2)$;
- $\{(\varphi_1 \wedge \varphi_2)\} \vDash_g (\varphi_1 \sqcap \varphi_2)$;

- $\{(\varphi_1 \sqcap \varphi_2)\} \vDash_g (\varphi_1 \wedge \varphi_2)$;
- $((\varphi_1 \wedge \varphi_2) \equiv (\varphi_1 \sqcap \varphi_2))$ is a valid global formula;
- $\{(\varphi_1 \sqcup \varphi_2)\} \vDash_g (\varphi_1 \vee \varphi_2)$;
- $((\varphi_1 \sqcup \varphi_2) \sqsupset (\varphi_1 \vee \varphi_2))$ is a valid global formula;
- $\{(\varphi_1 \vee \varphi_2)\} \nvDash_g (\varphi_1 \sqcup \varphi_2)$.

Proof: Direct consequences of Propositions 3.3, 3.4, 3.5 and metatheorem of deduction 3.12. QED

We now turn our attention towards discussing the existence of normal forms in the global propositional logic. Recall that the disjunctive normal form lemma holds in the local logic: that is, each local formula is equivalent to a disjunction of conjunctions of (local) literals (propositional symbols and their negations). An easy way to obtain the disjunctive normal form of φ is by picking up the valuations that satisfy the formula. Each component of the disjunction corresponds to such a valuation. If v satisfies φ and $\text{var}(\varphi) = \{\pi_1, \ldots, \pi_n\}$, then the component is a conjunction $\wedge_{i=1}^{n} \pi^*$ where π^* is π if $v(\pi) = 1$ and is $(\neg\, \pi)$ otherwise. In order to investigate the same problem for global logics we introduce global literals. A *global literal* of a global formula is either a local formula or the global negation of a local formula.

Proposition 3.14 Each global formula is is equivalent to a global disjunction of global conjunctions of global literals.

Proof: Let η be a global formula, $\text{gvar}(\eta) = \{\varphi_1, \ldots, \varphi_n)$ and $\text{gmod}(\eta) = \{V_1, \ldots, V_k\}$. Consider the following formula:

$$(\bigsqcup_{i=1}^{k} (\bigsqcap_{j=1}^{n} \varphi_{ij}^*))$$

where φ_{ij}^* is φ_j if $V_i \Vdash_g \varphi_j$ and is $(\boxminus\, \varphi_j)$ otherwise. Observe that

$$V_i \Vdash_g (\bigsqcap_{j=1}^{n} \varphi_{ij}^*)$$

and, moreover, $\text{gmod}(\sqcup_{i=1}^{k} (\sqcap_{j=1}^{n} \varphi_{ij}^*)) = \{V_1, \ldots, V_k\}$. QED

We can give a syntactic characterization of the disjunctive normal form. Let $\text{gmol}(\eta)$ be the set of all $\Phi \subseteq \text{gvar}(\eta)$ (molecules) such that the following global formula holds:

$$(((\bigsqcap_{\varphi \in \Phi} \varphi) \sqcap (\bigsqcap_{\varphi \in \text{gvar}(\eta) \setminus \Phi} (\boxminus\, \varphi))) \sqsupset \eta).$$

G. Sica (ed.) *Essays on the Foundations of Mathematics and Logic*

Proposition 3.15 Each global formula η is is equivalent to

$$(\bigsqcup_{\Phi\in\mathrm{gmol}(\eta)}((\bigsqcap_{\varphi\in\Phi}\varphi)\sqcap(\bigsqcap_{\varphi\in\mathrm{gvar}(\eta)\setminus\Phi}(\boxminus\varphi)))).$$

Proof: Assume that $V_i \Vdash_g \eta$. Then, by Proposition 3.14, there is $j \in \{1,\dots,n\}$ such that $V_i \Vdash_g (\bigsqcap_{j=1}^{n}\varphi^*_{ij})$. Take $\Phi = \{\varphi_j : \varphi^*_{ij} = \varphi_j\}$. Then, Φ is a molecule,

$$V_i \Vdash_g ((\bigsqcap_{\varphi\in\Phi}\varphi)\sqcap(\bigsqcap_{\varphi\in\mathrm{gvar}(\eta)\setminus\Phi}(\boxminus\varphi)))$$

and so V_i satisfies the disjunction. For the other part, assume that

$$V_i \Vdash_g (\bigsqcup_{\Phi\in\mathrm{gmol}(\eta)}((\bigsqcap_{\varphi\in\Phi}\varphi)\sqcap(\bigsqcap_{\varphi\in\mathrm{gvar}(\eta)\setminus\Phi}(\boxminus\varphi)))).$$

Then, there is a molecule Φ such that $V_i \Vdash_g ((\bigsqcap_{\varphi\in\Phi}\varphi)\sqcap(\bigsqcap_{\varphi\in\mathrm{gvar}(\eta)\setminus\Phi}(\boxminus\varphi)))$ and so $V_i \Vdash_g \eta$. QED

3.2 Calculus for the global propositional logic

It is well known that propositional logic (our local logic) is sound and complete. That means that $\Gamma^{\vdash_\ell} = \Gamma^{\vDash_\ell}$. When Γ is the empty set the elements of $\emptyset^{\vdash_\ell} = \emptyset^{\vDash_\ell}$ are the tautologies. As in the case of many other logics we want to import tautologies to the global logic. That is, we want to define tautological formula.

An *instantiation* is a map $\iota : L(C) \to gL(C)$ such that $\iota(\neg\varphi) = (\boxminus\iota(\varphi))$ and $\iota(\varphi_1 \Rightarrow \varphi_2) = (\iota(\varphi_1) \sqsupset \iota(\varphi_2))$. A global formula η is said to be a *tautological formula* if there are a tautology $\varphi \in L(C)$ and an instantiation ι such that $\iota(\varphi)$ is η. For instance, $(\eta_1 \sqsupset (\eta_2 \sqsupset \eta_1))$ is a tautological formula.

The set of *Hilbert global consequences* of a set Δ of global formulas is the set $\Delta^{\vdash_g}$ inductively defined as follows:

- Hypothesis, indicated by H:
  - $\delta \in \Delta^{\vdash_g}$ whenever $\delta \in \Delta$;
- Tautologies, indicated by T, and tautological formulas, indicated by gT:
  - $\varphi \in \Delta^{\vdash_g}$ whenever φ is a local tautology;
  - $\eta \in \Delta^{\vdash_g}$ whenever η is a tautological formula;
- Interplay axioms:
  - $((\varphi_1 \Rightarrow \varphi_2) \sqsupset (\varphi_1 \sqsupset \varphi_2)) \in \Delta^{\vdash_g}$, indicated by P1;
  - $((\varphi_1 \sqcap \varphi_2) \sqsupset (\varphi_1 \wedge \varphi_2)) \in \Delta^{\vdash_g}$, indicated by P2;

- Local modus ponens, indicated by MP, and global modus ponens, indicated by gMP:
  - If $\varphi_1, (\varphi_1 \Rightarrow \varphi_2) \in \Delta^{\vdash_g}$ then $\varphi_2 \in \Delta^{\vdash_g}$;
  - If $\delta_1, (\delta_1 \sqsupset \delta_2) \in \Delta^{\vdash_g}$ then $\delta_2 \in \Delta^{\vdash_g}$.

