Approximate reasoning about logic circuits with single-fan-out unreliable gates

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February 18, 2013 / revised July 8, 2013

Abstract

A complete extension of classical propositional logic is proposed for reasoning about circuits with unreliable gates. The pitfalls of extrapolating classical reasoning to such unreliable circuits are extensively illustrated. Several metatheorems are shown to hold with additional provisos. Applications are provided in verification of logic circuits and improving their reliability.

Keywords: probabilistic logic, unreliable circuits, combined connectives.

1 Introduction

It is well known that classical propositional logic is the right setting for the design and verification of logic circuits represented by formulas. Some examples of recent work in this broad area can be found in [4, 2].

However, logic circuits are built with unreliable gates that can produce the wrong output by fortuitous misfiring, hopefully with a very low probability. The major result in this area was established by John von Neumann [20]: with redundancy one can make the error probability δ of the overall circuit as close as needed to the error probability of the majority gate, independently of the error probability of the other logical gates.

Towards establishing this result while representing circuits by formulas, the following assumptions are required: the probability of a gate misfiring is independent of its inputs, gates misfire independently of each other, there is no sub-circuit reuse, and the circuit inputs are deterministic. For the sake of simplicity we also assume that every gate has the same probability of misfiring $\varepsilon < \frac{1}{2}$. Henceforth, we shall refer to this set of assumptions as the *standard application scenario*.

It is important to recall that the traditional representation of circuits by formulas (representing gates by connectives) has the following limitation: it disregards the possibility of reusing (the output of) a sub-circuit. This is immaterial in the case of circuits built with reliable gates that never misfire, because nothing is lost by repeating in the formula the subformula representing the sub-circuit. However, in the case of circuits built with unreliable gates, their representation by formulas is faithful only if no sub-circuit reuse is allowed. Clearly, otherwise, at the formula level, the independence hypothesis would be violated. Observe also that the no sub-circuit reuse hypothesis is equivalent to assuming that the circuit is built only with single-fan-out gates.

The interest on reasoning about logic circuits built with unreliable gates, in short *unreliable circuits*, was recently reawaken by developments towards nano-circuits, see for instance [7, 10, 16], where the extremely low level of energy carried by each gate leads to a higher probability of it being disturbed by the environment and, so, misfiring. Note also that in such nano-circuits, the single-fan-out hypothesis is warranted by the fact that it is not feasible to split the output of a gate without signal amplification.

Our objective is the development and study of key properties of a probabilistic extension of classical propositional logic, for the purpose of addressing the problems in the design and verification of logic circuits with unreliable gates. Other probabilistic and non-deterministic logics have been reported in the literature [1, 6, 9, 12, 13, 14, 15], but they do not address the specific problems of reasoning about unreliable circuits with gates that may misfire. The work on reasoning about probabilistic programs [3, 5] is even further away from the matter at hand.

Our approach was inspired by the notion of non-deterministic meet combination of connectives that appeared in the field of universal logic as a way of combining logics [17, 18]. Indeed, for instance, the unreliable AND gate can be seen as the meet combination of the AND and the NAND gates. The next obvious step is to probabilize the possible outcomes of the meet combination, the key idea explored in this paper. Observe that, in von Neumann's application scenario, given the input (b_1, b_2), the unreliable AND gate has the probability $\varepsilon < \frac{1}{2}$ of misfiring and, so, producing the wrong output $\mathrm{NAND}(b_1, b_2)$ and the probability $1 - \varepsilon$ of producing the correct output $\mathrm{AND}(b_1, b_2)$. Herein, we only address such misfiring errors.

The language, the semantics and the calculus of the proposed unreliable-circuit logic are presented in Section 2. An enriched calculus including some useful admissible rules is introduced in Section 3. Soundness and completeness results are established in Section 4. Although the proposed logic is a conservative extension of the classical propositional logic (as shown in Section 2), it is full of surprises for those used to the nice meta-properties of the latter, as illustrated throughout the paper. Nevertheless, many interesting meta-properties still hold as proved in Section 5. The application scenarios are discussed in Section 6. An assessment of what was achieved and the list of some still open problems are given in Section 7.

2 Unreliable-circuit logic

The envisaged unreliable-circuit logic is defined below as an extension of propositional logic. To this end, we need to adopt some notation concerning the latter.

Let PL denote the version of (classical) *propositional logic* with the following rich language. Its *signature* Σ contains the propositional constants tt (verum) and ff (falsum) plus the propositional connectives $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\supset$ (implication), $\equiv$ (equivalence) and M_{3+2k} (k-ary majority) for each $k \in \mathbb{N}$, as well as their negated-output counterparts $\overline{\neg}$ (identity), $\overline{\wedge}$ (negated conjunction), $\overline{\vee}$ (negated disjunction), $\overline{\supset}$ (negated implication), $\overline{\equiv}$ (negated equivalence) and $\overline{\mathsf{M}}_{3+2k}$ (k-ary negated majority) for each $k \in \mathbb{N}$. Each majority connective returns 1 if the majority of its inputs is 1 and, otherwise, it returns 0.

Observe that we could have introduced most of these signature elements as abbreviations from a small set of primitive connectives (e.g. falsum and implication). For instance,

$$\mathsf{M}_3(x_1, x_2, x_3)$$

could have been introduced as an abbreviation of

$$(x_1 \wedge x_2) \vee (x_2 \wedge x_3) \vee (x_3 \wedge x_1),$$

that is, starting only from falsum and implication, as an abbreviation of

$$(((\varphi_{12} \supset \mathsf{ff}) \supset \varphi_{23}) \supset \mathsf{ff}) \supset \varphi_{31}$$

where each φ_{ij} is the formula

$$(x_i \supset (x_j \supset \mathsf{ff})) \supset \mathsf{ff}.$$

Nevertheless, we adopt the rich signature above in order to simplify the presentation of PL, since no price has to be paid in its axiomatization (given that the set of tautologies is decidable).

Formulas in PL are built in the usual way with the elements of Σ and the *propositional variables* in $X = \{x_k : k \in \mathbb{N}\}$. The PL *language* (the set of its *formulas*) and the PL *ground language* (the set of its *ground formulas*[1]) are denoted by $L(X)$ and L, respectively.

Assuming that, as usual, the PL semantics is provided by valuations, we write

$$v \Vdash \varphi$$

for saying that valuation $v : X \to \{\bot, \top\}$ satisfies formula $\varphi \in L(X)$, and

$$\Lambda \vDash \varphi$$

for stating that φ is entailed from hypotheses in $\Lambda \subseteq L(X)$. Furthermore, assuming that PL is endowed with a Hilbert calculus including the tautologies (TAUT) as axioms plus modus ponens (MP) as the unique inference rule, we write

$$\Lambda \vdash \varphi$$

for stating that φ can be derived from Λ. Recall that this calculus is strongly sound and complete: $\Lambda \vDash \varphi$ if and only if $\Lambda \vdash \varphi$.

Before proceeding with the presentation of the envisaged unreliable-circuit logic, we also need to adopt some notation concerning the first-order theory of ordered real closed fields (denoted by ORCF), having in mind the use of its terms for denoting probabilities and other quantities.

Recall that the first-order signature of ORCF contains the constants 0 and 1, the unary function symbol $-$, the binary function symbols $+$ and $\times$, and the binary predicate symbols $=$ and $<$. As usual, we may write $t_1 \leq t_2$ for $(t_1 < t_2) \vee (t_1 = t_2)$, $t_1\, t_2$ for $t_1 \times t_2$ and t^n for

$$\underbrace{t \times \cdots \times t}_{n \text{ times}}.$$

Furthermore, we also use the following abbreviations for any given $m \in \mathbb{N}^+$ and $n \in \mathbb{N}$:

- m for $\underbrace{1 + \cdots + 1}_{\text{addition of } m \text{ units}}$;

[1] Formulas without variables.

- m^{-1} for the unique z such that $m \times z = 1$;[2]
- $\dfrac{n}{m}$ for $m^{-1} \times n$.

The last two abbreviations might be extended to other terms, but we need them only for numerals.

In order to avoid confusion with the other notions of satisfaction used herein, we adopt $\Vdash^{\mathsf{fo}}$ for denoting satisfaction in first-order logic.

Recall also that the theory ORCF is decidable [19]. This fact will be put to good use in the proposed axiomatization of gate misfiring in circuits. Furthermore, every model of ORCF satisfies the theorems and only the theorems of ORCF (Corollary 3.3.16 in [11]). We shall take advantage of this result in the semantics of UCL for adopting the ordered field $\mathbb{R}$ of the real numbers as the model of ORCF.

With this modicum of PL and of ORCF at hand, we are ready to present the syntax, the semantics and the calculus of the envisaged *unreliable-circuit logic* (denoted by UCL).

2.1 Syntax of UCL

The *signature* of UCL is the triple $(\Sigma^{\mathsf{uc}}, \nu, \mu)$ where:

- Σ^{uc} contains Σ and the following additional connectives used for representing the unreliable gates:

$$\widetilde{\neg},\ \widetilde{\overline{\neg}},\ \widetilde{\wedge},\ \widetilde{\overline{\wedge}},\ \widetilde{\vee},\ \widetilde{\overline{\vee}},\ \widetilde{\supset},\ \widetilde{\overline{\supset}},\ \widetilde{\equiv},\ \widetilde{\overline{\equiv}},\ \widetilde{\mathsf{M}}_{3+2k} \text{ and } \widetilde{\overline{\mathsf{M}}}_{3+2k};$$

- both ν and μ are symbols used for denoting probabilities.

Each unreliable gate is assumed to produce the correct output with probability ν, corresponding to $1-\varepsilon$ in von Neumann's scenario. A circuit is accepted as good if it produces the correct output with probability not less than μ, corresponding to $1-\delta$ in that scenario.

Observe that we could have introduced most of the connectives representing unreliable gates as abbreviations. Indeed, only the unreliable connective

[2] Indeed, in particular for each $m \in \mathbb{N}^+$,

$$\exists^1 z\,(m \times z = 1)$$

is a theorem of ORCF.

$\widetilde{\overline{\neg}}$ is needed as primitive in the scenario where every gate has the same probability of misfiring. The other unreliable connectives could be introduced as abbreviations, for instance,

$$x_1 \mathbin{\widetilde{\wedge}} x_2$$

as an abbreviation of

$$\widetilde{\overline{\neg}}(x_1 \wedge x_2).$$

Nevertheless, we adopted the rich signature above since no price has to be paid in setting-up UCL. As we shall see in due course, the presentation of the syntax, the semantics and the axiomatization of UCL is not burdened by the presence of so many connectives. Furthermore, proofs by induction on the structure of formulas are also not affected since they are carried out using a generic unreliable connective. We return to this issue at the end of Subsection 2.2 where we analyze the relationship between $x_1 \mathbin{\widetilde{\wedge}} x_2$ and $\widetilde{\overline{\neg}}(x_1 \wedge x_2)$.

We denote by $\widetilde{\Sigma}$ the subsignature of the unreliable connectives in Σ^{uc}. Thus,

$$\Sigma^{\mathsf{uc}} = \Sigma \cup \widetilde{\Sigma}.$$

For each $n \in \mathbb{N}$, we denote by Σ_n, Σ^{uc}_n and $\widetilde{\Sigma}_n$ the set of n-ary constructors in Σ, Σ^{uc} and $\widetilde{\Sigma}$, respectively. Plainly, $\widetilde{\Sigma}_0 = \emptyset$. For each $c \in \Sigma \setminus \Sigma_0$,

$$\overline{\overline{c}}$$

is taken to be c, namely in the following inductive definition. Given a PL formula φ and a formula ψ built with connectives in Σ^{uc} and propositional variables in X (an unreliable-circuit formula as explained below), we write

$$\varphi \sqsubseteq \psi$$

for saying that φ is a possible *outcome* of ψ. This outcome relation is inductively defined as expected:

- $\varphi \sqsubseteq \varphi$ provided that φ is a PL formula;
- $c(\varphi_1, \dots, \varphi_n) \sqsubseteq c(\psi_1, \dots, \psi_n)$ provided that $n \geq 1$, $c \in \Sigma_n$ and $\varphi_i \sqsubseteq \psi_i$ for $i = 1, ..., n$;
- $c(\varphi_1, \dots, \varphi_n) \sqsubseteq \widetilde{c}(\psi_1, \dots, \psi_n)$ provided that $\widetilde{c} \in \widetilde{\Sigma}_n$, and $\varphi_i \sqsubseteq \psi_i$ for $i = 1, ..., n$;
- $\overline{c}(\varphi_1, \dots, \varphi_n) \sqsubseteq \widetilde{c}(\psi_1, \dots, \psi_n)$ provided that $\widetilde{c} \in \widetilde{\Sigma}_n$, and $\varphi_i \sqsubseteq \psi_i$ for $i = 1, ..., n$.

For each such ψ, we denote by

$$\Omega_\psi$$

the set $\{\varphi : \varphi \sqsubseteq \psi\}$ of all possible outcomes of ψ. Clearly, $\Omega_\varphi = \{\varphi\}$ for each PL formula φ.

As already mentioned, terms are needed for denoting probabilities and other quantities. In UCL, by a *term* we mean a univariate polynomial written according to the term syntax of ORCF, using ν as the variable. For example,

$$\nu \times (1 - \nu)^2$$

that we may write

$$\nu(1 - \nu)^2$$

is a term of UCL. Symbol μ is also taken as a variable in the context of ORCF but it is not used in UCL terms.

Three kinds of formulas are needed for reasoning about circuits with unreliable gates:

- *Circuit formulas* or *c-formulas* that are propositional formulas built with the symbols in Σ^{uc} and X. These c-formulas can be used for representing unreliable circuits. For instance, the c-formula

  $$(x_1 \mathbin{\widetilde{\vee}} (\widetilde{\neg}\, x_2)) \mathbin{\widetilde{\wedge}} x_3$$

  represents the unreliable circuit in Figure 1. Circuit formulas can also be used for asserting relevant properties of unreliable circuits. For example, given the c-formula ψ and the PL formula φ, the c-formula

  $$\psi \equiv \varphi$$

  is intended to state that the unreliable circuit represented by ψ can be accepted as equivalent to the reliable circuit represented by φ, in the sense that the two circuits agree with probability of at least μ.

- *Outcome formulas* or *o-formulas* that are of the general form

  $$\Phi \sqsubseteq_P \psi$$

  where ψ is a c-formula, $\Phi \subseteq \Omega_\psi$ and P is a term. Such an o-formula is used with the intent of stating that the probability of the outcome of ψ being in Φ is at least P. For instance,

  $$\{(x_1 \vee (\neg\, x_2)) \wedge x_3, (x_1 \mathbin{\overline{\vee}} (\neg\, x_2)) \wedge x_3\} \sqsubseteq_{\nu^2} (x_1 \mathbin{\widetilde{\vee}} (\widetilde{\neg}\, x_2)) \mathbin{\widetilde{\wedge}} x_3$$

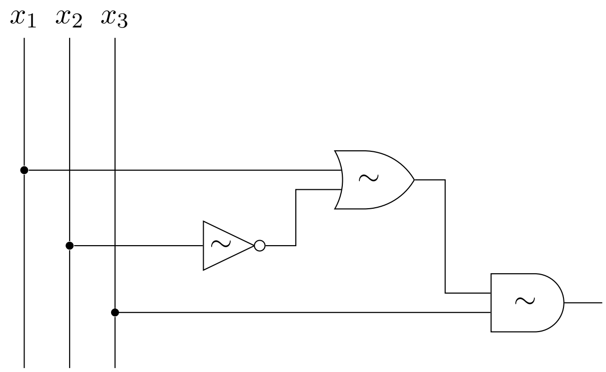

Figure 1: Circuit represented by the c-formula $(x_1 \mathbin{\widetilde{\vee}} (\widetilde{\neg}\, x_2)) \mathbin{\widetilde{\wedge}} x_3$.

should be true in any interpretation of UCL because $(x_1 \vee (\neg\, x_2)) \wedge x_3$ and $(x_1 \mathbin{\overline{\vee}} (\neg\, x_2)) \wedge x_3$ are both possible outcomes of $(x_1 \mathbin{\widetilde{\vee}} (\widetilde{\neg}\, x_2)) \mathbin{\widetilde{\wedge}} x_3$ (the former when all the unreliable gates perform perfectly and the latter when only the OR gate fails), the probability of the former is ν^3, the probability of the latter is $(1-\nu)\nu^2$, and $\nu^3 + (1-\nu)\nu^2 = \nu^2$.

- *Ambition formulas* or *a-formulas* that are of the general form

$$\mu \leq P$$

where P is a term. Such an a-formula can be used for constraining the envisaged non-failure probability μ of the overall circuit. For example, every interpretation of UCL where the a-formula

$$\mu \leq \nu^2 + (1-\nu)^2$$

holds should make

$$(\neg(x_1 \vee x_2)) \equiv (\widetilde{\neg}(x_1 \mathbin{\widetilde{\vee}} x_2))$$

true, since $\neg(x_1 \vee x_2)$ and $\overline{\neg}(x_1 \mathbin{\overline{\vee}} x_2)$ are the outcomes of the circuit at hand $\widetilde{\neg}(x_1 \mathbin{\widetilde{\vee}} x_2)$ that make it in agreement to the ideal one $\neg(x_1 \vee x_2)$, the probability of outcome $\neg(x_1 \vee x_2)$ is ν^2, the probability of outcome $\overline{\neg}(x_1 \mathbin{\overline{\vee}} x_2)$ is $(1-\nu)^2$, and, so, their aggregated probability is

$$\nu^2 + (1-\nu)^2.$$

Given m distinct formulas $\varphi_1, \ldots, \varphi_m$ in Ω_ψ, we may write

$$\varphi_1, \ldots, \varphi_m \sqsubseteq_P \psi$$

for $\{\varphi_1, \dots, \varphi_m\} \sqsubseteq_P \psi$. We denote by $L^{\mathsf{c}}(X)$, $L^{\mathsf{o}}(X)$ and L^{a} the set of c-formulas, o-formulas and a-formulas, respectively, and by $L^{\mathsf{uc}}(X)$ the set $L^{\mathsf{c}}(X) \cup L^{\mathsf{o}}(X) \cup L^{\mathsf{a}}$ of all UCL formulas. Observe that each of these sets is decidable.

Furthermore, we use L^{c} for the set of ground c-formulas. Evidently, L and $L(X)$ are decidable subsets of L^{c} and $L^{\mathsf{c}}(X)$, respectively.

Given a c-formula ψ and $\varphi \in \Omega_\psi$, we write

$$\mathfrak{P}[\psi \triangleright \varphi]$$

for the UCL term that provides the probability of outcome φ of ψ. This term is inductively defined as follows:

- $\mathfrak{P}[\varphi \triangleright \varphi]$ is 1 for each $\varphi \in L(X)$;
- $\mathfrak{P}[c(\psi_1, \dots, \psi_n) \triangleright c(\varphi_1, \dots, \varphi_n)]$ is $\displaystyle\prod_{i=1}^{n} \mathfrak{P}[\psi_i \triangleright \varphi_i]$ for each $n \geq 1$, $c \in \Sigma_n$ and $\varphi_i \sqsubseteq \psi_i$ for $i = 1, \dots, n$;
- $\mathfrak{P}[\widetilde{c}(\psi_1, \dots, \psi_n) \triangleright c(\varphi_1, \dots, \varphi_n)]$ is $\nu \displaystyle\prod_{i=1}^{n} \mathfrak{P}[\psi_i \triangleright \varphi_i]$ for each $\widetilde{c} \in \widetilde{\Sigma}_n$ and $\varphi_i \sqsubseteq \psi_i$ for $i = 1, \dots, n$;
- $\mathfrak{P}[\widetilde{c}(\psi_1, \dots, \psi_n) \triangleright \overline{c}(\varphi_1, \dots, \varphi_n)]$ is $(1 - \nu) \displaystyle\prod_{i=1}^{n} \mathfrak{P}[\psi_i \triangleright \varphi_i]$ for each $\widetilde{c} \in \widetilde{\Sigma}_n$ and $\varphi_i \sqsubseteq \psi_i$ for $i = 1, \dots, n$.

For instance,

$$\mathfrak{P}[\widetilde{\neg}(x_1 \mathbin{\widetilde{\triangledown}} x_2) \triangleright \neg(x_1 \triangledown x_2)]$$

is the polynomial

$$\nu(1 - \nu)$$

since, for the given input provided by x_1 and x_2, outcome $\neg(x_1 \triangledown x_2)$ happens when $\widetilde{\neg}$ behaves as it should and $\widetilde{\triangledown}$ fails, that is, when $\widetilde{\neg}$ produces the correct output and $\widetilde{\triangledown}$ misfires.

