Quantifier elimination via adjunction

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Abstract

Sufficient conditions are provided for quantifier elimination to hold in a first-order theory. The conditions have two main purposes: (1) to ensure that satisfaction of existential formulas is reflected by an embedding; and (2) to guarantee the existence of a "minimal" model of the theory extending a model of the universal formulas entailed by the theory. The first goal is obtained by requiring that a theory is ∃-adequate and the second by imposing the existence of an adjunction. Recognizing that, in some cases, a "minimal" model extending another can be obtained by iterating a construction, we also provide conditions that guarantee the existence of a ω-"limit" functor identifying such an extension, when the theories are in $\forall_2$. Examples are provided along the paper.

keywords quantifier elimination, model theory, adjunction, ∃-adequate.

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1 Introduction

Quantifier elimination is a key property to be investigated, namely when studying model completeness, completeness, minimality, definability or decidability of a first-order theory. It was first used by Alfred Tarski for proving the decidability of the theories of real closed fields, boolean algebras, and algebraically closed fiels of characteristic 0 or prime characteristic, among others (see [43]), and by Skolem, see [40]. Quantifier elimination has been used extensively by the model theoretic community when investigating mathematical theories, see for instance [21, 32, 10, 23, 17, 45, 27, 16, 2, 7, 28, 25, 5, 20, 43].

Nowadays quantifier elimination is also being used in theoretical computer science, namely in the areas of theorem proving and data abstractions [47, 15, 48]. Moreover, there is also a trend, motivated by applications, for proving

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elimination of quantifiers of arbitrary order and for analyzing the computational complexity of its decision problem, see [41, 14, 4, 31, 6, 13].

Quantifier elimination was first proved in a deductive way by showing how to eliminate quantifiers in formulas of the form $\exists x \varphi$ where φ is a conjunction of literals, see, for instance, [43, 35, 42, 46, 11]. On the other hand, as first advocated by Robinson, there are model theoretic ways to prove quantifier elimination, see for instance [26, 19, 18, 38, 8, 34, 33].

Herein, looking into different proofs of quantifier elimination in specific theories, we were able to abstract away the details and provide a sufficient (general) condition for a theory Θ to have quantifier elimination. Firstly, there should be an adjunction between the category $\mathrm{Mod}(\Theta)$ (of the models of Θ and their embeddings) and the category $\mathrm{Mod}(\Theta^{\forall})$ (where $\Theta^{\forall}$ is the theory composed by the formulas of the form $\forall \psi$ entailed by Θ, where ψ is a quantifier free formula) having as right adjoint the inclusion functor. Secondly, the theory should be $\exists$-adequate for some sets of literals (a condition imposing the reflection of satisfaction of relevant literals along specific embeddings).

The adjunction condition imposes a close relationship between $\mathrm{Mod}(\Theta)$ and $\mathrm{Mod}(\Theta^{\forall})$, see [24] or [1] for some background on category theory. To each model I of $\Theta^{\forall}$ we should be able to associate a "minimal" model I^* of Θ extending I, satisfying a "universal" property. Minimal in the sense that for every embedding of I in a model I' of Θ there is an embedding of I^* into I' that agrees in some way with the original embedding, and universal in the sense that this embedding is the unique that extends in this way the original embedding. On the other hand, the $\exists$-adequate condition guarantees the reflection along an embedding of the satisfaction of formulas of the form $\exists x \varphi$ where φ is a quantifier free formula. We illustrate our results by applying them to the theories of nontrivial torsion free divisible Abelian groups, Presburger arithmetic, real closed fields and algebraically closed fields.

Relying on the good property that composition of adjunctions is still an adjunction, the adjunction condition can be proved in a modular way. For instance, in the context of the theory Θ_{rcof} of real closed ordered fields we establish the adjunction between $\mathrm{Mod}(\Theta^{\forall}_{\mathrm{rcof}})$ (whose objects are interpretation structures induced by ordered integral domains) and $\mathrm{Mod}(\Theta_{\mathrm{rcof}})$, by first establishing an adjunction from $\mathrm{Mod}(\Theta^{\forall}_{\mathrm{rcof}})$ to $\mathrm{Mod}(\Theta_{\mathrm{of}})$ (whose objects are interpretation structures induced by ordered fields) through the ordered fraction field functor, and then an adjunction from $\mathrm{Mod}(\Theta_{\mathrm{of}})$ to $\mathrm{Mod}(\Theta_{\mathrm{rcof}})$.

Sometimes a "minimal" model extending another can be seen as obtained by iterating a certain construction, like in the theory of algebraically closed fields using the Artin construction [3]. Herein, we abstract away the details of obtaining in this way such an extension, and provide general results that can be used in several situations of this kind. More specifically, we start by considering an arbitrary functor E defining the one step construction, and from it we explicitly define a ω-"limit" functor E^{ω}. Then we prove that under some conditions over E, the functor E^{ω} can indeed be used for proving that the underlying theory enjoys quantifier elimination. It is worthwhile to mention that we cover all $\forall_2$ theories and that our approach is constructive, contrasting with the generalized use of Zorn's lemma for addressing these questions. Moreover we stress that the conditions are over E and not over E^ω. We explore and illustrate this technique in the context of algebraically closed fields.

As far as we know this more categorical perspective of quantifier elimination, in the terms presented herein, is new. We offer a collection of techniques and results that can be used to tackle the problem of whether a theory Θ enjoys quantifier elimination, and which provides a road map than can be followed: (1) Analyze the cardinality of the models of Θ; (2) Identify an exhaustive set of literals for Θ; (3) Prove the $\exists$-adequate property; (4) Find the left adjoint of the inclusion functor from $\mathrm{Mod}(\Theta)$ to $\mathrm{Mod}(\Theta^\forall)$.

We start with some preliminaries on Section 2 for recalling and introducing some basic facts and definitions needed throughout the paper. In Section 3 we discuss reflection of the satisfaction of existential formulas along an embedding and reduce this question to the simpler question of a theory be $\exists$-adequate. As illustration, the theories of non-trivial torsion free divisible Abelian groups, Presburger arithmetic, real closed fields and algebraically closed fields, are showed to be $\exists$-adequate. The sufficient conditions for quantifier elimination are established in Section 4 and illustrated in the context of some of the running examples. In the case of real closed fields we capitalize on the fact that the composition of adjunctions is still an adjunction to consider an intermediate category and two left adjoints functors whose composition is the left adjoint required. By observing that for some theories a model extending in a "minimal" way another is obtained by iterating ω-times a certain construction, we provide in Section 5 a categorial view of this process and investigate sufficient conditions over the functor for the one step construction, that guarantee that the functor corresponding to its iteration ω-times, is useful for proving quantifier elimination. The theory of algebraically closed fields is used to illustrate these results. Finally in Section 6 we draw some concluding remarks and highlight future work.

2 Preliminaries

In this section we recall and introduce some basic facts and definitions. We consider that first-order signatures include a binary predicate $\cong$ for equality, and assume that the denotation $\cong^{\mathsf{P}}$ of $\cong$ is $=$. As usual, a *literal* is either an atomic formula or a negation of an atomic formula, and a *theory* over a signature is a set of sentences (that is, formulas with no free variables). We write $\rho \equiv_x \sigma$ to indicate that assignments ρ and σ over a same interpretation structure are x-equivalent, and assume, with no loss of generality, that each quantifier free formula is presented as a finite disjunction of a finite conjunction of literals (that is, is presented in disjunctive normal form).

Given a signature Σ we say that a set Γ of formulas *locally entails* a formula φ, denoted by $\Gamma \vDash^l_\Sigma \varphi$, if for every interpretation structure I over Σ and assignment ρ over I, $I\rho \Vdash_\Sigma \varphi$ whenever $I\rho \Vdash_\Sigma \Gamma$, and Γ *entails* φ, denoted by $\Gamma \vDash_\Sigma \varphi$, if for every interpretation structure I over Σ, $I \Vdash_\Sigma \varphi$ whenever $I \Vdash_\Sigma \Gamma$. Entailment and local entailment coincide when Γ is a set of sentences and φ is a sentence. Given a set Γ of formulas we denote by $\Gamma^{\vDash_\Sigma}$ the set of formulas entailed by Γ. In the sequel, in order to simplify the presentation, we implicitly assume that an interpretation structure I is the tuple $(D, \cdot^{\mathsf{F}}, \cdot^{\mathsf{P}})$, an interpretation structure I' is the tuple $(D', \cdot^{\mathsf{F}'}, \cdot^{\mathsf{P}'})$, an interpretation structure I_1 is the tuple $(D_1, \cdot^{\mathsf{F}_1}, \cdot^{\mathsf{P}_1})$, and so on so forth.

A formula φ is said to be *universal* if φ is $\forall\psi$ for some quantifier free formula ψ, and given a theory Θ, we denote by $\Theta^{\forall}$ the set of all universal sentences in $\Theta^{\models_\Sigma}$. Given formulas φ_1 and φ_2, a theory Θ, and a set of literals Ω over Σ, we say that φ_1 is *equivalent to* φ_2 *in the context of* Θ, denoted by $\varphi \Leftrightarrow_\Theta \varphi_2$ whenever $\Theta \models_\Sigma \varphi_1 \Leftrightarrow \varphi_2$. Moreover, we say Ω is Θ-*exhaustive*, whenever for every literal ν in $L(\Sigma)$, there is a quantifier free formula φ using only literals in Ω such that $\varphi \Leftrightarrow_\Theta \nu$. A variable x is (Θ, Ω)-*essential* in a literal $\nu \in \Omega$ if whenever $\mu \Leftrightarrow_\Theta \nu$ then x occurs in μ, for every $\mu \in \Omega$. The concept of (Θ, Ω)-essential generalizes to any set of literals contained in Ω in the expected way.

Recall that $\{\forall_n\}_{n\in\mathbb{N}}$, $\{\forall_n^+\}_{n\in\mathbb{N}}$, $\{\exists_n\}_{n\in\mathbb{N}}$ and $\{\exists_n^+\}_{n\in\mathbb{N}}$ are the families of sets inductively defined as follows:

- $\forall_0$, $\exists_0$ are the set of quantifier free formulas;
- $\forall_{n+1}$ is composed by formulas of the form $\forall\psi$ where $\psi \in \exists_n^+$, and $\exists_n^+$ is the least set such that $\exists_n \subseteq \exists_n^+$ and is closed for conjunction and disjunction;
- $\exists_{n+1}$ is composed by formulas of the form $\exists\psi$ where $\psi \in \forall_n^+$, and $\forall_n^+$ is the least set such that $\forall_n \subseteq \forall_n^+$ and is closed for conjunction and disjunction;

and, given interpretation structures I and I' over a signature Σ, an *homomorphism* h from I to I', denoted by $h : I \to I'$, is a map $h : D \to D'$ such that

- $h(f_n^{\mathsf{F}}(d_1, \ldots, d_n)) = f_n^{\mathsf{F}'}(h(d_1), \ldots, h(d_n))$;
- if $p_n^{\mathsf{P}}(d_1, \ldots, d_n) = 1$ then $p_n^{\mathsf{P}'}(h(d_1), \ldots, h(d_n)) = 1$;

and an *embedding* h from I to I' is an injective homomorphism from I to I' such that $p_n^{\mathsf{P}}(d_1, \ldots, d_n) = 1$ if $p_n^{\mathsf{P}'}(h(d_1), \ldots, h(d_n)) = 1$.

We denote by Int_Σ the category whose objects are the interpretation structures over Σ and the morphisms are the embeddings between those interpretation structures.

Proposition 2.1 Let $h : I \to I'$ be an embedding, $\rho : X \to D$ an assignment and φ a quantifier free formula. Then

1. $I\rho \Vdash_\Sigma \varphi$ if and only if $I'\, h \circ \rho \Vdash_\Sigma \varphi$;
2. if $I\rho \Vdash_\Sigma \exists x\varphi$ then $I'\, h \circ \rho \Vdash_\Sigma \exists x\varphi$;
3. if $I' \Vdash_\Sigma \varphi$ then $I \Vdash_\Sigma \varphi$;
4. if $I' \Vdash_\Sigma \forall\varphi$ then $I \Vdash_\Sigma \forall\varphi$. $\diamond$

We omit the proof of the previous proposition, Proposition 2.1, since it follows straightforwardly. The result on item 2. of Proposition 2.1 can be extended to formulas in $\exists_1^+$ as we now state.

Proposition 2.2 Let $h : I \to I'$ be an embedding, $\rho : X \to D$ an assignment and φ a formula in $\exists_1^+$ with free variables $x_1, \ldots, x_n$. Then $I'\, h \circ \rho \Vdash_\Sigma \varphi$ whenever $I\rho \Vdash_\Sigma \varphi$. $\diamond$

Given a set Γ of formulas over Σ, we denote by $\text{Mod}(\Gamma)$ the category whose objects are the models of Γ and the morphisms are the embeddings between those models. We observe that $\text{Mod}(\Theta)$ is a subcategory of $\text{Mod}(\Theta^\forall)$ for any theory Θ. Whenever $\text{Mod}(\Upsilon)$ is a subcategory of $\text{Mod}(\Gamma)$, we denote the inclusion functor from $\text{Mod}(\Upsilon)$ to $\text{Mod}(\Gamma)$ by $J_{\Upsilon,\Gamma}$. Moreover, we denote the inclusion functor from $\text{Mod}(\Upsilon)$ to Int_Σ by $J_{\Upsilon,\Sigma}$.

Proposition 2.3 Let Θ be a theory over Σ and $h : I \to I'$ an embedding in $\text{Mod}(\Theta)$. Then $h(I)$ is also a model in $\text{Mod}(\Theta)$. $\diamond$

We omit the proof of the previous proposition, Proposition 2.3, since it follows straightforwardly.

Proposition 2.4 Let I and I' be interpretation structures over Σ, $h : I \to I'$ an embedding and Θ a theory over Σ. Then, $I \in \text{Mod}(\Theta^\forall)$ if $I' \in \text{Mod}(\Theta)$.

Proof: Assume that $I' \in \text{Mod}(\Theta)$. Let $(\forall\varphi) \in \Theta^\forall$ where φ is a quantifier free formula. Then, $I' \Vdash_\Sigma (\forall\varphi)$ and so, by Proposition 2.1, $I \Vdash_\Sigma (\forall\varphi)$. QED

3 Reflecting satisfaction of existential formulas

In item 2. of Proposition 2.1 we stated that satisfaction of $\exists\varphi$ formulas where φ is a quantifier free formula is preserved by embeddings. The reflection of the satisfaction of those formulas by an embedding does not hold in general. In this section we provide a sufficient condition for the reflection to happen. Such a sufficient condition is based on the notion of $(\exists, x)$-adequate.

Let Θ be a theory over a signature Σ, Ω a Θ-exhaustive set of literals, $\Psi \subseteq \Omega$ and x a (Θ, Ω)-essential variable in Ψ. We say that Θ is $(\exists, x)$-*adequate for* Ψ and Ω if for every

- embedding $h : I \to I'$ in $\text{Mod}(\Theta)$;
- assignment ρ' over I' with $\rho'(x)$ in $D' \setminus h(D)$ and $\rho'(y)$ in $h(D)$ for $y \neq x$;
- quantifier free formula $\bigvee_{j=1}^n \delta_j$ with literals in Ω and i in $1, \ldots, n$ such that $I'\rho' \Vdash \delta_i$;

there is an assignment ρ over I such that

- $\rho' \equiv_x h \circ \rho$;
- for every literal ν in δ_i, if $\nu \in \Psi$ then $I\rho \Vdash_\Sigma \nu$;
- if $\nu \in \Omega \setminus \Psi$ and x is (Θ, Ω)-essential in ν then ν is not in δ_i.

A theory Θ is $\exists$-*adequate* whenever it is $(\exists, x)$-*adequate* for some variable x and sets Ω and Ψ.

Observe that Ψ should be chosen so that when proposing such an assignment ρ we should look only for the satisfaction of the literals of δ_i in Ψ. Note also that if a theory Θ is $(\exists, x)$-adequate for some variable x and sets Ω and Ψ then for every variable y there is a set Ψ_y such that Θ is $(\exists, y)$-adequate for Ω and Ψ_y.

Proposition 3.1 Let Θ be an $\exists$-adequate theory and φ a quantifier free formula over a signature Σ, $h : I \to I'$ in Mod(Θ) and σ an assignment over I. Then $I\sigma \Vdash_\Sigma \exists x\varphi$ whenever $I'\, h \circ \sigma \Vdash_\Sigma \exists x\varphi$.