If $\eta \in \Delta^{\vdash_g}$ we say that η is a Hilbert *global consequence* of Δ. If Δ is the empty set we say that η is a theorem. In the sequel we may write $\Delta \vdash_g \eta$ when $\eta \in \Delta^{\vdash_g}$. For example, the theorem

$$\vdash_g ((\varphi_1 \wedge \varphi_2) \sqsupset (\varphi_1 \sqcap \varphi_2))$$

can be derived as follows:

1	$((\varphi_1 \wedge \varphi_2) \Rightarrow \varphi_1)$	T
2	$((\varphi_1 \wedge \varphi_2) \Rightarrow \varphi_2)$	T
3	$(((\varphi_1 \wedge \varphi_2) \Rightarrow \varphi_1) \sqsupset ((\varphi_1 \wedge \varphi_2) \sqsupset \varphi_1))$	$P1$
4	$(((\varphi_1 \wedge \varphi_2) \Rightarrow \varphi_2) \sqsupset ((\varphi_1 \wedge \varphi_2) \sqsupset \varphi_2))$	$P1$
5	$((\varphi_1 \wedge \varphi_2) \sqsupset \varphi_1)$	$gMP(1,3)$
6	$((\varphi_1 \wedge \varphi_2) \sqsupset \varphi_2)$	$gMP(2,4)$
7	$(((\varphi_1 \wedge \varphi_2) \sqsupset \varphi_1) \sqsupset$ $(((\varphi_1 \wedge \varphi_2) \sqsupset \varphi_2) \sqsupset ((\varphi_1 \wedge \varphi_2) \sqsupset (\varphi_1 \sqcap \varphi_2))))$	gT
8	$(((\varphi_1 \wedge \varphi_2) \sqsupset \varphi_2) \sqsupset ((\varphi_1 \wedge \varphi_2) \sqsupset (\varphi_1 \sqcap \varphi_2)))$	$gMP(5,7)$
9	$((\varphi_1 \wedge \varphi_2) \sqsupset (\varphi_1 \sqcap \varphi_2))$	$gMP(6,8)$

Hence, we can conclude

$$\vdash_g ((\varphi_1 \sqcap \varphi_2) \equiv (\varphi_1 \wedge \varphi_2)).$$

The interplay between the local and the global connectives is stated in the following result.

Proposition 3.16 Let $\varphi, \varphi_1, \varphi_2$ be local formulas. Then:

- $\{(\neg\, \varphi)\} \vdash_g (\boxminus \varphi)$;
- $((\neg\, \varphi) \sqsupset (\boxminus \varphi))$ is a global theorem;
- $\{(\varphi_1 \Rightarrow \varphi_2)\} \vdash_g (\varphi_1 \sqsupset \varphi_2)$;
- $((\varphi_1 \Rightarrow \varphi_2) \sqsupset (\varphi_1 \sqsupset \varphi_2))$ is a global theorem;
- $\{(\varphi_1 \wedge \varphi_2)\} \vdash_g (\varphi_1 \sqcap \varphi_2)$;
- $\{(\varphi_1 \sqcap \varphi_2)\} \vdash_g (\varphi_1 \wedge \varphi_2)$;

G. Sica (ed.) *Essays on the Foundations of Mathematics and Logic*

- $((\varphi_1 \wedge \varphi_2) \equiv (\varphi_1 \sqcap \varphi_2))$ is a global theorem;
- $\{(\varphi_1 \sqcup \varphi_2)\} \vdash_g (\varphi_1 \vee \varphi_2)$;
- $((\varphi_1 \sqcup \varphi_2) \sqsupset (\varphi_1 \vee \varphi_2))$ is a global theorem.

Lemma 3.17 The following are theorems of the global calculus:

i) $(\eta \equiv (\bigsqcup_{\Phi \in \text{gmol}(\eta)} ((\bigsqcap_{\varphi \in \Phi} \varphi) \sqcap (\bigsqcap_{\varphi \in \text{gvar}(\eta) \setminus \Phi} (\boxminus \varphi)))))$;

ii) $(\boxminus(\eta_1 \sqcup \eta_2) \equiv ((\boxminus \eta_1) \sqcap (\boxminus \eta_2)))$;

iii) $((\neg \varphi) \sqsupset (\boxminus \varphi))$.

That is, each global formula has a special disjunctive normal form and the De Morgan laws hold. Observe also that:

$$\Gamma \vdash_\ell \varphi \;\text{ implies }\; \Gamma \vdash_g \varphi$$

which means that the Hilbert local consequences are also Hilbert global consequences.

3.3 Soundness and completeness

The objective is to investigate soundness and completeness of the Hilbert calculus for global propositional logics. Let $\Delta^{\vDash_g} = \{\eta : \Delta \vDash_g \eta\}$. The Hilbert calculus is *sound* with respect to the semantics if $\Delta^{\vdash_g} \subseteq \Delta^{\vDash_g}$. And it is *complete* if $\Delta^{\vDash_g} \subseteq \Delta^{\vdash_g}$. The corresponding concepts of weakly sound and weakly complete appear when Δ is the empty set. In particular, the Hilbert calculus is weakly complete if $\emptyset^{\vDash_g} \subseteq \emptyset^{\vdash_g}$ or in other words, if φ is a valid formula then φ is a theorem.

Theorem 3.18 The global propositional calculus is sound.

Proof: Assume that $\Delta \vdash_g \eta$. The proof follows by induction on the length of a derivation of η from Δ observing that the axioms are valid global formulas and that gMP is sound. It is worthwhile to detail the proof of axiom gT since all the others are straightforward. Assume that η is such that there are an instantiation ι and a local tautology φ such that $\iota(\varphi)$ is η. Let V be a global valuation. Consider a set of valuations U_V where $u \in U_V$ is defined as follows: $u(\pi) = 1$ if $V \Vdash_\ell \iota(\pi)$ and $u(\pi) = 0$ otherwise, for every $\pi \in \text{var}(\varphi)$. Then, $U_V \subseteq \text{mod}(\psi)$ iff $V \in \text{gmod}(\iota(\psi))$ for every local formula ψ (easily shown by induction on the structure of ψ). Since φ is a tautology, $U_V \subseteq \text{mod}(\varphi)$ and so $V \in \text{gmod}(\eta)$. QED

Following the usual technique, we prove that if $\eta \notin \Delta^{\vdash_g}$ then $\eta \notin \Delta^{\vDash_g}$. The assertion $\eta \notin \Delta^{\vdash_g}$ means that the set Δ is consistent for global derivation. And the assertion $\eta \notin \Delta^{\vDash_g}$ indicates that we have to find a global valuation V such that $V \in \text{gmod}(\Delta)$ and $V \notin \text{gmod}(\eta)$.

Therefore, the main step of the usual construction is to find a model of a consistent set. Although the global propositional calculus can be shown to be complete, here we just consider weak completeness. As we shall see, the probabilistic extension of the global logic is only weak complete and the proof of this fact only requires the weak completeness of the global logic.

A global formula η is *g-consistent* if $\not\vdash_g (\boxminus \eta)$. Recall that a local formula φ is ℓ-consistent if $\not\vdash_\ell (\neg \varphi)$. We adapt the technique of [6] by showing that global consistency of a global formula is propagated to the global consistency of one of its molecules.

Lemma 3.19 If η is consistent then there is a molecule $\Phi \in \text{gmol}(\eta)$ such that $((\prod_{\varphi\in\Phi} \varphi) \sqcap (\prod_{\varphi\in\text{gvar}(\eta)\setminus\Phi} (\boxminus \varphi)))$ is consistent.

Proof: Assume that for each $\Phi \in \text{gmol}(\eta)$

$$(\boxminus((\prod_{\varphi\in\Phi} \varphi) \sqcap (\prod_{\varphi\in\text{gvar}(\eta)\setminus\Phi} (\boxminus \varphi))))$$

is a theorem (that is, no molecule is consistent). Then, using tautological reasoning, so is

$$(\prod_{\Phi\in\text{gmol}(\eta)} (\boxminus((\prod_{\varphi\in\Phi} \varphi) \sqcap (\prod_{\varphi\in\text{gvar}(\eta)\setminus\Phi} (\boxminus \varphi))))),$$

therefore

$$(\boxminus(\bigsqcup_{\Phi\in\text{gmol}(\eta)} ((\prod_{\varphi\in\Phi} \varphi) \sqcap (\prod_{\varphi\in\text{gvar}(\eta)\setminus\Phi} (\boxminus \varphi)))))$$

is a theorem by Lemma 3.17, hence $(\boxminus \eta)$ is also a theorem again by Lemma 3.17 and η is not g-consistent. QED

We are ready to give the main result for weak completeness. The one for completeness is a simple extension.

Theorem 3.20 The global propositional calculus is weakly complete.