2.2 Semantics of UCL

Each interpretation of UCL should provide a valuation to the variables in X, a model of ORCF and an assignment to the variables ν and μ. However, as already mentioned, the choice of the model of ORCF is immaterial since all such models are elementarily equivalent and, so, we adopt the ordered field

$\mathbb{R}$ of the real numbers. Accordingly, by an interpretation of UCL we mean a pair

$$I = (v, \rho)$$

where v is a propositional valuation and ρ is an assignment over $\mathbb{R}$ such that:[3]

$$\begin{cases} \frac{1}{2} < \rho(\mu) \leq 1 \\ \frac{1}{2} < \rho(\nu) \leq 1. \end{cases}$$

We now proceed to define *satisfaction*, by the interpretation $I = (v, \rho)$ at hand, of the three kinds of formulas in the language of UCL.

Starting with c-formulas, we write

$$I \Vdash^{\mathsf{uc}} \psi$$

for stating that

$$\mathbb{R}\,\rho \Vdash^{\mathsf{fo}} \sum_{\substack{\varphi \,\sqsubseteq\, \psi \\ v \,\Vdash\, \varphi}} \mathfrak{P}[\psi \triangleright \varphi] \geq \mu.$$

That is, the aggregated probability of the outcomes of ψ that are (classically) satisfied by v is at least the value of μ.

Concerning o-formulas, we write

$$I \Vdash^{\mathsf{uc}} \Phi \sqsubseteq_P \psi$$

for stating that

$$\mathbb{R}\,\rho \Vdash^{\mathsf{fo}} P \leq \sum_{\varphi \in \Phi} \mathfrak{P}[\psi \triangleright \varphi].$$

That is, the collection Φ of possible outcomes of ψ has aggregated probability greater than or equal to the value of P.

Finally, concerning a-formulas, we write

$$I \Vdash^{\mathsf{uc}} \mu \leq P$$

for stating that

$$\mathbb{R}\,\rho \Vdash^{\mathsf{fo}} \mu \leq P.$$

That is, the required probability $\rho(\mu)$ for the correct output being produced by the whole circuit does not exceed the value of P.

[3] Recall that we use ν and μ as variables in the language of ORCF.

Satisfaction is extended to mixed sets of o-formulas, c-formulas and a-formulas with no surprises. Given $\Gamma \subseteq L^{\mathsf{uc}}(X)$,

$$I \Vdash^{\mathsf{uc}} \Gamma$$

if $I \Vdash^{\mathsf{uc}} \gamma$ for each $\gamma \in \Gamma$. Then, entailment and validity in UCL are also defined as expected. Given $\{\theta\} \cup \Gamma \subseteq L^{\mathsf{uc}}(X)$, we write

$$\Gamma \vDash^{\mathsf{uc}} \theta$$

for stating that Γ *entails* θ in the following sense:

$$I \Vdash^{\mathsf{uc}} \theta \text{ whenever } I \Vdash^{\mathsf{uc}} \Gamma, \text{ for every interpretation } I.$$

Finally, we write

$$\vDash^{\mathsf{uc}} \theta$$

for $\emptyset \vDash^{\mathsf{uc}} \theta$, saying that formula θ is *valid*, in which case $I \Vdash^{\mathsf{uc}} \theta$ for every interpretation I.

As envisaged, the UCL entailment is an extension of the PL entailment. Furthermore, this extension is proved below to be conservative. To this end, we need the following lemma.

Proposition 2.1 Let $\varphi \in L(X)$ and I an interpretation with valuation v. Then,

$$I \Vdash^{\mathsf{uc}} \varphi \text{ if and only if } v \Vdash \varphi.$$

Proof: Let $I = (v, \rho)$. Then:

($\Rightarrow$) Assume that $I \Vdash^{\mathsf{uc}} \varphi$. Hence,

$$\mathbb{R}\,\rho \Vdash^{\mathsf{fo}} \sum_{\substack{\varphi' \in \Omega_\varphi \\ v \Vdash \varphi'}} \mathfrak{P}[\varphi \rhd \varphi'] \geq \mu.$$

Thus, $\{\varphi' \in \Omega_\varphi : v \Vdash \varphi'\} \neq \emptyset$ because $\rho(\mu) > 0$. On the other hand, $\Omega_\varphi = \{\varphi\}$. Therefore, $\varphi \in \{\varphi' \in \Omega_\varphi : v \Vdash \varphi'\}$ and, so, $v \Vdash \varphi$.

($\Leftarrow$) Assume that $v \Vdash \varphi$. Then,

$$\{\varphi\} = \{\varphi' \in \{\varphi\} : v \Vdash \varphi'\}.$$

On the other hand,

$$\{\varphi' \in \{\varphi\} : v \Vdash \varphi'\} = \{\varphi' \in \Omega_\varphi : v \Vdash \varphi'\}.$$

Therefore,

$$\sum_{\substack{\varphi' \in \Omega_\varphi \\ v \Vdash \varphi'}} \mathfrak{P}[\varphi \rhd \varphi'] \text{ is } \mathfrak{P}[\varphi \rhd \varphi]$$

and, so, it is the polynomial 1. Thus, in order to obtain $I \Vdash^{\mathsf{uc}} \varphi$, we have only to show that

$$\mathbb{R}\,\rho \Vdash^{\mathsf{fo}} 1 \geq \mu$$

which holds because $\rho(\mu) \leq 1$. QED

Theorem 2.2 (Model-theoretic conservativeness of UCL)
Let $\{\varphi\} \cup \Lambda \subseteq L(X)$. Then,

$$\Lambda \vDash^{\mathsf{uc}} \varphi \text{ if and only if } \Lambda \vDash \varphi.$$

Proof:
($\Rightarrow$) Assume that $\Lambda \vDash^{\mathsf{uc}} \varphi$. Let v be a valuation such that $v \Vdash \Lambda$. Let I be an interpretation with valuation v. Then, by Proposition 2.1, $I \Vdash^{\mathsf{uc}} \Lambda$ and, so, $I \Vdash^{\mathsf{uc}} \varphi$. Thus, again by Proposition 2.1, $v \Vdash \varphi$.

($\Leftarrow$) Assume that $\Lambda \vDash \varphi$. Let I be such that $I \Vdash^{\mathsf{uc}} \Lambda$. Let v be the valuation in I. Then, by Proposition 2.1, $v \Vdash \Lambda$ and, so, $v \Vdash \varphi$. Thus, once again by the same proposition, $I \Vdash^{\mathsf{uc}} \varphi$. QED

Before concluding this subsection, we look again at the possibility of introducing most of the unreliable connectives through abbreviations. To this end, consider the relationship between $x_1 \mathbin{\widetilde{\barwedge}} x_2$ and $\widetilde{\widetilde{\neg}}(x_1 \wedge x_2)$. As expected, it is straightforward to show

$$\begin{cases} x_1 \mathbin{\widetilde{\barwedge}} x_2 \vDash^{\mathsf{uc}} \widetilde{\widetilde{\neg}}(x_1 \wedge x_2) \\ \widetilde{\widetilde{\neg}}(x_1 \wedge x_2) \vDash^{\mathsf{uc}} x_1 \mathbin{\widetilde{\barwedge}} x_2 \end{cases}$$

which does warrant the idea of introducing $\widetilde{\barwedge}$ through an abbreviation. However, it may be surprising to ascertain that the formula

$$(x_1 \mathbin{\widetilde{\barwedge}} x_2) \equiv (\widetilde{\widetilde{\neg}}(x_1 \wedge x_2))$$

is not valid in UCL. For instance take $\rho(\mu) = 0.8$ and $\rho(\nu) = 0.51$. Then, for any valuation v, (v, ρ) does not satisfy the formula. In due course, many other examples will be provided of striking differences between PL and UCL.

2.3 Hilbert calculus

The calculus of UCL capitalizes on the decidability of the following problems which are used in some provisos:

- validity in PL;
- membership in $L(X)$;
- emptyness of intersection of two finite sets;
- theoremhood in ORCF.

The calculus contains the following axioms and rules:

- the PL tautologies as *axioms*:

$$\text{TAUT}\quad \frac{}{\varphi} \quad \text{provided that } \varphi \in L(X) \text{ and } \vDash \varphi;$$

- the *modus ponens rule*:

$$\text{MP}\quad \frac{\begin{array}{c}\psi \\ \psi \supset \varphi\end{array}}{\varphi} \quad \text{provided that } \varphi \in L(X);$$

- the following *o-axioms*:

$$\text{NO}\quad \frac{}{\emptyset \sqsubseteq_0 \psi};$$

$$\text{SO}\quad \frac{}{\varphi \sqsubseteq_{\mathfrak{P}[\psi \rhd \varphi]} \psi};$$

- the following *o-rules*:

$$\text{AO}\quad \frac{\Phi_i \sqsubseteq_{P_i} \psi \text{ for } i = 1,2}{\Phi_1 \cup \Phi_2 \sqsubseteq_{(P_1+P_2)} \psi} \quad \text{provided that } \Phi_1 \cap \Phi_2 = \emptyset;$$

$$\text{WO}\quad \frac{\Phi \sqsubseteq_{P_1} \psi}{\Phi \sqsubseteq_{P_2} \psi} \quad \text{provided that[4]}$$

$$\forall \nu \left(\left(\frac{1}{2} < \nu \leq 1 \right) \supset P_2 \leq P_1 \right) \in \mathsf{ORCF};$$

- the following *a-rule* for each $k \in \mathbb{N}$:

[4] Writing $P < z \leq P'$ for $P < z \wedge z \leq P'$.

$$\text{WA}\quad \frac{\mu \leq P_i \text{ for } i = 1, \ldots, k}{\mu \leq P} \quad \text{provided that}$$[5]

$$\forall\mu\forall\nu \left(\left(\frac{1}{2} < \mu, \nu \leq 1 \wedge \bigwedge_{i=1}^{k} \mu \leq P_i \right) \supset \mu \leq P \right) \in \mathsf{ORCF};$$

- the following *c-rule* for each $k \in \mathbb{N}^+$:[6]

$$\text{LFT}\quad \frac{\begin{array}{l} \displaystyle\bigvee_{i=1}^{k} \left(\bigwedge \Phi_i \right) \\ \begin{cases} \Phi_i \sqsubseteq_{P_i} \psi \\ \mu \leq P_i \end{cases} \text{for } i = 1, \ldots, k \end{array}}{\psi}.$$

The reader will wonder why we took the tautologies as axioms (TAUT) but not their unreliable instances. In fact, such instances are not valid in general. For example, the unreliable instance $(\tilde{\neg}\, x) \supset (\tilde{\neg}\, x)$ of the tautology $x \supset x$ is not valid. Indeed, take an interpretation $I = (v, \rho)$, such that $v(x) = \top$, $\rho(\nu) = 0.6$ and $\rho(\mu) = 0.8$. Consider

$$\Phi = \{(\neg\, x) \supset (\neg\, x), (\bar{\neg} x) \supset (\bar{\neg} x), (\neg\, x) \supset (\bar{\neg} x)\}.$$

Then,

- $v \Vdash \Phi$;
- $v \nVdash (\bar{\neg} x) \supset (\neg\, x)$.

Furthermore,

- $\mathfrak{P}[(\tilde{\neg}\, x) \supset (\tilde{\neg}\, x) \triangleright (\neg\, x) \supset (\neg\, x)] = \nu^2$;
- $\mathfrak{P}[(\tilde{\neg}\, x) \supset (\tilde{\neg}\, x) \triangleright (\bar{\neg} x) \supset (\bar{\neg} x)] = (1 - \nu)^2$;
- $\mathfrak{P}[(\tilde{\neg}\, x) \supset (\tilde{\neg}\, x) \triangleright (\neg\, x) \supset (\bar{\neg} x)] = \nu(1 - \nu)$.

[5]Writing $P < z, z' \leq P'$ for $P < z \leq P' \wedge P < z' \leq P'$.

[6]Given a finite set Φ of PL formulas, we write

$$\bigwedge \Phi$$

for the conjunction of the formulas in Φ. As usual, this conjunction is $\mathtt{tt}$ when $\Phi = \emptyset$.

Then,

$$\mathbb{R}\,\rho \Vdash^{\mathsf{fo}} \left(\sum_{\varphi\in\Phi} \mathfrak{P}[(\tilde{\neg}\, x) \supset (\tilde{\neg}\, x) \rhd \varphi] = \frac{19}{25}\right) \wedge \left(\frac{19}{25} < \mu\right)$$

and so

$$I \nVdash^{\mathsf{uc}} (\tilde{\neg}\, x) \supset (\tilde{\neg}\, x).$$

The reader will also wonder if MP is sound when the conclusion is a c-formula. It is not always so since, for example,

$$\tilde{\neg}\, x_1, (\tilde{\neg}\, x_1) \supset (\tilde{\neg}\, x_2) \nvDash^{\mathsf{uc}} \tilde{\neg}\, x_2.$$

Indeed, choose now an interpretation $I = (v, \rho)$ such that $v(x_1) = \perp$, $v(x_2) = \top$ and $\rho(\nu) = \rho(\mu) = 0.51$. Then, $\mathbb{R}$ together with ρ satisfies

$$\sum_{\substack{\varphi \,\sqsubseteq\, \tilde{\neg}\, x_2 \\ v \Vdash \varphi}} \mathfrak{P}[\tilde{\neg}\, x_2 \rhd \varphi] \;=\; \mathfrak{P}[\tilde{\neg}\, x_2 \rhd \overline{\neg} x_2] = 1 - \nu = \frac{49}{100} < \frac{51}{100} = \mu$$

and, so, $I \nVdash^{\mathsf{uc}} \tilde{\neg}\, x_2$. On the other hand, $\mathbb{R}$ together with ρ satisfies

$$\sum_{\substack{\varphi \,\sqsubseteq\, \tilde{\neg}\, x_1 \\ v \Vdash \varphi}} \mathfrak{P}[\tilde{\neg}\, x_1 \rhd \varphi] \;=\; \mathfrak{P}[\tilde{\neg}\, x_1 \rhd \neg\, x_1] = \nu = \frac{51}{100} \geq \frac{51}{100} = \mu$$

as well as

$$\sum_{\substack{\varphi \,\sqsubseteq\, (\tilde{\neg}\, x_1) \supset (\tilde{\neg}\, x_2) \\ v \Vdash \varphi}} \mathfrak{P}[(\tilde{\neg}\, x_1) \supset (\tilde{\neg}\, x_2) \rhd \varphi] \;=\; (1-\nu)^2 + 2\nu(1-\nu) = \frac{7399}{10000} \geq \mu$$

and, so, $I \Vdash^{\mathsf{uc}} \tilde{\neg}\, x_1$ and $I \Vdash^{\mathsf{uc}} (\tilde{\neg}\, x_1) \supset (\tilde{\neg}\, x_2)$.

Despite this counterexample, there are situations where MP is sound even in the presence of unreliable connectives in the conclusion. In fact, as shown in the next section, the dual of MP (conclusion in $L^{\mathsf{c}}(X)$ provided that the antecedent of the implication is in $L(X)$) is an admissible rule.

Most of the other axioms and rules are self-explanatory. Nevertheless, we provide some brief comments on each of them.

The additivity rule (AO) is as expected. It tells us that, if we know that the outcome of ψ is in Φ_1 with probability not smaller than P_1, the outcome of ψ is in Φ_2 with probability not smaller than P_2 and Φ_1 and Φ_2 are disjoint sets of possible outcomes of ψ, then we may conclude that the outcome of ψ is in $\Phi_1 \cup \Phi_2$ with probability not smaller than $P_1 + P_2$.

The weakening rule for o-formulas (WO) should also come as no surprise. It tells us that if we know that the outcome of ψ is in Φ with probability not smaller than P_1 and $P_1 \geq P_2$, then we may infer that the outcome of ψ is in Φ with probability not smaller than P_2. But how can we know at the symbolic level that $P_1 \geq P_2$? Here, we have to use the decision algorithm for ORCF with the additional knowledge that the value of ν is greater than $\frac{1}{2}$ and not greater than 1. To this end, it is enough to get as a theorem of ORCF the formula

$$\forall \nu \left(\left(\frac{1}{2} < \nu \leq 1 \right) \supset P_2 \leq P_1 \right).$$

The weakening rule for a-formulas (WA) is understood in a similar way.

Each empty outcome axiom (NO) expresses that the set of outcomes of ψ is empty with probability not smaller than 0.

Each single outcome axiom (SO) states the obvious fact that the outcome of ψ is φ with probability not smaller than $\mathfrak{P}[\psi \triangleright \varphi]$. Recall that $\mathfrak{P}[\psi \triangleright \varphi]$ was symbolically computed in Subsection 2.1 precisely as being the lower bound of the probability of the outcome of ψ being φ so as to reflect the behavior of the unreliable gates. We return to this issue in Section 3.

On the other hand, the lifting rule (LFT) does warrant a more detailed explanation. It was not easy to come-up with its formulation, notwithstanding the fact that a lifting rule was known to be necessary from our experience with the meet-combination of connectives [17].

As a motivating example, assume that we want to infer that

$$((\tilde{\neg}\, x_1) \wedge x_2) \supset (\tilde{\neg}\, x_1)$$

holds with probability not smaller than μ. We can do so provided that we constrain our ambition to μ being not greater than $\nu^2 - \nu + 1$. Indeed, it is easy to verify that

$$\left(\bigwedge \Phi_1 \right) \vee \left(\bigwedge \Phi_2 \right)$$

is a tautology as long as

$$\begin{cases} \Phi_1 = \{\varphi_1, \varphi_3, \varphi_4\} \\ \Phi_2 = \{\varphi_1, \varphi_2, \varphi_4\} \end{cases}$$

1	$((\neg\, x_1) \supset \mathtt{tt}) \wedge ((\bar{\neg}\, x_1) \supset \mathtt{tt})$	TAUT
2	$(\neg\, x_1) \supset \mathtt{tt} \sqsubseteq_{\nu} (\tilde{\neg}\, x_1) \supset \mathtt{tt}$	SO
3	$(\bar{\neg}\, x_1) \supset \mathtt{tt} \sqsubseteq_{1-\nu} (\tilde{\neg}\, x_1) \supset \mathtt{tt}$	SO
4	$(\neg\, x_1) \supset \mathtt{tt}, (\bar{\neg}\, x_1) \supset \mathtt{tt} \sqsubseteq_{\nu+(1-\nu)} (\tilde{\neg}\, x_1) \supset \mathtt{tt}$	AO : 2, 3
5	$(\neg\, x_1) \supset \mathtt{tt}, (\bar{\neg}\, x_1) \supset \mathtt{tt} \sqsubseteq_{1} (\tilde{\neg}\, x_1) \supset \mathtt{tt}$	WO : 4
6	$\mu \leq 1$	WA
7	$(\tilde{\neg}\, x_1) \supset \mathtt{tt}$	LFT : 1, 5, 6

Figure 2: $\vdash^{\mathsf{uc}} (\tilde{\neg}\, x_1) \supset \mathtt{tt}$.

where

$$\begin{cases} \varphi_1 \text{ is } ((\neg\, x_1) \wedge x_2) \supset (\neg\, x_1) \\ \varphi_2 \text{ is } ((\bar{\neg}\, x_1) \wedge x_2) \supset (\neg\, x_1) \\ \varphi_3 \text{ is } ((\neg\, x_1) \wedge x_2) \supset (\bar{\neg}\, x_1) \\ \varphi_4 \text{ is } ((\bar{\neg}\, x_1) \wedge x_2) \supset (\bar{\neg}\, x_1). \end{cases}$$

Furthermore, it is straightforward to verify that both

$$\Phi_1 \sqsubseteq_{\nu^2-\nu+1} ((\tilde{\neg}\, x_1) \wedge x_2) \supset (\tilde{\neg}\, x_1)$$

and

$$\Phi_2 \sqsubseteq_{\nu^2-\nu+1} ((\tilde{\neg}\, x_1) \wedge x_2) \supset (\tilde{\neg}\, x_1)$$

hold. Thus, assuming that

$$\mu \leq \nu^2 - \nu + 1$$

the LFT rule allows us to conclude that

$$((\tilde{\neg}\, x_1) \wedge x_2) \supset (\tilde{\neg}\, x_1)$$

holds with probability not smaller than μ. In fact, for each $i = 1, 2$, we were able to collect in Φ_i outcomes of $x_1 \,\tilde{\vee}\, (\tilde{\neg}\, x_1)$ with aggregated probability not smaller than the ambitioned upper bound for μ.