Proof:
Assume that Θ is $(\exists, x)$-adequate for a Θ-exhaustive set Ω of literals and for a set $\Psi \subseteq \Omega$ and that x is (Θ, Ω)-essential in Ψ. With no loss of generality, assume that φ is a quantifier free formula $\bigvee_{i=1}^{n} \delta_i$ whose literals are in Ω. Suppose that $I'\, h \circ \sigma \Vdash_\Sigma \exists x\varphi$. Then there is an assignment ρ' such that $\rho' \equiv_x h \circ \sigma$ and $I'\rho' \Vdash_\Sigma \varphi$. Assume that $I'\rho' \Vdash_\Sigma \delta_i$. Let ρ be an assignment over I satisfying the conditions in the definition of $(\exists, x)$-adequate and ν a literal in δ_i. We have two cases to consider:

(1) $\nu \in \Psi$ and so $I\rho \Vdash_\Sigma \nu$;

(2) $\nu \notin \Psi$. Observe that $\nu \in \Omega$. Then, x is not (Θ, Ω)-essential in ν. Indeed, otherwise ν would not occur in δ_i by definition of $(\exists, x)$-adequate. Denote by μ the literal equivalent to ν in which x does not occur. Therefore $I'\rho' \Vdash_\Sigma \mu$. Then $I'h \circ \rho \Vdash_\Sigma \mu$, hence by Proposition 2.1, $I\rho \Vdash_\Sigma \mu$ and so $I\rho \Vdash_\Sigma \nu$.

On the other hand, ρ is x-equivalent to σ: $\rho(y) = \sigma(y)$ for any variable $y \neq x$ since $h(\rho(y)) = \rho'(y) = h(\sigma(y))$ and h is injective. Therefore, $I\sigma \Vdash_\Sigma \exists x\varphi$. QED

The $\exists$-adequate property is satisfied by a wide range of theories. As illustration, we now show that the theories for non-trivial torsion free divisible Abelian groups, Presburger arithmetic, real closed fields and algebraically closed fields are $\exists$-adequate.

Non-trivial torsion free divisible Abelian groups

Let $\Sigma_{\rm tfdag}$ be a signature such that $F_0 = \{0\}$, $F_1 = \{-\}$, $F_2 = \{+\}$ and $P_2 = \{\cong\}$, and $\Theta_{\rm tfdag}$ be a theory for non-trivial abelian groups, see for instance [26], enriched with the following axioms, for any $n > 0$:

- $(\forall x((\neg(x \cong 0)) \Rightarrow (\neg(nx \cong 0))))$;
- $(\forall y(\exists x\, ((nx \cong y)))$;

where

$$zx \text{ is } \begin{cases} \underbrace{x + \cdots + x}_{z \text{ times}} & \text{if } z \geq 0 \\ \underbrace{-x + \cdots + -x}_{-z \text{ times}} & \text{otherwise.} \end{cases}$$

We can assume that each literal is either of the form

$$\neg(t \cong 0) \text{ or } t \cong 0$$

where t is $z_1 x_1 + \ldots + \ldots z_n x_n$ for non-zero integers $z_1, \ldots, z_n$ and pairwise distinct variables $x_1, \ldots, x_n$.

Given a model I of Θ_{tfdag}, a and b in D, and an integer z such that $za = b$, we denote such an element a by $\frac{b}{z}$. Observe that if also $c \in D$ is such that $zc = b$ then $a = c$. Given an embedding $h : I \to I'$ in $\text{Mod}(\Theta_{\text{tfdag}})$, if $\frac{b}{z}$ exists then $\frac{h(b)}{z}$ exists and $h(\frac{b}{z}) = \frac{h(b)}{z}$. Indeed $zh(\frac{b}{z}) = h(z\frac{b}{z}) = h(b)$.

The cardinality of the models of Θ_{tfdag} plays an important role in the existence of the assignment required by the $\exists$-adequate condition.

Proposition 3.2 Every model of Θ_{tfdag} has an infinite domain.

Proof: We show that for each model of Θ_{tfdag} there is an injective map from $\mathbb{N}$ to it. Let I be a model of Θ_{tfdag}. Then D has at least one more element besides 0^{F}. Let d be one such element. Consider the map $h : \mathbb{N} \to D$ such that

$$h(m) = dm.$$

and assume that $m_1 \neq m_2$. Suppose without loss of generality that $m_1 < m_2$. Then

$$\begin{array}{rcl} dm_1 & = & 0^{\mathsf{F}} +^{\mathsf{F}} dm_1 \\ & \neq & d(m_2 - m_1) +^{\mathsf{F}} dm_1 \quad (\dagger) \\ & = & dm_2 \end{array}$$

and so h is injective. Regarding ($\dagger$), let ρ be such that $\rho(x) = d$. Then $I\rho \Vdash_{\Sigma_{\text{tfdag}}} (\neg(x \cong 0))$ and so $I\rho \Vdash_{\Sigma_{\text{tfdag}}} (\neg((m_2 - m_1)x \cong 0))$ by Θ_{tfdag}. Therefore $0^{\mathsf{F}} \neq d(m_2 - m_1)$ as we wanted to show. QED

Let Ω_{tfdag} be the set of all literals over Σ_{tfdag} and Ψ_{tfdag} the set of literals of the form

$$\neg((t + zx) \cong 0)$$

where x does not occur in t and z is non-zero such that there is no equivalent literal in the context of Θ_{tfdag} in which x does not occur.

Proposition 3.3 The theory Θ_{tfdag} is $(\exists, x)$-adequate for Ψ_{tfdag} and Ω_{tfdag}.

Proof:
Let $h : I \to I'$ be an embedding in $\text{Mod}(\Theta_{\text{tfdag}})$, ρ' an assignment over I' such that $\rho'(x) \in D' \setminus h(D)$ and $\rho'(y) \in h(D)$ for every $y \neq x$, and $\bigvee_{i=1}^{n} \delta_i$ a quantifier free formula such that $I'\rho' \Vdash_{\Sigma_{\text{tfdag}}} \delta_i$ for some $i = 1, \ldots, n$.

Let $\neg((t_1 + z_1 x) \cong 0), \ldots, \neg((t_n + z_n x) \cong 0)$ be the literals of Ψ_{tfdag} in δ_i.

Consider the assignment ρ over I such that:

$$\rho(x) \notin \{-^{\mathsf{F}} \frac{h^{-1}([\![t_i]\!]^{I'\rho'})}{z_i} : i = 1, \ldots, n\}.$$

and $\rho(y) = h^{-1}(\rho'(y))$ for every $y \neq x$.

Note that $[\![t_i]\!]^{I'\rho'} \in h(D)$ since, by hypothesis, x does not occur in t_i. Observe also that there is such a value for $\rho(x)$ since D is infinite by Proposition 3.2. Then:

(a) $\rho' \equiv_x h \circ \rho$: Immediate by definition of ρ.

(b) Let $\nu \in \delta_i$ be such that $\nu \in \Psi_{\text{tfdag}}$. Hence ν is of the form $\neg((t_i + z_i x) \cong 0)$ for some $i = 1, \ldots, n$. Suppose, by contradiction, that $I\rho \not\Vdash_\Sigma \neg((t_i + z_i x) \cong 0)$, that is, $I\rho \Vdash_\Sigma ((t_i + z_i x) \cong 0)$. Then $[\![t_i]\!]^{I\rho} = -^{\mathsf{F}} z_i \rho(x)$ and so

$$\rho(x) = -^{\mathsf{F}} \frac{[\![t_i]\!]^{I\rho}}{z_i} = -^{\mathsf{F}} \frac{h^{-1}([\![t_i]\!]^{I'\rho'})}{z_i}$$

which is a contradiction with the definition of $\rho(x)$.

(c) Let $\nu \in \Omega_{\text{tfdag}} \setminus \Psi_{\text{tfdag}}$. Assume that x is $(\Theta_{\text{tfdag}}, \Omega_{\text{tfdag}})$-essential in ν. Hence ν is of the form $((t + zx) \cong 0)$ where z is a non-zero integer and x does not occur in t. Suppose, by contradiction, that ν occurs in δ_i. Then $I'\rho' \Vdash_\Sigma (t + zx) \cong 0$, hence $[\![t]\!]^{I'\rho'} = -^{\mathsf{F}'} z\rho'(x)$ and so

$$\rho'(x) = -^{\mathsf{F}'} \frac{[\![t]\!]^{I'\rho'}}{z} = h(-^{\mathsf{F}} \frac{[\![t]\!]^{I\rho}}{z})$$

which is a contradiction with the initial hypothesis on ρ'. QED

Presburger arithmetic

Consider the theory of arithmetic as proposed by Presburger, see [30, 26]. Let Σ_{pa} be the signature where: $F_0 = \{0, 1\}$, $F_1 = \{-\}$, $F_2 = \{+\}$, $P_1 = \{p_n : n = 2, 3, \ldots\}$ and $P_2 = \{\cong, <\}$, and Θ_{pa} be the theory composed by the sentences:

- $\forall x\, (\neg(x < x))$;
- $\forall x\, \forall y \forall z(((x < y) \wedge (y < z)) \to (x < z))$;
- $\forall x\, \forall y((x < y) \vee (x \cong y) \vee (y < x))$;
- $\forall x\, \forall y \forall z((x < y) \to ((x + z) < (y + z)))$;
- $\forall x\, \forall y((x + y) \cong (y + x))$;
- $\forall x\, ((0 + x) \cong x)$;
- $\forall x\, \forall y \forall z((x + (y + z)) \cong ((x + y) + z))$;
- $\forall x\, \forall y \forall z(((x + (-y)) \cong z) \Leftrightarrow (x \cong (y + z)))$;
- $0 < 1$;
- $\forall x\, ((x \leq 0) \vee (x \geq 1))$;
- $\forall x\, (p_n(x) \Leftrightarrow \exists y\, (x \cong ny))$ for each $n = 2, 3, \ldots$;

- $\forall x\, (\bigvee_{r=0}^{n-1}(p_n(x+r1) \wedge \bigwedge_{i \neq r} \neg p_n(x+i1)))$ for each $n = 2, 3, \ldots$.

Observe that p_n is interpreted as the elements divisible by n. We start by discussing the cardinality of the models of Θ_{pa}.

Proposition 3.4 Every model of Θ_{pa} has an infinite domain.

Proof: Let I be a model of Θ_{pa}. Consider the map $h : \mathbb{N} \to D$ such that

$$h(m) = 1^{\mathsf{F}} m.$$

Then:

(a) $0^{\mathsf{F}} <^{\mathsf{F}} 1^{\mathsf{F}} m$ for every positive natural m. Base: m is 1. Then the thesis holds immediately by the axiomatics in Θ_{pa}. Step: $m > 1$. Then:

$$\begin{array}{rcll} 0^{\mathsf{F}} & <^{\mathsf{F}} & 1^{\mathsf{F}} & \\ & = & 1^{\mathsf{F}} +^{\mathsf{F}} 0^{\mathsf{F}} & \\ & <^{\mathsf{F}} & 1^{\mathsf{F}} +^{\mathsf{F}} 1^{\mathsf{F}}(m-1) & \text{by the axioms in } \Theta_{\mathrm{pa}} \text{ since } 0^{\mathsf{F}} <^{\mathsf{F}} 1^{\mathsf{F}}(m-1) \text{ by HI} \\ & = & 1^{\mathsf{F}} m. & \end{array}$$

(b) h is injective. Assume that $m_1 \neq m_2$ and suppose with no loss of generality that $m_1 < m_2$. Then

$$\begin{array}{rcll} 1^{\mathsf{F}} m_1 & = & 1^{\mathsf{F}} m_1 +^{\mathsf{F}} 0^{\mathsf{F}} & \\ & <^{\mathsf{F}} & 1^{\mathsf{F}} m_1 +^{\mathsf{F}} 1^{\mathsf{F}}(m_2 - m_1) & \text{by } \Theta_{\mathrm{pa}} \text{ and (a) since } 0^{\mathsf{F}} <^{\mathsf{F}} 1^{\mathsf{F}}(m_2 - m_1) \\ & = & 1^{\mathsf{F}} m_2 & \end{array}$$

and so $h(m_1) \neq h(m_2)$. QED

Let Ω_{pa} be the set of literals of the form

$$p_n(t), \quad s < t \quad \text{and} \quad t \cong 0$$

where s and t are terms.

Lemma 3.5 The set Ω_{pa} is Θ_{pa}-exhaustive.

Proof:
It is enough to observe that

- $(\neg p_n(s)) \Leftrightarrow_{\Theta_{\mathrm{pa}}} \displaystyle\bigvee_{i=1,\ldots,n-1} p_n(s+i)$;
- $(\neg(s \cong t)) \Leftrightarrow_{\Theta_{\mathrm{pa}}} ((s < t) \vee (t < s))$;
- $(\neg(s < t)) \Leftrightarrow_{\Theta_{\mathrm{pa}}} ((s \cong t) \vee (t < s))$.

QED

Let Ψ_{pa} be the set of literals of the form

$$p_n(t + zx), \qquad t < zx \qquad \text{and} \qquad zx < t$$

where x does not occur in t and z is a non zero integer. It is obvious that x is $(\Theta_{\text{pa}}, \Omega_{\text{pa}})$-essential in Ψ_{pa}.

Proposition 3.6 The theory Θ_{pa} is $(\exists, x)$-adequate for Ψ_{pa} and Ω_{pa}.

Proof:
Let $h : I \to I'$ be an embedding in $\text{Mod}(\Theta_{\text{pa}})$, x a variable, ρ' an assignment over I' such that $\rho'(y) \in h(D)$ for every $y \neq x$ and $\rho'(x) \in D' \setminus h(D)$ and $\bigvee_{i=1}^{n} \delta_i$ a quantifier free formula whose literals are in Ω_{pa} such that $I'\rho' \Vdash_{\Sigma_{\text{pa}}} \delta_i$ for some $i = 1, \ldots, n$.

In the context of Θ_{pa}, observe that $u < v$ is equivalent to $-v < -u$ and $p_n(t)$ is equivalent to $p_n(-t)$, and given natural $n' > 0$, $p_n(t)$ is equivalent to $p_{n'n}(n't)$ and $u < v$ is equivalent to $n'u < n'v$. So with no loss of generality let $p_{n_1}(mx + t_1), \ldots, p_{n_{m_1}}(mx + t_{m_1})$, $s_1 < mx, \ldots, s_{m_2} < mx$ and $mx < v_1, \ldots, mx < v_{m_3}$ be the literals of Ψ_{pa} occurring in δ_i where m is a non-zero natural and x does not occur in $t_1, \ldots, t_{m_1}$, $s_1, \ldots, s_{m_2}$, and $v_1, \ldots, v_{m_3}$.

Let $\hat{\rho}'$ be an assignment x-equivalent to ρ' such that $\hat{\rho}'(x) = m\rho'(x)$ and $\hat{\delta}_i$ the formula

$$(\bigwedge_{i=1}^{m_1} p_{n_i}(x + t_i)) \bigwedge (\bigwedge_{j=1}^{m_2} s_j < x) \bigwedge (\bigwedge_{k=1}^{m_3} x < v_k).$$

Then $I'\hat{\rho}' \Vdash_{\Sigma_{\text{pa}}} \hat{\delta}_i$.