Proof: Assume that $\not\vdash_g \eta$. Then, $\not\vdash_g (\boxminus(\boxminus \eta))$, hence $(\boxminus \eta)$ is g-consistent and so by Lemma 3.19 there is $\Phi \in \text{gmol}(\eta)$ such that

$$((\prod_{\varphi\in\Phi} \varphi) \sqcap (\prod_{\varphi\in\text{gvar}(\eta)\setminus\Phi} (\boxminus \varphi)))$$

is g-consistent. Moreover,

$$(\prod_{\varphi\in\Phi} \varphi)$$

is also g-consistent: if not then $\vdash_g (\boxminus(\bigsqcap_{\varphi\in\Phi}\varphi))$, hence

$$\vdash_g ((\boxminus(\bigsqcap_{\varphi\in\Phi}\varphi))\bigsqcup(\boxminus(\bigsqcap_{\varphi\in \mathrm{gvar}(\eta)\setminus\Phi}\varphi)))$$

and so $\vdash_g (\boxminus((\bigsqcap_{\varphi\in\Phi}\varphi)\sqcap(\bigsqcap_{\varphi\in \mathrm{gvar}(\eta)\setminus\Phi}\varphi)))$ which is a contradiction. Observe that

$$(\bigwedge_{\varphi\in\Phi}\varphi)$$

is also g-consistent. If not then $\vdash_g (\boxminus(\bigwedge_{\varphi\in\Phi}\varphi))$ and so, by P2, $\vdash_g (\boxminus(\bigsqcap_{\varphi\in\Phi}\varphi))$ against the g-consistency of $(\bigsqcap_{\varphi\in\Phi}\varphi)$. Moreover, by Lemma 3.17.3, $(\bigwedge_{\varphi\in\Phi}\varphi)$ is ℓ-consistent and, by the existence model property for the local logic there is a valuation that satisfies it. Let V be the largest set of valuations such that

$$V \Vdash_\ell (\bigwedge_{\varphi\in\Phi}\varphi).$$

Then, $V \Vdash_g (\bigsqcap_{\varphi\in\Phi}\varphi)$. It remains to show that $V \not\Vdash_\ell \varphi$ for every $\varphi \in \mathrm{gvar}(\eta)\setminus\Phi$. Assume that there is $\psi \in \mathrm{gvar}(\eta)\setminus\Phi$ such that $V \Vdash_\ell \psi$. Then, there is $\psi \in \mathrm{gvar}(\eta)\setminus\Phi$ such that

$$((\bigwedge_{\varphi\in\Phi}\varphi)\Rightarrow\psi)$$

is a tautology and so, by axiom gT,

$$\vdash_g ((\bigwedge_{\varphi\in\Phi}\varphi)\Rightarrow\psi).$$

Using, axiom P1 and rule gMP, there is $\psi \in \mathrm{gvar}(\eta)\setminus\Phi$ such that

$$((\bigwedge_{\varphi\in\Phi}\varphi)\sqsupset\psi)$$

is also a global valid formula. By, P2 and qT, there is $\psi \in \mathrm{gvar}(\eta)\setminus\Phi$ such that

$$\vdash_g ((\bigsqcap_{\varphi\in\Phi}\varphi)\sqsupset\psi).$$

Hence, by successive applications of gT

$$\vdash_g ((\bigsqcap_{\varphi\in\Phi}\varphi)\sqsupset(\bigsqcup_{\varphi\in \mathrm{gvar}(\eta)\setminus\Phi}\varphi)),$$

hence

$$\vdash_g ((\bigsqcap_{\varphi\in\Phi}\varphi)\sqcap(\bigsqcap_{\varphi\in \mathrm{gvar}(\eta)\setminus\Phi}(\boxminus\varphi)))$$

and so

$$(\boxminus((\prod_{\varphi\in\Phi}\varphi)\sqcap(\prod_{\varphi\in\text{gvar}(\eta)\setminus\Phi}(\boxminus\varphi))))$$

which is a contradiction. QED

Observe that this proof relies on the fact that the disjunctive normal form lemma holds in the global propositional logic.

The objective of the next subsection is to investigate whether the global propositional logics still has other well known properties of the local logic. Among those properties we choose the finite model property and Craig interpolation. We refrain here from considering properties like decidability and complexity.

3.4 Finite-model and interpolation

Global logic has the *finite model property*. That is, if $V \Vdash_g \delta$ then there is a finite structure (a finite set of valuations) that also satisfies the formula.

Proposition 3.21 Assume that $\text{gmod}(\delta) \neq \emptyset$. Then, in $\text{gmod}(\delta)$ there is a a finite global valuation.

Proof: Assume that $V \Vdash_g \delta$. Consider the set U composed by the valuations in V such that:

- for every valuation $v \in V$ there is a valuation $u \in U$ such that $v \Vdash_g \varphi$ iff $u \Vdash_g \varphi$ for every $\varphi \in \text{gvar}(\delta)$;
- there are no $u_1, u_2 \in U$ such that $u_1 \Vdash_g \varphi$ iff $u_2 \Vdash_g \varphi$ for every $\varphi \in \text{gvar}(\delta)$.

The set U is finite since $\text{gvar}(\eta)$ is a finite set. Moreover, $U \in [V]_{\approx_\delta}$ and so, by Proposition 3.8, we conclude that $U \in \text{gmod}(\delta)$ since $V \in \text{gmod}(\delta)$. QED

We now investigate interpolation. Let $\text{var}(\delta)$ be the set of propositional symbols that occur in global formula δ. The global logic has *interpolation* if the following holds: $\vdash_g (\delta_1 \sqsupset \delta_2)$ iff there is a global formula δ such that $\text{var}(\delta) \subseteq \text{var}(\delta_1) \cap \text{var}(\delta_2)$, $\vdash_g (\delta_1 \sqsupset \delta)$ and $\vdash_g (\delta \sqsupset \delta_2)$. Since the global logic is complete we can analyze Craig interpolation via semantics. That is, we can show that $\vDash_g (\delta_1 \sqsupset \delta_2)$ iff there is a global formula δ such that $\text{var}(\delta) \subseteq \text{var}(\delta_1) \cap \text{var}(\delta_2)$, $\vDash_g (\delta_1 \sqsupset \delta)$ and $\vDash_g (\delta \sqsupset \delta_2)$. We say that δ is a Craig interpolant for δ_1 and δ_2. We start by proving some auxiliary results. Before we need the notion of substitution. A substitution over $gL(C)$ is a map $\sigma : \pi \to gL(C)$. Substitutions can be extended to global formulas. We denote by $\sigma(\delta)$ the formula in $gL(C)$ obtained by replacing each $\pi \in \text{var}(\delta)$ by $\sigma(\pi)$.

Lemma 3.22 Let σ be a substitution over $gL(C)$, V a structure and $V_\sigma = \{v_\sigma : v \in V\}$ a structure where $v_\sigma(\pi) = (v \Vdash_g \sigma(\pi))$. Then, $V_\sigma \Vdash \delta$ iff $V \Vdash \sigma(\delta)$.

G. Sica (ed.) *Essays on the Foundations of Mathematics and Logic*

Proof: The statement is proved easily by induction on δ. QED

Proposition 3.23 Let $\vDash_g (\delta_1 \sqsupset \delta_2)$ and σ a substitution over $gL(C)$. Then, $\vDash_g (\sigma(\delta_1) \sqsupset \sigma(\delta_2))$.

Proof: The proof uses Lemma 3.22. QED

The statement above means that global logic is closed for substitution.

Lemma 3.24 Let V and V' be structures such that for each $v \in V$ there is $v' \in V'$ such that $v(\pi) = v'(\pi)$ for every $\pi \in \text{var}(\delta)$ and for each $v' \in V'$ there is $v \in V$ such that $v(\pi) = v'(\pi)$ for every $\pi \in \text{var}(\delta)$. Then, $V \Vdash_g \delta$ iff $V' \Vdash_g \delta$.

Proof: The statement is proved easily by induction on δ. QED

Lemma 3.25 Assume that $\text{var}(\delta_1) \cap \text{var}(\delta_2) = \emptyset$. Then, the following holds: $\vDash_g (\delta_1 \sqsupset \delta_2)$ iff $\vDash_g (\boxminus \delta_1)$ or $\vDash_g \delta_2$.