Derivability and theoremhood in this calculus are defined as usual. Given $\{\theta\} \cup \Gamma \subseteq L^{\mathsf{uc}}(X)$, we write

$$\Gamma \vdash^{\mathsf{uc}} \theta$$

for stating that θ is *derivable* from Γ, that is, for stating that there is a derivation sequence for obtaining θ from the elements of Γ (as hypotheses) and the axioms, using the rules of the calculus. Furthermore, when $\emptyset \vdash^{\mathsf{uc}} \psi$, written $\vdash^{\mathsf{uc}} \psi$, we say that ψ is a *theorem* of UCL.

$$
\begin{array}{lll}
1 & x_1 \vee (\neg\, x_1) & \text{TAUT} \\
2 & x_1 \vee (\neg\, x_1) \sqsubseteq_{\nu^2} x_1 \,\tilde{\vee}\, (\tilde{\neg}\, x_1) & \text{SO} \\
3 & \mu \leq \nu^2 & \text{HYP} \\
4 & x_1 \,\tilde{\vee}\, (\tilde{\neg}\, x_1) & \text{LFT} : 1, 2, 3
\end{array}
$$

Figure 3: $\mu \leq \nu^2 \vdash^{\mathsf{uc}} x_1 \,\tilde{\vee}\, (\tilde{\neg}\, x_1)$.

For example, the derivation sequence in Figure 2 establishes that $(\tilde{\neg}\, x_1) \supset \mathtt{tt}$ is a theorem of UCL.

Recall that a formula may be derivable only when imposing some conditions on μ and ν. For instance, the derivation sequence in Figure 3 derives $(\tilde{\neg}\, x_1) \,\tilde{\vee}\, x_1$ in UCL as long as $\mu \leq \nu^2$. This hypothesis is needed at step 4 for applying the LFT rule.

$$
\begin{array}{lll}
1 & ((((\neg\, x_1) \wedge x_2) \supset (\neg\, x_1)) \wedge (((\neg\, x_1) \wedge x_2) \supset (\bar{\neg}\, x_1)) \wedge & \\
 & \qquad (((\bar{\neg}\, x_1) \wedge x_2) \supset (\bar{\neg}\, x_1))) & \\
 & \qquad\qquad \vee & \\
 & (((\neg\, x_1) \wedge x_2) \supset (\neg\, x_1)) \wedge (((\bar{\neg}\, x_1) \wedge x_2) \supset (\neg\, x_1)) \wedge & \\
 & \qquad (((\bar{\neg}\, x_1) \wedge x_2) \supset (\bar{\neg}\, x_1))) & \text{TAUT} \\
2 & ((\neg\, x_1) \wedge x_2) \supset (\neg\, x_1) \sqsubseteq_{\nu^2} ((\tilde{\neg}\, x_1) \wedge x_2) \supset (\tilde{\neg}\, x_1) & \text{SO} \\
3 & ((\bar{\neg}\, x_1) \wedge x_2) \supset (\neg\, x_1) \sqsubseteq_{\nu(1-\nu)} ((\tilde{\neg}\, x_1) \wedge x_2) \supset (\tilde{\neg}\, x_1) & \text{SO} \\
4 & ((\neg\, x_1) \wedge x_2) \supset (\bar{\neg}\, x_1) \sqsubseteq_{\nu(1-\nu)} ((\tilde{\neg}\, x_1) \wedge x_2) \supset (\tilde{\neg}\, x_1) & \text{SO} \\
5 & ((\bar{\neg}\, x_1) \wedge x_2) \supset (\bar{\neg}\, x_1) \sqsubseteq_{(1-\nu)^2} ((\tilde{\neg}\, x_1) \wedge x_2) \supset (\tilde{\neg}\, x_1) & \text{SO} \\
6 & ((\neg\, x_1) \wedge x_2) \supset (\neg\, x_1), ((\neg\, x_1) \wedge x_2) \supset (\bar{\neg}\, x_1), & \\
 & ((\bar{\neg}\, x_1) \wedge x_2) \supset (\bar{\neg}\, x_1) \sqsubseteq_{\nu^2-\nu+1} ((\tilde{\neg}\, x_1) \wedge x_2) \supset (\tilde{\neg}\, x_1) & \text{AO} : 2, 3, 5 \\
7 & ((\neg\, x_1) \wedge x_2) \supset (\neg\, x_1), ((\bar{\neg}\, x_1) \wedge x_2) \supset (\neg\, x_1), & \\
 & ((\bar{\neg}\, x_1) \wedge x_2) \supset (\bar{\neg}\, x_1) \sqsubseteq_{\nu^2-\nu+1} ((\tilde{\neg}\, x_1) \wedge x_2) \supset (\tilde{\neg}\, x_1) & \text{AO} : 2, 4, 5 \\
8 & \mu \leq \nu^2 - \nu + 1 & \text{HYP} \\
9 & ((\tilde{\neg}\, x_1) \wedge x_2) \supset (\tilde{\neg}\, x_1) & \text{LFT}{:}1, 6, 7, 8
\end{array}
$$

Figure 4: $\mu \leq \nu^2 - \nu + 1 \vdash^{\mathsf{uc}} ((\tilde{\neg}\, x_1) \wedge x_2) \supset (\tilde{\neg}\, x_1)$.

Returning to the motivating example given above for the LFT rule, a derivation is presented in Figure 4 for deriving $((\tilde{\neg}\, x_1) \wedge x_2) \supset (\tilde{\neg}\, x_1)$ from the hypothesis $\mu \leq \nu^2 - \nu + 1$.

As a last example, we present in Figure 5 a derivation showing that inconsistency at the level of ambition formulas propagates to propositional inconsistency. Observe that the weakening made at step 4 is legitimate because

$$\forall\mu\forall\nu\left(\left(\frac{1}{2}<\mu,\nu\leq 1\wedge\mu\leq\frac{1}{2}\right)\supset\mu\leq 0\right)$$

is in ORCF and, so, we can apply WA.

1	$\bigwedge\emptyset$	TAUT
2	$\emptyset\sqsubseteq_0 \mathsf{ff}$	NO
3	$\mu\leq\frac{1}{2}$	HYP
4	$\mu\leq 0$	WA : 3
5	ff	LFT : 1, 2, 4

Figure 5: $\mu\leq\frac{1}{2}\vdash^{\mathsf{uc}}\mathsf{ff}$.

To conclude this subsection, we look at the relationship between the PL and the UCL calculi. Clearly, UCL is an extension of the PL also at the proof-theoretic level. Furthermore, it is straightforward to prove that this extension is conservative.

Theorem 2.3 (Proof-theoretic conservativeness of UCL)
Let $\{\varphi\}\cup\Lambda\subseteq L(X)$. Then,

$$\Lambda\vdash^{\mathsf{uc}}\varphi \text{ if and only if } \Lambda\vdash\varphi.$$

This result is the proof-theoretic counterpart of Theorem 2.2. They are both used in the sequel.

3 Enriched calculus

The following redundant axioms and rules allow the connective-wise derivation of o-formulas, instead of relying on the SO axioms and the inductive definition of $\mathfrak{P}[\psi\rhd\varphi]$. This has the advantage of making explicit in the derivation the calculations that are needed concerning $\mathfrak{P}[\psi\rhd\varphi]$ and that otherwise would be written on the margin.

$$\Omega\mathrm{O}\ \frac{}{\Omega_\psi\sqsubseteq_1\psi};$$

$$\text{CO} \quad \frac{\varphi_i \sqsubseteq_{P_i} \psi_i \text{ for } i=1,\dots,n}{c(\varphi_1,\dots,\varphi_n) \sqsubseteq_{(\prod_{i=1}^n P_i)} c(\psi_1,\dots,\psi_n)};$$

$$\text{UCO}\uparrow \quad \frac{\varphi_i \sqsubseteq_{P_i} \psi_i \text{ for } i=1,\dots,n}{c(\varphi_1,\dots,\varphi_n) \sqsubseteq_{(\nu\prod_{i=1}^n P_i)} \widetilde{c}(\psi_1,\dots,\psi_n)};$$

$$\text{UCO}\downarrow \quad \frac{\varphi_i \sqsubseteq_{P_i} \psi_i \text{ for } i=1,\dots,n}{\overline{c}(\varphi_1,\dots,\varphi_n) \sqsubseteq_{((1-\nu)\prod_{i=1}^n P_i)} \widetilde{c}(\psi_1,\dots,\psi_n)}.$$

Moreover, the following redundant rule allows, in some cases, the use of *modus ponens* in the presence of unreliable connectives in the conclusion.

$$\text{MP}^\bullet \quad \frac{\begin{array}{c}\varphi \\ \varphi \supset \psi\end{array}}{\psi} \quad \text{provided that } \varphi \in L(X).$$

Observe that this enriched calculus brings to the foreground the behavior of the unreliable gates. Nevertheless, it is not burdened by the presence in the signature of a rich variety of connectives representing them.

In the sequel, we write

$$\Gamma \vdash^{\mathsf{uc}^+} \theta$$

for stating that θ can be derived from Γ in the UCL calculus enriched with the axioms and rules above.

As an illustration of the use of the additional axioms and rules on o-formulas, consider the derivation in Figure 6 of

$$(\neg\, x_1) \equiv \mathsf{M}_3(\widetilde{\neg}\, x_1, \widetilde{\neg}\, x_2, \widetilde{\neg}\, x_3)$$

from hypothesis $\mu \leq \nu^3 + 3\nu^2(1-\nu)$. This amounts to say that, by using three unreliable negations and the perfect majority connective, we can mimic reliable negation modulo the indicated proviso on μ and ν. A weaker proviso would be sufficient if we used more unreliable negations in parallel. These realistic examples are instances of the pioneering result by John von Neumann on minimizing overall circuit error rate [20] when using unreliable gates. We shall return to this topic in Section 6.

Clearly, for instance as in Figure 7, $(\neg\, x_1) \equiv \mathsf{M}_3(\widetilde{\neg}\, x_1, \widetilde{\neg}\, x_2, \widetilde{\neg}\, x_3)$ can be derived from $\mu \leq \nu^3 + 3\nu^2(1-\nu)$ in the original UCL calculus (without the additional axioms and rules). This derivation is much shorter but leaves out the calculations of the probabilities of the unreliable connective outcomes by folding them in the use of the SO axiom.

1	$\varphi_1 : (\neg\, x_1) \equiv \mathsf{M}_3(\neg\, x_1, \neg\, x_1, \neg\, x_1)$	TAUT
2	$\varphi_2 : (\neg\, x_1) \equiv \mathsf{M}_3(\neg\, x_1, \neg\, x_1, \bar{\neg}\, x_1)$	TAUT
3	$\varphi_3 : (\neg\, x_1) \equiv \mathsf{M}_3(\neg\, x_1, \bar{\neg}\, x_1, \neg\, x_1)$	TAUT
4	$\varphi_4 : (\neg\, x_1) \equiv \mathsf{M}_3(\bar{\neg}\, x_1, \neg\, x_1, \neg\, x_1)$	TAUT
5	$\varphi_1 \wedge \ldots \wedge \varphi_4$	TAUT : $1, 2, 3, 4$
6	$x_1 \sqsubseteq_1 x_1$	ΩO
7	$\neg\, x_1 \sqsubseteq_\nu \tilde{\neg}\, x_1$	UCO$\uparrow$: 6
8	$\mathsf{M}_3(\neg\, x_1, \neg\, x_1, \neg\, x_1) \sqsubseteq_{\nu^3} \mathsf{M}_3(\tilde{\neg}\, x_1, \tilde{\neg}\, x_1, \tilde{\neg}\, x_1)$	CO : $7, 7, 7$
9	$\bar{\neg}\, x_1 \sqsubseteq_{1-\nu} \tilde{\neg}\, x_1$	UCO$\downarrow$: 6
10	$\mathsf{M}_3(\neg\, x_1, \neg\, x_1, \bar{\neg}\, x_1) \sqsubseteq_{\nu^2(1-\nu)} \mathsf{M}_3(\tilde{\neg}\, x_1, \tilde{\neg}\, x_1, \tilde{\neg}\, x_1)$	CO : $7, 7, 9$
11	$\mathsf{M}_3(\neg\, x_1, \bar{\neg}\, x_1, \neg\, x_1) \sqsubseteq_{\nu^2(1-\nu)} \mathsf{M}_3(\tilde{\neg}\, x_1, \tilde{\neg}\, x_1, \tilde{\neg}\, x_1)$	CO : $7, 9, 7$
12	$\mathsf{M}_3(\bar{\neg}\, x_1, \neg\, x_1, \neg\, x_1) \sqsubseteq_{\nu^2(1-\nu)} \mathsf{M}_3(\tilde{\neg}\, x_1, \tilde{\neg}\, x_1, \tilde{\neg}\, x_1)$	CO : $9, 7, 7$
13	$\neg\, x_1 \sqsubseteq_1 \neg\, x_1$	CO : 6
14	$\varphi_1 \sqsubseteq_{\nu^3} (\neg\, x_1) \equiv \mathsf{M}_3(\tilde{\neg}\, x_1, \tilde{\neg}\, x_1, \tilde{\neg}\, x_1)$	CO : $13, 8$
15	$\varphi_2 \sqsubseteq_{\nu^2(1-\nu)} (\neg\, x_1) \equiv \mathsf{M}_3(\tilde{\neg}\, x_1, \tilde{\neg}\, x_1, \tilde{\neg}\, x_1)$	CO : $13, 10$
16	$\varphi_3 \sqsubseteq_{\nu^2(1-\nu)} (\neg\, x_1) \equiv \mathsf{M}_3(\tilde{\neg}\, x_1, \tilde{\neg}\, x_1, \tilde{\neg}\, x_1)$	CO : $13, 11$
17	$\varphi_4 \sqsubseteq_{\nu^2(1-\nu)} (\neg\, x_1) \equiv \mathsf{M}_3(\tilde{\neg}\, x_1, \tilde{\neg}\, x_1, \tilde{\neg}\, x_1)$	CO : $13, 12$
18	$\varphi_1, \ldots, \varphi_4 \sqsubseteq_{\nu^3+3\nu^2(1-\nu)} (\neg\, x_1) \equiv \mathsf{M}_3(\tilde{\neg}\, x_1, \tilde{\neg}\, x_1, \tilde{\neg}\, x_1)$	AO : 14–17
19	$\mu \leq \nu^3 + 3\nu^2(1-\nu)$	HYP
20	$(\neg\, x_1) \equiv \mathsf{M}_3(\tilde{\neg}\, x_1, \tilde{\neg}\, x_1, \tilde{\neg}\, x_1)$	LFT : $5, 18, 19$

Figure 6: $\mu \leq \nu^3 + 3\nu^2(1-\nu) \vdash^{\mathsf{uc}^+} (\neg\, x_1) \equiv \mathsf{M}_3(\tilde{\neg}\, x_1, \tilde{\neg}\, x_1, \tilde{\neg}\, x_1)$.

A similar elimination of the additional axioms and rules can be made for every derivation with hypotheses in L^{a}, as the results below show.

Proposition 3.1 Let $\psi \in L^{\mathsf{c}}(X)$. Then,

$$\left(\sum_{\varphi \in \Omega_\psi} \mathfrak{P}[\psi \rhd \varphi] = 1 \right) \in \mathsf{ORCF}.$$

Proof: The proof follows by induction on ψ.
(Basis) $\psi \in L(X)$. Then, $\Omega_\psi = \{\psi\}$. So

$$\sum_{\varphi \in \Omega_\psi} \mathfrak{P}[\psi \rhd \varphi] \quad \text{is} \quad \mathfrak{P}[\psi \rhd \psi].$$

1	$\varphi_1 : (\neg\, x_1) \equiv \mathsf{M}_3(\neg\, x_1, \neg\, x_1, \neg\, x_1)$	TAUT
2	$\varphi_2 : (\neg\, x_1) \equiv \mathsf{M}_3(\neg\, x_1, \neg\, x_1, \overline{\neg}\, x_1)$	TAUT
3	$\varphi_3 : (\neg\, x_1) \equiv \mathsf{M}_3(\neg\, x_1, \overline{\neg}\, x_1, \neg\, x_1)$	TAUT
4	$\varphi_4 : (\neg\, x_1) \equiv \mathsf{M}_3(\overline{\neg}\, x_1, \neg\, x_1, \neg\, x_1)$	TAUT
5	$\varphi_1 \wedge \ldots \wedge \varphi_4$	$\mathrm{TAUT} : 1, 2, 3, 4$
6	$\varphi_1 \sqsubseteq_{\nu^3} (\neg\, x_1) \equiv \mathsf{M}_3(\widetilde{\neg}\, x_1, \widetilde{\neg}\, x_1, \widetilde{\neg}\, x_1)$	SO
7	$\varphi_2 \sqsubseteq_{\nu^2(1-\nu)} (\neg\, x_1) \equiv \mathsf{M}_3(\widetilde{\neg}\, x_1, \widetilde{\neg}\, x_1, \widetilde{\neg}\, x_1)$	SO
8	$\varphi_3 \sqsubseteq_{\nu^2(1-\nu)} (\neg\, x_1) \equiv \mathsf{M}_3(\widetilde{\neg}\, x_1, \widetilde{\neg}\, x_1, \widetilde{\neg}\, x_1)$	SO
9	$\varphi_4 \sqsubseteq_{\nu^2(1-\nu)} (\neg\, x_1) \equiv \mathsf{M}_3(\widetilde{\neg}\, x_1, \widetilde{\neg}\, x_1, \widetilde{\neg}\, x_1)$	SO
10	$\varphi_1, \ldots, \varphi_4 \sqsubseteq_{\nu^3+3\nu^2(1-\nu)} (\neg\, x_1) \equiv \mathsf{M}_3(\widetilde{\neg}\, x_1, \widetilde{\neg}\, x_1, \widetilde{\neg}\, x_1)$	$\mathrm{AO} : 6{-}9$
11	$\mu \leq \nu^3 + 3\nu^2(1-\nu)$	HYP
12	$(\neg\, x_1) \equiv \mathsf{M}_3(\widetilde{\neg}\, x_1, \widetilde{\neg}\, x_1, \widetilde{\neg}\, x_1)$	$\mathrm{LFT} : 5, 10, 11$

Figure 7: $\mu \leq \nu^3 + 3\nu^2(1-\nu) \vdash^{\mathsf{uc}} (\neg\, x_1) \equiv \mathsf{M}_3(\widetilde{\neg}\, x_1, \widetilde{\neg}\, x_1, \widetilde{\neg}\, x_1)$.

The thesis follows since $\mathfrak{P}[\psi \triangleright \psi]$ is 1 by definition.

(Step) Consider the following cases:

(a) ψ is $c(\psi_1, \ldots, \psi_n)$. Then, by definition of Ω_ψ,

$$\sum_{\varphi \in \Omega_\psi} \mathfrak{P}[\psi \triangleright \varphi] = \sum_{\varphi_1 \in \Omega_{\psi_1}} \cdots \sum_{\varphi_n \in \Omega_{\psi_n}} \mathfrak{P}[\psi_1 \triangleright \varphi_1] \times \cdots \times \mathfrak{P}[\psi_n \triangleright \varphi_n].$$

On the other hand,

$$\sum_{\varphi_1 \in \Omega_{\psi_1}} \cdots \sum_{\varphi_n \in \Omega_{\psi_n}} \mathfrak{P}[\psi_1 \triangleright \varphi_1] \times \cdots \times \mathfrak{P}[\psi_n \triangleright \varphi_n] =$$

$$\left(\sum_{\varphi_1 \in \Omega_{\psi_1}} \mathfrak{P}[\psi_1 \triangleright \varphi_1] \right) \times \cdots \times \left(\sum_{\varphi_n \in \Omega_{\psi_n}} \mathfrak{P}[\psi_n \triangleright \varphi_n] \right)$$

is in ORCF, since distributivity holds in ORCF. So, the thesis follows, since

$$\left(\sum_{\varphi_i \in \Omega_{\psi_i}} \mathfrak{P}[\psi_i \triangleright \varphi_i] = 1 \right) \in \mathsf{ORCF}$$

by induction hypothesis, for every $i = 1, \ldots, n$.