Denote by q'_i the element of D' such that

$$\hat{\rho}'(x) +^{\mathsf{F}'} [\![t_i]\!]^{I'\hat{\rho}'} = n_i q'_i.$$

Observe that for every assignment σ over I such that $h \circ \sigma \equiv_x \rho'$ there is no natural number n such that

$$\max([\![s_1]\!]^{I\sigma}, \ldots, [\![s_{m_2}]\!]^{I\sigma}) -^{\mathsf{F}} \min([\![v_1]\!]^{I\sigma}, \ldots, [\![v_{m_3}]\!]^{I\sigma}) = n1^{\mathsf{F}}$$

as we now show: let σ be an assignment over I such that $h \circ \sigma \equiv_x \rho'$ and suppose, by contradiction, that there is such a natural number n. Then

$$\max([\![s_1]\!]^{I'\rho'}, \ldots, [\![s_{m_2}]\!]^{I'\rho'}) -^{\mathsf{F}'} \min([\![v_1]\!]^{I'\rho'}, \ldots, [\![v_{m_3}]\!]^{I'\rho'}) = n1^{\mathsf{F}'}.$$

Note that there is no natural number n' less than n with

$$\max([\![s_1]\!]^{I'\rho'}, \ldots, [\![s_{m_2}]\!]^{I'\rho'}) +^{\mathsf{F}'} n'1^{\mathsf{F}'} <^{\mathsf{F}'} m\rho'(x)$$

and

$$m\rho'(x) <^{\mathsf{F}'} \max([\![s_1]\!]^{I'\rho'}, \ldots, [\![s_{m_2}]\!]^{I'\rho'}) +^{\mathsf{F}'} (n' + 1)1^{\mathsf{F}'}$$

since otherwise

$$0^{\mathsf{F}'} < m\rho'(x) -^{\mathsf{F}'} \max([\![s_1]\!]^{I'\rho'}, \ldots, [\![s_{m_2}]\!]^{I'\rho'}) -^{\mathsf{F}'} n'1^{\mathsf{F}'} < 1^{\mathsf{F}'}$$

which cannot happen by Θ_{pa}. Therefore, since $<$ is a total order and

$$\max([\![s_1]\!]^{I'\rho'}, \ldots, [\![s_{m_2}]\!]^{I'\rho'}) <^{\mathsf{F}'} m\rho'(x) <^{\mathsf{F}'} \min([\![v_1]\!]^{I'\rho'}, \ldots, [\![v_{m_3}]\!]^{I'\rho'})$$

there is a natural number n' with

$$\max([\![s_1]\!]^{I'\rho'}, \ldots, [\![s_{m_2}]\!]^{I'\rho'}) +^{\mathsf{F}'} n'1^{\mathsf{F}'} = m\rho'(x)$$

which implies that $\rho'(x) \in h(D)$ which is a contradiction;

Let $0 \leq r' < n_1 \ldots n_k$ be a natural and q' an element of D' such that

$$-^{\mathsf{F}'} \hat{\rho}'(x) +^{\mathsf{F}'} r'1^{\mathsf{F}'} = mn_1 \ldots n_k q'$$

and $0 \leq r < mn_1 \ldots n_k$ a natural and q an element of D with

$$h^{-1}(\max([\![s_1]\!]^{I'\rho'}, \ldots, [\![s_{m_2}]\!]^{I'\rho'})) +^{\mathsf{F}} r1^{\mathsf{F}} = mn_1 \ldots n_k q.$$

So, take an assignment $\hat{\rho}$ over I with $h \circ \hat{\rho} \equiv_x \rho'$ such that:

$$\hat{\rho}(x) = h^{-1}(\max([\![s_1]\!]^{I'\rho'}, \ldots, [\![s_{m_2}]\!]^{I'\rho'})) +^{\mathsf{F}} (r + r' + mn_1 \ldots n_k)1^{\mathsf{F}}.$$

Observe that D is infinite by Proposition 3.4. Then:

(1) $I\hat{\rho} \Vdash_\Sigma \hat{\delta}_i$. Let $\hat{\nu}$ be a literal in $\hat{\delta}_i$. We have to consider the following cases:

(i) $\hat{\nu}$ is $s_j < x$ for j in $\{1, \ldots, m_2\}$. Then

$$\begin{array}{rcl} [\![s_j]\!]^{I\hat{\rho}} & \leq^{\mathsf{F}} & h^{-1}(\max([\![s_1]\!]^{I'\rho'}, \ldots, [\![s_{m_2}]\!]^{I'\rho'}) \\ & <^{\mathsf{F}} & h^{-1}(\max([\![s_1]\!]^{I'\rho'}, \ldots, [\![s_{m_2}]\!]^{I'\rho'})) +^{\mathsf{F}} (r + r' + mn_1 \ldots n_k)1^{\mathsf{F}} \\ & = & \hat{\rho}(x) \end{array}$$

and so $I\hat{\rho} \Vdash_\Sigma \hat{\nu}$;

(ii) $\hat{\nu}$ is $x < v_k$ for i in $\{1, \ldots, m_3\}$. Suppose, by contradiction, that

$$h^{-1}(\min([\![v_1]\!]^{I'\rho'}, \ldots, [\![v_{m_3}]\!]^{I'\rho'})) <^{\mathsf{F}} \hat{\rho}(x).$$

Then

$$h^{-1}(\min([\![v_1]\!]^{I'\rho'}, \ldots, [\![v_{m_3}]\!]^{I'\rho'})) -^{\mathsf{F}} \hat{\rho}(x) <^{\mathsf{F}} 0^{\mathsf{F}}.$$

Hence

$$\begin{array}{c} h^{-1}(\min([\![v_1]\!]^{I'\rho'}, \ldots, [\![v_{m_3}]\!]^{I'\rho'})) -^{\mathsf{F}} h^{-1}(\max([\![s_1]\!]^{I'\rho'}, \ldots, [\![s_{m_2}]\!]^{I'\rho'})) \\ <^{\mathsf{F}} \\ (r + r' + mn_1 \ldots n_k)1^{\mathsf{F}} \end{array}$$

which contradicts the fact that for any assignment σ over I with $h \circ \sigma \equiv_x \rho'$ there is no such natural number. Similarly if $h^{-1}(\min([\![v_1]\!]^{I'\rho'}, \ldots, [\![v_{m_3}]\!]^{I'\rho'})) = \hat{\rho}(x)$. Hence

$$\begin{array}{rcl} \hat{\rho}(x) & <^{\mathsf{F}} & h^{-1}(\min([\![v_1]\!]^{I'\rho'}, \ldots, [\![v_{m_3}]\!]^{I'\rho'})) \\ & \leq^{\mathsf{F}} & [\![v_k]\!]^{I\hat{\rho}} \end{array}$$

and so $I\hat{\rho} \Vdash_{\Sigma_{\mathrm{pa}}} \hat{\nu}$;

(iii) $\hat{\nu}$ is $p_{n_i}(x + t_i)$ for i in $\{1, \dots, m_1\}$. Observe that

$$\begin{array}{rcl} \hat{\rho}(x) & = & h^{-1}(\max([\![s_1]\!]^{I'\rho'}, \dots, [\![s_{m_2}]\!]^{I'\rho'})) +^{\mathsf{F}} (r + r' + mn_1 \dots n_k)1^{\mathsf{F}} \\ & = & mn_1 \dots n_k q +^{\mathsf{F}} r'1^{\mathsf{F}} +^{\mathsf{F}} mn_1 \dots n_k 1^{\mathsf{F}} \\ & = & r'1^{\mathsf{F}} +^{\mathsf{F}} mn_1 \dots n_k (q +^{\mathsf{F}} 1^{\mathsf{F}}) \end{array}$$

and so

$$h(\hat{\rho}(x)) = r'1^{\mathsf{F}'} +^{\mathsf{F}'} mn_1 \dots n_k (h(q) +^{\mathsf{F}'} 1^{\mathsf{F}'}).$$

Since

$$\begin{array}{rcl} h(\hat{\rho}(x) +^{\mathsf{F}} [\![t_i]\!]^{I\hat{\rho}}) & = & r'1^{\mathsf{F}'} +^{\mathsf{F}'} mn_1 \dots n_k (h(q) +^{\mathsf{F}'} 1^{\mathsf{F}'}) +^{\mathsf{F}'} [\![t_i]\!]^{I'\hat{\rho}'} \\ & = & mn_1 \dots n_k q' +^{\mathsf{F}'} \hat{\rho}'(x) +^{\mathsf{F}'} mn_1 \dots n_k (h(q) +^{\mathsf{F}'} 1^{\mathsf{F}'}) +^{\mathsf{F}'} [\![t_i]\!]^{I'\hat{\rho}'} \\ & = & mn_1 \dots n_k q' +^{\mathsf{F}'} n_i q_i' +^{\mathsf{F}'} mn_1 \dots n_k (h(q) +^{\mathsf{F}'} 1^{\mathsf{F}'}) \end{array}$$

then

$$p_{n_i}^{\mathsf{F}'}(h(\hat{\rho}(x) +^{\mathsf{F}} [\![t_i]\!]^{I\hat{\rho}})) = 1$$

and so, since h is an embedding,

$$p_{n_i}^{\mathsf{F}}(\hat{\rho}(x) +^{\mathsf{F}} [\![t_i]\!]^{I\hat{\rho}}) = 1.$$

Therefore $I\hat{\rho} \Vdash_{\Sigma_{\mathrm{pa}}} \hat{\nu}$.

(2) $I\hat{\rho} \Vdash_{\Sigma_{\mathrm{pa}}} p_m(x)$. Capitalizing in some parts of (iii)

$$\begin{array}{rcl} h(\hat{\rho}(x)) & = & r'1^{\mathsf{F}'} +^{\mathsf{F}'} mn_1 \dots n_k (h(q) +^{\mathsf{F}'} 1^{\mathsf{F}'}) \\ & = & mn_1 \dots n_k q' +^{\mathsf{F}'} \hat{\rho}'(x) +^{\mathsf{F}'} mn_1 \dots n_k (h(q) +^{\mathsf{F}'} 1^{\mathsf{F}'}) \\ & = & mn_1 \dots n_k q' +^{\mathsf{F}'} m\rho'(x) +^{\mathsf{F}'} mn_1 \dots n_k (h(q) +^{\mathsf{F}'} 1^{\mathsf{F}'}). \end{array}$$

Hence

$$p_m^{\mathsf{F}'}(h(\hat{\rho}(x)))$$

and so, since h is an embedding,

$$p_m^{\mathsf{F}}(\hat{\rho}(x)) = 1.$$

as we wanted to show.

So, let $d \in D$ be such that

$$\hat{\rho}(x) = md$$

and let ρ be an assignment x-equivalent to $\hat{\rho}(x)$ such that $\rho(x) = d$. Then

$$\hat{\rho}(x) = m\rho(x)$$

and since $I\hat{\rho} \Vdash_{\Sigma_{\mathrm{pa}}} \hat{\delta}_i$ and taking into account the definition of δ_i we can conclude that

$$I\rho \Vdash_{\Sigma_{\mathrm{pa}}} \delta_i$$

as we wanted to show. Moreover:

(a) It is immediate to see that $\rho' \equiv_x h \circ \rho$.

(b) Let $\nu \in \delta_i$ be such that $\nu \in \Psi_{\text{pa}}$. Then $I\rho \Vdash_{\Sigma_{\text{pa}}} \nu$ since $I\rho \Vdash_{\Sigma_{\text{pa}}} \delta_i$ as we showed above.

(c) Let $\nu \in \Omega_{\text{pa}} \setminus \Psi_{\text{pa}}$ be such that x is $(\Theta_{\text{pa}}, \Omega_{\text{pa}})$-essential in ν. Then ν is of the form $((s + wx) \cong 0)$ where w is a non zero integer and s is a term not containing x. Assume, by contradiction, that $\nu \in \delta_i$. Then $I'\rho' \Vdash_{\Sigma_{\text{pa}}} \nu$, hence $\rho'(x) \in h(D)$, which is a contradiction. QED

Algebraically closed fields

Consider algebraically closed fields, see [22, 26]. Let Σ_{f} be the signature for fields where: $F_0 = \{0, 1\}$, $F_1 = \{-\}$, $F_2 = \{+, \times\}$, and $P_2 = \{\cong\}$, and Θ_{f} the theory containing the field axioms. Let $\Sigma_{\text{acf}} = \Sigma_{\text{f}}$ be the signature for algebraically closed fields, and Θ_{acf} an enrichment of Θ_{f} with the following sentences:

$$(\dagger) \quad \forall x_1 \ldots \forall x_n \exists y\, (y^n + x_1 y^{n-1} + \cdots + x_n \cong 0),$$

for every $n > 0$.

Proposition 3.7 Every model of Θ_{acf} has an infinite domain.

Proof: Let I be a model of Θ_{acf}. Assume by contradiction that it is finite. Consider the map $q : D \to D$ such that

$$q(d) = d^2 +^{\mathsf{F}} (-^{\mathsf{F}} 1^{\mathsf{F}})d.$$

Then $q(0^{\mathsf{F}}) = 0^{\mathsf{F}}$ and $q(1^{\mathsf{F}}) = 0^{\mathsf{F}}$. Hence q is not surjective and so there is $a \in D$ such that $q(d) \neq a$ for every d in D. That is, $d^2 +^{\mathsf{F}} (-^{\mathsf{F}} 1^{\mathsf{F}})d \neq a$ for every d in D. Hence $d^2 +^{\mathsf{F}} (-^{\mathsf{F}} 1^{\mathsf{F}})d +^{\mathsf{F}} (-^{\mathsf{F}} a) \neq 0^{\mathsf{F}}$ for every d in D. Therefore there are values of x_1 and x_2 for which the polynomial equation $y^2 + x_1 y + x_2 \cong 0$ does not have a root in D, and so, I does not satisfy the axiom $(\dagger)$ in Θ_{acf}. QED

Observe that each atomic formula can be seen as a polynomial equation

$$q(x_1, \ldots, x_n, x) \cong 0.$$

Let Ω_{acf} be the set of all literals and Ψ_{acf} be the set of all negations of atomic formulas

$$\neg(q(x_1, \ldots, x_n, x) \cong 0)$$

where x is $(\Theta_{\text{acf}}, \Omega_{\text{acf}})$-essential.

Proposition 3.8 The theory Θ_{acf} is $(\exists, x)$-adequate for Ψ_{acf} and Ω_{acf}.

Proof:
Let $h : I \to I'$ be an embedding in $\text{Mod}(\Theta_{\text{acf}})$, ρ' an assignment over I' such that $\rho'(y) \in h(D)$ for every $y \neq x$ and $\rho'(x) \in D' \setminus h(D)$ and $\bigvee_{i=1}^{n} \delta_i$ a quantifier free formula such that $I'\rho' \Vdash_{\Sigma_{\text{acf}}} \delta_i$ for some $i = 1, \ldots, n$.
Assume that $\neg q_1(x_{11}, \ldots, x_{1n_1}, x) \cong 0, \ldots, \neg q_k(x_{k1}, \ldots, x_{kn_k}, x) \cong 0$ are the

literals of Ψ_{acf} in δ_i.

Let ρ be an assignment over I such that $h \circ \rho \equiv_x \rho'$ and $\rho(x)$ is not a root of the polynomial equation $q_i^{\mathsf{F}}(h^{-1}(\rho'(x_{i1})), \dots, h^{-1}(\rho'(x_{in_i})), x) = 0^{\mathsf{F}}$ for $i = 1, \dots, k$ which exists since D is infinite, see Proposition 3.7, and the number of such roots are finite. Then:

(i) Let ν in δ_i be such that $\nu \in \Psi_{\text{acf}}$. Then $I\rho \Vdash_{\Sigma_{\text{acf}}} \nu$ by definition of ρ.

(ii) Let $\nu \in \Omega_{\Theta_{\text{acf}}} \setminus \Psi_{\text{acf}}$. Assume that x is $(\Theta_{\text{acf}}, \Omega_{\text{acf}})$-essential in ν with respect to $\Omega_{\Theta_{\text{acf}}}$. Then ν is of the form $q(x_1, \dots, x_n, x) \cong 0$. Assume, by contradiction, that

$$q(x_1, \dots, x_n, x) \cong 0 \in \delta_i.$$

Then $I'\rho' \Vdash_{\Sigma_{\text{acf}}} q(x_1, \dots, x_n, x) \cong 0$ and so $\rho'(x)$ is a solution of the polynomial equation $q^{\mathsf{F}'}(\rho'(x_1), \dots, \rho'(x_n), x) = 0^{\mathsf{F}'}$. Let m be the number of roots of the polynomial equation $q^{\mathsf{F}}(\rho(x_1), \dots, \rho(x_n), x) = 0^{\mathsf{F}}$ in D and $d_1, \dots, d_m$ those roots. Note that $h(d_1), \dots, h(d_m)$ are also the m roots in D' of the equation $q^{\mathsf{F}'}(h(\rho(x_1)), \dots, h(\rho(x_n)), x) = 0^{\mathsf{F}'}$. So $\rho'(x) = h(d_j)$ for some j in $\{1, \dots, m\}$ since $\rho'(x_i) = h(\rho(x_i))$ for $i = 1, \dots, n$, which contradicts the fact that $\rho'(x) \in D' \setminus h(D)$. QED

Real closed fields

Let Σ_{of} be the enrichment of the signature Σ_{f} with a new binary predicate symbol $<$, and Θ_{rcof}, the *theory of real closed ordered fields*, see [26, 19, 6], an enrichment of the theory Θ_{of}, for ordered fields, with the following axioms:

- $\forall x_1 \dots \forall x_n \exists y\, ((y^n + x_1 y^{n-1} + \cdots + x_{n-1} y + x_n) \cong 0)$ for every odd natural number n;
- $\forall x \exists y\, ((y^2 \cong x) \vee (y^2 \cong (-x))$;
- $\forall x_1 \dots \forall x_n\, \neg((x_1^2 + \cdots + x_n^2 + 1) \cong 0)$.

Recall that an *ordered field* is a pair $(R, <)$ where R is a field and $< \subseteq R^2$ is a linear order such that for every elements r, r_1 and r_2 of R: (1) $r_1 + r < r_2 + r$ whenever $r_1 < r_2$; and (2) $0 < r_1 \times r_2$ whenever $0 < r_1$ and $0 < r_2$. So Θ_{of} extends Θ_{f} with the axioms for linearity as well as axioms representing the properties above. Observe that, in an ordered field, for every elements r, r_1, r_2, r_3 and r_4 of the field: (1) either $-r < 0 < r$ or $r = 0$ or $r < 0 < -r$; (2) $r_1 + r_3 < r_2 + r_4$ whenever $r_1 < r_2$ and $r_3 < r_4$; and (3) $r_1 \times r < r_2 \times r$ whenever $r_1 < r_2$ and $0 < r$.

A real closed field can be ordered by a binary relation defined such that the relation holds between any two of its elements whenever their difference is a nonzero square. That ordering is the only possible ordering of a real closed field, see [26]. The models of Θ_{rcof} are exactly the models induced by real closed fields with that unique ordering. Observe that

$$\Theta_{\text{rcof}} \vDash_{\Sigma_{\text{of}}} \forall x \forall y ((x < y) \leftrightarrow \exists z\, ((\neg(z \cong 0)) \wedge ((x + z^2) \cong y))).$$

In fact, for every real closed field there is a model of Θ_{rcof} with the same domain such that a set is definable over the real closed field if and only if it is definable over the Θ_{rcof} model.