Proof: (1i) Suppose $\vDash_g (\boxminus \delta_1)$. Then, for every structure V, $V \Vdash_g (\boxminus \delta_1)$, hence $V \nVdash_g \delta_1$ and so $V \Vdash_g (\delta_1 \sqsupset \delta_2)$. (1ii) Suppose that $\vDash_g \delta_2$. Then, for every structure V, $V \Vdash_g \delta_2$ and so $V \Vdash_g (\delta_1 \sqsupset \delta_2)$. (2) Suppose that $\nvDash_g (\boxminus \delta_1)$ and $\nvDash_g \delta_2$. Then, there are structures V_1 and V_2 such that $V_1 \nVdash_g (\boxminus \delta_1)$ and $V_2 \nVdash_g \delta_2$. Consider the set of all valuations V such that for every $v \in V$, $v(\pi) = v_1(\pi)$ for some $v_1 \in V_1$ and $\pi \in \text{var}(\delta_1)$ and $v(\pi) = v_2(\pi)$ for some $v_2 \in V_2$ and $\pi \in \text{var}(\delta_2)$. Using Lemma 3.24, $V \nVdash_g (\boxminus \delta_1)$ and $V \nVdash_g \delta_2$, hence $V \Vdash_g \delta_1$ and $V \nVdash_g \delta_2$ and so $V \nVdash_g (\delta_1 \sqsupset \delta_2)$. QED

Theorem 3.26 The global logic has Craig interpolation.

Proof: (1) Suppose that $\vDash_g (\delta_1 \sqsupset \delta)$, $\vDash_g (\delta \sqsupset \delta_2)$ and $\text{var}(\delta) \subseteq \text{var}(\delta_1) \cap \text{var}(\delta_2)$. Let V be a structure. We consider two cases. (a) $V \nVdash_g \delta$. Then, $V \nVdash_g \delta_1$ and so $V \Vdash_g (\delta_1 \sqsupset \delta_2)$. (b) $V \Vdash_g \delta$. Then, $V \Vdash_g \delta_2$ and so $V \Vdash_g (\delta_1 \sqsupset \delta_2)$. (2) Assume that $\vDash_g (\delta_1 \sqsupset \delta_2)$. We have two cases. If $\text{var}(\delta_1) \cap \text{var}(\delta_2) = \emptyset$ then by Lemma 3.25 either $\vDash_g (\boxminus \delta_1)$ or $\vDash_g \delta_2$. In the first case take as interpolant a valid global formula. In the second case take as δ a contradiction. Let $\text{var}(\delta_1) \cap \text{var}(\delta_2) \neq \emptyset$. We proceed by induction on the number n of propositional symbols that occur in δ_1 but not in δ_2. Base. Assume that $n = 0$. Then, $\text{var}(\delta_1) \subseteq \text{var}(\delta_2)$. Take δ as δ_1. Then, $\text{var}(\delta) = \text{var}(\delta_1)$ and so $\text{var}(\delta) \subseteq \text{var}(\delta_1) \cup \text{var}(\delta_2)$. Moreover, $\vDash_g (\delta_1 \sqsupset \delta_1)$ and $\vDash_g (\delta_1 \sqsupset \delta_2)$. Step. Assume that $\text{var}(\delta_1) = \{\pi_1, \ldots, \pi_n, \pi'_1, \ldots, \pi'_m, \pi'_{m+1}\} \in \text{var}(\delta_1)$ and $\pi'_1, \ldots, \pi'_m, \pi'_{m+1} \notin \text{var}(\delta_2)$. Let σ_1 and σ_2 be substitutions such that $\sigma_1(\pi'_{m+1}) = \pi_1$ and $\sigma_1(\pi) = \pi$ for the other propositional symbols, $\sigma_2(\pi'_{m+1}) = (\boxminus \pi_1)$ and $\sigma_2(\pi) = \pi$ for the other propositional symbols. Observe that $\sigma_1(\delta_2) = \delta_2$ and $\sigma_2(\delta_2) = \delta_2$. Using Proposition 3.23, $\vDash_g (\sigma_1(\delta_1) \sqsupset \delta_2)$ and $\vDash_g (\sigma_2(\delta_1) \sqsupset \delta_2)$ and, moreover, $\vDash_g ((\sigma_1(\delta_1) \sqsupset \delta_2) \sqcap (\sigma_2(\delta_1) \sqsupset \delta_2))$ and $\vDash_g ((\sigma_1(\delta_1) \sqcup \sigma_2(\delta_1)) \sqsupset \delta_2)$. Using

G. Sica (ed.) *Essays on the Foundations of Mathematics and Logic*

the induction hypothesis over the last global formula, there is a global formula δ such that $\text{var}(\delta) \subseteq \text{var}((\sigma_1(\delta_1) \sqcup \sigma_2(\delta_1))) \cap \text{var}(\delta_2)$, $\vDash_g ((\sigma_1(\delta_1) \sqcup \sigma_2(\delta_1)) \sqsupset \delta)$ and $\vDash_g (\delta \sqsupset \delta_2)$. It remains to prove that $\vDash_g (\delta_1 \sqsupset (\sigma_1(\delta_1) \sqcup \sigma_2(\delta_1)))$. Let V be a structure such that $V \Vdash_g \delta_1$. We have to consider four cases. (a) $V \Vdash_g \pi'_{m+1}$ and $V \Vdash_g \pi_1$. Then, $V \Vdash_g \sigma_1(\delta_1)$. (b) $V \Vdash_g \pi'_{m+1}$ and $V \nVdash_g \pi_1$. Then, $V \Vdash_g \pi'_{m+1}$ and $V \Vdash_g (\boxminus \pi_1)$ and so $V \Vdash_g \sigma_2(\delta_1)$. (c) $V \nVdash_g \pi'_{m+1}$ and $V \Vdash_g \pi_1$. Then, $V \Vdash_g (\boxminus \pi'_{m+1})$ and $V \Vdash_g \pi_1$ and so $V \Vdash_g \sigma_1(\delta_1)$. (d) $V \nVdash_g \pi'_{m+1}$ and $V \nVdash_g \pi_1$. Then, $V \Vdash_g \sigma_1(\delta_1)$. QED

4 Exogenous probability logic

The denotation of terms and satisfaction of global formulas require a richer structure with probabilities. We consider that each interpretation structure has a probability space whose outcome space is a set of valuations. Moreover, we need an assignment for interpreting real variables.

Recall that a probability space is a triple $\langle \Omega, \mathcal{B}, P \rangle$ where Ω is a non empty set, $\mathcal{B} \subseteq \wp\Omega$ is a Borel field (that is, $\mathcal{B}$ includes Ω and is closed for complements and countable unions) and $P : \mathcal{B} \to [0, 1]$ is a non negative function such that:

- $P(\bigcup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} P(B_i)$ whenever $B_i \cap B_j = \emptyset$ for every $i \neq j$.
- $P(\Omega) = 1$.

The elements of Ω are the outcomes, the elements of $\mathcal{B}$ are called events and $P(B)$ is the probability of event B. In short, P is a measure (additive map) with mass 1. For example, given a countable Ω, it is usual to adopt $\wp\Omega$ for $\mathcal{B}$. Observe that, in this case, the probability P is determined by the probability assigned to the singletons. More precisely, assume that $p : \{\{\omega\} : \omega \in \Omega\} \to [0, 1]$ is a non negative map. Then, the probability map P induced by p is defined as follows:

$$P(B) = \sum_{\omega \in B} p(\{\omega\}).$$

It is easy to see that P satisfies the probability axioms and that all probabilities are, in this case, generated by the singletons.