(b) ψ is $\widetilde{c}(\psi_1,\ldots,\psi_n)$. Let

$$\begin{cases}\Omega^1_\psi = \{c(\varphi_1,\ldots,\varphi_n) : \varphi_i \in \Omega_{\psi_i}, i = 1,\ldots,n\}\\ \Omega^2_\psi = \{\overline{c}(\varphi_1,\ldots,\varphi_n) : \varphi_i \in \Omega_{\psi_i}, i = 1,\ldots,n\}.\end{cases}$$

Then,

$$\sum_{\varphi\in\Omega_\psi} \mathfrak{P}[\psi \triangleright \varphi] = \sum_{\varphi\in\Omega^1_\psi} \mathfrak{P}[\psi \triangleright \varphi] + \sum_{\varphi\in\Omega^2_\psi} \mathfrak{P}[\psi \triangleright \varphi]$$

and since distributivity holds in ORCF then,

$$\sum_{\varphi\in\Omega_\psi} \mathfrak{P}[\psi \triangleright \varphi] =$$

$$\left(\sum_{\varphi_1\in\Omega_{\psi_1}} \mathfrak{P}[\psi_1 \triangleright \varphi_1]\right) \times \cdots \times \left(\sum_{\varphi_n\in\Omega_{\psi_n}} \mathfrak{P}[\psi_n \triangleright \varphi_n]\right) \times \nu$$

$$+$$

$$\left(\sum_{\varphi_1\in\Omega_{\psi_1}} \mathfrak{P}[\psi_1 \triangleright \varphi_1]\right) \times \cdots \times \left(\sum_{\varphi_n\in\Omega_{\psi_n}} \mathfrak{P}[\psi_n \triangleright \varphi_n]\right) \times (1-\nu)$$

is in ORCF. The thesis follows since

$$(((1 \times \cdots \times 1) \times \nu) + ((1 \times \cdots \times 1) \times (1-\nu))) = 1 \in \mathsf{ORCF}$$

and

$$\left(\sum_{\varphi_i\in\Omega_{\psi_i}} \mathfrak{P}[\psi_i \triangleright \varphi_i] = 1\right) \in \mathsf{ORCF},$$

by induction hypothesis, for every $i = 1,\ldots,n$. QED

Proposition 3.2 Let $\psi \in L^{\mathsf{c}}(X)$. Then,

$$\vdash^{\mathsf{uc}} \Omega_\psi \sqsubseteq_1 \psi.$$

Proof: The thesis follows immediately taking into account that

$$\vdash^{\mathsf{uc}} \Omega_\psi \sqsubseteq_{\left(\sum_{\varphi\in\Omega_\psi} \mathfrak{P}[\psi\triangleright\varphi]\right)} \psi$$

using rules SO and AO, and capitalizing on Proposition 3.1. QED

Proposition 3.3 Let $\Gamma \subseteq L^{\mathsf{a}}$ and $\theta \in L^{\mathsf{uc}}(X)$. Then,

$$\text{if} \quad \Gamma \vdash^{\mathsf{uc}} \varphi \sqsubseteq_P \psi \quad \text{then} \quad \forall \nu \left(\left(\frac{1}{2} < \nu \leq 1 \right) \supset P \leq \mathfrak{P}[\psi \triangleright \varphi] \right) \in \mathsf{ORCF}.$$

Proof: Let $\theta_1 \dots \theta_m$ be a derivation of $\varphi \sqsubseteq_P \psi$ from Γ in UCL. We prove the result by induction on m.
(Basis) θ_m is obtained by SO. Then, the thesis follows immediately since P is $\mathfrak{P}[\psi \triangleright \varphi]$.
(Step) θ_m is obtained by WO. Let θ_i be the premise $\varphi \sqsubseteq_{P'} \psi$ of the rule. Then,

$$\forall \nu \left(\left(\frac{1}{2} < \nu \leq 1 \right) \supset P \leq P' \right) \in \mathsf{ORCF}.$$

On the other hand, by the induction hypothesis,

$$\forall \nu \left(\left(\frac{1}{2} < \nu \leq 1 \right) \supset P' \leq \mathfrak{P}[\psi \triangleright \varphi] \right) \in \mathsf{ORCF}$$

and so the thesis follows. QED

Proposition 3.4 Let $\Gamma \subseteq L^{\mathsf{a}}$ and $\theta \in L^{\mathsf{uc}}(X)$. Then,

$$\text{if} \quad \Gamma \vdash^{\mathsf{uc}^+} \theta \quad \text{then} \quad \Gamma \vdash^{\mathsf{uc}} \theta.$$

Proof: Let $\theta_1 \dots \theta_m$ be a derivation of θ from Γ in UCL enriched with the additional rules above. We prove the result by induction on m.

(Basis) Then, either θ is in Γ or θ is a tautology or θ is obtained by NO or by SO. In all cases, $\Gamma \vdash^{\mathsf{uc}} \theta$.

(Step) There are two cases to consider:

(a) θ is obtained by a rule in UCL. Then, the result follows straightforwardly by the induction hypothesis.

(b) θ is obtained either by CO or UCO$\downarrow$ or UCO$\uparrow$. We only consider the case where it follows by UCO$\downarrow$. Assume that θ is of the form

$$\overline{c}(\varphi_1, \dots, \varphi_n) \sqsubseteq_{(1-\nu)\prod_{i=1}^n P_i} \widetilde{c}(\psi_1, \dots, \psi_n).$$

Let the premises be $\varphi_i \sqsubseteq_{P_i} \psi_i$ for $i = 1, \dots, n$. Then, by the induction hypothesis,

$$\Gamma \vdash^{\mathsf{uc}} \varphi_i \sqsubseteq_{P_i} \psi_i \quad \text{for } i = 1, \dots, n.$$

Thus, by Proposition 3.3, for $i = 1, \dots, n$,

$$\forall \nu \left(\left(\frac{1}{2} < \nu \leq 1 \right) \supset P_i \leq \mathfrak{P}[\psi_i \triangleright \varphi_i] \right) \in \mathsf{ORCF}.$$

Hence,

$$\forall \nu \left(\left(\frac{1}{2} < \nu \leq 1 \right) \supset \left((1-\nu) \prod_{i=1}^{n} P_i \right) \leq \left((1-\nu) \prod_{i=1}^{n} \mathfrak{P}[\psi_i \triangleright \varphi_i] \right) \right) \in \mathsf{ORCF}.$$

Observe that

$$\mathfrak{P}[\widetilde{c}(\psi_1, \ldots, \psi_n) \triangleright \overline{c}(\varphi_1, \ldots, \varphi_n)] = (1-\nu) \prod_{i=1}^{n} \mathfrak{P}[\psi_i \triangleright \varphi_i]$$

and so

$$\vdash^{\mathsf{uc}} \overline{c}(\varphi_1, \ldots, \varphi_n) \sqsubseteq_{(1-\nu)\prod_{i=1}^{n} \mathfrak{P}[\psi_i \triangleright \varphi_i]} \widetilde{c}(\psi_1, \ldots, \psi_n)$$

by rule SO. Hence

$$\vdash^{\mathsf{uc}} \overline{c}(\varphi_1, \ldots, \varphi_n) \sqsubseteq_{(1-\nu)\prod_{i=1}^{n} P_i} \widetilde{c}(\psi_1, \ldots, \psi_n)$$

by rule WO.

(c) θ is obtained by $\mathrm{MP}^{\bullet}$ from θ_j and $\theta_j \supset \theta$. Thus, $\theta_j \in L(X)$. The subcase where $\theta \in L(X)$ follows immediately. Assume that θ is not in $L(X)$. Then, $\theta_j \supset \theta$ is obtained by LFT from

- $\bigvee_{i=1}^{k} \left(\bigwedge \{\theta_j \supset \varphi : \varphi \in \Phi_i\} \right)$ where $\Phi_i \subseteq \Omega_\theta$ for $i = 1, \ldots, k$;
- $\{\theta_j \supset \varphi : \varphi \in \Phi_i\} \sqsubseteq_{P_i} \theta_j \supset \theta$ for $i = 1, \ldots, k$;
- $\mu \leq P_i$ for $i = 1, \ldots, k$.

Therefore, using the induction hypothesis, propositional reasoning and o-reasoning

- $\vdash^{\mathsf{uc}} \bigvee_{i=1}^{k} \left(\bigwedge \Phi_i \right)$ where $\Phi_i \subseteq \Omega_\theta$ for $i = 1, \ldots, k$;
- $\vdash^{\mathsf{uc}} \Phi_i \sqsubseteq_{P_i} \theta$ for $i = 1, \ldots, k$;
- $\vdash^{\mathsf{uc}} \mu \leq P_i$ for $i = 1, \ldots, k$;

and, so, by LFT, $\vdash^{\mathsf{uc}} \theta$. QED

4 Soundness and completeness

The objective now is to assess how far the proposed calculus captures the semantics of UCL. First, we prove strong soundness, that is:

$$\text{if} \quad \Gamma \vdash^{\mathsf{uc}} \theta \quad \text{then} \quad \Gamma \vDash^{\mathsf{uc}} \theta$$

for every $\Gamma \cup \{\theta\} \subseteq L^{\mathsf{uc}}(X)$. Afterward, with respect to completeness, we prove the following results:

- Weak completeness for o-formulas:

$$\text{if} \quad \vDash^{\mathsf{uc}} \Phi \sqsubseteq_P \psi \quad \text{then} \quad \vdash^{\mathsf{uc}} \Phi \sqsubseteq_P \psi.$$

- Constrained strong completeness for c-formulas and a-formulas:

$$\begin{cases} \text{if} \quad \Gamma \vDash^{\mathsf{uc}} \psi \quad \text{then} \quad \Gamma \vdash^{\mathsf{uc}} \psi \\ \text{if} \quad \Gamma \vDash^{\mathsf{uc}} \mu \leq P \quad \text{then} \quad \Gamma \vdash^{\mathsf{uc}} \mu \leq P \end{cases}$$

provided that Γ is a finite subset of L^{a}.

Observe that strong completeness for o-formulas is out of question. Indeed, it is easy to ascertain that entailment for o-formulas is not compact. For instance,

$$\left\{\neg\, x_1 \sqsubseteq_{\frac{2}{3}-\frac{1}{10+k}} \widetilde{\neg}\, x_1 \,:\, k \in \mathbb{N}\right\} \vDash^{\mathsf{uc}} \neg\, x_1 \sqsubseteq_{\frac{2}{3}} \widetilde{\neg}\, x_1$$

but there is no finite subset of the hypothesis that would entail the conclusion. Therefore, given the obvious compactness of the UCL calculus and its strong soundness, $\neg\, x_1 \sqsubseteq_{\frac{2}{3}} \widetilde{\neg}\, x_1$ is not derivable from the infinite set of hypotheses at hand because it is not derivable from any finite subset of hypotheses.

Strong completeness for c-formulas and a-formulas is also out of question for the same reason. In fact, if we allow an infinite set of a-formulas as hypotheses we can also find an example where no finite subset would entail the same conclusion. For instance,

$$\left\{\mu \leq \nu^2 + \frac{1}{n} \,:\, n \in \mathbb{N}^+\right\} \vDash^{\mathsf{uc}} x_1 \mathbin{\widetilde{\vee}} (\widetilde{\neg}\, x_1)$$

and

$$\left\{\mu \leq \nu^2 + \frac{1}{n} \,:\, n \in \mathbb{N}^+\right\} \vDash^{\mathsf{uc}} \mu \leq \nu^2.$$

Nevertheless, it is important to allow a finite number of a-formulas as hypotheses, since we do not have connectives in the UCL language for combining a-formulas and for combining them with c-formulas.

These completeness results are enough for the practical use we have in mind for UCL. But, from a theoretical point of view, it would be interesting to allow also a finite number of o-formulas as hypotheses and, furthermore, the use of a finite number of c-formulas as hypotheses. We leave theses issues for possible future work.

4.1 Soundness

It is enough to show soundness of tautologies and soundness of each rule in UCL. Then, the result follows by a straightforward induction.

Proposition 4.1 TAUT and MP are sound.

Proof:
(TAUT) Assume that φ is a tautology in PL. Let $I = (v, \rho)$. Observe that $v \Vdash \varphi$. Hence, by Proposition 2.1, $I \Vdash^{\mathsf{uc}} \varphi$.

(MP) Assume, by contradiction, that there is $I = (v, \rho)$ such that

$$(\dagger) \qquad I \Vdash^{\mathsf{uc}} \psi, \quad I \Vdash^{\mathsf{uc}} \psi \supset \varphi \quad \text{and} \quad I \nVdash^{\mathsf{uc}} \varphi.$$

Hence, by Proposition 2.1, $v \nVdash \varphi$. Therefore,

$$\{\varphi' \in \Omega_\psi : v \nVdash \varphi'\} = \{\varphi' \in \Omega_\psi : v \Vdash \varphi' \supset \varphi\}.$$

Then,

$$\begin{aligned}
1 &= \sum_{\varphi' \in \Omega_\psi} \mathfrak{P}[\psi \triangleright \varphi'] \\
&= \sum_{\substack{\varphi' \in \Omega_\psi \\ v \Vdash \varphi'}} \mathfrak{P}[\psi \triangleright \varphi'] + \sum_{\substack{\varphi' \in \Omega_\psi \\ v \nVdash \varphi'}} \mathfrak{P}[\psi \triangleright \varphi'] \\
&= \sum_{\substack{\varphi' \in \Omega_\psi \\ v \Vdash \varphi'}} \mathfrak{P}[\psi \triangleright \varphi'] + \sum_{\substack{\varphi' \in \Omega_\psi \\ v \Vdash \varphi' \supset \varphi}} \mathfrak{P}[\psi \supset \varphi \triangleright \varphi' \supset \varphi] \\
&\geq 2\mu > 1.
\end{aligned}$$

QED

Proposition 4.2 NO, SO, WO and AO are sound.

Proof:
(NO) Observe that

$$\sum_{\varphi \in \emptyset} \mathfrak{P}[\psi \triangleright \varphi] = 0.$$

Let $I = (v, \rho)$. Then,

$$\mathbb{R}\rho \Vdash^{\mathsf{fo}} 0 \leq \sum_{\varphi \in \emptyset} \mathfrak{P}[\psi \triangleright \varphi].$$

(SO) Straightforward by definition.

(WO) Let $I = (v, \rho)$. Assume $\forall \nu \, ((\frac{1}{2} < \nu \leq 1) \supset P_2 \leq P_1) \in \mathsf{ORCF}$ and $I \Vdash^{\mathsf{uc}} \Phi \sqsubseteq_{P_1} \psi$. Observe that

$$\mathbb{R}\,\rho \Vdash^{\mathsf{fo}} P_2 \leq P_1.$$

On the other hand,

$$\mathbb{R}\,\rho \Vdash^{\mathsf{fo}} \sum_{\varphi \in \Phi} \mathfrak{P}[\psi \triangleright \varphi] \geq P_1.$$

Therefore,

$$\mathbb{R}\,\rho \Vdash^{\mathsf{fo}} \sum_{\varphi \in \Phi} \mathfrak{P}[\psi \triangleright \varphi] \geq P_2.$$

(AO) We omit the proof of soundness for this rule since it follows straightforwardly. QED

Proposition 4.3 WA is sound.

Proof:
(WA) Assume that

$$\forall \mu \forall \nu \left(\left(\frac{1}{2} < \mu, \nu \leq 1 \wedge \mu \leq P_1 \wedge \ldots \wedge \mu \leq P_k \right) \supset \mu \leq P \right) \in \mathsf{ORCF}$$

and that $I \Vdash^{\mathsf{uc}} \mu \leq P_i$ for $i = 1, \ldots, k$. Then,

$$\mathbb{R}\,\rho \Vdash^{\mathsf{fo}} \mu \leq P_i, \text{ for } i = 1, \ldots, k.$$

Thus $\mathbb{R}\,\rho \Vdash^{\mathsf{fo}} \mu \leq P$ and so $I \Vdash^{\mathsf{uc}} \mu \leq P$. QED

Proposition 4.4 The lifting rule is sound.

Proof: Let $I = (v, \rho)$. Assume that

$$\begin{cases} I \Vdash^{\mathsf{uc}} \bigvee_{i=1}^{k} \bigwedge \Phi_i \\ \begin{cases} I \Vdash^{\mathsf{uc}} \Phi_i \sqsubseteq_{P_i} \psi \\ I \Vdash^{\mathsf{uc}} \mu \leq P_i \end{cases} \quad \text{for } i = 1, \ldots, k. \end{cases}$$

Then, there is $i \in \{1, \ldots, k\}$ such that $v \Vdash \bigwedge \Phi_i$ by Proposition 2.1 since $\bigvee_{i=1}^{k} \bigwedge \Phi_i \in L(X)$. Observe that

$$\{\varphi \in \Omega_\psi : v \Vdash \varphi\} \supseteq \Phi_i$$

and, so,

$$\mathbb{R}\,\rho \Vdash^{\mathsf{fo}} \sum_{\substack{\varphi \in \Omega_\psi \\ v \Vdash \varphi}} \mathfrak{P}[\psi \triangleright \varphi] \geq \sum_{\varphi \in \Phi_i} \mathfrak{P}[\psi \triangleright \varphi].$$

By hypothesis,

$$\mathbb{R}\,\rho \Vdash^{\mathsf{fo}} \sum_{\varphi \in \Phi_i} \mathfrak{P}[\psi \triangleright \varphi] \geq P_i$$

and

$$\mathbb{R}\,\rho \Vdash^{\mathsf{fo}} P_i \geq \mu.$$

Therefore,

$$\mathbb{R}\,\rho \Vdash^{\mathsf{fo}} \sum_{\substack{\varphi \in \Omega_\psi \\ v \Vdash \varphi}} \mathfrak{P}[\psi \triangleright \varphi] \geq \mu$$

and, so, the thesis follows. QED

Theorem 4.5 (Strong soundness of UCL)
Let $\Gamma \cup \{\theta\} \subseteq L^{\mathsf{uc}}(X)$. Then,

$$\Gamma \vDash^{\mathsf{uc}} \theta \quad \text{whenever} \quad \Gamma \vdash^{\mathsf{uc}} \theta.$$

Proof: Straightforward induction on the length of the derivation given for $\Gamma \vdash^{\mathsf{uc}} \theta$. QED

4.2 Completeness

The objective now is to investigate completeness of UCL. In the sequel, given $\psi \in L^{\mathsf{c}}(X)$ and $\Phi \subseteq \Omega_\psi$, we may use

$$\mathfrak{P}[\psi \rhd \Phi] \quad \text{for} \quad \sum_{\varphi \in \Phi} \mathfrak{P}[\psi \rhd \varphi].$$

First, we establish the envisaged constrained strong completeness result concerning a-formulas.

Proposition 4.6 Let Γ be a finite subset of L^{a} and P a term. Then,

$$\Gamma \vdash^{\mathsf{uc}} \mu \leq P \quad \text{whenever} \quad \Gamma \vDash^{\mathsf{uc}} \mu \leq P.$$

Proof: Assume that $\Gamma \vDash^{\mathsf{uc}} \mu \leq P$ and suppose without loss of generality that Γ is the set $\{\mu \leq P_1, \ldots, \mu \leq P_k\}$. We now show that

$$(\dagger) \qquad \mathbb{R}\,\rho \Vdash^{\mathsf{fo}} \forall \mu \forall \nu \left(\left(\frac{1}{2} < \mu, \nu \leq 1 \wedge \bigwedge_{i=1}^{k} \mu \leq P_i \right) \supset \mu \leq P \right)$$

for every assignment ρ over $\mathbb{R}$. So, let ρ be an assignment over $\mathbb{R}$ such that

$$\mathbb{R}\,\rho \Vdash^{\mathsf{fo}} \frac{1}{2} < \mu, \nu \leq 1 \wedge \bigwedge_{i=1}^{k} \mu \leq P_i.$$

Let v be an arbitrary valuation. Then,

$$(v, \rho) \Vdash^{\mathsf{uc}} \bigwedge_{i=1}^{k} \mu \leq P_i$$

and, so,

$$(v, \rho) \Vdash^{\mathsf{uc}} \mu \leq P$$

since $\Gamma \vDash^{\mathsf{uc}} \mu \leq P$. Hence,

$$\mathbb{R}\,\rho \Vdash^{\mathsf{fo}} \mu \leq P$$

and, thus, we have proved $(\dagger)$. Therefore,

$$\begin{array}{rll} 1 & \mu \leq P_1 & \text{HYP} \\ \vdots & \vdots & \vdots \\ k & \mu \leq P_k & \text{HYP} \\ k+1 & \mu \leq P & \text{WA} : 1, \ldots, k \end{array}$$

is a derivation for $\Gamma \vdash^{\mathsf{uc}} \mu \leq P$. QED

Next, we obtain the weak completeness of the UCL calculus concerning o-formulas.