Proposition 3.9 Every model of Θ_{rcof} has an infinite domain.

Proof: Let I be a model of Θ_{rcof}. Assume by contradiction that it is finite. Consider the map $q : D \to D$ such that $q(d) = d^3 +^{\mathsf{F}} (-^{\mathsf{F}} 1^{\mathsf{F}})d$. Then $q(0^{\mathsf{F}}) = 0^{\mathsf{F}}$ and $q(1^{\mathsf{F}}) = 0^{\mathsf{F}}$. Hence q is not surjective and so there is $a \in D$ such that $q(d) \neq a$ for every d in D. That is, $d^3 +^{\mathsf{F}} (-^{\mathsf{F}} 1^{\mathsf{F}})d \neq a$ for every d in D. Hence $d^3 +^{\mathsf{F}} (-^{\mathsf{F}} 1^{\mathsf{F}})d +^{\mathsf{F}} (-^{\mathsf{F}} a) \neq 0^{\mathsf{F}}$ for every d in D. Therefore there are values of x_1, x_2 and x_3 for which the polynomial equation $y^3 + x_1 y^2 + x_2 y + x_3 \cong 0$ does not have a root in D, and so, I cannot satisfy all the axioms in Θ_{rcof}. QED

Let Ω_{rcof} be the set of all atomic formulas, that is, with no loss of generality, formulas of the form

$$q(x_1, \dots, x_n, x) \cong 0 \qquad \text{or} \qquad 0 < q(x_1, \dots, x_n, x).$$

Lemma 3.10 The set Ω_{rcof} is Θ_{rcof}-exhaustive.

Proof:
It is enough to observe that

- $\neg(q(x_1, \dots, x_n, x) \cong 0)$ is equivalent to $(0 < -q(x_1, \dots, x_n, x)) \vee (0 < q(x_1, \dots, x_n, x))$ in Θ_{rcof};
- $\neg(0 < q(x_1, \dots, x_n, x))$ is equivalent to $(q(x_1, \dots, x_n, x) \cong 0) \vee (0 < -q(x_1, \dots, x_n, x))$ in Θ_{rcof}.

QED

Let Ψ_{rcof} be the set of atomic formulas of the form

$$0 < q(x_1, \dots, x_n, x)$$

where x is $(\Theta_{\text{rcof}}, \Omega_{\text{rcof}})$-essential.

Proposition 3.11 The theory Θ_{rcof} is $(\exists, x)$-adequate for Ψ_{rcof} and Ω_{rcof}.

Proof:
Let $h : I \to I'$ be an embedding in $\text{Mod}(\Theta_{\text{rcof}})$, ρ' an assignment over I' such that $\rho'(y) \in h(D)$ for every $y \neq x$ and $\rho'(x) \in D' \setminus h(D)$ and $\bigvee_{i=1}^{n} \delta_i$ a quantifier free formula such that $I'\rho' \Vdash_{\Sigma_{\text{of}}} \delta_i$ for some $i = 1, \dots, n$.

Assume that $0 < q_1(x_{11}, \dots, x_{1n_1}, x), \dots, 0 < q_k(x_{k1}, \dots, x_{kn_k}, x)$ are the literals of Ψ_{rcof} in δ_i. Since $I'\rho' \Vdash_{\Sigma_{\text{of}}} \delta_i$ then $\rho'(x)$ is a solution of the polynomial inequalities $0^{\mathsf{F}'} <^{\mathsf{F}'} q_1^{\mathsf{F}'}(\rho'(x_{11}), \dots, \rho'(x_{1n_1}), x), \dots, 0^{\mathsf{F}'} <^{\mathsf{F}'} q_k^{\mathsf{F}'}(\rho'(x_{k1}), \dots, \rho'(x_{kn_k}), x)$.

Observe that each root of $q_1^{\mathsf{F}'}(\rho'(x_{11}), \dots, \rho'(x_{1n_1}), x)$ in D' is in $h(D)$. Suppose by contradiction that there is a root d' of $q_1^{\mathsf{F}'}(\rho'(x_{11}), \dots, \rho'(x_{1n_1}), x)$ in D' which is not in $h(D)$. Then I' is a proper algebraic extension of $h(I)$. But this can not happen since $h(I)$ is also a model of Θ_{rcof}, by Proposition 2.3, and so does not have any proper formally real algebraic extension.

Let a'_i be the root of $q_i^{\mathsf{F}'}(\rho'(x_{i1}),\ldots,\rho'(x_{in_i}),x)$ less than $\rho'(x)$ closest to $\rho'(x)$ if the polynomial has a root less than $\rho'(x)$, otherwise let a'_i be undefined. Moreover let b'_i be the root of $q_1^{\mathsf{F}'}(\rho'(x_{i1}),\ldots,\rho'(x_{in_i}),x)$ greater than $\rho'(x)$ closest to $\rho'(x)$ if the polynomial has a root greater than $\rho'(x)$, otherwise let b'_i be undefined. Let a' be the maximum of the a'_i that are not undefined, or be undefined if all the a'_i are undefined. Similarly, let b' be the minimum of the b'_i that are not undefined, or be undefined if all the b'_i are undefined.

Let ρ be an assignment over I such that $h\circ\rho \equiv_x \rho'$. Observe that D and D' are both infinite by Proposition 3.9. In order to define $\rho(x)$ consider the following four cases:

(i) a' and b' are defined. Then a' and b' are in $h(D)$, $a' <^{\mathsf{F}'} \rho'(x)$ and $\rho'(x) <^{\mathsf{F}'} b'$. Hence $a' \neq b'$ and so $h^{-1}(a') \neq h^{-1}(b')$. So there are an infinite number of elements of D strictly between $h^{-1}(a')$ and $h^{-1}(b')$ since I is a real closed field. Let $\rho(x)$ be one such element;

(ii) a' is undefined and b' is defined. Then for every $i = 1,\ldots,n$ there is no root in D of $q_i^{\mathsf{F}}(h^{-1}(\rho'(x_{i1})),\ldots,h^{-1}(\rho'(x_{in_i})),x)$ whose image by h is less than $\rho'(x)$. Observe that b' is in $h(D)$. Let $\rho(x)$ be an element in D less than $h^{-1}(b')$;

(iii) a' is defined and b' is undefined. Let $\rho(x)$ be an element in D greater than $h^{-1}(a')$;

(iv) a' and b' are undefined. Let $\rho(x)$ be any element of D.

Hence by the intermediate value theorem and since h is an embedding, $\rho(x)$ is a solution of the inequalities $0^{\mathsf{F}} <^{\mathsf{F}} q_i^{\mathsf{F}}(h^{-1}(\rho'(x_{i1})),\ldots,h^{-1}(\rho'(x_{in_i})),x)$ for $i = 1,\ldots,n$.

Therefore:

(i) Let ν in δ_i be such that $\nu \in \Psi_{\text{rcof}}$. Then $I\rho \Vdash_\Sigma \nu$ by definition of ρ.

(ii) Let $\nu \in \Omega_{\text{rcof}} \backslash \Psi_{\text{rcof}}$ be such that x is $(\Theta_{\text{rcof}},\Omega_{\text{rcof}})$-essential in ν. Then ν is of the form $q(x_1,\ldots,x_n,x) \cong 0$. Assume, by contradiction, that $q(x_1,\ldots,x_n,x) \cong 0 \in \delta_i$. Then $I'\rho' \Vdash_\Sigma q(x_1,\ldots,x_n,x) \cong 0$ and so $\rho'(x)$ is a solution of the polynomial equation $q^{\mathsf{F}'}(\rho'(x_1),\ldots,\rho'(x_n),x) = 0^{\mathsf{F}'}$. Hence, by the observation at the beginning of the proof, $\rho'(x) \in h(D)$ which can not happen by the initial hypothesis over ρ'. So $q(x_1,\ldots,x_n,x) \cong 0 \notin \delta_i$. QED

4 Quantifier elimination via adjunction

In this section we establish sufficient conditions for a theory to have quantifier elimination. We start by recalling and introducing some useful notions and stating some needed results.

A theory Θ over Σ *has quantifier elimination* providing that for every formula φ there is a quantifier free formula φ^* such that $\Theta \vDash_\Sigma \varphi \Leftrightarrow \varphi^*$ and φ and φ^* have the same set of free variables.

Proposition 4.1 Let Θ be a theory over Σ. Assume that for every quantifier free formula ψ, with $\{x, x_1, \ldots, x_n\}$ as the set of free variables, there is a quantifier free formula $\bar{\psi}$, with $\{x_1, \ldots, x_n\}$ as the set of free variables, such that

$$\Theta \vDash_\Sigma (\exists x \psi) \Rightarrow \bar{\psi}.$$

Then, Θ has elimination of quantifiers. $\diamond$

The following result establishes a relationship between the existence of a quantifier free formula equivalent to a given formula in the scope of a theory and the satisfaction of that formula by models of the theory.

Proposition 4.2 Let Σ be a signature, Θ a theory over Σ and φ a formula over Σ. Then the following statements are equivalent:

- there is a quantifier free formula ψ over Σ such that $\text{fv}_\Sigma(\psi) = \text{fv}_\Sigma(\varphi)$ and $\Theta \vDash_\Sigma (\varphi \Leftrightarrow \psi)$;
- $I_1\, g_1 \circ \rho \Vdash_\Sigma \varphi$ if and only if $I_2\, g_2 \circ \rho \Vdash_\Sigma \varphi$ for all ρ over I whenever I_1 and I_2 are models of Θ and I is an interpretation structure such that there is an embedding $g_j : I \to I_j$ for $j = 1, 2$. $\diamond$

For the proof of Proposition 4.2 see [26].

Let Θ be a theory over Σ, $g_j : I \to I_j$ an embedding in $\text{Mod}(\Theta^\forall)$ for $j = 1, 2$ with $I_1, I_2 \in \text{Mod}(\Theta)$. We say that I_1 is *g_1, g_2-equivalent* to I_2 in Θ, written $I_1 \approx^\Theta_{g_1,g_2} I_2$ if

$$I_1 g_1 \circ \rho \Vdash_\Sigma \exists x \varphi \text{ if and only if } I_2 g_2 \circ \rho \Vdash_\Sigma \exists x \varphi$$

for every $x \in \text{fv}_\Sigma(\varphi)$, assignment ρ over I and quantifier free formula φ over Σ.

Proposition 4.3 Let Θ be a theory over Σ and $g_j : I \to I_j$ an embedding in $\text{Mod}(\Theta^\forall)$ for $j = 1, 2$ with $I_1, I_2 \in \text{Mod}(\Theta)$ such that

$$I_1 \approx^\Theta_{g_1,g_2} I_2.$$

Then Θ has quantifier elimination.

Proof:
Let φ be a quantifier free formula such that $x \in \text{fv}_\Sigma(\varphi)$ and ρ an assignment over I. From the hypothesis we can infer:

$$I_1\, g_1 \circ \rho \Vdash (\exists\, x \varphi) \text{ if and only if } I_2\, g_2 \circ \rho \Vdash (\exists\, x \varphi).$$

By Proposition 4.2, we conclude that there is a quantifier free formula ψ with $\text{fv}_\Sigma(\psi) = \text{fv}_\Sigma(\varphi) \setminus \{x\}$ and such that

$$\Theta \vDash_\Sigma (\psi \Leftrightarrow (\exists\, x \varphi)).$$

Finally, by Proposition 4.1, Θ has quantifier elimination. QED

Given functors $F, H : \mathbf{C} \to \mathbf{D}$, a *natural transformation* $\alpha : F \to H$ is a family

$$\alpha = \{\alpha_c : F(c) \to H(c)\}_{c \in |\mathbf{C}|}$$

of morphisms in $\mathbf{D}$ such that

$$H(f) \circ \alpha_{c_1} = \alpha_{c_2} \circ F(f)$$

for every morphism $f : c_1 \to c_2$ in $\mathbf{C}$.

Quantifier elimination can be characterized in a more algebraic way, using the concept of natural transformation, as we now establish in Lemma 4.4.

Lemma 4.4 Let Θ be an $\exists$-adequate theory over Σ. Assume that there is a natural transformation $\eta_{\Theta^\forall} : \mathrm{id}_{\Theta^\forall} \to J_{\Theta,\Theta^\forall} \circ E_{\Theta^\forall,\Theta}$ such that given any embedding $h : I \to J_{\Theta,\Theta^\forall}(I')$ in $\mathrm{Mod}(\Theta^\forall)$, there is a morphism $\bar{h} : E_{\Theta^\forall,\Theta}(I) \to I'$ in $\mathrm{Mod}(\Theta)$ such that $J_{\Theta,\Theta^\forall}(\bar{h}) \circ \eta_{\Theta^\forall I} = h$. Then Θ has quantifier elimination.

Proof: We show that under the hypothesis of the lemma the conditions of Proposition 4.3 hold. Let I'_1 and I'_2 be models in $\mathrm{Mod}(\Theta)$, I an interpretation structure over Σ, ρ an assignment over I, $h_1 : I \to I'_1$ and $h_2 : I \to I'_2$ embeddings and φ a quantifier free formula such that $x \in \mathrm{fv}_\Sigma(\varphi)$. Let $\eta_{\Theta^\forall}$ be a natural transformation satisfying the conditions of the lemma.

Observe that $I \in \mathrm{Mod}(\Theta^\forall)$, by Proposition 2.4, and I'_1 and I'_2 are $J_{\Theta,\Theta^\forall}(I'_1)$ and $J_{\Theta,\Theta^\forall}(I'_2)$, respectively. Therefore, $h_1 : I \to J_{\Theta,\Theta^\forall}(I'_1)$ and $h_2 : I \to J_{\Theta,\Theta^\forall}(I'_2)$ are embeddings in $\mathrm{Mod}(\Theta^\forall)$. Then, using the hypothesis, there are embeddings $\bar{h}_1 : E_{\Theta^\forall,\Theta}(I) \to I'_1$ and $\bar{h}_2 : E_{\Theta^\forall,\Theta}(I) \to I'_2$ in $\mathrm{Mod}(\Theta)$ such that $\bar{h}_1 \circ (\eta_{\Theta^\forall})_I = h_1$ and $\bar{h}_2 \circ (\eta_{\Theta^\forall})_I = h_2$.

According to Proposition 4.3, to conclude that Θ has quantifier elimination it is enough to show that

$$I'_1 h_1 \circ \rho \Vdash_\Sigma \exists x \varphi \ \text{ if and only if } \ I'_2 h_2 \circ \rho \Vdash_\Sigma \exists x \varphi.$$

Suppose that

$$I'_1\, h_1 \circ \rho \Vdash_\Sigma (\exists x\, \varphi).$$

Then, since $\bar{h}_1 \circ (\eta_{\Theta^\forall})_I = h_1$,

$$I'_1\, \bar{h}_1 \circ (\eta_{\Theta^\forall})_I \circ \rho \Vdash_\Sigma (\exists x\, \varphi).$$

Hence, by Proposition 3.1, since Θ is $\exists$-adequate

$$E_{\Theta^\forall,\Theta}(I)\, (\eta_{\Theta^\forall})_I \circ \rho \Vdash_\Sigma (\exists x\, \varphi).$$

Therefore

$$I'_2\, \bar{h}_2 \circ (\eta_{\Theta^\forall})_I \circ \rho \Vdash_\Sigma (\exists x\, \varphi)$$

by Proposition 2.1, and so

$$I'_2\, h_2 \circ \rho \Vdash_\Sigma \exists x \varphi$$

since $\bar{h}_2 \circ (\eta_{\Theta^\forall})_I = h_2$. Similarly for the other direction. QED

It is possible to go further in the algebraic characterization of quantifier elimination by connecting it with the existence of an adjunction between $\mathrm{Mod}(\Theta^\forall)$ and $\mathrm{Mod}(\Theta)$. We now recall the concept of adjunction.

Let $F : \mathbf{C} \to \mathbf{D}$ and $H : \mathbf{D} \to \mathbf{C}$ be functors. Functor F is said to be *left adjoint* of functor H, denoted by

$$F \dashv H$$

if there is a natural transformation

$$\eta : \mathrm{id}_{\mathbf{C}} \to H \circ F,$$

called the *unit* of the adjunction satisfying the following universal property: given any morphism $h : c \to H(d)$ in $\mathbf{C}$, there is a unique morphism $\bar{h} : F(c) \to d$ in $\mathbf{D}$ such that

$$H(\bar{h}) \circ \eta_c = h.$$

We say that there is an *adjunction between categories* $\mathbf{C}$ and $\mathbf{D}$ if there are functors $F : \mathbf{C} \to \mathbf{D}$ and $H : \mathbf{D} \to \mathbf{C}$ such that $F \dashv H$.

Using Lemma 4.4 it is possible to conclude that a theory enjoys quantifier elimination whenever there is a left adjoint for the inclusion functor $J_{\Theta,\Theta^\forall}$ from $\mathrm{Mod}(\Theta)$ to $\mathrm{Mod}(\Theta^\forall)$ and Θ is $\exists$-adequate. We omit its proof since it follows straightforwardly.