4.1 Semantics

An *interpretation structure* is a pair $I = \langle V, P \rangle$ where V is global valuation (a set of local valuations) and $P = \langle V, \wp V, \mu \rangle$ is a probability space. An *assignment* ρ is a map such that $\rho(x) \in \mathbb{R}$ for each $x \in X$. The *denotation of probability terms* over the interpretation structure I and assignment ρ is the map

$$[\![\cdot]\!]_{I\rho} : T(C) \to \mathbb{R}$$

G. Sica (ed.) *Essays on the Foundations of Mathematics and Logic*

inductively defined as follows:

- $[\![x]\!]_{I\rho} = \rho(x)$;
- $[\![r]\!]_{I\rho} = r$;
- $[\![(\int\varphi)]\!]_{I\rho} = \mu(\text{mod}(\varphi) \cap V)$;
- $[\![(t_1 + t_2)]\!]_{I\rho} = [\![t_1]\!]_{I\rho} + [\![t_2]\!]_{I\rho}$ and $[\![(t_1 t_2)]\!]_{I\rho} = [\![t_1]\!]_{I\rho} \times [\![t_2]\!]_{I\rho}$.

The denotation of $(\int\varphi)$ is the probability, given by μ of the subset of V composed by the models of φ. If $\text{mod}(\varphi) \cap V = V$ then the probability of φ over I is one even if $V \neq \mathcal{V}$. On the other hand, the probability of φ in a particular structure can be zero even if the formula is not a valid one.

The satisfaction of probability formulas given an interpretation structure and an assignment is inductively as follows:

- $I\rho \Vdash \varphi$ iff $V \Vdash_g \varphi$;
- $I\rho \Vdash (t_1 \leq t_2)$ iff $[\![t_1]\!]_{I\rho} \leq [\![t_2]\!]_{I\rho}$;
- $I\rho \Vdash (\boxminus\,\delta)$ iff $I\rho \nVdash \delta$;
- $I\rho \Vdash (\delta_1 \sqsupset \delta_2)$ iff $I\rho \nVdash \delta_1$ or $I\rho \Vdash \delta_2$.

Several useful abbreviations can be introduced like the following:

- $(t_1 < t_2)$ for $((t_1 \leq t_2) \sqcap (\boxminus(t_2 \leq t_1)))$;
- $(t_1 = t_2)$ for $((t_1 \leq t_2) \sqcap (t_2 \leq t_1))$;
- $(\Diamond\varphi)$ for $(0 < (\int\varphi))$;
- $(\Box\varphi)$ for $(1 = (\int\varphi))$;

We investigate some of the properties of the connectives starting with $\Box$ and $\Diamond$ concluding that they are normal modalities.

Proposition 4.1 The operator $\Box$ is normal. That is:

$$I\rho \Vdash (\Box(\varphi_1 \wedge \varphi_2)) \ \text{ iff } \ I\rho \Vdash ((\Box\varphi_1) \sqcap (\Box\varphi_2)).$$

Proof: (i) Assume that $I\rho \Vdash (\Box(\varphi_1 \wedge \varphi_2))$. Hence, $\mu(\text{gmod}(\varphi_1 \wedge \varphi_2) \cap V) = 1$. Since $\text{gmod}((\varphi_1 \wedge \varphi_2)) \subseteq \text{gmod}(\varphi_i)$ for $i = 1, 2$, $\mu(\text{gmod}(\varphi_1) \cap V) = \mu(\text{gmod}(\varphi_2) \cap V) = 1$ and so $I\rho \Vdash (\Box\varphi_i)$ for $i = 1, 2$. (ii) Assume now that $I\rho \Vdash ((\Box\varphi_1) \sqcap (\Box\varphi_2))$. Then, $I\rho \Vdash (\Box\varphi_1)$ and $I\rho \Vdash (\Box\varphi_2)$ and so $\mu(\text{gmod}(\varphi_1) \cap V) = 1$ and $\mu(\text{gmod}(\varphi_2) \cap V) = 1$. Since $\text{gmod}(\varphi_1 \wedge \varphi_2) = \text{gmod}(\varphi_1) \cap \text{gmod}(\varphi_2)$, $\mu(\text{gmod}(\varphi_1 \wedge \varphi_2) \cap V) = \mu(\text{gmod}(\varphi_1) \cap \text{gmod}(\varphi_2) \cap V)$. We show that $\text{gmod}(\varphi_1) \cap V = \text{gmod}(\varphi_2) \cup V$. Assume that $v \in \text{gmod}(\varphi_1) \cap V$ and $v \notin \text{gmod}(\varphi_2) \cap V$. Then, $\mu(v) = 0$. Therefore, $\mu(\text{gmod}(\varphi_1) \cap \text{gmod}(\varphi_2) \cap V) = \mu(\text{gmod}(\varphi_i) \cap V)$ for $i = 1, 2$, hence is 1 and so $I\rho \Vdash (\Box(\varphi_1 \wedge \varphi_2))$. QED

Proposition 4.2 The operator $\diamond$ is normal. That is:

$$I\rho \Vdash ((\diamond\varphi_1) \sqcup (\diamond\varphi_2)) \text{ iff } I\rho \Vdash (\diamond(\varphi_1 \vee \varphi_2)).$$

Proof: (i) Assume that $I\rho \Vdash ((\diamond\varphi_1) \sqcup (\diamond\varphi_2))$ and assume without loss of generality that $I\rho \Vdash (\diamond\varphi_1)$. Then, $\mu(\text{gmod}(\varphi_1) \cap V) > 0$ and so $\mu(\text{gmod}(\varphi_1 \vee \varphi_2) \cap V) > 0$ since $\text{gmod}(\varphi_1) \subseteq \text{gmod}(\varphi_1 \vee \varphi_2)$. (ii) Assume now that $I\rho \Vdash (\diamond(\varphi_1 \vee \varphi_2))$. Then, $\mu(\text{gmod}(\varphi_1 \vee \varphi_2) \cap V) > 0$ and so $\mu((\text{gmod}(\varphi_1) \cap V) \cup (\text{gmod}(\varphi_2) \cup V)) > 0$ since $\text{gmod}(\varphi_1 \vee \varphi_2) = \text{gmod}(\varphi_1) \cup \text{gmod}(\varphi_2)$. QED

The following result shows that $\diamond$ can be defined as an abbreviation using $\Box$ and both the local and the global negations.

Proposition 4.3 For every interpretation structure and assignment ρ and local formula φ,

$$I\rho \Vdash (\boxminus(\Box(\neg\varphi))) \text{ iff } I\rho \Vdash (\diamond\varphi).$$

Proof: $I\rho \Vdash (\boxminus(\Box(\neg\varphi)))$ iff $I\rho \nVdash (\Box(\neg\varphi))$ iff $\mu(\{v \in V : v \Vdash_\ell (\neg\varphi)\}) \neq 1$ iff $\mu(\{v \in V : v \Vdash_\ell \varphi\}) > 0$ iff $I\rho \Vdash (\diamond\varphi)$. QED

Proposition 4.4 Assume that $\mu(v) > 0$ for every valuation. Then, $\Box$ is a T modality.

Proof: Assume that $I\rho \Vdash (\Box\varphi)$. Then, $\mu(\{v \in V : v \Vdash_\ell \varphi\}) = 1$. On the other hand, using the fact that $\mu(v) > 0$ for every valuation, $\{v \in V : v \Vdash_\ell \varphi\}$ is the set of all valuations and so $I\rho \Vdash \varphi$. QED

We can extend the semantic consequence to the full logic. We say that $I\rho \Vdash \Delta$ where Δ is a set of probability formulas if $I\rho \Vdash \delta$ for every $\delta \in \Delta$. Given an interpretation structure and an assignment, we say that the set of probability formulas Δ *entails* the probability formula η, written $\Delta \vDash_{I\rho} \eta$ if $I\rho \Vdash \Delta$ implies that $I\rho \Vdash \eta$.

Proposition 4.5 The metatheorem of deduction holds:

$$\Delta, \eta_1 \vDash_{I\rho} \eta_2 \quad \text{iff} \quad \Delta \vDash_{I\rho} (\eta_1 \sqsupset \eta_2).$$

The proof of the metatheorem of deduction is strightforward. The following are useful properties of the semantic consequence.

Proposition 4.6 Let $\varphi, \varphi_1, \varphi_2$ are local formulas. Then:

- $\varphi \vDash_{I\rho} (\Box\varphi)$;
- $(\varphi_1 \Rightarrow \varphi_2) \vDash ((\int\varphi_1) \leq (\int\varphi_2))$.