Proposition 4.7 Let $\psi \in L^{\mathsf{c}}(X)$, $\Phi \subseteq \Omega_\psi$ and P a term. Then,

$$\vdash^{\mathsf{uc}} \Phi \sqsubseteq_P \psi \quad \text{whenever} \quad \vDash^{\mathsf{uc}} \Phi \sqsubseteq_P \psi.$$

Proof: Assume that $\vDash^{\mathsf{uc}} \Phi \sqsubseteq_P \psi$. We want to show that

$$\forall \nu \left(\left(\frac{1}{2} < \nu \leq 1 \right) \supset (P \leq \mathfrak{P}[\psi \triangleright \Phi]) \right) \in \mathsf{ORCF}.$$

Let ρ be an assignment over $\mathbb{R}$ such that

$$\mathbb{R}\, \rho \Vdash^{\mathsf{fo}} \frac{1}{2} < \nu \leq 1.$$

Let ρ' be an assignment such that $\rho'(\nu) = \rho(\nu)$ and $\frac{1}{2} < \rho'(\mu) \leq 1$. Then,

$$\mathbb{R}\, \rho' \Vdash^{\mathsf{fo}} \frac{1}{2} < \nu \leq 1.$$

Let v be a valuation and $I = (v, \rho')$. Then,

$$I \Vdash^{\mathsf{uc}} \Phi \sqsubseteq_P \psi$$

and, so,

$$\mathbb{R}\, \rho' \Vdash^{\mathsf{fo}} \sum_{\varphi \in \Phi} \mathfrak{P}[\psi \triangleright \varphi] \geq P.$$

Therefore,

$$\mathbb{R}\, \rho \Vdash^{\mathsf{fo}} P \leq \mathfrak{P}[\psi \triangleright \Phi].$$

On the other hand, by rules NO, SO and AO, $\vdash^{\mathsf{uc}} \Phi \sqsubseteq_{\mathfrak{P}[\psi \triangleright \Phi]} \psi$ and, so, $\vdash^{\mathsf{uc}} \Phi \sqsubseteq_P \psi$ by rule WO. QED

Finally, we proceed to prove the envisaged constrained strong completeness concerning c-formulas. To this end, given a valuation v and $\psi \in L^{\mathsf{c}}(X)$, it becomes handy to write

$$\Omega_\psi^v \quad \text{for} \quad \{\varphi \in \Omega_\psi : v \Vdash \varphi\}.$$

We start by showing two auxiliary results.

Proposition 4.8 Let $\Gamma \subseteq L^{\mathsf{a}}$, $\psi \in L^{\mathsf{c}}(X)$ and v be a valuation. Then,

$$\Gamma \vDash^{\mathsf{uc}} \mu \leq \mathfrak{P}[\psi \triangleright \Omega_\psi^v] \quad \text{whenever} \quad \Gamma \vDash^{\mathsf{uc}} \psi.$$

Proof: Let v be a valuation. Assume that $\Gamma \vDash^{\mathsf{uc}} \psi$. Let $I = (v, \rho)$ be an interpretation such that $I \Vdash^{\mathsf{uc}} \Gamma$. We want to show that

$$I \Vdash^{\mathsf{uc}} \mu \leq \mathfrak{P}[\psi \rhd \Omega^v_\psi].$$

Observe that

$$\mathfrak{P}[\psi \rhd \Omega^v_\psi] = \sum_{\varphi \in \Omega^v_\psi} \mathfrak{P}[\psi \rhd \varphi] = \sum_{\substack{\varphi \,\in\, \Omega_\psi \\ v \,\Vdash\, \varphi}} \mathfrak{P}[\psi \rhd \varphi].$$

Moreover, since $I \Vdash^{\mathsf{uc}} \psi$,

$$\mathbb{R}\, \rho \Vdash^{\mathsf{fo}} \sum_{\substack{\varphi \,\in\, \Omega_\psi \\ v \,\Vdash\, \varphi}} \mathfrak{P}[\psi \rhd \varphi] \geq \mu$$

and the thesis follows. QED

Proposition 4.9 Let $\psi \in L^{\mathsf{c}}(X)$. Then,

$$\{\mu \leq \mathfrak{P}[\psi \rhd \Omega^v_\psi] : v \text{ is a valuation}\} \vdash^{\mathsf{uc}} \psi.$$

Proof: Observe that $\{\Omega^v_\psi : v \text{ is a valuation}\}$ is a finite set. The thesis follows by LFT since:

(a) $\vdash \bigvee_v \bigwedge \Omega^v_\psi$. Indeed $\vDash \bigvee_v \bigwedge \Omega^v_\psi$ and so, by completeness of PL, the thesis follows.

(b) $\vdash^{\mathsf{uc}} \Omega^v_\psi \sqsubseteq_{\mathfrak{P}[\psi \rhd \Omega^v_\psi]} \psi$ for every valuation v. Immediate by rules NO, SO and AO.

(c) $\{\mu \leq \mathfrak{P}[\psi \rhd \Omega^v_\psi] : v \text{ is a valuation}\} \vdash^{\mathsf{uc}} \mu \leq \mathfrak{P}[\psi \rhd \Omega^v_\psi]$ for every valuation v. This follows by extensivity of $\vdash^{\mathsf{uc}}$. QED

Theorem 4.10 (Constrained strong completeness of UCL)
Let Γ be a finite subset of L^{a} and $\psi \in L^{\mathsf{c}}(X)$. Then,

$$\Gamma \vdash^{\mathsf{uc}} \psi \quad \text{whenever} \quad \Gamma \vDash^{\mathsf{uc}} \psi.$$

Proof: Assume that $\Gamma \vDash^{\mathsf{uc}} \psi$. Then, by Proposition 4.8, $\Gamma \vDash^{\mathsf{uc}} \mu \leq \mathfrak{P}[\psi \rhd \Omega^v_\psi]$ for every valuation v. We now show that

$$\forall \mu \forall \nu \ \left(\left(\frac{1}{2} < \mu, \nu \leq 1 \wedge \left(\bigwedge \Gamma \right) \right) \supset \mu \leq \mathfrak{P}[\psi \rhd \Omega^v_\psi] \right) \in \mathsf{ORCF}$$

for every valuation v. Let ρ be an assignment over $\mathbb{R}$. Assume that

$$\mathbb{R}\,\rho \Vdash^{\mathsf{fo}} \frac{1}{2} < \mu, \nu \leq 1 \wedge \left(\bigwedge \Gamma\right).$$

Let $I = (v, \rho)$ be an interpretation for some valuation v. Observe that I is in fact an interpretation since $\mathbb{R}\,\rho \Vdash^{\mathsf{fo}} \frac{1}{2} < \mu, \nu \leq 1$. Moreover, $I \Vdash^{\mathsf{uc}} \Gamma$ and so $I \Vdash^{\mathsf{uc}} \mu \leq \mathfrak{P}[\psi \triangleright \Omega_\psi^v]$. Therefore,

$$\mathbb{R}\,\rho \Vdash^{\mathsf{fo}} \mu \leq \mathfrak{P}[\psi \triangleright \Omega_\psi^v].$$

Then, since Γ is assumed to be finite, by WA,

$$(\dagger) \quad \Gamma \vdash^{\mathsf{uc}} \mu \leq \mathfrak{P}[\psi \triangleright \Omega_\psi^v]$$

for every valuation v. On the other hand, by Proposition 4.9,

$$(\ddagger) \quad \{\mu \leq \mathfrak{P}[\psi \triangleright \Omega_\psi^v] : v \text{ is a valuation}\} \vdash^{\mathsf{uc}} \psi.$$

Thus, by ($\dagger$) and ($\ddagger$), the result follows by idempotence of $\vdash^{\mathsf{uc}}$. QED

5 Metatheorems

In this section we investigate some properties of the UCL calculus, namely the metatheorems of deduction, equivalence, interderivability and substitution of equivalents, as well as closure for substitutions. Together with the positive results that we were able to establish, we also present counterexamples illustrating the differences between UCL and PL.

5.1 Metatheorem of deduction

We start by showing some auxiliary results that are needed in the proof of the metatheorem of deduction.

Proposition 5.1 Let $\psi \in L^{\mathsf{c}}(X)$ and $\Phi \subseteq \Omega_\psi$. Then,

$$\vdash^{\mathsf{uc}} \Phi \sqsubseteq_P \psi \quad \text{iff} \quad \forall \nu \left(\left(\frac{1}{2} < \nu \leq 1 \right) \supset P \leq \mathfrak{P}[\psi \triangleright \Phi] \right) \in \mathsf{ORCF}.$$

Proof: The implication from left to right follows immediately from the soundness of UCL, see Theorem 4.5. The other implication is a direct consequence of NO, SO, AO and WO. QED

Proposition 5.2 Let $\psi, \psi' \in L^{\mathsf{c}}(X)$ and $\varphi' \in \Omega_{\psi'}$. Then,

$$\vdash^{\mathsf{uc}} \{\varphi \supset \varphi' : \varphi \in \Omega_\psi\} \sqsubseteq_{\mathfrak{P}[\psi' \rhd \varphi']} \psi \supset \psi'.$$

Proof: Observe that, by rule SO, for every $\varphi \in \Omega_\psi$,

$$\vdash^{\mathsf{uc}} \varphi \supset \varphi' \sqsubseteq_{\mathfrak{P}[\psi \rhd \varphi] \times \mathfrak{P}[\psi' \rhd \varphi']} \psi \supset \psi'$$

and, so, by rule AO,

$$\vdash^{\mathsf{uc}} \{\varphi \supset \varphi' : \varphi \in \Omega_\psi\} \sqsubseteq_{(\sum_{\varphi \in \Omega_\psi} \mathfrak{P}[\psi \rhd \varphi] \times \mathfrak{P}[\psi' \rhd \varphi'])} \psi \supset \psi'.$$

Note that, in ORCF,

$$\begin{aligned} \textstyle\sum_{\varphi \in \Omega_\psi} \mathfrak{P}[\psi \rhd \varphi] \times \mathfrak{P}[\psi' \rhd \varphi'] &= \mathfrak{P}[\psi' \rhd \varphi'] \times \textstyle\sum_{\varphi \in \Omega_\psi} \mathfrak{P}[\psi \rhd \varphi] \\ &= \mathfrak{P}[\psi' \rhd \varphi'] \end{aligned}$$

by Proposition 3.1. Hence, by rule WO, the thesis follows. QED

Proposition 5.3 Let $\Gamma \subseteq L^{\mathsf{a}} \cup L(X)$, $\psi \in L^{\mathsf{c}}(X)$ and $\Phi \subseteq \Omega_\psi$. Then,

$$\vdash^{\mathsf{uc}} \Phi \sqsubseteq_P \psi \quad \text{whenever} \quad \Gamma \vdash^{\mathsf{uc}} \Phi \sqsubseteq_P \psi.$$

Proof: The result follows by induction on a derivation of $\Phi \sqsubseteq_P \psi$ from Γ. In the basis, the conclusion follows by NO and SO. Since NO and SO are axioms, the thesis follows immediately. Regarding the step, observe that the rules used to conclude $\Phi \sqsubseteq_P \psi$ all have o-formulas as premises. Hence, by applying the induction hypothesis to each premise, we can conclude that they are theorems. Using the same rule the thesis follows. QED

Proposition 5.4 Let $\Gamma_1 \subseteq L^{\mathsf{a}}$, $\Gamma_2 \subseteq L(X)$ and a term P. Then,

$$\Gamma_1 \vdash^{\mathsf{uc}} \mu \leq P \quad \text{whenever} \quad \Gamma_1, \Gamma_2 \vdash^{\mathsf{uc}} \mu \leq P.$$

Proof: The result follows by straightforward induction on the given derivation of $\mu \leq P$ from $\Gamma_1 \cup \Gamma_2$. In the basis, the conclusion is either an hypothesis or follows by WA over an empty set of premises. So the same derivation is also a derivation of $\mu \leq P$ from Γ_1. Regarding the step, the conclusion follows by rule WA over a non-empty set of premises in the derivation. Hence, the thesis follows by the induction hypothesis and by applying the same rule. QED

As usual in proof theory, given a derivation $\theta_1 \ldots \theta_k$ of ψ from Γ, we say that θ_i *depends* on $\gamma \in \Gamma$ in this derivation if either θ_i is γ or θ_i is obtained using a rule with at least one of the premises depending on γ. Moreover, an application of MP is said to be *classical* if both premises are in $L(X)$.

Proposition 5.5 Let $\Gamma \subseteq L^{\mathsf{a}}$ and ψ and ψ' be distinct formulas in $L^{\mathsf{c}}(X)$. Assume that there is a derivation of ψ' from $\Gamma \cup \{\psi\}$ where ψ' depends on ψ and all the applications of MP over dependents of ψ are classical. Then, $\psi \in L(X)$.

Proof: Let $\theta_1 \ldots \theta_k$ be a derivation of $\Gamma, \psi \vdash^{\mathsf{uc}} \psi'$ where ψ' depends on ψ and all the applications of MP over dependents of ψ are classical. The proof follows by induction on k. Since ψ' depends on ψ, ψ' is not obtained by TAUT. Moreover, since ψ' is not ψ, ψ' does not appear as an hypothesis. Hence, we have only to consider two cases:

(1) ψ' is obtained by LFT from an $L(X)$ formula θ_{i_1}, an o-formula θ_{i_2} and an a-formula θ_{i_3}. Since a-formulas and o-formulas do not depend on c-formulas in any derivation, the premise that depends on ψ must be θ_{i_1}. We need to consider two possibilities:

(i) ψ is θ_{i_1}. Then, ψ is in $L(X)$ as we wanted to show;

(ii) ψ is not θ_{i_1}. Then, by the induction hypothesis, $\psi \in L(X)$.

(2) ψ' is obtained by MP from θ_i and $\theta_i \supset \psi'$ where either θ_i or $\theta_i \supset \psi'$ depends on ψ in the given derivation. Since ψ' depends on ψ, then, both θ_i and $\theta_i \supset \psi'$ are in $L(X)$. If $\theta_i \supset \psi'$ depends on ψ in the given derivation, again we need to consider two possibilities:

(i) ψ is $\theta_i \supset \psi'$. Then, ψ is in $L(X)$ as we wanted to show;

(ii) ψ is not $\theta_i \supset \psi'$. Then, by the induction hypothesis, $\psi \in L(X)$.

If θ_i depends on ψ, yet again we need to consider two possibilities and apply the same reasoning as above in (1). QED

Theorem 5.6 (Metatheorem of deduction - MTD)
Let $\Gamma \subseteq L^{\mathsf{a}}$ and $\psi, \psi' \in L^{\mathsf{c}}(X)$. Assume that ψ' fulfills the following proviso: either ψ' is distinct from ψ or $\psi' \in L(X)$. Then,

$$\Gamma \vdash^{\mathsf{uc}} \psi \supset \psi'$$

whenever there is a derivation establishing

$$\Gamma, \psi \vdash^{\mathsf{uc}} \psi'$$

where all the applications of MP over dependents of ψ are classical.

Proof: Let $\theta_1 \dots \theta_k$ be a derivation of ψ' from $\Gamma \cup \{\psi\}$ where all the applications of MP over dependents of ψ are classical. The proof follows by induction on k.

(Basis) Consider two cases.

(1) ψ' is obtained by TAUT. Then, $\psi' \in L(X)$. Take

$$\Phi = \{\varphi \supset \psi' : \varphi \in \Omega_\psi\}.$$

Then, by tautological reasoning

$$\vdash \varphi \supset \psi'$$

for every $\varphi \in \Omega_\psi$, and so,

$$\vdash^{\mathsf{uc}} \bigwedge \Phi.$$

On the other hand, by Proposition 5.2, since $\mathfrak{P}[\psi' \triangleright \psi'] = 1$,

$$\vdash^{\mathsf{uc}} \Phi \sqsubseteq_1 \psi \supset \psi'.$$

Hence, the thesis follows by rule LFT.

(2) ψ' is ψ. Then, by hypothesis, $\psi' \in L(X)$. The proof is omitted since it is similar to case (1).

(Step) There are two cases to consider.

(3) ψ' is obtained by LFT from $\bigvee_{i=1}^n \bigwedge \Phi_i$, $\Phi_i \sqsubseteq_{P_i} \psi'$ and $\mu \leq P_i$ for $i = 1, \dots, n$. Then,

$$\Gamma, \psi \vdash^{\mathsf{uc}} \bigvee_{i=1}^{n} \bigwedge \Phi_i$$

$$\begin{cases} \Gamma, \psi \vdash^{\mathsf{uc}} \Phi_i \sqsubseteq_{P_i} \psi' \\ \Gamma, \psi \vdash^{\mathsf{uc}} \mu \leq P_i \end{cases} \quad \text{for } i = 1, \dots, n.$$

We want to show that there are $\Phi'_i \subseteq \Omega_{\psi \supset \psi'}$ and P'_i for $i = 1, \dots, n$ such that

$$\text{(a) } \Gamma \vdash^{\mathsf{uc}} \bigvee_{i=1}^{n} \bigwedge \Phi'_i$$

$$\begin{cases} \text{(b) } \Gamma \vdash^{\mathsf{uc}} \Phi'_i \sqsubseteq_{P'_i} \psi \supset \psi' \\ \text{(c) } \Gamma \vdash^{\mathsf{uc}} \mu \leq P'_i \end{cases} \quad \text{for } i = 1, \dots, n.$$

There are three cases to consider:

(a) ψ' depends on ψ in $\theta_1 \ldots \theta_k$, and ψ' and ψ are distinct. Then, by Proposition 5.5, $\psi \in L(X)$. Take

$$\Phi_i' = \{\psi \supset \varphi_i : \varphi_i \in \Phi_i\}$$

and $P_i' = P_i$ for every $i = 1, \ldots, n$. Then:

(i) Observe that, by the induction hypothesis,

$$\Gamma \vdash^{\mathsf{uc}} \psi \supset \bigvee_{i=1}^{n} \bigwedge \Phi_i.$$

Since $\{\psi\} \cup \bigcup_{i=1}^{n} \Phi_i \subseteq L(X)$, then, by tautological reasoning,

$$\Gamma \vdash^{\mathsf{uc}} \bigvee_{i=1}^{n} \bigwedge \Phi_i'.$$

(ii) Observe that, by rule SO,

$$\vdash^{\mathsf{uc}} \psi \supset \varphi \sqsubseteq_{\mathfrak{P}[\psi' \triangleright \varphi]} \psi \supset \psi'$$

for every $\varphi \in \Omega_{\psi'}$. Thus, either by rule NO when $\Phi_i = \emptyset$ or by rule AO otherwise,

$$\vdash^{\mathsf{uc}} \Phi_i' \sqsubseteq_{\mathfrak{P}[\psi' \triangleright \Phi_i]} \psi \supset \psi'$$

for $i = 1, \ldots, n$. On the other hand, by Proposition 5.3, $\vdash^{\mathsf{uc}} \Phi_i \sqsubseteq_{P_i} \psi'$ and so by Proposition 5.1,

$$\forall \nu \left(\left(\frac{1}{2} < \nu \leq 1 \right) \supset P_i \leq \mathfrak{P}[\psi' \triangleright \Phi_i] \right) \in \mathsf{ORCF}.$$

Thus,

$$\vdash^{\mathsf{uc}} \Phi_i' \sqsubseteq_{P_i} \psi \supset \psi'$$

by rule WO.

(iii) $\Gamma \vdash^{\mathsf{uc}} \mu \leq P_i'$, by Proposition 5.4 since $P_i = P_i'$.

Therefore, by LFT, $\Gamma \vdash^{\mathsf{uc}} \psi \supset \psi'$.