Theorem 4.5 Let Θ be an $\exists$-adequate theory over Σ. Assume that there is a left adjoint $E_{\Theta^\forall,\Theta}$ of $J_{\Theta,\Theta^\forall}$. Then Θ has quantifier elimination. $\diamond$

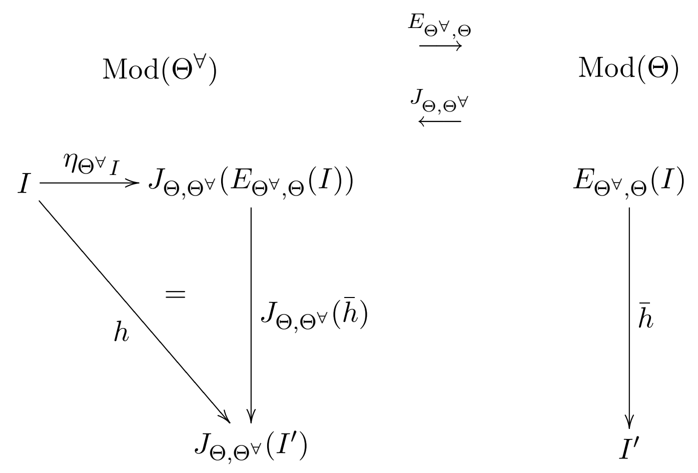

Figure 1: Adjunction between $\mathrm{Mod}(\Theta^\forall)$ and $\mathrm{Mod}(\Theta)$

We now apply Theorem 4.5 to the theories of non-trivial torsion free divisible Abelian groups, Presburger arithmetic and real closed fields. The elimination of quantifiers for Θ_{acf} is used as illustration of the results in Section 5.

Non-trivial torsion free divisible Abelian groups

Due to Theorem 4.5 since Θ_{tfdag} is $\exists$-adequate by Proposition 3.3 it is enough to show that $J_{\Theta_{\text{tfdag}},\Theta^{\forall}_{\text{tfdag}}}$ has a left adjoint in order to conclude that Θ_{tfdag} has quantifier elimination.

Proposition 4.6 Functor $J_{\Theta_{\text{tfdag}},\Theta^{\forall}_{\text{tfdag}}}$ has a left adjoint.

Proof:
Let $E_{\Theta^{\forall}_{\text{tfdag}},\Theta_{\text{tfdag}}}$ be a functor from $\text{Mod}(\Theta^{\forall}_{\text{tfdag}})$ to $\text{Mod}(\Theta_{\text{tfdag}})$ such that:

- $E_{\Theta^{\forall}_{\text{tfdag}},\Theta_{\text{tfdag}}}(I) = \bar{I}$ where $\bar{I}$, denoted by $\bar{I} = (D_\sim, \cdot^{\bar{\mathsf{F}}}, \cdot^{\bar{\mathsf{P}}})$ is the interpretation structure over Σ_{tfdag} defined as follows:
  - the domain $D_\sim$ of $\bar{I}$ is the quotient of the set $\{(d,n) : d \in D, n \in \mathbb{N}, n > 0\}$ by the binary relation $\sim$ such that $(d_1,n_1) \sim (d_2,n_2)$ if $n_2 d_1 = n_1 d_2$;
  - $[(d_1,n_1)] +^{\bar{\mathsf{F}}} [(d_2,n_2)] = [(n_2 d_1 +^{\mathsf{F}} n_1 d_2, n_1 n_2)]$;
  - $-^{\bar{\mathsf{F}}}[(d_1,n_1)] = [(-^{\mathsf{F}} d_1, n_1)]$;
  - $0^{\bar{\mathsf{F}}} = [(0^{\mathsf{F}},1)]$;
- $E_{\Theta^{\forall}_{\text{tfdag}},\Theta_{\text{tfdag}}}(h)([(d_1,n)]) = [(h(d_1),n)]$ for any $h : I_1 \to I_2$;

Observe that $\bar{I}$ is a model of Θ_{tfdag} and $E_{\Theta^{\forall}_{\text{tfdag}},\Theta_{\text{tfdag}}}(h)$ is well defined. That is,

$$E_{\Theta^{\forall}_{\text{tfdag}},\Theta_{\text{tfdag}}}(h)([(d_1,n_1)]) = E_{\Theta^{\forall}_{\text{tfdag}},\Theta_{\text{tfdag}}}(h)([(d_2,n_2)])$$

whenever $(d_2,n_2) \in [(d_1,n_1)]$.

Consider the map $\eta_I : D \to D_\sim$ such that

$$\eta_I(d) = [(d,1)]$$

for each $d \in D$.

1. η_I is a group homomorphism. For instance

$$\begin{aligned}\eta_I(d_1 +^{\mathsf{F}} d_2) &= [(d_1 +^{\mathsf{F}} d_2, 1)] \\ &= [(d_1,1)] +^{\bar{\mathsf{F}}} [(d_2,1)] \\ &= \eta_I(d_1) +^{\bar{\mathsf{F}}} \eta_I(d_2)\end{aligned}$$

The other conditions are omitted since they follow similarly.

2. η_I is injective.
Assume that $\eta_I(d_1) = \eta_I(d_2)$. Then $[(d_1,1)] = [(d_2,1)]$ and by definition of $\sim$ $d_1 = d_2$.

3. η is a natural transformation from $\mathrm{id}_{\mathrm{Mod}(\Theta^{\forall}_{\mathrm{tfdag}})}$ to $J_{\Theta_{\mathrm{tfdag}},\Theta^{\forall}_{\mathrm{tfdag}}} \circ E_{\Theta^{\forall}_{\mathrm{tfdag}},\Theta_{\mathrm{tfdag}}}$.
In fact, given $h : I_1 \to I_2$,

$$\begin{aligned}
((J_{\Theta_{\mathrm{tfdag}},\Theta^{\forall}_{\mathrm{tfdag}}} \circ E_{\Theta^{\forall}_{\mathrm{tfdag}},\Theta_{\mathrm{tfdag}}}(h)) \circ \eta_I)(d_1) &= ((J_{\Theta_{\mathrm{tfdag}},\Theta^{\forall}_{\mathrm{tfdag}}} \circ E_{\Theta^{\forall}_{\mathrm{tfdag}},\Theta_{\mathrm{tfdag}}}(h))(\eta_I(d_1)) \\
&= (J_{\Theta_{\mathrm{tfdag}},\Theta^{\forall}_{\mathrm{tfdag}}} \circ E_{\Theta^{\forall}_{\mathrm{tfdag}},\Theta_{\mathrm{tfdag}}}(h))([(d_1,1)]) \\
&= [(h(d_1),1)] \\
&= \eta_{I_2}(h(d_1)) \\
&= (\eta_{I_2} \circ h)(d_1) \\
&= (\eta_{I_2} \circ \mathrm{id}(h))(d_1)
\end{aligned}$$

4. Universal property.
Let $h : I \to J_{\Theta_{\mathrm{tfdag}},\Theta^{\forall}_{\mathrm{tfdag}}}(I')$ be an embedding in $\mathrm{Mod}(\Theta^{\forall}_{\mathrm{tfdag}})$. Let $\bar{h} : D_{\sim} \to D'$ be a map such that

$$\bar{h}([(d,n)]) = d'$$

where d' is the unique element of I' such that $nd' = h(d)$.
(a) $\bar{h}$ is a group homomorphism.

$$\begin{aligned}
\bar{h}([(d_1,n_1)] +^{\bar{\mathsf{F}}} [(d_2,n_2)]) &= \bar{h}([n_2d_1 +^{\mathsf{F}} n_1d_2, n_2n_1]) \\
&= \frac{h(n_2d_1 +^{\mathsf{F}} n_1d_2)}{n_1n_2} \\
&= \frac{h(n_2d_1)}{n_1n_2} +^{\mathsf{F}'} \frac{h(n_1d_2)}{n_1n_2} \\
&= \frac{h(d_1)}{n_1} +^{\mathsf{F}'} \frac{h(d_2)}{n_2} \\
&= \bar{h}([(d_1,n_1)] +^{\mathsf{F}'} \bar{h}([(d_2,n_2)].
\end{aligned}$$

(b) $\bar{h}$ is injective.
Assume that $\bar{h}([(d_1,n_1)]) = \bar{h}([(d_2,n_2)])$. Then

$$\frac{h(d_1)}{n_1} = \frac{h(d_2)}{n_2},$$

hence $n_2h(d_1) = n_1h(d_1)$ and so, by the injectivity of h, $n_2d_1 = n_1d_2$. Therefore, $(d_1,n_1) \sim (d_2,n_2)$ and so $[(d_1,n_1)] = [(d_2,n_2)]$.
(c) $\bar{h} \circ \eta_{I_1} = h$. In fact:

$$\begin{aligned}
(\bar{h} \circ \eta_{I_1})(d_1) &= \bar{h}(\eta_{I_1}(d_1)) \\
&= \bar{h}([(d_1,1)]) \\
&= h(d_1)
\end{aligned}$$

(d) if $g' \circ \eta_{I_1} = h$ where g' is a embedding in $\text{Mod}(\Theta_{\text{tfdag}})$ from $E_{\Theta^{\forall}_{\text{tfdag}},\Theta_{\text{tfdag}}}(I_1)$ to I' then $g' = \bar{h}$. In fact:

$$\begin{array}{rcl} g'([(d_1,n_1)]) & = & n_1 g'([(d,1)]) \text{ where } n_1[(d,1)] = [(d_1,n_1)] \text{ by } \Theta_{\text{tfdag}} \\ & = & n_1((g' \circ \eta_{I_1})(d)) \\ & = & n_1 h(d) \\ & = & n_1 \bar{h}([(d,1)]) \\ & = & \bar{h}(n_1[(d,1)]) \\ & = & \bar{h}([(d_1,n_1)]). \end{array}$$

Hence, η is the unit of the adjunction. QED

So we can now conclude that Θ_{tfdag} has quantifier elimination.

Theorem 4.7 The theory Θ_{tfdag} has quantifier elimination. $\diamond$

The proof of Theorem 4.7 is omitted since it follows immediately by Theorem 4.5 taking into account that Θ_{tfdag} is $\exists$-adequate by Proposition 3.3 and that $J_{\Theta_{\text{tfdag}},\Theta^{\forall}_{\text{tfdag}}}$ has a left adjoint by Proposition 4.6.

Presburger arithmetic

We start by defining how to obtain a model of Θ_{pa} extending in a minimal way a model of $\Theta^{\forall}_{\text{pa}}$. The minimality of the extension is confirmed when establishing the adjunction in Proposition 4.9. Given $I \in \text{Mod}(\Theta^{\forall}_{\text{pa}})$, let

$$\bar{I} = (\bar{D}, \cdot^{\bar{\mathsf{F}}}, \cdot^{\bar{\mathsf{P}}})$$

be an interpretation structure over Σ_{pa} defined as follows:

- $\bar{D}$ is the quotient of the set $\{(d,n) : d \in D, n \in \mathbb{N} \text{ and } (n = 1 \text{ or } P_n^{\mathsf{P}}(d) = 1)\}$ by the equivalence relation $\sim$ where $(d_1,n_1) \sim (d_2,n_2)$ if and only if $0^{\mathsf{F}} = n_2(-^{\mathsf{F}} d_1) +^{\mathsf{F}} n_1 d_2$;
- $[(d_1,n_1)] +^{\bar{\mathsf{F}}} [(d_2,n_2)] = [(n_2 d_1 +^{\mathsf{F}} n_1 d_2, n_1 n_2)]$;
- $-^{\bar{\mathsf{F}}}[(d_1,n_1)] = [(-^{\mathsf{F}} d_1, n_1)]$;
- $0^{\bar{\mathsf{F}}} = [(0^{\mathsf{F}},1)]$;
- $1^{\bar{\mathsf{F}}} = [(1^{\mathsf{F}},1)]$;
- $[(d_1,n_1)] <^{\bar{\mathsf{P}}} [(d_2,n_2)]$ if and only if $0^{\mathsf{F}} <^{\mathsf{P}} n_2(-^{\mathsf{F}} d_1) +^{\mathsf{F}} n_1 d_2$;
- $P_n{}^{\bar{\mathsf{P}}}(\bar{d}) = \begin{cases} 1 & \text{if } \bar{d} \text{ is } n\bar{d}_1 \text{ for some } n \in \mathbb{N} \text{ and } \bar{d}_1 \in \bar{D} \\ 0 & \text{otherwise} \end{cases}$.

We now show that the interpretation structure $\bar{I}$ defined in this way is indeed a model of Θ_{pa}.

Proposition 4.8 Given $I \in \text{Mod}(\Theta^{\forall}_{\text{pa}})$ then $\bar{I} \in \text{Mod}(\Theta_{\text{pa}})$.

Proof: The proof follows by showing that $\bar{I} \Vdash \theta$ for each θ in Θ_{pa}. For instance:

- $\bar{I} \Vdash \forall x((x + (-x)) \cong 0)$. Let ρ be an assignment over $\bar{I}$. Suppose $\rho(x) = [(d, n)]$. Then

$$\begin{array}{rcl} [(d,n)] +^{\bar{\mathsf{F}}} (-^{\bar{\mathsf{F}}}[(d,n)]) & = & [(d,n)] +^{\bar{\mathsf{F}}} [(-^{\mathsf{F}}d, n)] \\ & = & [(nd +^{\mathsf{F}} n(-^{\mathsf{F}}d), n^2)] \\ & = & [(0, n^2)] \\ & = & 0^{\bar{\mathsf{F}}} \end{array}$$

- $\bar{I} \Vdash 0 < 1$. In fact

$$\begin{array}{rcl} 1(-^{\mathsf{F}}0^{\mathsf{F}}) +^{\mathsf{F}} 1(1^{\mathsf{F}}) & = & 0^{\mathsf{F}} +^{\mathsf{F}} 1^{\mathsf{F}} \\ & = & 1^{\mathsf{F}} \\ & >^{\mathsf{P}} & 0^{\mathsf{F}} \end{array}$$

The proofs of the other cases are omitted since they follow similarly. QED

Proposition 4.9 The functor $J_{\Theta_{\text{pa}},\Theta^{\forall}_{\text{pa}}}$ has a left adjoint.

Proof:
Let $E_{\Theta_{\text{pa}}}$ be the functor from $\text{Mod}(\Theta^{\forall}_{\text{pa}})$ to $\text{Mod}(\Theta_{\text{pa}})$ such that:

- $E_{\Theta_{\text{pa}}}(I) = \bar{I}$;
- $E_{\Theta_{\text{pa}}}(h)([(d_1, n)]) = [(h(d_1), n)]$ for any $h : I_1 \to I_2$;

and η a family $\{\eta_I\}_{I \in \text{Mod}(\Theta^{\forall}_{\text{pa}})}$ of embeddings in $\text{Mod}(\Theta^{\forall}_{\text{pa}})$ where $\eta_I : I \to J_{\Theta_{\text{pa}},\Theta^{\forall}_{\text{pa}}}(E_{\Theta_{\text{pa}}}(I))$ is such that $\eta_I(d) = [(d, 1)]$. Then:

1. η is a natural transformation. Indeed: let $h \in \text{Mod}(\Theta^{\forall}_{\text{pa}})(I_1, I_2)$. Then:

$$\begin{array}{rcl} (\eta_{I_2} \circ \text{id}_{\text{Mod}(\Theta^{\forall}_{\text{pa}})}(h))(d_1) & = & (\eta_{I_2} \circ h)(d_1) \\ & = & \eta_{I_2}(h(d_1)) \\ & = & [(h(d_1), 1)] \\ & = & E_{\Theta_{\text{pa}}}(h)([(d_1, 1)]) \\ & = & J_{\Theta_{\text{pa}},\Theta^{\forall}_{\text{pa}}}(E_{\Theta_{\text{pa}}}(h))([(d_1, 1)]) \\ & = & (J_{\Theta_{\text{pa}},\Theta^{\forall}_{\text{pa}}} \circ E_{\Theta_{\text{pa}}})(h)([(d_1, 1)]) \\ & = & (J_{\Theta_{\text{pa}},\Theta^{\forall}_{\text{pa}}} \circ E_{\Theta_{\text{pa}}})(h)(\eta_{I_1}(d_1)) \\ & = & ((J_{\Theta_{\text{pa}},\Theta^{\forall}_{\text{pa}}} \circ E_{\Theta_{\text{pa}}})(h) \circ \eta_{I_1})(d_1) \end{array}$$

2. let $h : I \to J_{\Theta_{\text{pa}},\Theta^{\forall}_{\text{pa}}}(I')$ be an embedding in $\text{Mod}(\Theta^{\forall}_{pa})$. Consider the embedding $\bar{h}^q$ in $\text{Mod}(\Theta_{pa})$ from $E_{\Theta_{\text{pa}}}(I)$ to I' such that

$$\bar{h}^q([(d,n)]) = \begin{cases} h(d) \text{ if } n = 1 \\ h(y) \text{ otherwise, where } y \in D \text{ such that } d = ny \end{cases}$$

Then $\bar{h}^q \circ \eta_I = h$. In fact

$$\begin{array}{rcl}(\bar{h}^q \circ \eta_I)(d) & = & \bar{h}^q([(d,1)]) \\ & = & h(d)\end{array}$$

3. there is a unique embedding g' in $\mathrm{Mod}(\Theta_{pa})$ from $E_{\Theta_{\mathrm{pa}}}(I)$ to I' such that $J_{\Theta_{\mathrm{pa}},\Theta^{\forall}_{\mathrm{pa}}}(g') \circ \eta_I = h$. Let g' be an embedding in $\mathrm{Mod}(\Theta_{pa})$ from $E_{\Theta_{\mathrm{pa}}}(I)$ to I' such that $J_{\Theta_{\mathrm{pa}},\Theta^{\forall}_{\mathrm{pa}}}(g') \circ \eta_I = h$, and $[(d,n)]$ be an element of $\bar{D}$. Consider two cases:

- $n = 1$. Then

$$\begin{array}{rcl}g'([(d,n)]) & = & g'(\eta_I(d)) \\ & = & J_{\Theta_{\mathrm{pa}},\Theta^{\forall}_{\mathrm{pa}}}(g')(\eta_I(d)) \\ & = & (J_{\Theta_{\mathrm{pa}},\Theta^{\forall}_{\mathrm{pa}}}(g') \circ \eta_I)(d) \\ & = & h(d) \\ & = & \bar{h}^q([(d,n)])\end{array}$$

as we wanted to show;

- $n \neq 1$. Let $y \in D$ be such that $d = ny$. Then $[(d,n)] = [(y,1)]$. So

$$\begin{array}{rcl}g'([(d,n)]) & = & g'([(y,1)]) \\ & = & g'(\eta_I(y)) \\ & = & J_{\Theta_{\mathrm{pa}},\Theta^{\forall}_{\mathrm{pa}}}(g')(\eta_I(y)) \\ & = & (J_{\Theta_{\mathrm{pa}},\Theta^{\forall}_{\mathrm{pa}}}(g') \circ \eta_I)(y) \\ & = & h(y) \\ & = & \bar{h}^q([(d,n)])\end{array}$$

as we wanted to show. QED

Theorem 4.10 The theory Θ_{pa} has quantifier elimination.