Proof: (i) Assume that $I\rho \Vdash \varphi$. Then, $V \Vdash_g \varphi$ and so $\mu(\text{gmod}(\varphi) \cap V) = \mu(V) = 1$. (ii) Assume that $I\rho \Vdash (\varphi_1 \Rightarrow \varphi_2)$. Then, $V \Vdash_g (\varphi_1 \Rightarrow \varphi_2)$, hence $(\text{gmod}(\varphi_1) \cap V) \subseteq (\text{gmod}(\varphi_2) \cap V)$, therefore $\mu(\text{gmod}(\varphi_1) \cap V) \leq \mu(\text{gmod}(\varphi_2) \cap V)$ and so $I\rho \Vdash ((\int \varphi_1) \leq (\int \varphi_2))$. QED

The first statement is the necessitation rule for $\Box$ and the second is a kind of probability monotonicity. Observe that $I\rho \Vdash_g \nu$ iff $J\rho \Vdash_g \nu$ for every interpretation structures I and J for every oracle formula. So, it makes sense to write $\rho \Vdash_g \nu$ in this case. Also for local propositional formula φ we have the following: $I\rho \Vdash_g \rho_1$ iff $I\rho_2 \Vdash_g \varphi$ for every assignments ρ_1 and ρ_2. Both I and ρ are relevant when dealing with formulas that involve probability terms.

4.2 Calculus

The Hilbert calculus for the global logic including the real assertions includes three more axioms for the probability reasoning plus an axiom the oracle reasoning.

PM $\vdash_g (\Box \mathbf{t})$

FA $\vdash_g (\Box(\neg(\varphi_1 \wedge \varphi_2))) \sqsupset ((\int(\varphi_1 \vee \varphi_2)) = ((\int \varphi_1) + (\int \varphi_2))))$

PMon $\vdash_g ((\varphi_1 \Rightarrow \varphi_2) \sqsupset ((\int \varphi_1) \leq (\int \varphi_2)))$

We also need an oracle for inequations on real numbers. For this purpose we add the following axiom:

$$\text{Oracle} \quad \vdash_g \nu \ \text{ if } \ \rho \Vdash_g \nu \ \text{ for every } \ \rho$$

whenever ν is an oracle formula. The PM axiom states that the true has probability one. Axiom FA is the counterpart of finite additivity in probability spaces. Axiom PMon states the interplay between implication and probability: it is a kind of monotonicity. The oracle axiom says that if in the field of real numbers we can prove an assertion we assume that the assertion can also be proved in the calculus.
An example consider the following derivation for $\vdash_g ((\int \varphi) \leq 1)$:

1	$(\varphi \Rightarrow \mathbf{t})$	T
2	$((\int \varphi) \leq (\int \mathbf{t}))$	PMon(1)
3	$((\int \mathbf{t}) = 1)$	PM
4	$(((\int \varphi) \leq (\int \mathbf{t})) \sqsupset (((\int \mathbf{t}) = 1) \sqsupset ((\int \varphi) \leq 1)))$	Oracle
5	$(((\int \mathbf{t}) = 1) \sqsupset ((\int \varphi) \leq 1))$	gMP(2,4)
6	$((\int \varphi) \leq 1)$	gMP(3,5)

G. Sica (ed.) *Essays on the Foundations of Mathematics and Logic*

4.3 Application

A zero-knowledge protocol is a protocol that allows a prover P to show to a verifier V that he has a secret S without revealing it (for more details on zero-knowledge protocols see [10]). The protocol consists of three steps:

- First, the prover sends a value (commitment) to the verifier such that, if he has the secret, for any challenge put to him by the verifier, he is able to send a response convincing the verifier that he has the secret. If he his cheating then he cannot produce a response at least with probability $\frac{1}{2}$;
- Second, the verifier sends a random bit (challenge) to the prover;
- Finally, the prover sends a response according to the bit received.

In principle there are three objectives:

i) V has probability 1 of verifying the secret, if indeed the prover has it (soundness);

ii) V has probability less than 1 of verifying the secret, if the prover is cheating (completeness);

iii) V can not learn the secret (security).

Let Π be $\{s, a, c\}$ where s is the propositional symbol for stating that the prover has a secret, a is the propositional symbol for stating the verifier accepts the secret of the prover and finally, c is the propositional symbol for stating that the commitment is compatible with the challenge. The specification S of the zero-knowledge protocol is as follows:

$$
\begin{array}{ll}
S_1 & ((s \wedge c) \Rightarrow a) \\
S_2 & ((s \wedge (\neg c)) \Rightarrow a) \\
S_3 & (((\neg s) \wedge c) \Rightarrow (\neg a)) \\
S_4 & (((\neg s) \wedge (\neg c)) \Rightarrow a) \\
S_5 & ((\textstyle\int (\neg c)) = \frac{1}{2})
\end{array}
$$

From S, we prove

$$
\begin{array}{ll}
O_1 & (s \sqsupset ((\textstyle\int a) = 1)) \\
O_2 & ((\neg s) \sqsupset ((\textstyle\int a) < 1))
\end{array}
$$

corresponding to objectives 1 and 2 respectively. The derivation of O_1 is as follows:

G. Sica (ed.) *Essays on the Foundations of Mathematics and Logic*

1	$((s \wedge c) \Rightarrow a)$	S_1
2	$((s \wedge (\neg c)) \Rightarrow a)$	S_2
3	$(s \Rightarrow a)$	$T(1,2)$
4	$(s \sqsupset a)$	$P_1(3)$
5	$(a \sqsupset ((\int a) = 1))$	Teo
6	$(s \sqsupset ((\int a) = 1))$	$gT(4,5)$

The derivation of O_2 is as follows:

1	$(((\neg s) \wedge c) \Rightarrow (\neg a))$	S_3
2	$(((\neg s) \wedge (\neg c)) \Rightarrow a)$	S_4
3	$((\neg s) \Rightarrow (a \Rightarrow (\neg c)))$	$T(1,2)$
4	$((\neg s) \sqsupset (a \Rightarrow (\neg c)))$	$P_1(3)$
5	$((a \Rightarrow (\neg c)) \sqsupset ((\int a) \leq (\int (\neg c))))$	PMon
6	$((\neg s) \sqsupset ((\int a) \leq (\int (\neg c))))$	$gT(4,5)$
7	$((\int (\neg c)) = \frac{1}{2})$	S_5
8	$((\neg s) \sqsupset (((\int a) \leq (\int (\neg c))) \sqcap ((\int (\neg c)) = \frac{1}{2}))$	$gT(6,7)$
9	$((((\int a) \leq (\int (\neg c))) \sqcap ((\int (\neg c)) = \frac{1}{2})) \sqsupset (\int a) < 1)$	Oracle
10	$((\neg s) \sqsupset ((\int a) < 1))$	$gT(8,9)$

4.4 Soundness and completeness

The objective is to investigate soundness and completeness of the Hilbert calculus for the global logics including the probability component. We can only consider weak versions of soundness and completeness. Unfortunately completeness is not provable in the presence of hypotheses as the following example shows. Observe that

$$\{(r \leq x_k) : r < \frac{1}{2}\} \vDash_g \ (\frac{1}{2} \leq x_k)$$

holds but that is not the case with $\{(r \leq x_k) : r < \frac{1}{2}\} \ \vdash_g \ (\frac{1}{2} \leq x_k)$ since the rules in our Hilbert calculus are finitary, the notion of derivation is a finite sequence and this logic is not compact. That is, what we derive from an infinite set of hypotheses cannot always be derived from a finite subset.

Theorem 4.7 For every global formula δ, if $\vdash_g \delta$ then $\vDash_g \delta$.

Proof: Taking into account the soundness of the global propositional logic proved in Theorem 3.18 we only have to prove that the three probability axioms are sound.