(b) ψ' does not depend on ψ in $\theta_1 \ldots \theta_k$. Take

$$\Phi_i' = \{\varphi \supset \varphi' : \varphi \in \Omega_{\psi}, \varphi' \in \Phi_i\}$$

and $P_i' = P_i$ for every $i = 1, \ldots, n$. Then:

(i) Observe that

$$\Gamma \vdash^{\mathsf{uc}} \bigvee_{i=1}^{n} \bigwedge \Phi_i.$$

Then, by tautological reasoning

$$\Gamma \vdash^{\mathsf{uc}} \left(\bigvee \Omega_\psi\right) \supset \bigvee_{i=1}^{n} \bigwedge \Phi_i$$

and, so, again by tautological reasoning,

$$\Gamma \vdash^{\mathsf{uc}} \bigvee_{i=1}^{n} \left(\left(\bigvee \Omega_\psi\right) \supset \bigwedge \Phi_i\right).$$

Thus, once again by tautological reasoning,

$$\Gamma \vdash^{\mathsf{uc}} \bigvee_{i=1}^{n} \bigwedge_{\varphi \in \Omega_\psi} \left(\varphi \supset \bigwedge \Phi_i\right)$$

and, so,

$$\Gamma \vdash^{\mathsf{uc}} \bigvee_{i=1}^{n} \bigwedge \Phi'_i.$$

(ii) By Proposition 5.2,

$$\vdash^{\mathsf{uc}} \{\varphi \supset \varphi' : \varphi \in \Omega_\psi\} \sqsubseteq_{\mathfrak{P}[\psi' \triangleright \varphi']} \psi \supset \psi'$$

for every $\varphi' \in \Omega_{\psi'}$. So either by rule NO when $\Phi_i = \emptyset$ or by rule AO otherwise,

$$\vdash^{\mathsf{uc}} \Phi'_i \sqsubseteq_{\mathfrak{P}[\psi' \triangleright \Phi_i]} \psi \supset \psi'$$

for $i = 1, \ldots, n$. On the other hand, by Proposition 5.3, $\vdash^{\mathsf{uc}} \Phi_i \sqsubseteq_{P_i} \psi'$ and so by Proposition 5.1,

$$\forall \nu \left(\left(\frac{1}{2} < \nu \leq 1\right) \supset P_i \leq \mathfrak{P}[\psi' \triangleright \Phi_i]\right) \in \mathsf{ORCF}.$$

Thus,

$$\vdash^{\mathsf{uc}} \Phi'_i \sqsubseteq_{P_i} \psi \supset \psi'$$

by rule WO.

(iii) $\Gamma \vdash^{\mathsf{uc}} \mu \leq P'_i$, by Proposition 5.4 since $P_i = P'_i$.

Therefore, by LFT, $\Gamma \vdash^{\mathsf{uc}} \psi \supset \psi'$.

(c) ψ' is ψ. Then $\psi' \in L(X)$ by hypothesis. The proof is omitted since it is similar to case (1).

(4) ψ' is obtained by MP from ψ'' and $\psi'' \supset \psi'$ where $\psi' \in L(X)$. We have three cases:

(a) ψ' depends on ψ in $\theta_1 \dots \theta_k$, and ψ' and ψ are distinct. Then, by Proposition 5.5, $\psi \in L(X)$. Moreover, $\psi'' \in L(X)$ since all the applications of MP on dependents of ψ are classical. By the induction hypothesis

$$\begin{cases} \Gamma \vdash^{\mathsf{uc}} \psi \supset \psi'' \\ \Gamma \vdash^{\mathsf{uc}} \psi \supset (\psi'' \supset \psi'). \end{cases}$$

Then, by tautological reasoning, $\Gamma \vdash^{\mathsf{uc}} \psi \supset \psi'$.

(b) ψ' does not depend on ψ in $\theta_1 \dots \theta_k$. Then,

$$\Gamma \vdash^{\mathsf{uc}} \psi'.$$

By tautological reasoning,

$$\Gamma \vdash^{\mathsf{uc}} \varphi \supset \psi'$$

for every $\varphi \in \Omega_\psi$. The rest of the proof is similar to the one in (1).

(c) ψ' is ψ. Then, $\psi' \in L(X)$ by hypothesis. The proof is omitted since it is similar to case (1). QED

Observe that one would have no difficulty in violating the metatheorem of deduction without the proviso on ψ'. For instance,

$$\widetilde{\neg}\, x \vdash^{\mathsf{uc}} \widetilde{\neg}\, x$$

by extensivity. On the other hand,

$$\nvDash^{\mathsf{uc}} (\widetilde{\neg}\, x) \supset (\widetilde{\neg}\, x)$$

as we saw at the end of Subsection 2.3. Hence, by soundness of UCL,

$$\nvdash^{\mathsf{uc}} (\widetilde{\neg}\, x) \supset (\widetilde{\neg}\, x).$$

5.2 Metatheorems of equivalence and interderivability

Since

$$\vDash^{\mathsf{uc}} (\psi \equiv \psi') \equiv ((\psi \supset \psi') \wedge (\psi' \supset \psi))$$

does not hold in general for arbitrary $\psi, \psi' \in L^{\mathsf{c}}(X)$, it is worthwhile to investigate the relationship between implication and equivalence, as well as the relationship between interderivability and equivalence.

Proposition 5.7 Let $\Gamma \subseteq L^{\mathsf{a}}$ be a finite set and $\psi, \psi' \in L^{\mathsf{c}}(X)$ be such that

- $\Gamma \vdash^{\mathsf{uc}} \psi \supset \psi'$;
- $\varphi \supset \varphi' \vdash \varphi' \supset \varphi$ for every $\varphi \in \Omega_\psi$ and $\varphi' \in \Omega_{\psi'}$.

Then,

$$\Gamma \vdash^{\mathsf{uc}} \psi \equiv \psi'.$$

Proof: Observe that by soundness of FCL (see Theorem 4.5), $\Gamma \vDash^{\mathsf{uc}} \psi \supset \psi'$ and by soundness of PL, $\varphi \supset \varphi' \vDash \varphi' \supset \varphi$ for every $\varphi \in \Omega_\psi$ and $\varphi' \in \Omega_{\psi'}$. We proceed to show that $\Gamma \vDash^{\mathsf{uc}} \psi \equiv \psi'$. Let $I = (v, \rho)$ be an interpretation such that $\mathbb{R}\,\rho \Vdash^{\mathsf{fo}} \Gamma$. We must show that

$$\mathbb{R}\,\rho \Vdash^{\mathsf{fo}} \sum_{\substack{\varphi \equiv \varphi' \ \sqsubseteq\ \psi \equiv \psi' \\ v \Vdash \varphi \equiv \varphi'}} \mathfrak{P}[\psi \equiv \psi' \triangleright \varphi \equiv \varphi'] \geq \mu.$$

Indeed, $\mathbb{R}$ together with ρ satisfies

$$\begin{aligned}
\sum_{\substack{\varphi \equiv \varphi' \ \sqsubseteq\ \psi \equiv \psi' \\ v \Vdash \varphi \equiv \varphi'}} \mathfrak{P}[\psi \equiv \psi' \triangleright \varphi \equiv \varphi'] \quad &= \quad \sum_{\substack{\varphi \equiv \varphi' \ \sqsubseteq\ \psi \equiv \psi' \\ v \Vdash \varphi' \supset \varphi \\ v \Vdash \varphi \supset \varphi'}} \mathfrak{P}[\psi \equiv \psi' \triangleright \varphi \equiv \varphi'] \\
&= \quad \sum_{\substack{\varphi \equiv \varphi' \ \sqsubseteq\ \psi \equiv \psi' \\ v \Vdash \varphi \supset \varphi'}} \mathfrak{P}[\psi \equiv \psi' \triangleright \varphi \equiv \varphi'] \\
&= \quad \sum_{\substack{\varphi \supset \varphi' \ \sqsubseteq\ \psi \supset \psi' \\ v \Vdash \varphi \supset \varphi'}} \mathfrak{P}[\psi \supset \psi' \triangleright \varphi \supset \varphi'] \\
&\geq \quad \mu,
\end{aligned}$$

taking into account that

$$\{v : v \Vdash \varphi' \equiv \varphi\} = \{v : v \Vdash \varphi \supset \varphi'\} \cap \{v : v \Vdash \varphi' \supset \varphi\}$$

and, since $\varphi \supset \varphi' \vdash \varphi' \supset \varphi$, $\{v : v \Vdash \varphi \supset \varphi'\} = \{v : v \Vdash \varphi' \equiv \varphi\}$. So, by completeness of FCL (see Theorem 4.10), $\Gamma \vdash^{\mathsf{uc}} \psi \equiv \psi'$. QED

Proposition 5.8 Let $\psi \in L^{\mathsf{c}}(X)$, $\varphi \in \Omega_\psi$, and ρ an assignment over $\mathbb{R}$ such that $\frac{1}{2} < \rho(\nu) < 1$. Then,

$$\mathbb{R}\,\rho \Vdash^{\mathsf{fo}} \mathfrak{P}[\psi \triangleright \varphi] > 0.$$

Proof: The proof is carried out by induction on the structure of ψ.

(Basis) ψ is φ. Then, $\mathfrak{P}[\psi \triangleright \varphi] = 1$ and, so, $\mathbb{R}\,\rho \Vdash^{\mathsf{fo}} \mathfrak{P}[\psi \triangleright \varphi] > 0$.

(Step) We need to consider three cases:
(1) ψ is $c(\psi_1, \dots, \psi_n)$ and φ is $c(\varphi_1, \dots, \varphi_n)$. Then,

$$\mathfrak{P}[\psi \triangleright \varphi] = \prod_{i=1}^{n} \mathfrak{P}[\psi_i \triangleright \varphi_i]$$

and the thesis follows since, by the induction hypothesis, $\mathbb{R}\,\rho \Vdash^{\mathsf{fo}} \mathfrak{P}[\psi_i \triangleright \varphi_i] > 0$ for each $i = 1, \dots, n$.
(2) ψ is $\widetilde{c}(\psi_1, \dots, \psi_n)$ and φ is $c(\varphi_1, \dots, \varphi_n)$. Then,

$$\mathfrak{P}[\psi \triangleright \varphi] = \nu \prod_{i=1}^{n} \mathfrak{P}[\psi_i \triangleright \varphi_i]$$

and the thesis follows since $\mathbb{R}\,\rho \Vdash^{\mathsf{fo}} \nu > 0$ by hypothesis and, again by the induction hypothesis, $\mathbb{R}\,\rho \Vdash^{\mathsf{fo}} \mathfrak{P}[\psi_i \triangleright \varphi_i] > 0$ for each $i = 1, \dots, n$.
(3) ψ is $\widetilde{c}(\psi_1, \dots, \psi_n)$ and φ is $\overline{c}(\varphi_1, \dots, \varphi_n)$. Then, replacing ν by $1 - \nu$, the thesis follows as in (2). QED

Proposition 5.9 Let $\psi \in L^{\mathsf{c}}(X)$ and $\varphi \in \Omega_\psi$ be such that $\vDash^{\mathsf{uc}} \psi$. Then, φ is a tautology.

Proof: Let v be a valuation. Consider an assignment ρ over $\mathbb{R}$ such that $\rho(\mu) = 1$ and $\frac{1}{2} < \rho(\nu) < 1$. Observe that

$$(v, \rho) \Vdash^{\mathsf{uc}} \psi.$$

Hence,

$$\mathbb{R}\,\rho \Vdash^{\mathsf{fo}} \sum_{\substack{\varphi \,\sqsubseteq\, \psi \\ v \Vdash \varphi}} \mathfrak{P}[\psi \triangleright \varphi] \geq \mu.$$

Since, $\rho(\mu) = 1$,

$$\mathbb{R}\,\rho \Vdash^{\mathsf{fo}} \sum_{\substack{\varphi \,\sqsubseteq\, \psi \\ v \,\Vdash\, \varphi}} \mathfrak{P}[\psi \triangleright \varphi] = \sum_{\varphi \,\sqsubseteq\, \psi} \mathfrak{P}[\psi \triangleright \varphi]$$

by Proposition 3.1. Moreover, $\mathbb{R}\,\rho \Vdash^{\mathsf{fo}} \mathfrak{P}[\psi \triangleright \varphi] > 0$ for each $\varphi \in \Omega_\psi$ by Proposition 5.8 since $\frac{1}{2} < \rho(\nu) < 1$. Therefore,

$$\{\varphi \in \Omega_\psi : v \Vdash \varphi\} = \Omega_\psi.$$

Hence, $v \Vdash \varphi$. QED

Theorem 5.10 (1st Metatheorem of equivalence - MTE1)
Let $\psi, \psi' \in L^{\mathsf{c}}(X)$. Then,

$$\vdash^{\mathsf{uc}} \psi \equiv \psi' \quad \text{whenever} \quad \begin{cases} \vdash^{\mathsf{uc}} \psi \supset \psi' \\ \vdash^{\mathsf{uc}} \psi' \supset \psi. \end{cases}$$

Proof: Observe that by soundness of FCL (see Theorem 4.5), $\vDash^{\mathsf{uc}} \psi \supset \psi'$ and $\vDash^{\mathsf{uc}} \psi' \supset \psi$. Hence, by Proposition 5.9, $\vDash \varphi \supset \varphi'$ and $\vDash \varphi' \supset \varphi$. for every $\varphi \supset \varphi' \in \Omega_{\psi \supset \psi'}$ and $\varphi' \supset \varphi \in \Omega_{\psi' \supset \psi}$. Thus, by completeness of PL,

$$\varphi \supset \varphi' \vdash \varphi' \supset \varphi$$

for every $\varphi \supset \varphi' \in \Omega_{\psi \supset \psi'}$ and $\varphi' \supset \varphi \in \Omega_{\psi' \supset \psi}$ and, so, by Proposition 5.7, the thesis follows. QED

Given our application scenario, the result above is not very useful since hypotheses are not allowed in the derivations. The following result does allow hypotheses but it requires that at least one of the formulas is classical.

Theorem 5.11 (2nd Metatheorem of equivalence - MTE2)
Let $\Gamma \subseteq L^{\mathsf{a}}$ be a finite set, $\psi \in L^{\mathsf{c}}(X)$ and $\varphi \in L(X)$. Then,

$$\Gamma \vdash^{\mathsf{uc}} \psi \equiv \varphi \quad \text{whenever} \quad \begin{cases} \Gamma \vdash^{\mathsf{uc}} \psi \supset \varphi \\ \Gamma \vdash^{\mathsf{uc}} \varphi \supset \psi. \end{cases}$$

Proof: Observe that by soundness of FCL (see Theorem 4.5), $\Gamma \vDash^{\mathsf{uc}} \psi \supset \varphi$ and $\Gamma \vDash^{\mathsf{uc}} \varphi \supset \psi$. Let $I = (v, \rho)$ be an interpretation such that $I \Vdash^{\mathsf{uc}} \Gamma$. We have two cases:

(a) $v \Vdash \varphi$. Then, $\mathbb{R}$ together with ρ satisfies

$$\sum_{\substack{\varphi' \equiv \varphi \ \sqsubseteq \ \psi \equiv \varphi \\ v \Vdash \varphi \equiv \varphi'}} \mathfrak{P}[\psi \equiv \varphi \rhd \varphi' \equiv \varphi] \ = \sum_{\substack{\varphi' \ \sqsubseteq \ \psi \\ v \Vdash \varphi \supset \varphi'}} \mathfrak{P}[\psi \rhd \varphi'] \ \geq \ \mu.$$

(b) $v \not\Vdash \varphi$. Then, $\mathbb{R}$ together with ρ satisfies

$$\sum_{\substack{\varphi' \equiv \varphi \ \sqsubseteq \ \psi \equiv \varphi \\ v \Vdash \varphi \equiv \varphi'}} \mathfrak{P}[\psi \equiv \varphi \rhd \varphi' \equiv \varphi] \ = \sum_{\substack{\varphi' \ \sqsubseteq \ \psi \\ v \Vdash \varphi' \supset \varphi}} \mathfrak{P}[\psi \rhd \varphi'] \ \geq \ \mu.$$

Hence, $I \Vdash^{\mathsf{uc}} \psi \equiv \varphi$, and, so, $\Gamma \vDash^{\mathsf{uc}} \psi \equiv \varphi$. Therefore, the thesis follows by Theorem 4.10. QED

It is also interesting to look at the converses of the metatheorems of equivalence.

Theorem 5.12 (Converse of the MTE1)
Let $\psi, \psi' \in L^{\mathsf{c}}(X)$. Then,

$$\vdash^{\mathsf{uc}} \psi \supset \psi' \quad \text{whenever} \quad \vdash^{\mathsf{uc}} \psi \equiv \psi'.$$

Proof: By Theorem 4.5, $\vDash^{\mathsf{uc}} \psi \equiv \psi'$. Let $I = (v, \rho)$ be an interpretation of UCL. Then,

$$\mathbb{R}\,\rho \Vdash^{\mathsf{fo}} \sum_{\substack{\varphi \equiv \varphi' \ \sqsubseteq \ \psi \equiv \psi' \\ v \Vdash \varphi \equiv \varphi'}} \mathfrak{P}[\psi \equiv \psi' \rhd \varphi \equiv \varphi'] \geq \mu.$$

Observe that

- $\{v : v \Vdash \varphi \supset \varphi'\} \supseteq \{v : v \Vdash \varphi \equiv \varphi'\}$;
- $\varphi \supset \varphi' \in \Omega_{\psi \supset \psi'}$ iff $\varphi \equiv \varphi' \in \Omega_{\psi \equiv \psi'}$;
- $\mathfrak{P}[\psi \supset \psi' \rhd \varphi \supset \varphi'] = \mathfrak{P}[\psi \equiv \psi' \rhd \varphi \equiv \varphi']$.

Hence, $\mathbb{R}$ together with ρ satisfies

$$\sum_{\substack{\varphi \supset \varphi' \ \sqsubseteq \ \psi \supset \psi' \\ v \Vdash \varphi \supset \varphi'}} \mathfrak{P}[\psi \supset \psi' \rhd \varphi \supset \varphi'] \geq \sum_{\substack{\varphi \equiv \varphi' \ \sqsubseteq \ \psi \equiv \psi' \\ v \Vdash \varphi \equiv \varphi'}} \mathfrak{P}[\psi \equiv \psi' \rhd \varphi \equiv \varphi'] \ \geq \ \mu.$$

Hence, $\mathbb{R}\,\rho \Vdash^{\mathsf{fo}} \psi \supset \psi'$. Therefore, the thesis follows, by Theorem 4.10. QED

Theorem 5.13 (Converse of the MTE2)
Let $\Gamma \subseteq L^{\mathsf{a}}$ be a finite set, $\psi \in L^{\mathsf{c}}(X)$ and $\varphi \in L(X)$. Then,

$$\begin{cases} \Gamma \vdash^{\mathsf{uc}} \psi \supset \varphi \\ \Gamma \vdash^{\mathsf{uc}} \varphi \supset \psi \end{cases} \qquad \text{whenever} \qquad \Gamma \vdash^{\mathsf{uc}} \psi \equiv \varphi.$$

Proof: By Theorem 4.5, $\Gamma \vDash^{\mathsf{uc}} \psi \equiv \varphi$. Let $I = (v, \rho)$ be an interpretation of UCL. Observe that

$$\{\varphi' \in \Omega_\psi : v \Vdash \varphi' \equiv \varphi\} \subseteq \{\varphi' \in \Omega_\psi : v \Vdash \varphi' \supset \varphi\}$$

and, so,

$$\sum_{\substack{\varphi' \supset \varphi \ \sqsubseteq \ \psi \supset \varphi \\ v \Vdash \varphi' \supset \varphi}} \mathfrak{P}[\psi \supset \varphi \rhd \varphi' \supset \varphi] \geq \sum_{\substack{\varphi \equiv \varphi' \ \sqsubseteq \ \psi \equiv \psi' \\ v \Vdash \varphi' \equiv \varphi}} \mathfrak{P}[\psi \equiv \varphi \rhd \varphi' \equiv \varphi] \ \geq \ \mu.$$

Therefore, $I \Vdash^{\mathsf{uc}} \psi \supset \varphi$ and, so, $\Gamma \vDash^{\mathsf{uc}} \psi \supset \varphi$. The thesis follows from Theorem 4.10. Similarly, the second consequence also holds. QED

Concerning the relationship between interderivability and equivalence, the following results have provisos similar to those in the MTD.

Theorem 5.14 (1st Metatheorem of interderivability - MTI1)
Let $\psi, \psi' \in L^{\mathsf{c}}(X)$. Assume ψ' fulfills the following proviso: either ψ' is distinct from ψ or $\psi' \in L(X)$. Then,

$$\vdash^{\mathsf{uc}} \psi \equiv \psi'$$

whenever there are derivations establishing

$$\begin{cases} \psi \vdash^{\mathsf{uc}} \psi' \\ \psi' \vdash^{\mathsf{uc}} \psi \end{cases}$$

where all the applications of MP over dependents of ψ are classical.