Proof:
The result follows by Theorem 4.5 since Θ_{pa} is $\exists$-adequate by Proposition 3.6 and since, by Proposition 4.9, $J_{\Theta_{\mathrm{pa}},\Theta^{\forall}_{\mathrm{pa}}}$ has a left adjoint. QED

Real closed fields

In order to use Theorem 4.5 to prove that Θ_{rcof} is a theory with quantifier elimination, we show in this section that the functor $J_{\Theta_{\mathrm{rcof}},\Theta^{\forall}_{\mathrm{rcof}}}$ has a left adjoint, denoted by $E_{\Theta^{\forall}_{\mathrm{rcof}},\Theta_{\mathrm{rcof}}}$. We capitalize on the fact that the composition of adjunctions is also an adjunction and define the functor $E_{\Theta^{\forall}_{\mathrm{rcof}},\Theta_{\mathrm{rcof}}}$ as the composition of the functor RC with the functor oFF as depicted here:

$$\mathrm{Mod}(\Theta^{\forall}_{\mathrm{rcof}}) \underbrace{\overset{\mathrm{oFF}}{\to} \mathrm{Mod}(\Theta_{\mathrm{of}}) \overset{\mathrm{RC}}{\to}}_{E_{\Theta^{\forall}_{\mathrm{rcof}},\Theta_{\mathrm{rcof}}}} \mathrm{Mod}(\Theta_{\mathrm{rcof}})$$

where oFF is introduced in Proposition 4.11 and RC is introduced in Proposition 4.15.

We recall that a field is said to be *real* or *formally real* if -1 is not a sum of squares, and is *real closed* if and only if it is real and has no proper algebraic extension which is real. It can be proven that a field is orderable if and only if it is real. Moreover a orderable field has characteristic zero. As a consequence, finite fields cannot be ordered. Recall also the basic concept of zero divisors, integral domain and ordered integral domain. More specifically, elements r_1 and r_2 of a ring are said to be *zero divisors* whenever r_1 and r_2 are not zero and its product is zero. An *integral domain* is a commutative ring with no zero divisors such that the zero and the unit are distinct. An *ordered integral domain* is an integral domain with a linear order such that for every elements r, r_1 and r_2 of the ring: (1) $r_1 + r < r_2 + r$ whenever $r_1 < r_2$; and (2) $0 < r_1 \times r_2$ whenever $0 < r_1$ and $0 < r_2$.

Observe that the models of the theory $\Theta^{\forall}_{\mathrm{rcof}}$ are precisely the interpretation structures induced by ordered integral domains.

Proposition 4.11 Let $\mathrm{oFF} = (\mathrm{oFF}_0, \mathrm{oFF}_1)$ be such that, given a model I of $\Theta^{\forall}_{\mathrm{rcof}}$,

$$\mathrm{oFF}_0(I) = (D^*, \cdot^{\mathsf{F}^*}, \cdot^{\mathsf{P}^*}),$$

where:

- D^* is the quotient of the set $(D \times (D \setminus \{0^{\mathsf{F}}\})) \times (D \times (D \setminus \{0^{\mathsf{F}}\}))$ induced by the equivalence relation $\approx$ where $(d_1, d_2) \approx (d_3, d_4)$ if and only if $d_1 \times^{\mathsf{F}} d_4 = d_3 \times^{\mathsf{F}} d_2$;
- $+^{\mathsf{F}^*}([(d_1, d_2)], [(d_3, d_4)]) = [(d_1 \times^{\mathsf{F}} d_4 +^{\mathsf{F}} d_3 \times d_2, d_2 \times^{\mathsf{F}} d_4)]$;
- $-^{\mathsf{F}^*}([(d_1, d_2)]) = [(-^{\mathsf{F}} d_1, d_2)]$;
- $\times^{\mathsf{F}^*}([(d_1, d_2)], [(d_3, d_4)]) = [(d_1 \times^{\mathsf{F}} d_3, d_2 \times^{\mathsf{F}} d_4)]$;
- $0^{\mathsf{F}^*} = [(0^{\mathsf{F}}, d)]$ for some $d \in D \setminus \{0^{\mathsf{F}}\}$;
- $1^{\mathsf{F}^*} = [(1^{\mathsf{F}}, 1^{\mathsf{F}})]$;
- $\cong^{\mathsf{F}^*}([(d_1, d_2)], [(d_3, d_4)]) = 1$ if and only if $[(d_1, d_2)] = [(d_3, d_4)]$;
- $[(d_1, d_2)] <^{\mathsf{P}^*} [(d_3, d_4)]$ whenever
  - either $0^{\mathsf{F}} <^{\mathsf{P}} d_3 \times^{\mathsf{F}} d_2 +^{\mathsf{F}} (-^{\mathsf{F}} d_1) \times^{\mathsf{F}} d_4$ and $0^{\mathsf{F}} <^{\mathsf{P}} d_2 \times^{\mathsf{F}} d_4$;
  - or $d_3 \times^{\mathsf{F}} d_2 +^{\mathsf{F}} (-^{\mathsf{F}} d_1) \times^{\mathsf{F}} d_4 <^{\mathsf{P}} 0^{\mathsf{F}}$ and $d_2 \times^{\mathsf{F}} d_4 <^{\mathsf{P}} 0^{\mathsf{F}}$;

and

$$\mathrm{oFF}_1(h : I \to I') : \mathrm{oFF}(I) \to \mathrm{oFF}(I')$$

is such that

$$\mathrm{oFF}(h)([(d_1, d_2)]) = [(h(d_1), h(d_2))]$$

for every $[(d_1, d_2)]$ in $\mathrm{oFF}(I)$. Then oFF is a functor from $\mathrm{Mod}(\Theta^{\forall}_{\mathrm{rcof}})$ to $\mathrm{Mod}(\Theta_{\mathrm{of}})$.

Proof:

(1) $\mathrm{oFF}_0(I)$ is well defined. In particular, the operations are well defined. For instance, consider the case of $-^{\mathsf{F}^*}$, that is, we show that if $(d_3, d_4) \in [(d_1, d_2)]$ then $(-^{\mathsf{F}} d_3, d_4) \in [(-^{\mathsf{F}} d_1, d_2)]$. Assume that $(d_3, d_4) \in [(d_1, d_2)]$. Then, by definition of $\approx$,

$$d_1 \times^{\mathsf{F}} d_4 = d_3 \times^{\mathsf{F}} d_2$$

and so

$$-^{\mathsf{F}}(d_1 \times^{\mathsf{F}} d_4) = -^{\mathsf{F}}(d_3 \times^{\mathsf{F}} d_2).$$

On the other hand,

$$(d_1 \times^{\mathsf{F}} d_4) +^{\mathsf{F}} ((-^{\mathsf{F}} d_1) \times^{\mathsf{F}} d_4) = 0^{\mathsf{F}},$$

hence

$$((-^{\mathsf{F}} d_1) \times^{\mathsf{F}} d_4) = ((-^{\mathsf{F}} d_3) \times^{\mathsf{F}} d_2)$$

and, therefore, $(-^{\mathsf{F}} d_3, d_4) \in [(-^{\mathsf{F}} d_1, d_2)]$.

(2) $\mathrm{oFF}_0(I)$ is a model of Θ_{of}. For instance:

(i) $\mathrm{oFF}_0(I) \Vdash_\Sigma (\forall x\, ((x + 0) \cong x))$. Indeed, let ρ be an assignment over $\mathrm{oFF}_0(I)$ and $\rho' \equiv_x \rho$. Then

$$\begin{array}{lll} \mathrm{oFF}_0(I)\rho' \Vdash_\Sigma ((x+0) \cong x)) & \text{iff} & \rho(x) +^{\mathsf{F}'} 0^{\mathsf{F}'} = \rho(x) \\ & \text{iff} & +^{\mathsf{F}'}([(d_1, d_2)], [(0^{\mathsf{F}}, d)]) = [(d_1, d_2)] \\ & \text{iff} & [(d_1 \times^{\mathsf{F}} d, d_2 \times^{\mathsf{F}} d)] = [(d_1, d_2)] \\ & \text{iff} & [(d_1, d_2)] = [(d_1, d_2)] \\ & \text{iff} & \text{true} \end{array}$$

(ii) $\mathrm{oFF}_0(I) \Vdash_\Sigma (\forall x \exists y\, ((x \times y) \cong 1))$. In fact, let ρ be an assignment over $\mathrm{oFF}_0(I)$, and assume that $\rho(x) = [(d_1, d_2)]$. Consider $\rho' \equiv_x \rho$ such that $\rho'(x) = (d_2, d_1)$. Then

$$\begin{array}{lll} \mathrm{oFF}_0(I)\rho' \Vdash_\Sigma ((x \times y) \cong 1) & \text{iff} & (\rho'(x) \times^{\mathsf{F}'} \rho'(y)) = [(1^{\mathsf{F}}, 1^{\mathsf{F}})] \\ & \text{iff} & ([(d_1, d_2)] \times^{\mathsf{F}'} [(d_2, d_1)]) = [(1^{\mathsf{F}}, 1^{\mathsf{F}})] \\ & \text{iff} & [(d_1 \times^{\mathsf{F}} d_2, d_2 \times^{\mathsf{F}} d_1)] = [(1^{\mathsf{F}}, 1^{\mathsf{F}})] \\ & \text{iff} & [(d_1 \times^{\mathsf{F}} d_2, d_1 \times^{\mathsf{F}} d_2)] = [(1^{\mathsf{F}}, 1^{\mathsf{F}})] \\ & \text{iff} & \text{true} \end{array}$$

(3) $\mathrm{oFF}_1(h)$ is injective. Indeed:

$$\begin{array}{llll} \mathrm{oFF}_1(h)([(d_1, d_2)]) & = & \mathrm{oFF}_1(h)([(d_3, d_4)]) & (\text{def of } \mathrm{oFF}_1(h)) \text{ iff} \\ [(h(d_1), h(d_2))] & = & [(h(d_3), h(d_4))] & (\text{def of } \approx) \text{ iff} \\ h(d_1) \times^{\mathsf{F}^{*\prime}} h(d_4) & = & h(d_3) \times^{\mathsf{F}^{*\prime}} h(d_2) & h \text{ homomorphism iff} \\ h(d_1 \times^{\mathsf{F}^*} d_4) & = & h(d_3 \times^{\mathsf{F}^*} d_2) & h \text{ injective iff} \\ d_1 \times^{\mathsf{F}^*} d_4 & = & d_3 \times^{\mathsf{F}^*} d_2 & (\text{def of } \approx) \text{ iff} \\ [(d_1, d_2)] & = & [(d_3, d_4)] & \end{array}$$

(4) $\mathrm{oFF}_1(h)$ is an embedding. By (3) and taking into account, for instance, that:

(i) $\mathrm{oFF}_1(h)(0^{\mathsf{F}^*}) = 0^{\mathsf{F}^{*\prime}}$. Indeed:

$$\begin{array}{rcll} \mathrm{oFF}_1(h)(0^{\mathsf{F}^*}) & = & \mathrm{oFF}_1(h)([(0^{\mathsf{F}}, d)]) & \text{def of } 0^{\mathsf{F}^*} \\ & = & [(h(0^{\mathsf{F}}), h(d))] & \text{def of } \mathrm{oFF}_1(h) \\ & = & [(0^{\mathsf{F}'}, h(d))] & h \text{ embedding} \\ & = & 0^{\mathsf{F}^{*\prime}} & \text{def of } 0^{\mathsf{F}^{*\prime}} \end{array}$$

(ii) oFF_1 preserves the order. Assume that $h(d_2) \times^{\mathsf{F}'} h(d_4) \geq^{\mathsf{P}'} 0^{\mathsf{F}'}$ and that $[(d_1, d_2)] <^{\mathsf{P}^*} [(d_3, d_4)]$. Since h is an embedding, then $d_2 \times^{\mathsf{F}} d_4 \geq^{\mathsf{P}} 0^{\mathsf{F}}$. Then

$$0^{\mathsf{F}} <^{\mathsf{P}} d_3 \times^{\mathsf{F}} d_2 +^{\mathsf{F}} (-^{\mathsf{F}} d_1) \times^{\mathsf{F}} d_4$$

and so $0^{\mathsf{F}'} <^{\mathsf{P}'} h(d_3 \times^{\mathsf{F}} d_2 +^{\mathsf{F}} (-^{\mathsf{F}} d_1) \times^{\mathsf{F}} d_4)$. Since, h is an homomorphism

$$0^{\mathsf{F}'} <^{\mathsf{P}'} h(d_3) \times^{\mathsf{F}'} h(d_2) +^{\mathsf{F}'} (-^{\mathsf{F}'} h(d_1)) \times^{\mathsf{F}'} h(d_4))$$

and so $[h(d_1), h(d_2))] <^{\mathsf{P}^{*\prime}} [h(d_3), h(d_4))]$ as we wanted to show.

Similarly for the other function and predicate symbols.

(5) oFF_1 preserves identities and composition. The proof is omitted since it follows straightforwardly by case analysis. QED

Given a model I of $\Theta^{\forall}_{\mathrm{rcof}}$ the interpretation structure $\mathrm{oFF}_0(I)$ is called the *ordered field of fractions* induced by the ordered integral domain I. Observe that $\mathrm{Mod}(\Theta_{\mathrm{of}}) \subseteq \mathrm{Mod}(\Theta^{\forall}_{\mathrm{rcof}})$ since every ordered field is an ordered integral domain. Let $J_{\Theta_{\mathrm{of}},\Theta^{\forall}_{\mathrm{rcof}}} : \mathrm{Mod}(\Theta_{\mathrm{of}}) \to \mathrm{Mod}(\Theta^{\forall}_{\mathrm{rcof}})$ be the inclusion functor.

Proposition 4.12 Functor oFF is left adjoint of the inclusion functor $J_{\Theta_{\mathrm{of}},\Theta^{\forall}_{\mathrm{rcof}}}$.