G. Sica (ed.) *Essays on the Foundations of Mathematics and Logic*

(i) PM. $\mathcal{V} \Vdash_g \mathbf{t}$ and $\mu(\mathcal{V}) = 1$.
(ii) FA. Assume that $I\rho \Vdash_g (\Box(\neg(\varphi_1 \wedge \varphi_2)))$. We start by showing that

$$(*)\ \mu((\text{mod}(\varphi_1 \wedge \varphi_2) \cap V) = 0.$$

By the hypothesis, $[\![(\int(\neg(\varphi_1 \wedge \varphi_2)))]\!]_{I\rho} = 1$, that is, $\mu(\text{mod}(\neg(\varphi_1 \wedge \varphi_2))) \cap V) = 1$. But, on the other hand,

$$\mu(V) = \mu((\text{mod}(\neg(\varphi_1 \wedge \varphi_2)) \cap V) \cup (\text{mod}(\varphi_1 \wedge \varphi_2) \cap V))$$

and, since $\text{mod}(\neg(\varphi_1 \wedge \varphi_2)) \cap V) \cap (\text{mod}(\varphi_1 \wedge \varphi_2) \cap V) = \emptyset$,

$$\mu(V) = \mu(\text{mod}(\neg(\varphi_1 \wedge \varphi_2)) \cap V) + \mu(\text{mod}(\varphi_1 \wedge \varphi_2) \cap V).$$

Therefore, $\mu(\text{mod}(\varphi_1 \wedge \varphi_2) \cap V) = 0$. Now we are ready to show that $I\rho \Vdash_g ((\int(\varphi_1 \vee \varphi_2)) = ((\int\varphi_1) + (\int\varphi_2)))$, that is,

$$[\![(\int(\varphi_1 \vee \varphi_2))]\!]_{I\rho} = [\![(\int\varphi_1) + (\int\varphi_2))]\!]_{I\rho}.$$

But,

$$[\![(\int(\varphi_1 \vee \varphi_2))]\!]_{I\rho} = \mu(\{v \in V : v \Vdash_\ell (\varphi_1 \vee \varphi_2)\})$$

and

$$\mu(\{v \in V : v \Vdash_\ell (\varphi_1 \vee \varphi_2)\}) = \mu(\bigcup_{i=1}^{2}\{v \in V : v \Vdash_\ell \varphi_i\}).$$

Using finite additivity

$$\mu(\bigcup_{i=1}^{2}\{v \in V : v \Vdash_\ell \varphi_i\}) = \sum_{i=1}^{2}\mu(\{v \in V : v \Vdash_\ell \varphi_i\}) - \mu((\text{mod}(\varphi_1 \wedge \varphi_2) \cap V)$$

and so by (*)

$$\mu(\bigcup_{i=1}^{2}\{v \in V : v \Vdash_\ell \varphi_i\}) = \sum_{i=1}^{2}\mu(\{v \in V : v \Vdash_\ell \varphi_i\}).$$

(iii) PMon. Assume that $I\rho \Vdash_g (\varphi_1 \Rightarrow \varphi_2)$. Then,

$$\{v \in V : v \Vdash_\ell \varphi_1\} \subseteq \{v \in V : v \Vdash_\ell \varphi_2\},$$

hence $\mu(\{v \in V : v \Vdash_\ell \varphi_1\}) \subseteq \mu(\{v \in V : v \Vdash_\ell \varphi_2\})$ and so $(\int\varphi_1) \subseteq (\int\varphi_2)$. QED

In what concerns weak completeness we are going to use the fact that every global formula has a disjunctive normal and capitalize on the technique in [6]. Let Gatom : $G(C) \to \wp L(C) \cup P(C)$ be a map inductively defined as follows: $\text{Gatom}(\varphi) = \{\varphi\}$, $\text{Gatom}(\nu) = \{\nu\}$; $\text{Gatom}(\boxminus \eta) = \text{Gatom}(\eta)$ and $\text{Gatom}(\eta_1 \sqsupset$

G. Sica (ed.) *Essays on the Foundations of Mathematics and Logic*

$\eta_2) = \text{Gatom}(\eta_1) \cup \text{Gatom}(\eta_2)$. Let $\text{Gmol}(\eta)$ as the set of all $A \subseteq \text{Gatom}(\alpha)$ (global molecules) such that the following global formula holds:

$$(((\bigsqcap_{\alpha \in A} \alpha) \sqcap (\bigsqcap_{\alpha \in \text{Gatom}(\eta)\setminus A} (\boxminus \alpha))) \sqsupset \eta).$$

Note that a global molecule includes global propositional molecules in $\text{gmol}(\eta)$ and formulas of the form $(t_1 \leq t_2)$ and $(\boxminus(t_1 \leq t_2))$. We call $\text{gmol}(A)$ the set of global propositional molecules in $\text{gmol}(\eta)$ that occur in A and by $\text{pmol}(A)$ the set of probability formulas that occur in A. Let $\text{ipmol}(A)$ be the set of inequalities in $\text{pmol}(A)$ not involving probabilities.

Theorem 4.8 For every global formula δ, if $\vDash_g \delta$ then $\vdash_g \delta$.

Proof: The proof is by contraposition. Assume that $\not\vdash_g \delta$. Then, $\vdash_g (\boxminus \delta)$ is globally consistent. The objective is to find a structure $I = \langle V, P \rangle$, a set of valuations and a probability space $P = \langle V, \wp V, \mu \rangle$, and an assignment ρ such that $I\rho \Vdash_g (\boxminus \delta)$. The strategy of the proof is as follows: (i) The problem can be stated in a more concrete way saying that we have to find I and ρ such that $I\rho \Vdash A$ for some global molecule A of $(\boxminus \eta)$; (ii) For such an A: let V as the set $\text{gmod}(\text{gmol}(A))$; for each $\alpha \in \text{pmol}(A) \setminus \text{ipmol}(A)$;

i) let each term $(\int \varphi)$ be replaced by $\sum_{v \in \text{mod}(\varphi)} z_v$ (each variable z_v represents the probability of valuation v);

ii) if $t_1 \leq t_2$ is an inequation with probability terms get an inequation $u_1 \leq u_2$ by replacing in both t_1 and t_2 the probability terms as indicated in 1.

In this way we obtain a system of inequalities including:

- (a) the inequalities in $\text{ipmol}(A)$;
- (b) the inequalities in 2;
- (c) the inequalities $\sum_{v \in V} z_v \leq 1$ and $1 \leq \sum_{v \in V} z_v$.

The unknown variables are the real variables and the variables z_v. If the system of inequations has a solution then we get:

- ρ by saying that $\rho(x)$ is the solution x of the system of inequations;
- μ by defining $\mu(v)$ as the solution z_v;
- the value of a variable that does not come as a solution of the induced system of inequalities is any computable real number;
- $\mu(v)$ for a $v \in V$ not in the scope of a probability formula is free providing that the sum of all valuations for $v \in V$ is less than 1.

Assume that the system of equations does not have a solution. That is, there are no values for the real variables in X that occur in the system and also there are no values for the z variables. Then, for each assignment ρ we can conclude that $\rho \not\Vdash_g \alpha$ for some $\alpha \in \mathrm{ipmol}(A)$. In other words, every assignment ρ is such that $\rho \Vdash_g (\boxminus \alpha)$ for some $\alpha \in \mathrm{ipmol}(A)$. Hence, using the oracle axiom we can conclude that

$$\vdash_g (\bigsqcup_{\alpha \in \mathrm{ipmol}(A)} (\boxminus \alpha)),$$

therefore

$$\vdash_g ((\bigsqcup_{\alpha \in \mathrm{ipmol}(A)} (\boxminus \alpha)) \sqcup (\bigsqcup_{\alpha \in \mathrm{pmol}(A) \setminus \mathrm{ipmol}(A)} (\boxminus \alpha))$$

and, so,

$$\vdash_g (\boxminus \bigsqcap_{\alpha \in A} \alpha)$$

contradicting the consistency hypothesis of A. QED

We consider a small example of the construction. Assume that A is composed by three formulas:

- $x \leq 0.4$;
- $(2(\int(\pi_1 \vee (\pi_2 \wedge \pi_3))) + (3(\int((\neg\, \pi_1) \wedge \pi_2)))) \leq x$;
- $((\pi_2 \vee \pi_3) \vee (\pi_1 \wedge (\neg\, \pi_2) \wedge (\neg\, \pi_3)))$.