Proof: Consider two cases.
(a) ψ' is distinct from ψ. Then, by Theorem 5.6, $\vdash^{\mathsf{uc}} \psi \supset \psi'$ and $\vdash^{\mathsf{uc}} \psi' \supset \psi$ and, so, the thesis follows by Theorem 5.10.
(b) ψ' is ψ. Hence, $\psi, \psi' \in L(X)$. Thus, $\psi \vdash \psi'$ and $\psi' \vdash \psi$ by Theorem 2.3. Therefore, by the metatheorem of deduction in PL, $\vdash \psi \supset \psi'$ and $\vdash \psi' \supset \psi$ and, so, $\vdash \psi \equiv \psi'$. Therefore, $\vdash^{\mathsf{uc}} \psi \equiv \psi'$. QED

For illustrating the role of the proviso in the MTI1, recall that

$$\tilde{\neg}\, x \vdash^{\mathsf{uc}} \tilde{\neg}\, x \quad \text{and} \quad \nvdash^{\mathsf{uc}} (\tilde{\neg}\, x) \supset (\tilde{\neg}\, x)$$

as shown after the proof of the MTD. Then, $\nvdash^{\mathsf{uc}} (\tilde{\neg}\, x) \equiv (\tilde{\neg}\, x)$, by the converse of the MTE1 (Theorem 5.12).

Theorem 5.15 (2nd Metatheorem of interderivability - MTI2)
Let $\Gamma \subseteq L^{\mathsf{a}}$ be a finite set, $\psi \in L^{\mathsf{c}}(X)$ and $\varphi \in L(X)$. Then,

$$\Gamma \vdash^{\mathsf{uc}} \psi \equiv \varphi$$

whenever there are derivations establishing

$$\begin{cases} \Gamma, \psi \vdash^{\mathsf{uc}} \varphi \\ \Gamma, \varphi \vdash^{\mathsf{uc}} \psi \end{cases}$$

where all the applications of MP over dependents of ψ are classical.

Proof: Observe that

$$\Gamma \vdash^{\mathsf{uc}} \psi \supset \varphi \quad \text{and} \quad \Gamma \vdash^{\mathsf{uc}} \varphi \supset \psi$$

by Theorem 5.6. Then, the thesis follows by Theorem 5.11. QED

The converse of the MTI2 is expected (see below). But, the converse of the MTI1 raises problems. With the calculus at hand it is only possible to establish the following result. Indeed, showing that $\psi \vdash^{\mathsf{uc}} \varphi$ holds whenever $\varphi \equiv \psi$ would require a stronger calculus that takes advantage of hypotheses in $L^{\mathsf{c}}(X)$.

Theorem 5.16 (Converse of the MTI1)
Let $\varphi \in L(X)$ and $\psi \in L^{\mathsf{c}}(X)$. Then,

$$\varphi \vdash^{\mathsf{uc}} \psi \quad \text{whenever} \quad \vdash^{\mathsf{uc}} \varphi \equiv \psi.$$

Proof: By Theorem 4.5, $\vDash^{\mathsf{uc}} \varphi \equiv \psi$. Then, by Proposition 5.9, $\varphi \equiv \varphi'$ is a tautology for each $\varphi' \in \Omega_\psi$, and so, by completeness of PL, $\vdash \varphi \equiv \varphi'$ for each $\varphi' \in \Omega_\psi$. Therefore, $\varphi \vdash \varphi'$ for each $\varphi' \in \Omega_\psi$. Moreover, $\varphi \vdash^{\mathsf{uc}} \varphi'$ for each $\varphi' \in \Omega_\psi$, by Theorem 2.3. The thesis follows by LFT, since $\varphi \vdash^{\mathsf{uc}} \bigwedge \Omega_\psi$, $\varphi \vdash^{\mathsf{uc}} \Omega_\psi \sqsubseteq_1 \psi$ and $\varphi \vdash^{\mathsf{uc}} \mu \leq 1$. QED

Theorem 5.17 (Converse of the MTI2)
Let $\Gamma \subseteq L^{\mathsf{a}}$ be a finite set, $\psi \in L^{\mathsf{c}}(X)$ and $\varphi \in L(X)$. Then,

$$\begin{cases} \Gamma, \psi \vdash^{\mathsf{uc}} \varphi \\ \Gamma, \varphi \vdash^{\mathsf{uc}} \psi \end{cases} \quad \text{whenever} \quad \Gamma \vdash^{\mathsf{uc}} \psi \equiv \varphi.$$

Proof: By Theorem 5.13, $\Gamma \vdash^{\mathsf{uc}} \psi \supset \varphi$ and $\Gamma \vdash^{\mathsf{uc}} \varphi \supset \psi$. Hence, using MP, $\Gamma, \psi \vdash^{\mathsf{uc}} \varphi$ and, by MP$^{\bullet}$, $\Gamma, \varphi \vdash^{\mathsf{uc}^{+}} \psi$. Therefore, $\Gamma, \varphi \vdash^{\mathsf{uc}} \psi$, using Proposition 3.4. QED

5.3 Metatheorem of substitution for equivalent

Interchanging equivalent formulas in UCL is a risky endeavor in the presence of unreliable connectives. As an illustration, consider what happens when we substitute a propositional variable by itself. Notwithstanding that x is equivalent to x in UCL, that is, $x \equiv x$ is a theorem of UCL, the reader may be surprised by the fact that $(\tilde{\neg}\, x) \equiv (\tilde{\neg}\, x)$ is not a theorem of UCL. Indeed, this fact follows, by the converse of the MTE1, from $\not\vdash^{\mathsf{uc}} (\tilde{\neg}\, x) \supset (\tilde{\neg}\, x)$ which we established after the proof of the MTI1.

Given the extreme simplicity of this counterexample, one might think that the principle of substitution for equivalent will not go very far beyond the classical fragment of UCL. However, the next theorem shows that, even in the presence of unreliable connectives, replacing a single instance of a subformula by an equivalent one may still result in an equivalent formula.

Theorem 5.18 (Metatheorem of substitution for equivalent)
Let $\eta, \eta', \psi, \psi' \in L^{\mathsf{c}}(X)$ be such that

- $\vdash^{\mathsf{uc}} \eta \equiv \eta$;
- ψ has at most one occurrence of η;
- there are no occurrences of unreliable connectives in ψ outside η;
- either ψ' is ψ or ψ' is obtained from ψ by replacing η by η'.

Then,

$$\vdash^{\mathsf{uc}} \psi \equiv \psi' \quad \text{whenever} \quad \vdash^{\mathsf{uc}} \eta \equiv \eta'.$$

Proof: Assume that $\vdash^{\mathsf{uc}} \eta \equiv \eta'$. Then, $\vDash^{\mathsf{uc}} \eta \equiv \eta'$ by soundness, Theorem 4.5. We now show that $\vDash^{\mathsf{uc}} \psi \equiv \psi'$ by structural induction on ψ:

(Basis) Directly from the hypothesis.

(Step) ψ is not a propositional variable. If η does not occur in ψ then ψ' is ψ. Moreover ψ has no occurrences of unreliable connectives. Hence $\vDash \psi \equiv \psi'$ and therefore $\vDash^{\mathsf{uc}} \psi \equiv \psi'$ by Theorem 2.2. So suppose that η occurs in ψ. Consider two cases:

(a) ψ is η. This case follows directly by hypothesis.

(b) ψ is not η. Let ψ be of the form $c(\psi_1, \ldots, \psi_n)$, and ψ' of the form $c(\psi'_1, \ldots, \psi'_n)$ where $n \geq 1$. Then, for each $i = 1, \ldots, n$, each ψ_i has at most one occurrence of η, there are no occurrences of unreliable connectives in ψ_i outside η and either ψ'_i is ψ_i or is obtained from ψ_i by replacing η by η'. So by induction hypothesis

$$\vDash^{\mathsf{uc}} \psi_i \equiv \psi'_i$$

for $i = 1, \ldots, n$. Assume without loss of generality that η occurs in ψ_1. We now show that $I \Vdash^{\mathsf{uc}} \psi \equiv \psi'$ for every interpretation I. Let $I = (v, \rho)$ be an interpretation. Then, $I \Vdash^{\mathsf{uc}} \psi_i \equiv \psi'_i$ for every $i = 1, \ldots, n$ and so $v \Vdash \psi_i \equiv \psi'_i$ for every $i = 2, \ldots, n$ by Proposition 2.1. Observe that, for $i = 2, \ldots, n$,

$$\mathfrak{P}[\psi_i \equiv \psi'_i \triangleright \varphi_i \equiv \varphi'_i] = \mathfrak{P}[\psi_i \equiv \psi'_i \triangleright \psi_i \equiv \psi'_i] = 1$$

by definition since $\psi_i \equiv \psi'_i \in L(X)$. On the other hand,

$$\mathbb{R}\,\rho \Vdash^{\mathsf{fo}} \sum_{\substack{\varphi_1 \equiv \varphi'_1 \;\sqsubseteq\; \psi_1 \equiv \psi'_1 \\ v \Vdash \varphi_1 \equiv \varphi'_1}} \mathfrak{P}[\psi_1 \equiv \psi'_1 \triangleright \varphi_1 \equiv \varphi'_1] \geq \mu$$

since $I \Vdash^{\mathsf{uc}} \psi_1 \equiv \psi'_1$. Observe that, in ORCF,

$$\begin{aligned}
&\mathfrak{P}[c(\psi_1, \ldots, \psi_n) \equiv c(\psi'_1, \ldots, \psi'_n) \triangleright c(\varphi_1, \ldots, \varphi_n) \equiv c(\varphi'_1, \ldots, \varphi'_n)] = \\
&\quad \mathfrak{P}[c(\psi_1, \ldots, \psi_n) \triangleright c(\varphi_1, \ldots, \varphi_n)] \times \mathfrak{P}[c(\psi'_1, \ldots, \psi'_n) \triangleright c(\varphi'_1, \ldots, \varphi'_n)] = \\
&\quad \mathfrak{P}[\psi_1 \triangleright \varphi_1] \times \ldots \times \mathfrak{P}[\psi_n \triangleright \varphi_n] \times \mathfrak{P}[\psi'_1 \triangleright \varphi'_1] \times \ldots \times \mathfrak{P}[\psi'_n \triangleright \varphi'_n] = \\
&\quad \mathfrak{P}[\psi_1 \triangleright \varphi_1] \times \mathfrak{P}[\psi'_1 \triangleright \varphi'_1] \times \ldots \times \mathfrak{P}[\psi_n \triangleright \varphi_n] \times \mathfrak{P}[\psi'_n \triangleright \varphi'_n] = \\
&\quad \mathfrak{P}[\psi_1 \equiv \psi'_1 \triangleright \varphi_1 \equiv \varphi'_1] \times \ldots \times \mathfrak{P}[\psi_n \equiv \psi'_n \triangleright \varphi_n \equiv \varphi'_n]
\end{aligned}$$

for every $c(\varphi_1, \ldots, \varphi_n) \equiv c(\varphi'_1, \ldots, \varphi'_n) \in \Omega_{c(\psi_1,\ldots,\psi_n) \equiv c(\psi'_1,\ldots,\psi'_n)}$. On the other hand

$$c(\varphi_1, \ldots, \varphi_n) \equiv c(\varphi'_1, \ldots, \varphi'_n) \in \Omega_{c(\psi_1,\ldots,\psi_n) \equiv c(\psi'_1,\ldots,\psi'_n)}$$

iff

$$\varphi_i \equiv \varphi'_i \in \Omega_{\psi_i \equiv \psi'_i} \quad \text{for every } i = 1, \ldots, n.$$

So, in ORCF,

$$\begin{aligned}
&\sum_{\substack{\varphi \equiv \varphi' \ \sqsubseteq \ \psi \equiv \psi' \\ v \Vdash \varphi \equiv \varphi'}} \mathfrak{P}[\psi \equiv \psi' \rhd \varphi \equiv \varphi'] \ = \\
&\qquad \sum_{\substack{\varphi \equiv \varphi' \ \sqsubseteq \ \psi \equiv \psi' \\ v \Vdash \varphi \equiv \varphi'}} \Pi_{i=1}^{n} \, \mathfrak{P}[\psi_i \equiv \psi_i' \rhd \varphi_i \equiv \varphi_i'] \ = \\
&\qquad \sum_{\substack{\varphi \equiv \varphi' \ \sqsubseteq \ \psi \equiv \psi' \\ v \Vdash \varphi \equiv \varphi'}} \mathfrak{P}[\psi_1 \equiv \psi_1' \rhd \varphi_1 \equiv \varphi_1'] \ \geq \\
&\qquad \sum_{\substack{\varphi_i \equiv \varphi_i' \ \sqsubseteq \ \psi_i \equiv \psi_i' \\ v \Vdash \varphi_i \equiv \varphi_i' \\ i = 1, \ldots, n}} \mathfrak{P}[\psi_1 \equiv \psi_1' \rhd \varphi_1 \equiv \varphi_1'] \ = \\
&\qquad \sum_{\substack{\varphi_i \equiv \varphi_i' \ \sqsubseteq \ \psi_i \equiv \psi_i' \\ i = 1, \ldots, n \\ v \Vdash \varphi_1 \equiv \varphi_1'}} \mathfrak{P}[\psi_1 \equiv \psi_1' \rhd \varphi_1 \equiv \varphi_1'] \ = \\
&\qquad \sum_{\substack{\varphi_1 \equiv \varphi_1' \ \sqsubseteq \ \psi_1 \equiv \psi_1' \\ v \Vdash \varphi_1 \equiv \varphi_1'}} \mathfrak{P}[\psi_1 \equiv \psi_1' \rhd \varphi_1 \equiv \varphi_1'].
\end{aligned}$$

Then,

$$\mathbb{R}\, \rho \Vdash^{\mathsf{fo}} \sum_{\substack{\varphi \equiv \varphi' \ \sqsubseteq \ \psi \equiv \psi' \\ v \Vdash \varphi \equiv \varphi'}} \mathfrak{P}[\psi \equiv \psi' \rhd \varphi \equiv \varphi'] \geq \mu.$$

Hence, the thesis follows. QED

As an interesting application of the metatheorem of substitution for equivalent in UCL, let

- η be $\mathtt{tt}$;
- η' be $(\tilde{\neg}\, x_1) \supset \mathtt{tt}$;
- ψ be $\mathtt{tt} \supset x_1$;
- ψ' be $((\tilde{\neg}\, x_1) \supset \mathtt{tt}) \supset x_1$;

then,

- $\vdash^{\mathsf{uc}}\ \mathsf{tt} \equiv ((\tilde{\neg}\, x_1) \supset \mathsf{tt})$, see Figure 8;
- $\vdash^{\mathsf{uc}}\ \mathsf{tt} \equiv \mathsf{tt}$;
- ψ has at most one occurrence of η;
- there are no occurrences of unreliable connectives in ψ outside η;
- ψ' is obtained from ψ by replacing η by η'.

Thus,

$$\vdash^{\mathsf{uc}}\ (\mathsf{tt} \supset x_1) \equiv (((\tilde{\neg}\, x_1) \supset \mathsf{tt}) \supset x_1)$$

by the metatheorem of substitution for equivalent.

1	$\varphi_1 : \mathsf{tt} \equiv ((\neg\, x_1) \supset \mathsf{tt})$	TAUT
2	$\varphi_2 : \mathsf{tt} \equiv ((\bar{\neg}\, x_1) \supset \mathsf{tt})$	TAUT
3	$\varphi_1 \wedge \varphi_2$	TAUT
4	$\varphi_1\ \sqsubseteq_{\nu}\ \mathsf{tt} \equiv ((\tilde{\neg}\, x_1) \supset \mathsf{tt})$	SO
5	$\varphi_2\ \sqsubseteq_{1-\nu}\ \mathsf{tt} \equiv ((\tilde{\neg}\, x_1) \supset \mathsf{tt})$	SO
6	$\varphi_1, \varphi_2\ \sqsubseteq_{\nu+(1-\nu)}\ \mathsf{tt} \equiv ((\tilde{\neg}\, x_1) \supset \mathsf{tt})$	$\mathrm{AO} : 4, 5$
7	$\varphi_1, \varphi_2\ \sqsubseteq_{1}\ \mathsf{tt} \equiv ((\tilde{\neg}\, x_1) \supset \mathsf{tt})$	$\mathrm{WO} : 6$
8	$\mu \leq 1$	WA
9	$\mathsf{tt} \equiv ((\tilde{\neg}\, x_1) \supset \mathsf{tt})$	$\mathrm{LFT} : 3, 7, 8$

Figure 8: $\vdash^{\mathsf{uc}}\ \mathsf{tt} \equiv ((\tilde{\neg}\, x_1) \supset \mathsf{tt})$.

5.4 Closure for substitutions

Recall that closure for substitutions does not hold in general. As we saw at the end of Subsection 2.3, an unreliable instance of a tautology can be not valid. However, substitutions not involving unreliable connectives do not raise surprises. Indeed, we prove below that theoremhood of c-formulas is preserved by substitution provided that propositional variables are replaced only by PL formulas. To this end we need some terminology and notation.

In UCL, as expected, a *substitution* is a map $\sigma : X \to L^{\mathsf{c}}(X)$. Given $\psi \in L^{\mathsf{c}}(X)$ and a substitution σ, we denote by $\sigma(\psi)$ the formula in $L^{\mathsf{c}}(X)$ obtained from ψ by replacing each propositional variable x by $\sigma(x)$. By a *reliable substitution* we mean a substitution such that $\sigma(X) \subseteq L(X)$. It is also convenient to define two auxiliary maps. Let $\psi \in L^{\mathsf{c}}(X)$. The maps

$$\mathrm{nc}_{\psi}, \overline{\mathrm{nc}}_{\psi} : \Omega_{\psi} \to \mathbb{N}$$

(that count the number of positive and negative outcomes, respectively, of unreliable connectives in obtaining the argument formula as an outcome of ψ) are inductively defined as follows:

- $\mathrm{nc}_\psi(\psi) = 0$ and $\overline{\mathrm{nc}}_\psi(\psi) = 0$ whenever $\psi \in L(X)$;
- $\mathrm{nc}_{c(\psi_1,\ldots,\psi_n)}(c(\varphi_1,\ldots,\varphi_n)) = \sum_{i=1}^n \mathrm{nc}_{\psi_i}(\varphi_i)$ whenever $c \in \Sigma_n$;
- $\overline{\mathrm{nc}}_{c(\psi_1,\ldots,\psi_n)}(c(\varphi_1,\ldots,\varphi_n)) = \sum_{i=1}^n \overline{\mathrm{nc}}_{\psi_i}(\varphi_i)$ whenever $c \in \Sigma_n$;
- $\mathrm{nc}_{\widetilde{c}(\psi_1,\ldots,\psi_n)}(c(\varphi_1,\ldots,\varphi_n)) = 1 + \sum_{i=1}^n \mathrm{nc}_{\psi_i}(\varphi_i)$ whenever $c \in \Sigma_n$;
- $\overline{\mathrm{nc}}_{\widetilde{c}(\psi_1,\ldots,\psi_n)}(c(\varphi_1,\ldots,\varphi_n)) = \sum_{i=1}^n \overline{\mathrm{nc}}_{\psi_i}(\varphi_i)$ whenever $c \in \Sigma_n$;
- $\mathrm{nc}_{\widetilde{c}(\psi_1,\ldots,\psi_n)}(\overline{c}(\varphi_1,\ldots,\varphi_n)) = \sum_{i=1}^n \mathrm{nc}_{\psi_i}(\varphi_i)$ whenever $c \in \Sigma_n$;
- $\overline{\mathrm{nc}}_{\widetilde{c}(\psi_1,\ldots,\psi_n)}(\overline{c}(\varphi_1,\ldots,\varphi_n)) = 1 + \sum_{i=1}^n \overline{\mathrm{nc}}_{\psi_i}(\varphi_i)$ whenever $c \in \Sigma_n$.

Observe that

$$\forall \nu \left(\left(\frac{1}{2} < \nu \leq 1 \right) \supset \left(\mathfrak{P}[\psi \rhd \varphi] = \nu^{\mathrm{nc}_\psi(\varphi)} \times (1-\nu)^{\overline{\mathrm{nc}}_\psi(\varphi)} \right) \right) \in \mathsf{ORCF}.$$

Proposition 5.19 Let $\psi \in L^{\mathsf{c}}(X)$, $\Phi \subseteq \Omega_\psi$ and σ be a reliable substitution. Then,

$$\mathrm{nc}_\psi(\Phi) = \mathrm{nc}_{\sigma(\psi)}(\sigma(\Phi))$$

and similarly for $\overline{\mathrm{nc}}$.