Proof: Given I in $\mathrm{Mod}(\Theta^{\forall}_{\mathrm{rcof}})$ take $\eta_I : D \to D^*$ such that $\eta_I(d) = [(d, 1^{\mathsf{F}})]$ for every $d \in D$. Then:

(a) η_I is injective. Indeed:

$$\begin{array}{rcll} \eta_I(d_1) & = & \eta_I(d_2) & \text{if and only if} \\ d_1 \times^{\mathsf{F}} 1^{\mathsf{F}} & = & d_2 \times^{\mathsf{F}} 1^{\mathsf{F}} & \text{if and only if} \\ d_1 & = & d_2. & \end{array}$$

(b) $\eta_I : I \to J_{\Theta_{\mathrm{of}},\Theta^{\forall}_{\mathrm{rcof}}}(\mathrm{oFF}(I))$ is an embedding. By (a) and taking into account, for instance, that:

(i) $\eta_I(0^{\mathsf{F}}) = 0^{\mathsf{F}^*}$. Indeed:

$$\eta_I(0^{\mathsf{F}}) = [(0^{\mathsf{F}}, 1^{\mathsf{F}})] = 0^{\mathsf{F}^*}.$$

(ii) $\eta_I(d_1 \times^{\mathsf{F}} d_2) = \eta_I(d_1) \times^{\mathsf{F}^*} \eta_I(d_2)$. Indeed:

$$\begin{array}{rcl} \eta_I(d_1 \times^{\mathsf{F}} d_2) & = & [(d_1 \times^{\mathsf{F}} d_2, 1^{\mathsf{F}})] \\ & = & [(d_1, 1^{\mathsf{F}})] \times^{\mathsf{F}^*} [(d_2, 1^{\mathsf{F}})] \\ & = & \eta_I(d_1) \times^{\mathsf{F}^*} \eta_I(d_2). \end{array}$$

(iii) η_I preserves the order. That is, if $d_1 <^{\mathsf{P}} d_2$ then $\eta_I(d_1) <^{\mathsf{P}^*} \eta_I(d_2)$. Assume that $d_1 <^{\mathsf{P}} d_2$. Then

$$d_2 \times^{\mathsf{F}} 1^{\mathsf{F}} +^{\mathsf{F}} (-^{\mathsf{F}} d_1) \times^{\mathsf{F}} 1^{\mathsf{F}} = d_2 +^{\mathsf{F}} (-^{\mathsf{F}} d_1) > 0^{\mathsf{F}}$$

as we wanted to show.

(c) the family $\eta = \{\eta_I\}_{I \in \mathrm{Mod}(\Theta_{\mathrm{rcof}^{\forall}})}$ is a natural transformation. The proof is omitted since it follows straightforwardly.

We now show that oFF is left adjoint of $J_{\Theta_{\mathrm{of}},\Theta^{\forall}_{\mathrm{rcof}}}$ having η as the unit of the adjunction. Let I be a model of $\Theta^{\forall}_{\mathrm{rcof}}$, I' a model of Θ_{of} and $h : I \to J_{\Theta_{\mathrm{of}},\Theta^{\forall}_{\mathrm{rcof}}}(I')$ a morphism in $\mathrm{Mod}(\Theta^{\forall}_{\mathrm{rcof}})$. Consider the map $\bar{h} : D^* \to D'$ such that:

$$\bar{h}([(d_1, d_2)]) = h(d_1) \times^{\mathsf{F}'} h(d_2)^{-1}.$$

Then:

(1) $\bar{h}$ is injective. Indeed:

$$\begin{array}{lcll}
\bar{h}([(d_1,d_2)]) & = & \bar{h}([(d_3,d_4)]) & \text{iff} \\
h(d_1) \times^{\mathsf{F}'} h(d_2)^{-1} & = & h(d_3) \times^{\mathsf{F}'} h(d_4)^{-1} & \text{iff} \\
h(d_1) \times^{\mathsf{F}'} h(d_4) & = & h(d_2) \times^{\mathsf{F}'} h(d_3) & \text{(inverse) iff} \\
h(d_1 \times^{\mathsf{F}} d_4) & = & h(d_2 \times^{\mathsf{F}} d_3) & (h \text{ homomorphism) iff} \\
d_1 \times^{\mathsf{F}} d_4 & = & d_2 \times^{\mathsf{F}} d_3 & (h \text{ injective) iff} \\
[(d_1,d_2)] & = & [(d_3,d_4)] &
\end{array}$$

(2) $\bar{h}$ is an embedding. By (1) and taking into account that:

$$\begin{array}{lcl}
\bar{h}([(d_1,d_2)] \times^{\mathsf{F}^*} [(d_3,d_4)]) & = & \bar{h}([d_1 \times^{\mathsf{F}^*} d_3, d_2 \times^{\mathsf{F}^*} d_4]) \\
& = & h(d_1 \times^{\mathsf{F}^*} d_3) \times^{\mathsf{F}'} h(d_2 \times^{\mathsf{F}^*} d_4)^{-1} \\
& = & h(d_1) \times^{\mathsf{F}'} h(d_3) \times^{\mathsf{F}'} h(d_4)^{-1} \times^{\mathsf{F}'} h(d_2)^{-1} \\
& = & h(d_1) \times^{\mathsf{F}'} h(d_2)^{-1} \times^{\mathsf{F}'} h(d_3) \times^{\mathsf{F}'} h(d_4)^{-1} \\
& = & \bar{h}([(d_1,d_2)]) \times^{\mathsf{F}'} \bar{h}([(d_3,d_4)])
\end{array}$$

and similarly for the other conditions for function and predicate symbols.

(3) $J_{\Theta_{\mathrm{of}},\Theta^{\forall}_{\mathrm{rcof}}}(\bar{h})(\eta_I(d)) = J_{\Theta_{\mathrm{of}},\Theta^{\forall}_{\mathrm{rcof}}}(\bar{h})([(d, 1^{\mathsf{F}})]) = h(d) \times^{\mathsf{F}'} 1^{\mathsf{F}'} = h(d)$.

(4) Unicity. Assume that $g' : \mathrm{oFF}_0(I) \to I'$ is an embedding such that

$$J_{\Theta_{\mathrm{of}},\Theta^{\forall}_{\mathrm{rcof}}}(g') \circ \eta_I(d) = h(d).$$

Then

$$\begin{array}{lcl}
g'([d_1,d_2]) & = & g'([d_1, 1^{\mathsf{F}}] \times^{\mathsf{F}^*} [d_2, 1^{\mathsf{F}}]^{-1}) \\
& = & g'([d_1, 1^{\mathsf{F}}]) \times^{\mathsf{F}'} g'([d_2, 1^{\mathsf{F}}])^{-1} \\
& = & h(d_1) \times^{\mathsf{F}'} h(d_2)^{-1} \\
& = & \bar{h}([d_1, 1^{\mathsf{F}}]) \times^{\mathsf{F}'} \bar{h}([d_2, 1^{\mathsf{F}}])^{-1} \\
& = & \bar{h}([d_1,d_2])
\end{array}$$

as we wanted to show. QED

The objective now is to define the functor

$$\text{Mod}(\Theta_{\text{of}}) \overset{\text{RC}}{\to} \text{Mod}(\Theta_{\text{rcof}}).$$

For this purpose, we need some results related to the real closure of an ordered field (for more details, see [9]). Recall first that a field F' is called a *real closure* of an ordered field $(F, <)$ whenever: (1) $F' \supseteq F$ is an algebraic extension of F; (2) F' is real closed; and (3) the unique ordering of F' extends the ordering of F, i.e., the inclusion of F into F' preserves the ordering. The results are the following:

Proposition 4.13 Every ordered field has a real closure. Given two real closures of an ordered field there exists a unique isomorphism between them coinciding on the elements of the ordered field. $\diamond$

Proposition 4.14 Let $(F, \leq)$ be an ordered field, R a real closure of $(F, \leq)$ and R' a real closed extension of F whose ordering extends that of F. Then there exists a unique F-homomorphism from R to R'. $\diamond$

Observe that any field homomorphism is either identically zero or is injective, see for instance [12].

Proposition 4.15 The pair $\text{RC} = (\text{RC}_0, \text{RC}_1)$ such that

- $\text{RC}_0(I)$ is a real closure of I;
- $\text{RC}_1(h : I_1 \to I_2)$ is the unique embedding from $\text{RC}_0(I_1)$ to $\text{RC}_0(I_2)$ extending h;

is a functor $\text{RC} : \text{Mod}(\Theta_{\text{of}}) \to \text{Mod}(\Theta_{\text{rcof}})$.

Proof:
The element $\text{RC}_0(I)$ is a real closed field by definition.

The proof that RC_1 preserves identities and composition follows immediately using the uniqueness condition of the embedding, see Proposition 4.14. QED

We now show that RC is left adjoint to the inclusion functor $J_{\Theta_{\text{rcof}},\Theta_{\text{of}}}$.

Proposition 4.16 The functor RC is left adjoint to the inclusion functor $J_{\Theta_{\text{rcof}},\Theta_{\text{of}}}$ from $\text{Mod}(\Theta_{\text{rcof}})$ to $\text{Mod}(\Theta_{\text{of}})$.

Proof:
For each model I of Θ_{of} let η_I be a map from I to $\text{RC}(I)$ such that $\eta_I(d) = d$ for each d in I. It is immediate to conclude that η_I is an embedding. Denote by η the family $\{\eta_I\}_{I \in \text{Mod}(\Theta_{\text{of}})}$. Then η is a natural transformation from $\text{id}_{\text{Mod}(\Theta_{\text{of}})}$ to $J_{\Theta_{\text{rcof}},\Theta_{\text{of}}} \circ \text{RC}$, that is, $\text{RC}(h) \circ \eta_{I_1} = \eta_{I_2} \circ h$. In fact

$$\begin{aligned} \text{RC}(h) \circ \eta_{I_1}(d) &= \text{RC}(h)(d) \\ &= h(d) \\ &= \eta_{I_2}(h(d)) \end{aligned}$$

as we wanted to show;

Let $h : I \to J_{\Theta_{\mathrm{rcof}},\Theta_{\mathrm{of}}}(I')$ be an embedding in $\mathrm{Mod}(\Theta_{\mathrm{of}})$. Then by Proposition 4.14 there is an embedding $\bar{h}$ from $\mathrm{RC}(I)$ to I' that extends h to $\mathrm{RC}(I)$. So $\bar{h} \circ \eta_I = h$. The uniqueness condition is satisfied since by Proposition 4.14 there is a unique embedding from $\mathrm{RC}(I)$ to I' extending h. QED

Theorem 4.17 The theory Θ_{rcof} has elimination of quantifiers.

Proof:
The result follows by Theorem 4.5 since Θ_{rcof} is $\exists$-adequate by Proposition 3.11 and since, by Proposition 4.16, $J_{\Theta_{\mathrm{rcof}},\Theta_{\mathrm{of}}}$ has a left adjoint, by Proposition 4.12, $J_{\Theta_{\mathrm{of}},\Theta^{\forall}_{\mathrm{rcof}}}$ has a left adjoint, and the composition of those left adjoints is the left adjoint of the appropriate composition of the inclusion functors. QED

5 Adjunction from a one step endofunctor

There are cases where a model of Θ extending in a "minimal" way a model of Υ, where Υ is contained in Θ and contains $\Theta^{\forall}$, is obtained from successive applications of a certain construction. This construction applied to a model of Υ produces a model of Υ extending the given one while being "closer", in some sense, to be a model of Θ, such that, by applying the construction at most ω times, it is possible to obtain a model of Θ. Observe that the models of Θ are also models of Υ.

In this section we explicitly define what we consider the ω-"limit" functor E^{ω}_{Υ} of another functor E_{Υ} and provide sufficient conditions over the base functor E_{Υ} in order for the "limit" functor to satisfy the appropriate conditions of our results of quantifier elimination.

Let Υ be a theory over a signature Σ, $E_{\Upsilon} : \mathrm{Mod}(\Upsilon) \to \mathrm{Mod}(\Upsilon)$ an endofunctor and $\eta_{\Upsilon} : \mathrm{id}_{\mathrm{Mod}(\Upsilon)} \to E_{\Upsilon}$ a natural transformation. For $0 \leq \alpha \leq \omega$, consider

$$E^{\alpha}_{\Upsilon} : \mathrm{Mod}(\Upsilon) \to \mathrm{Int}_{\Sigma}$$

and

$$\eta^{\alpha}_{\Upsilon} : J_{\Upsilon,\Sigma} \to E^{\alpha}_{\Upsilon}$$

where $\eta^{\alpha}_{\Upsilon} = \{(\eta^{\alpha}_{\Upsilon})_I : I \to E^{\alpha}_{\Upsilon}(I)\}_{I \in \mathrm{Mod}(\Upsilon)}$, defined as follows:

- $E^{0}_{\Upsilon} = \mathrm{id}_{\mathrm{Mod}(\Upsilon)}$ and $(\eta^{0}_{\Upsilon})_I = \mathrm{id}_I$;
- $E^{\alpha+1}_{\Upsilon} = E_{\Upsilon} \circ E^{\alpha}_{\Upsilon}$ and $(\eta^{\alpha+1}_{\Upsilon})_I = (\eta_{\Upsilon})_{E^{\alpha}_{\Upsilon}(I)} \circ (\eta^{\alpha}_{\Upsilon})_I$;
- $E^{\omega}_{\Upsilon}(I) = (D_{\omega}, \cdot^{\mathsf{F}_{\omega}}, \cdot^{\mathsf{P}_{\omega}})$, where
  - D_{ω} is composed by all sets

    $$[d] = \{d' : (\eta^{k,m}_{\Upsilon})_I(d) = d' \text{ or } (\eta^{k,m}_{\Upsilon})_I(d') = d,\ k, m \in \mathbb{N}\},$$

    for

    $$d \in \bigcup_{0 \leq \gamma < \omega} D_{\gamma}$$

where

$$(\eta_\Upsilon^{k,m})_I : E_\Upsilon^k(I) \to E_\Upsilon^{k+m}(I)$$

is inductively defined as follows: $(\eta_\Upsilon^{k,0})_I$ is $\mathrm{id}_{E_\Upsilon^k(I)}$ and for any positive natural m

$$(\eta_\Upsilon^{k,m})_I = (\eta_\Upsilon)_{E_\Upsilon^{k+m-1}(I)} \circ \cdots \circ (\eta_\Upsilon)_{E_\Upsilon^k(I)};$$

– $f^{\mathsf{F}_\omega}([d_1], \ldots, [d_n]) = [f^{\mathsf{F}_k}((\eta_\Upsilon^{k_1,k-k_1})_I(\mu_{[d_1]}), \ldots, (\eta_\Upsilon^{k_n,k-k_n})_I(\mu_{[d_n]}))];$

– $p^{\mathsf{P}_\omega}([d_1], \ldots, [d_n]) = p^{\mathsf{P}_k}((\eta_\Upsilon^{k_1,k-k_1})_I(\mu_{[d_1]}), \ldots, (\eta_\Upsilon^{k_n,k-k_n})_I(\mu_{[d_n]}));$

where $\mu_{[d_i]} \in [d_i] \cap D_{k_i}$ is such that for every $d_i' \in [d_i]$ there is m_i' such that $d_i' = (\eta_\Upsilon^{k_i,m_i'})_I(\mu_{[d_i]})$ for $i = 1, \ldots, n$ and $k = \max(k_1, \ldots, k_n)$;

- $E_\Upsilon^\omega(h)([d]) = [E_\Upsilon^k(h)(d)]$ where $d \in D_k$;
- $(\eta_\Upsilon^\omega)_I(d) = [d]$.

Observe that

$$(\eta_\Upsilon^{k,n})_I = (\eta_\Upsilon)_{E_\Upsilon^{k+n-1}(I)} \circ (\eta_\Upsilon^{k,n-1})_I.$$

We omit the proof of the next result, since it follows immediately by a straightforward induction taking into account that the composition of functors is also a functor.

Lemma 5.1 For every natural number k, E_Υ^k is a functor from $\mathrm{Mod}(\Upsilon)$ to $\mathrm{Mod}(\Upsilon)$. ◊

We now show that the families $\eta_\Upsilon^{k,n}$ are indeed natural transformations.

Lemma 5.2 For every naturals k and n, the family $\eta_\Upsilon^{k,n}$ is a natural transformation from E_Υ^k to E_Υ^{k+n}.

Proof: The proof follows by induction on n. The base is immediate. Step: Assume that $\eta_\Upsilon^{k,n-1}$ is a natural transformation. Then

$$\begin{aligned} E_\Upsilon^{k+n}(h) \circ (\eta_\Upsilon^{k,n})_I &= E_\Upsilon^{k+n}(h) \circ (\eta_\Upsilon)_{E_\Upsilon^{k+n-1}(I)} \circ (\eta_\Upsilon^{k,n-1})_I \\ &= (\eta_\Upsilon)_{E_\Upsilon^{k+n-1}(I')} \circ E_\Upsilon^{k+n-1}(h) \circ (\eta_\Upsilon^{k,n-1})_I \\ &= (\eta_\Upsilon)_{E_\Upsilon^{k+n-1}(I')} \circ (\eta_\Upsilon^{k,n-1})_{I'} \circ E_\Upsilon^k(h) \\ &= (\eta_\Upsilon^{k,n})_{I'} \circ E_\Upsilon^k(h) \end{aligned}$$

where $h : I \to I'$ is an embedding in $\mathrm{Mod}(\Upsilon)$. QED

The denotation of a term is preserved by the appropriate natural transformation $\eta_\Upsilon^{k,m}$ over the image of an interpretation structure I by functor E_Υ^k, as we now show.