We start by observing that we have to consider eight valuations using Proposition 3.24 since η has three propositional symbols. Let v_{ijk} be the valuation v such that $v(\pi_1) = i$, $v(\pi_2) = j$ and $v(\pi_3) = k$. Then, $\mathrm{mod}(\pi_1 \vee (\pi_2 \wedge \pi_3)) = \{v_{100}, v_{011}, v_{111}\}$ and $\mathrm{mod}((\neg\, \pi_1) \wedge \pi_2) = \{v_{010}, v_{011}\}$. The induced system of inequations is as follows:

$$\begin{cases} x \leq 0.4 \\ 2z_{v_{100}} + 5z_{v_{011}} + 2z_{v_{111}} + 3z_{v_{010}} \leq x \\ \sum_{v \in \mathrm{mod}(\alpha_3)} z_v \leq 1 \\ 1 \leq \sum_{v \in \mathrm{mod}(\alpha_3)} z_v \end{cases}$$

with the following possible solution:

$$\begin{cases} x = 0.4 \\ z_{v_{100}} = z_{v_{011}} = 0.02, z_{v_{111}} = 0.1, z_{v_{010}} = 0.02 \end{cases}$$

Therefore, the structure I and the assignment ρ are as follows:

G. Sica (ed.) *Essays on the Foundations of Mathematics and Logic*

- V is $\mathcal{V} \setminus \{v_{000}\}$;
- $\rho(x) = 0.4$;
- $\mu(v_{100}) = \mu(v_{011}) = 0.02$, $\mu(v_{111}) = 0.1$, $\mu(v_{010}) = 0.02$;
- $\rho(y) = 0$ for every $y \in X \setminus \{y\}$;
- $\mu(v_{110}) = 0.05$, $\mu(v_{001}) = 0.15$, $\mu(v_{101}) = 0.4$.

5 Conclusions

A probability logic resulting from enriching propositional logic, the local logic, was presented using an exogenous semantics. A model of the probability logic is a global valuation, that is, a set of propositional valuations, endowed with a probability measure. Syntactically, probability appears as a constructor that when applied to a local formula returns a probability term. A probability assertion is an inequality between probability terms. The global logic is a propositional logic whose atomic formulas are the local formulas and the probability assertions. A Hilbert calculus was provided and shown to be sound and weakly complete (generalizing the technique in [6]). The interplay between the local and the global propositional logics was also explored.

The work presented in this paper raises several new problems. The most challenging one is how to take a general base logic and make it probabilistic. That is: how to define a new probabilization operator on logics that for any argument base logic returns the corresponding probability logic. The work on probabilization of logics is already under way [15]. Main issues are: characterization of the domain of such an operator (namely, what are the minimal requirements for the base logic); new techniques for proving weak completeness not assuming that the disjunctive normal form lemma holds in the base logic.

The work on probabilization of logics will also be relevant in giving pointers to the quantization of logics as an operator on logics: what are the requirements for the base logic in order to be possible to extend it to a quantum logic (following the first quantization examples in [13, 14]).

More concretely, we also want to consider first-order and modal logics as base logics so that we can use the corresponding probability logics for specifying and reasoning about more realistic secure protocols.

Acknowledgment

This work was partially supported by FCT and FEDER through POCTI, namely via CLC and the QuantLog POCTI/MAT/55796/2004 Project.

Bibliography

[1] Abadi, M. and J. Y. Halpern. [1994] Decidability and expressiveness for first-order logics of probability. Information and Computation, 112, n1, 1-36.

[2] Adams, E. [1998] *A Primer of Probability Logic*, volume 68 of *CSLI Lecture Notes.* CSLI Publications.

[3] Burgess, J. [1969] Probability logic. Journal of Symbolic Logic, 34, n1, 264-274.

[4] Carnap, R. [1950] *Logical Foundations of Probability.* The University of Chicago Press. Chicago.

[5] Fagin, R. and J. Y. Halpern. [1994] Reasoning about knowledge and probability. Journal of the Association for Computing Machinery, 41, n2, 340-367.

[6] Fagin, R., J. Y. Halpern, and N. Megiddo. [1990] A logic for reasoning about probabilities. Information and Computation, 87, n1-2, 78-128.

[7] Fariñas del Cerro, L. and A. Herzig. [1991] A modal analysis of possibility theory. *Fundamentals of artificial intelligence research (Smolenice, 1991)*, volume 535 of *Lecture Notes in Computer Science*, pages 11-18. Springer.

[8] Fattorosi-Barnaba, M. and F. De Caro. [1985] Graded modalities. I. Studia Logica, 44, n2, 197-221.

[9] Fox, J. [2003] Probability, logic and the cognitive foundations of rational belief. Journal of Applied Logic, 1, n3-4, 197-224.

[10] Goldreich, O. and Y. Oren. [1994] Definitions and properties of zero-knowledge proof systems. Journal of Cryptology, 7, n1, 1-32.

[11] Hájek, A. [2001] Probability, logic, and probability logic. *The Blackwell Guide to Philosophical Logic*, pages 362-384. Blackwell. Oxford.

[12] Mateus, P., A. Pacheco, J. Pinto, A. Sernadas, and C. Sernadas. [2001] Probabilistic situation calculus. Annals of Mathematics and Artificial Intelligence, 32, n1-4, 393-431.

[13] Mateus, P. and A. Sernadas. [2004] Reasoning about quantum systems. *Logics in Artificial Intelligence, Ninth European Conference, JELIA'04*, volume 3229 of *Lecture Notes in Artificial Intelligence*, pages 239-251. Edited by J. Alferes and J. Leite. Springer-Verlag.

[14] Mateus P. and A. Sernadas. [2005] Weakly complete axiomatization of exogenous quantum propositional logic. Preprint, CLC, Department of Mathematics, Instituto Superior Técnico, 1049-001 Lisboa, Portugal. ArXiv math.LO/0503453. Submitted for publication.

G. Sica (ed.) *Essays on the Foundations of Mathematics and Logic*

[15] Mateus P., A. Sernadas, and C. Sernadas. [2005] Probabilization of logics. Technical report, CLC, Department of Mathematics, Instituto Superior Técnico, 1049-001 Lisboa, Portugal. In preparation.

[16] Nilsson, N. J. [1986] Probabilistic logic. Artificial Intelligence, 28, n1, 71-87.

[17] Nilsson, N. J. [1993] Probabilistic logic revisited. Artificial Intelligence, 59, n1-2, 39-42.

[18] Ognjanović Z. and M. Rašković. [1999] Some probability logics with new types of probability operators. Journal of Logic and Computation, 9, n2, 181-195.

[19] Ognjanovic Z. and M. Raškovic. [2000] Some first-order probability logics. Theoretical Computer Science, 247, n1-2, 191-212.

[20] Rutten J., M. Kwiatkowska, G. Norman, and D. Parker. [2004] *Mathematical Techniques for Analyzing Concurrent and Probabilistic Systems*, volume 23 of *CRM Monograph Series*. AMS.

[21] Segerberg, K. [1971] Qualitative probability in a modal setting. *Proceedings of the Second Scandinavian Logic Symposium* volume 63 of *Studies in Logic and the Foundations of Mathematics*, pages 341-352. North-Holland.

[22] van der Hoek, W. [1992] On the semantics of graded modalities. Journal of Applied Non-Classical Logics, 2, n1, 81-123.

[23] Williamson, J. [2002] Probability logic. *Handbook of the Logic of Argument and Inference*, volume 1, pages 397-424. North-Holland.

[24] Williamson, J. and D. Gabbay, editors. [2003] *Combining Probability and Logic*. Elsevier Science B.V., Amsterdam. Papers from the 4th Augustus de Morgan Workshop held at King's College, London, November 4-6, 2002. Journal of Applied Logic, 1, n3-4.

Paulo Mateus, Amílcar Sernadas, and Cristina Sernadas
Center for Logic and Computation (CLC)
Department of Mathematics
Instituto Superior Técnico
Av. Rovisco Pais
1049-001 Lisboa - Portugal
e-mail: {pmat,acs,css}@math.ist.utl.pt
web-site: http://clc.math.ist.utl.pt/

G. Sica (ed.) *Essays on the Foundations of Mathematics and Logic*