Proof: We prove by induction that $\mathrm{nc}_\psi(\varphi) = \mathrm{nc}_{\sigma(\psi)}(\sigma(\varphi))$.
(Basis) $\psi \in L(X)$. Then, $\sigma(\psi) \in L(X)$ and so $\mathrm{nc}_\psi(\varphi) = 0 = \mathrm{nc}_{\sigma(\psi)}(\sigma(\varphi))$.
(Step) We just consider the case where ψ and φ are $\widetilde{c}(\psi_1,\ldots,\psi_n)$ and $c(\varphi_1,\ldots,\varphi_n)$, respectively. Then,

$$\begin{array}{rcl} \mathrm{nc}_\psi(\varphi) & = & 1 + \sum_{i=1}^n \mathrm{nc}_{\psi_i}(\varphi_i) \\ & = & 1 + \sum_{i=1}^n \mathrm{nc}_{\sigma(\psi_i)}(\sigma(\varphi_i)) \\ & = & \mathrm{nc}_{\sigma(\psi)}(\sigma(\varphi)). \end{array}$$

The thesis follows easily for Φ. QED

Proposition 5.20 Let $\psi \in L^{\mathsf{c}}(X)$, $\Phi \subseteq \Omega_\psi$ and σ be a reliable substitution. Then, in ORCF,

$$\mathfrak{P}[\psi \rhd \Phi] = \mathfrak{P}[\sigma(\psi) \rhd \sigma(\Phi)].$$

Proof: Observe that, in ORCF,

$$\begin{array}{rcl}\mathfrak{P}[\psi \triangleright \Phi] & = & \sum_{\varphi\in\Phi} \mathfrak{P}[\psi \triangleright \varphi] \\ & = & \sum_{\varphi\in\Phi} \nu^{\mathrm{nc}_\psi(\varphi)} \times (1-\nu)^{\overline{\mathrm{nc}}_\psi(\varphi)} \\ & = & \sum_{\varphi\in\Phi} \nu^{\mathrm{nc}_{\sigma(\psi)}(\sigma(\varphi))} \times (1-\nu)^{\overline{\mathrm{nc}}_{\sigma(\psi)}(\sigma(\varphi))} \\ & = & \sum_{\varphi\in\Phi} \mathfrak{P}[\sigma(\psi) \triangleright \sigma(\varphi)] \\ & = & \mathfrak{P}[\sigma(\psi) \triangleright \sigma(\Phi)].\end{array}$$

QED

Theorem 5.21 (Closure for reliable substitutions)
Let $\psi \in L^{\mathsf{c}}(X)$ and σ be a reliable substitution. Then,

$$\vdash^{\mathsf{uc}} \sigma(\psi) \quad \text{whenever} \quad \vdash^{\mathsf{uc}} \psi.$$

Proof: The proof is by induction on a derivation of ψ.

(Basis) ψ is obtained by TAUT. Then, $\vdash^{\mathsf{uc}} \sigma(\psi)$ using closure for substitution for PL and since $\vdash \subseteq \vdash^{\mathsf{uc}}$, by Theorem 2.2.

(Step) We have two cases.

(1) ψ is obtained by MP from ψ_1 and $\psi_1 \supset \psi$ where $\psi \in L(X)$ and $\psi_1 \in L^{\mathsf{c}}(X)$. Thus, by the induction hypothesis, $\vdash^{\mathsf{uc}} \sigma(\psi_1)$ and $\vdash^{\mathsf{uc}} \sigma(\psi_1) \supset \sigma(\psi)$ and so by MP $\vdash^{\mathsf{uc}} \sigma(\psi)$, since $\sigma(\psi) \in L(X)$.

(2) ψ is obtained by LFT from

$$\vdash^{\mathsf{uc}} \bigvee_{i=1}^{n} \bigwedge \Phi_i$$

$$\left\{\begin{array}{ll}(*) & \vdash^{\mathsf{uc}} \Phi_i \sqsubseteq_{P_i} \psi \\ & \vdash^{\mathsf{uc}} \mu \leq P_i\end{array}\right. \text{ for } i = 1, \ldots, n.$$

Hence, by closure for substitution for PL,

$$\vdash^{\mathsf{uc}} \bigvee_{i=1}^{n} \bigwedge \sigma(\Phi_i).$$

By $(*)$ and Proposition 5.1,

$$\forall \nu \left(\left(\frac{1}{2} < \nu \leq 1 \right) \supset P_i \leq \mathfrak{P}[\psi \triangleright \Phi_i] \right) \in \mathsf{ORCF}.$$

Hence, by Proposition 5.20,

$$\forall \nu \left(\left(\frac{1}{2} < \nu \leq 1 \right) \supset P_i \leq \mathfrak{P}[\sigma(\psi) \triangleright \sigma(\Phi_i)] \right) \in \mathsf{ORCF}$$

and so, again by Proposition 5.1,

$$\vdash^{\mathsf{uc}} \sigma(\Phi_i) \sqsubseteq_{P_i} \sigma(\psi).$$

By rule LFT the thesis follows. QED

Observe that even substitutions involving unreliable gates can preserve validity. The following result provides a useful sufficient condition.

Theorem 5.22 (Closure for almost reliable substitutions)
Let $\varphi \in L(X)$ and σ be a substitution such that $\sigma(x) \in L(X)$ for each x occurring more than once in φ. Then,

$$\vdash^{\mathsf{uc}} \sigma(\varphi) \quad \text{whenever} \quad \varphi \in \text{TAUT}.$$

Proof: We start by showing that $\varphi' \in \Omega_{\sigma(\varphi)}$ is a tautology. Let v be a valuation. Let σ' be a substitution such that:

- $\sigma'(x) = \sigma(x)$ whenever x occurs more than once in φ;
- $\sigma'(x) \in L(X)$ whenever x does not occur in φ;
- otherwise $\sigma'(x)$ is the outcome of $\sigma(x)$ that occurs in φ'.

Thus, φ' is $\sigma'(\varphi)$ and so φ' is a PL instance of a tautology and, so, is a tautology. Let $I = (v, \rho)$ be an interpretation of UCL. Then, $\mathbb{R}$ together with ρ satisfies

$$\sum_{\substack{\varphi' \in \Omega_{\sigma(\varphi)} \\ v \Vdash \varphi'}} \mathfrak{P}[\sigma(\varphi) \triangleright \varphi'] = \sum_{\varphi' \in \Omega_{\sigma(\varphi)}} \mathfrak{P}[\sigma(\varphi) \triangleright \varphi'] = 1.$$

The thesis follows by Theorem 4.10. QED

6 Application scenarios

Recall that, within the standard application scenario described in Section 1 using unreliable gates with misfiring probability smaller than $1 - \nu$, the main engineering problem is the design of a circuit ψ that behaves like the envisaged ideal circuit φ with probability not smaller than μ, possibly constraining the value of μ vis-à-vis the value of ν.

The proposed unreliable-circuit logic (UCL) should help at least in the task of checking that ψ and φ are indeed equivalent with probability not smaller than μ assuming that $\gamma_1, \dots, \gamma_n \in L^{\mathsf{a}}$ hold. To this end, it is enough to establish that

$$\gamma_1, \dots, \gamma_n \vdash^{\mathsf{uc}} \psi \equiv \varphi.$$

Capitalizing on the MTE2 (Theorem 5.11), it is equivalent to establish

$$\begin{cases} \gamma_1, \dots, \gamma_n \vdash^{\mathsf{uc}} \psi \supset \varphi \\ \gamma_1, \dots, \gamma_n \vdash^{\mathsf{uc}} \varphi \supset \psi \end{cases}$$

which can be easier.

In alternative, capitalizing on the MTI2 (Theorem 5.15), it is enough to establish

$$\begin{cases} \gamma_1, \dots, \gamma_n, \psi \vdash^{\mathsf{uc}} \varphi \\ \gamma_1, \dots, \gamma_n, \varphi \vdash^{\mathsf{uc}} \psi \end{cases}$$

while avoiding non-classical applications of MP to dependents of ψ and φ, respectively.

In addition, UCL can also help the designer in getting a better grasp of the logical properties of unreliable circuits. Namely, the provisos in the metatheorems established in the previous section should always be kept in mind in order to avoid the common pitfalls of extrapolating from classical reasoning to reasoning about unreliable circuits.

It is also worth mentioning that we can formalize von Neumann's theorem in UCL. Assuming that the majority gates do not misfire with probability not smaller than ν_{M} and that all the other unreliable gates do not misfire with probability not smaller than $\nu > \frac{1}{2}$, von Neumann proved in [20] that for any circuit φ built only with perfect gates and for every $d > 0$ there is a circuit ψ built only with unreliable gates that behaves as the ideal circuit φ with probability not smaller that $\nu_{\mathsf{M}} - d$.

As an illustration of this result, recall the derivation of

$$\mu \leq \nu^3 + 3\nu^2(1 - \nu) \vdash^{\mathsf{uc}} (\neg\, x_1) \equiv \mathsf{M}_3(\tilde{\neg}\, x_1, \tilde{\neg}\, x_1, \tilde{\neg}\, x_1)$$

presented in Figure 7. It amounts to stating that perfect negation can be implemented only with unreliable gates and the ideal majority gate, ensuring that the probability of output error is smaller than $1 - d$ with

$$d = \nu^3 + 3\nu^2(1-\nu).$$

Note that in this example we use the perfect M_3 or, equivalently, we assume that $\nu_{\mathsf{M}} = 1$.

In general, falling into the standard application scenario, that is, assuming that $\nu_{\mathsf{M}} = \nu$, by implementing negation with

$$\widetilde{\mathsf{M}}_{3+2k}(\tilde{\neg}\, x_1, \ldots, \tilde{\neg}\, x_1)$$

we ensure that the output is correct with probability not smaller than

$$\nu \sum_{i=0}^{k+1} \binom{3+2k}{3+2k-i} \nu^{3+2k-i}(1-\nu)^i$$

$$+$$

$$(1-\nu) \sum_{i=k+2}^{3+2k} \binom{3+2k}{3+2k-i} \nu^{3+2k-i}(1-\nu)^i.$$

The first term computes the probability of the majority gate and most of the negation gates correctly firing, while the second term computes the probability of the majority gate and most of the negation gates misfiring. Clearly, the output is the correct one in both cases.

Observe that if the probability ν is greater than $\frac{1}{2}$ then as k goes to infinity the value the whole expression above goes to ν. Hence, for any $d > 0$ there is k such that the value of the expression above is not smaller than $\nu - d$, in accordance with von Neumann's result.

Within the standard application scenario, von Neumann's result can be stated as follows.

Theorem 6.1 For any $\varphi \in L(\{x_1, \ldots, x_n\})$ and real number $d > 0$, there is $\psi \in L^{\mathsf{c}}(\{x_1, \ldots, x_n\})$ such that

$$\mu \leq \nu - d \vdash^{\mathsf{uc}} \varphi \equiv \psi.$$

The proof is straightforward. Take von Neumann's proof to obtain

$$\mu \leq \nu - d \vDash^{\mathsf{uc}} \varphi \equiv \psi$$

and invoke the completeness of UCL.

The reader may wonder how robust is the proposed logic with respect to possible changes in the assumptions of the standard application scenario:

- (a1) every gate has the same probability $1-\nu < \frac{1}{2}$ of misfiring which is independent of its inputs;
- (a2) gates misfire independently of each other;
- (a3) there is no sub-circuit reuse or, equivalently, every gate has single fan-out;
- (a4) the circuit inputs are deterministic.

Of these assumptions, only (a1) can be relaxed in a straightforward way. For instance, the probability ν of a gate not misfiring may depend on the type of gate at hand. Recall that von Neumann assumed that the misfire probability of the majority gates could be different from the other unreliable gates. It would be quite easy to adapt UCL to the scenario where each unreliable gate $\widetilde{c}$ misfires with probability $1-\nu_{\widetilde{c}}$.

A little bit more complicated would be to adapt UCL to the scenario where we know that

$$\check{\nu} \leq \mathsf{Prob}[\widetilde{c} \triangleright c] \leq \hat{\nu}$$

instead of knowing that

$$\mathsf{Prob}[\widetilde{c} \triangleright c] = \nu,$$

but still assume that (a2), (a3) and (a4) hold.

In this relaxed scenario, given a c-formula ψ and $\varphi \in \Omega_\psi$, we write

$$\check{\mathfrak{P}}[\psi \triangleright \varphi]$$

for a minorant of the exact value of the probability of the outcome φ of ψ defined as follows:

- $\check{\mathfrak{P}}[\varphi \triangleright \varphi]$ is 1 for each $\varphi \in L(X)$;
- $\check{\mathfrak{P}}[c(\psi_1, \ldots, \psi_n) \triangleright c(\varphi_1, \ldots, \varphi_n)]$ is $\displaystyle\prod_{i=1}^{n} \check{\mathfrak{P}}[\psi_i \triangleright \varphi_i]$;
- $\check{\mathfrak{P}}[\widetilde{c}(\psi_1, \ldots, \psi_n) \triangleright c(\varphi_1, \ldots, \varphi_n)]$ is $\displaystyle\check{\nu} \prod_{i=1}^{n} \check{\mathfrak{P}}[\psi_i \triangleright \varphi_i]$;

- $\check{\mathfrak{P}}[\widetilde{c}(\psi_1, \ldots, \psi_n) \triangleright \overline{c}(\varphi_1, \ldots, \varphi_n)]$ is $(1-\hat{\nu}) \displaystyle\prod_{i=1}^{n} \check{\mathfrak{P}}[\psi_i \triangleright \varphi_i]$.

Similarly, we write

$$\hat{\mathfrak{P}}[\psi \triangleright \varphi]$$

for a majorant of the exact value of the probability of the outcome φ of ψ defined as follows:

- $\hat{\mathfrak{P}}[\varphi \triangleright \varphi]$ is 1 for each $\varphi \in L(X)$;
- $\hat{\mathfrak{P}}[c(\psi_1, \ldots, \psi_n) \triangleright c(\varphi_1, \ldots, \varphi_n)]$ is $\displaystyle\prod_{i=1}^{n} \hat{\mathfrak{P}}[\psi_i \triangleright \varphi_i]$ for each $n \geq 1$, $c \in \Sigma_n$ and $\varphi_i \sqsubseteq \psi_i$ for $i = 1, \ldots, n$;
- $\hat{\mathfrak{P}}[\widetilde{c}(\psi_1, \ldots, \psi_n) \triangleright c(\varphi_1, \ldots, \varphi_n)]$ is $\hat{\nu} \displaystyle\prod_{i=1}^{n} \hat{\mathfrak{P}}[\psi_i \triangleright \varphi_i]$ for each $\widetilde{c} \in \widetilde{\Sigma}_n$ and $\varphi_i \sqsubseteq \psi_i$ for $i = 1, \ldots, n$;
- $\hat{\mathfrak{P}}[\widetilde{c}(\psi_1, \ldots, \psi_n) \triangleright \overline{c}(\varphi_1, \ldots, \varphi_n)]$ is $(1-\check{\nu}) \displaystyle\prod_{i=1}^{n} \hat{\mathfrak{P}}[\psi_i \triangleright \varphi_i]$ for each $\widetilde{c} \in \widetilde{\Sigma}_n$ and $\varphi_i \sqsubseteq \psi_i$ for $i = 1, \ldots, n$.

Moreover, given a c-formula ψ and one of its outcomes φ in Ω_ψ, the exact probability of outcome φ of ψ, written

$$\mathsf{Prob}[\psi \triangleright \varphi]$$

is inductively defined as follows:

- $\mathsf{Prob}[\varphi \triangleright \varphi]$ is 1 for each $\varphi \in L(X)$;
- $\mathsf{Prob}[c(\psi_1, \ldots, \psi_n) \triangleright c(\varphi_1, \ldots, \varphi_n)]$ is $\displaystyle\prod_{i=1}^{n} \mathsf{Prob}[\psi_i \triangleright \varphi_i]$;
- $\mathsf{Prob}[\widetilde{c}(\psi_1, \ldots, \psi_n) \triangleright c(\varphi_1, \ldots, \varphi_n)]$ is $\mathsf{Prob}[\widetilde{c} \triangleright c] \displaystyle\prod_{i=1}^{n} \mathsf{Prob}[\psi_i \triangleright \varphi_i]$;
- $\mathsf{Prob}[\widetilde{c}(\psi_1, \ldots, \psi_n) \triangleright \overline{c}(\varphi_1, \ldots, \varphi_n)]$ is $(1-\mathsf{Prob}[\widetilde{c} \triangleright c]) \displaystyle\prod_{i=1}^{n} \mathsf{Prob}[\psi_i \triangleright \varphi_i]$.

Then as expected, given $\psi \in L^{\mathsf{c}}(X)$ and $\varphi \in \Omega_\psi$, we have:

$$\forall \check{\nu} \forall \hat{\nu} \left(\left(\frac{1}{2} < \nu \leq \hat{\nu} \leq 1 \right) \supset \left(\check{\mathfrak{P}}[\psi \rhd \varphi] \leq \mathsf{Prob}[\psi \rhd \varphi] \leq \hat{\mathfrak{P}}[\psi \rhd \varphi] \right) \right) \in \mathsf{ORCF}.$$

For this relaxed scenario we need to enrich the o-formulas. Given an interpretation $I = (v, \rho)$, terms P and Q, a c-formula ψ and $\Phi \subseteq \Omega_\psi$, we define

$$I \Vdash^{\mathsf{uc}} \Phi \sqsubseteq_P^Q \psi$$

as

$$\mathbb{R}\, \rho \Vdash^{\mathsf{fo}} P \leq \sum_{\varphi \in \Phi} \mathsf{Prob}[\psi \rhd \varphi] \leq Q.$$

Observe that

$$I \Vdash^{\mathsf{uc}} \psi$$

still means

$$\mathbb{R}\, \rho \Vdash^{\mathsf{fo}} \sum_{\substack{\varphi \,\sqsubseteq\, \psi \\ v \Vdash \varphi}} \mathsf{Prob}[\psi \rhd \varphi] \geq \mu.$$

Concerning the calculus, only the axioms NO and SO should be adapted as follows:

$$\text{NO} \quad \frac{}{\emptyset \sqsubseteq_0^0 \psi};$$

$$\text{SO} \quad \frac{}{\Omega_\psi \sqsubseteq_{\check{\mathfrak{P}}[\psi \rhd \varphi]}^{\hat{\mathfrak{P}}[\psi \rhd \varphi]} \psi}.$$

The resulting logic $\mathsf{UCL}_{\check{\nu}}^{\hat{\nu}}$ is expected to inherit without surprises the metatheorems of UCL.

Of the remaining assumptions, only (a4) seems to be immediately tractable. A first approach to probabilistic inputs would be to receive the inputs through unreliable double negations. However, the more interesting scenario of possibly correlated inputs would require a more sophisticated method, e.g. by bringing to bear the techniques of exogenous enrichment [12] to UCL. Dropping the independence assumption (a2) or the single-fan-out hypothesis (a3) would require major changes to UCL.

7 Outlook

Starting from von Neumann's assumptions concerning circuits built with unreliable gates that are prone to fortuitous, misfiring errors, we were able to set up a logic (UCL) appropriate for reasoning about such circuits, as a conservative extension of classical propositional logic (PL). For the axiomatization we capitalized on the decidability of the first-order theory of ordered real closed fields. Useful completeness results were established in due course.

The pitfalls of extrapolating classical reasoning to the realm of unreliable circuits were extensively illustrated. Several metatheorems were established with additional provisos that once again show the striking differences between PL and UCL. On the other hand, we decided to leave to a forthcoming paper the decidability and other algorithmic issues of UCL.

Possible changes to von Neumann's scenario were also considered. We sketched with some detail only the adaptation of UCL that would be required by not knowing precisely the gate-misfiring probabilities.

The next step should concentrate on allowing probabilistic inputs, by applying the techniques of exogenous enrichment to UCL. Afterwards, it will be possible to address the problem of reasoning about quantum circuits with unreliable gates.

The problems raised by circuit reuse and by dropping the gate independence hypothesis also seem interesting lines of work, albeit requiring major changes to the approach taken in this paper.

In another direction, it seems worthwhile to carry out the logical study of persistency and recurrence of errors in gates. To this end, one will need a temporal probabilistic logic, either adopting a shallow Markov model of the errors in the gates (following [8]) or a deeper model of their errors taking into account the precise functional nature of each gate collapse (following the proposal of one of the referees).

Acknowledgments

The authors are grateful to both referees who suggested several improvements to the paper, namely concerning the justification of the adopted model of unreliable gates and the motivation of the proposed axiomatization. This work was partially supported, under the MCL (Meet-Combination of Logics) and PQDR (Probabilistic, Quantum and Differential Reasoning) initiatives of SQIG at IT, by FCT and EU FEDER, namely via the FCT PEst-

OE/EEI/LA0008/2013 and AMDSC UTAustin/MAT/0057/2008 projects, as well as by the European Union's Seventh Framework Programme for Research (FP7), namely through project LANDAUER (GA 318287).

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