Lemma 5.3 Let t be a term, I a model of Υ, and k and k' natural numbers with $k' \leq k$. Then

$$(\eta_\Upsilon^{k',k-k'})_I(\llbracket t \rrbracket^{E_\Upsilon^{k'}(I)\rho^{k'}}) = \llbracket t \rrbracket^{E_\Upsilon^{k}(I)\rho^{k}}$$

where ρ^k and $\rho^{k'}$ are assignments over $E_\Upsilon^k(I)$ and $E_\Upsilon^{k'}(I)$ respectively such that $\rho^k(x) = (\eta_\Upsilon^{k',k-k'})_I(\rho^{k'}(x))$ for every x occurring in t.

Proof: By induction on t.

Base: t is $x \in X$. Indeed,

$$(\eta_\Upsilon^{k',k-k'})_I(\llbracket x \rrbracket^{E_\Upsilon^{k'}(I)\rho^{k'}}) = (\eta_\Upsilon^{k',k-k'})_I(\rho^{k'}(x)) = \rho^k(x) = \llbracket x \rrbracket^{E_\Upsilon^{k}(I)\rho^{k}}.$$

Step: t is $f(t_1, \ldots, t_n)$. Then,

$$\begin{array}{l}(\eta_\Upsilon^{k',k-k'})_I(\llbracket t \rrbracket^{E_\Upsilon^{k'}(I)\rho^{k'}}) = \\ \quad (\eta_\Upsilon^{k',k-k'})_I(f^{\mathsf{F}_{k'}}(\llbracket t_1 \rrbracket^{E_\Upsilon^{k'}(I)\rho^{k'}}, \ldots, \llbracket t_n \rrbracket^{E_\Upsilon^{k'}(I)\rho^{k'}})) = \\ \quad f^{\mathsf{F}_k}((\eta_\Upsilon^{k',k-k'})_I(\llbracket t_1 \rrbracket^{E_\Upsilon^{k'}(I)\rho^{k'}}), \ldots, (\eta_\Upsilon^{k',k-k'})_I(\llbracket t_n \rrbracket^{E_\Upsilon^{k'}(I)\rho^{k'}})) = \\ \quad f^{\mathsf{F}_k}(\llbracket t_1 \rrbracket^{E_\Upsilon^{k}(I)\rho^{k}}, \ldots, \llbracket t_n \rrbracket^{E_\Upsilon^{k}(I)\rho^{k}}) = \\ \quad \llbracket f(t_1, \ldots, t_n) \rrbracket^{E_\Upsilon^{k}(I)\rho^{k}}\end{array}$$

as we wanted to show. QED

With respect to E_Υ^ω we can only show by now that it is a functor from $\mathrm{Mod}(\Upsilon)$ to Int_Σ and that η_Υ^ω is a natural transformation from $J_{\Upsilon,\Sigma}$ to E_Υ^ω. Later on, we will show in Proposition 5.10 that E_Υ^ω is indeed a functor from $\mathrm{Mod}(\Upsilon)$ to $\mathrm{Mod}(\Upsilon)$ if Υ is contained in $\forall_2$. In this case, η_Υ^ω is in fact a natural transformation from $\mathrm{id}_{\mathrm{Mod}(\Upsilon)}$ to E_Υ^ω.

Furthermore, in Proposition 5.11, we prove that E_Υ^ω is a functor from $\mathrm{Mod}(\Upsilon)$ to $\mathrm{Mod}(\Theta)$ whenever Θ contained in $\forall_2$ contains Υ, and an additional condition is satisfied. Then η_Υ^ω is even a natural transformation from $\mathrm{id}_{\mathrm{Mod}(\Upsilon)}$ to $J_{\Theta,\Upsilon} \circ E_\Upsilon^\omega$ (see Proposition 5.11).

Proposition 5.4 The map E_Υ^ω is a functor from $\mathrm{Mod}(\Upsilon)$ to Int_Σ and η_Υ^ω is a natural transformation from $J_{\Upsilon,\Sigma}$ to E_Υ^ω.

Proof: In fact:

(1) $E_\Upsilon^\omega(I)$ is an interpretation structure. We have to show that f^{F_ω} and p^{P_ω} are well defined. We only show that the first is well defined. Assume that $d_i' \in [d_i]$ for $i = 1, \ldots, n$. We show that

$$f^{\mathsf{F}_\omega}([d_1], \ldots, [d_n]) = f^{\mathsf{F}_\omega}([d_1'], \ldots, [d_n']).$$

Indeed

$$\begin{array}{rcl}f^{\mathsf{F}_\omega}([d_1], \ldots, [d_n]) & = & [f^{\mathsf{F}_k}((\eta_\Upsilon^{k_1,k-k_1})_I(\mu_{[d_1]}), \ldots, (\eta_\Upsilon^{k_n,k-k_n})_I(\mu_{[d_n]}))] \\ & = & [f^{\mathsf{F}_{\beta+k}}((\eta_\Upsilon^{k_1,k-k_1})_I(\mu_{[d_1']}), \ldots, (\eta_\Upsilon^{k_n,k-k_n})_I(\mu_{[d_n']}))] \\ & = & f^{\mathsf{F}_\omega}([d_1'], \ldots, [d_n']).\end{array}$$

(2) $E^\omega_\Upsilon(h)$ is in Int_Σ for every $h : I \to I'$ in $\mathrm{Mod}(\Upsilon)$. In fact:

(i) $E^\omega_\Upsilon(h)$ is well defined. Let $h : I \to I'$ be an embedding in $\mathrm{Mod}(\Upsilon)$ and $[d_1] = [d_2] \in D_\omega$. Assume with no loss of generality that $d_1 \in D_{k_1}$, $d_2 \in D_{k_2}$ and $k_1 \leq k_2$. So $(\eta_\Upsilon^{k_1,k_2-k_1})_I(d_1) = d_2$. Then:

$$\begin{array}{rcll} E^\omega_\Upsilon(h)([d_1]) & = & [E^{k_1}_\Upsilon(h)(d_1)] & \\ & = & [(\eta_\Upsilon^{k_1,k_2-k_1})_I(E^{k_1}_\Upsilon(h)(d_1))] & \\ & = & [E^{k_2}_\Upsilon(h)((\eta_\Upsilon^{k_1,k_2-k_1})_I(d_1))] & \text{by Lemma 5.2} \\ & = & [E^{k_2}_\Upsilon(h)(d_2)] & \\ & = & E^\omega_\Upsilon(h)([d_2]); & \end{array}$$

(ii) $E^\omega_\Upsilon(h)(f^{\mathsf{F}_\omega}([d_1],\ldots,[d_n])) = f^{\mathsf{F}'_\omega}(E^\omega_\Upsilon(h)([d_1]),\ldots,E^\omega_\Upsilon(h)([d_n]))$. Let $h : I \to I'$ be an embedding in $\mathrm{Mod}(\Upsilon)$. Then:

$$\begin{array}{l} E^\omega_\Upsilon(h)(f^{\mathsf{F}_\omega}([d_1],\ldots,[d_n])) = \\ \quad E^\omega_\Upsilon(h)([f^{\mathsf{F}_k}((\eta_\Upsilon^{k_1,k-k_1})_I(\mu_{[d_1]}),\ldots,(\eta_\Upsilon^{k_n,k-k_n})_I(\mu_{[d_n]}))]) = \\ \quad [E^k_\Upsilon(h)(f^{\mathsf{F}_k}((\eta_\Upsilon^{k_1,k-k_1})_I(\mu_{[d_1]}),\ldots,(\eta_\Upsilon^{k_n,k-k_n})_I(\mu_{[d_n]})))] = \\ \quad [f^{\mathsf{F}'_k}(E^k_\Upsilon(h)((\eta_\Upsilon^{k_1,k-k_1})_I(\mu_{[d_1]})),\ldots,E^k_\Upsilon(h)((\eta_\Upsilon^{k_n,k-k_n})_I(\mu_{[d_n]})))] = \\ \quad [f^{\mathsf{F}'_k}((\eta_\Upsilon^{k_1,k-k_1})_{I'}(E^{k_1}_\Upsilon(h)(\mu_{[d_1]})),\ldots,(\eta_\Upsilon^{k_n,k-k_n})_{I'}(E^{k_n}_\Upsilon(h)(\mu_{[d_n]}))))] = \\ \quad [f^{\mathsf{F}'_k}((\eta_\Upsilon^{k'_1,k-k'_1})_{I'}(\mu_1),\ldots,(\eta_\Upsilon^{k'_n,k-k'_n})_{I'}(\mu_n))] = \\ \quad [f^{\mathsf{F}'_k}((\eta_\Upsilon^{k',k-k'})_{I'}((\eta_\Upsilon^{k'_1,k'-k'_1})_{I'}(\mu_1)),\ldots,(\eta_\Upsilon^{k',k-k'})_{I'}((\eta_\Upsilon^{k'_n,k'-k'_n})_{I'}(\mu_n)))] = \\ \quad [(\eta_\Upsilon^{k',k-k'})_{I'}(f^{\mathsf{F}'_{k'}}((\eta_\Upsilon^{k'_1,k'-k'_1})_{I'}(\mu_1),\ldots,(\eta_\Upsilon^{k'_n,k'-k'_n})_{I'}(\mu_n)))] = \\ \quad [f^{\mathsf{F}'_{k'}}((\eta_\Upsilon^{k'_1,k'-k'_1})_{I'}(\mu_1),\ldots,(\eta_\Upsilon^{k'_n,k'-k'_n})_{I'}(\mu_n))] = \\ \quad f^{\mathsf{F}'_\omega}([E^{k_1}_\Upsilon(h)(\mu_{[d_1]})],\ldots,[E^{k_n}_\Upsilon(h)(\mu_{[d_n]})]) = \\ \quad f^{\mathsf{F}'_\omega}(E^\omega_\Upsilon(h)([\mu_{[d_1]}]),\ldots,E^\omega_\Upsilon(h)([\mu_{[d_n]}])) = \\ \quad f^{\mathsf{F}'_\omega}(E^\omega_\Upsilon(h)([d_1]),\ldots,E^\omega_\Upsilon(h)([d_n])). \end{array}$$

where μ_i is $\mu_{[E^{k_i}_\Upsilon(h)(\mu_{[d_i]})]}$ for $i = 1,\ldots,n$. Similarly, for $p \in P$.

(3) E^ω_Υ is a functor from $\mathrm{Mod}(\Upsilon)$ to Int_Σ. Taking into account (1) and (2) it remains to show that:

(i) $E^\omega_\Upsilon(\mathrm{id}_I)([d]) = \mathrm{id}_{E^\omega_\Upsilon(I)}([d])$. In fact:

$$\begin{array}{rcl} E^\omega_\Upsilon(\mathrm{id}_I)([d]) & = & [E^k_\Upsilon(\mathrm{id}_I)(d)] \\ & = & [\mathrm{id}_{E^k_\Upsilon(I)}(d)] \\ & = & [d] \\ & = & \mathrm{id}_{E^\omega_\Upsilon(I)}([d]) \end{array}$$

(ii) $E^\omega_\Upsilon(h' \circ h) = E^\omega_\Upsilon(h') \circ E^\omega_\Upsilon(h)$. Indeed:

$$\begin{array}{rcl} E^\omega_\Upsilon(h' \circ h)([d]) & = & [E^k_\Upsilon(h' \circ h)(d)] \\ & = & [E^k_\Upsilon(h')(E^k_\Upsilon(h)(d)] \\ & = & E^\omega_\Upsilon(h')([E^k_\Upsilon(h)(d)]) \\ & = & E^\omega_\Upsilon(h')(E^\omega_\Upsilon(h)([d])). \end{array}$$

(4) $(\eta^\omega_\Upsilon)_I$ is an embedding in Int_Σ from I to $E^\omega_\Upsilon(I)$ for every I in $\mathrm{Mod}(\Upsilon)$. In fact $(\eta^\omega_\Upsilon)_I(f^{\mathsf{F}}(d_1,\dots,d_n)) = f^{\mathsf{F}_\omega}((\eta^\omega_\Upsilon)_I(d_1),\dots,(\eta^\omega_\Upsilon)_I(d_n))$ as we now show:

$$\begin{array}{l} (\eta^\omega_\Upsilon)_I(f^{\mathsf{F}}(d_1,\dots,d_n)) = \\ \quad [f^{\mathsf{F}}(d_1,\dots,d_n)] = \\ \quad [f^{\mathsf{F}}((\eta^{0,0}_\Upsilon)_I(\mu_{[d_1]}),\dots,(\eta^{0,0}_\Upsilon)_I(\mu_{[d_n]}))] = \\ \quad f^{\mathsf{F}_\omega}([d_1],\dots,[d_n]) = \\ \quad f^{\mathsf{F}_\omega}((\eta^\omega_\Upsilon)_I(d_1),\dots,(\eta^\omega_\Upsilon)_I(d_n)) \end{array}$$

and similarly for p in P.

(5) η^ω_Υ is a natural transformation from $J_{\Upsilon,\Sigma}$ to E^ω_Υ. Taking into account result (4) above it remains to show that:

$$\begin{array}{rcl} E^\omega_\Upsilon(h)((\eta^\omega_\Upsilon)_I(d)) & = & E^\omega_\Upsilon(h)([(\eta_\Upsilon)_I(d)]) \\ & = & [E_\Upsilon(h)((\eta_\Upsilon)_I(d))] \\ & = & [(\eta_\Upsilon)_{I'}(h(d))] \\ & = & (\eta^\omega_\Upsilon)_{I'}(h(d)) \end{array}$$

as we wanted to show. QED

Our next goal is to refine the class where the images of E^ω_Υ belong. For that we will now prove some lemmas relating the behavior of E^ω_Υ with the behavior of E^k_Υ with respect to denotation of terms and satisfaction of formulas.

Lemma 5.5 Let I be a model of Υ, ρ^ω an assignment over $E^\omega_\Upsilon(I)$, t a term with variables $x_1,\dots,x_n$, k_i such that $\mu_{\rho^\omega(x_i)} \in D_{k_i}$, $k = \max(k_1,\dots,k_n)$, and ρ^k an assignment over $E^k_\Upsilon(I)$ such that $\rho^k(x_i) = (\eta^{k_i,k-k_i}_\Upsilon)_I(\mu_{\rho^\omega(x_i)})$. Then

$$[\![t]\!]^{E^\omega_\Upsilon(I)\rho^\omega} = [[\![t]\!]^{E^k_\Upsilon(I)\rho^k}].$$

Proof: By induction on t.
Base: t is x_i for i in $\{1,\dots,n\}$. Indeed,

$$[[\![x_i]\!]^{E^k_\Upsilon(I)\rho^k}] = [\rho^k(x_i)] = [(\eta^{k_i,k-k_i}_\Upsilon)_I(\mu_{\rho^\omega(x_i)})] = [\mu_{\rho^\omega(x_i)}] = \rho^\omega(x_i) = [\![x_i]\!]^{E^\omega_\Upsilon(I)\rho^\omega}.$$

Step: t is $f(t_1,\ldots,t_n)$. Then

$$
\begin{array}{l}
[\llbracket f(t_1,\ldots,t_n)\rrbracket^{E^k_\Upsilon(I)\rho^k}] = \\
\quad [f^{\mathsf{F}_k}(\llbracket t_1\rrbracket^{E^k_\Upsilon(I)\rho^k},\ldots,\llbracket t_n\rrbracket^{E^k_\Upsilon(I)\rho^k})] = \\
\quad [f^{\mathsf{F}_k}((\eta^{k'_1,k-k'_1}_\Upsilon)_I(\llbracket t_1\rrbracket^{E^{k'_1}_\Upsilon(I)\rho^{k'_1}}),\ldots,(\eta^{k'_n,k-k'_n}_\Upsilon)_I(\llbracket t_n\rrbracket^{E^{k'_n}_\Upsilon(I)\rho^{k'_n}})] = \\
\quad [f^{\mathsf{F}_k}((\eta^{k'_1,k-k'_1}_\Upsilon)_I((\eta^{k''_1,k'_1-k''_1}_\Upsilon)_I(\mu_{[d_1]})),\ldots,(\eta^{k'_n,k-k'_n}_\Upsilon)_I((\eta^{k''_n,k'_n-k''_n}_\Upsilon)_I(\mu_{[d_n]}))] = \\
\quad [f^{\mathsf{F}_k}((\eta^{k''_1,k-k''_1}_\Upsilon)_I(\mu_{[d_1]}),\ldots,(\eta^{k''_n,k-k''_n}_\Upsilon)_I(\mu_{[d_n]}))] = \\
\quad [f^{\mathsf{F}_k}((\eta^{k'',k-k''}_\Upsilon)_I((\eta^{k''_1,k''-k''_1}_\Upsilon)_I(\mu_{[d_1]})),\ldots,(\eta^{k'',k-k''}_\Upsilon)_I((\eta^{k''_n,k''-k''_n}_\Upsilon)_I(\mu_{[d_n]}))] = \\
\quad [(\eta^{k'',k-k''}_\Upsilon)_I(f^{\mathsf{F}_{k''}}((\eta^{k''_1,k''-k''_1}_\Upsilon)_I(\mu_{[d_1]}),\ldots,(\eta^{k''_n,k''-k''_n}_\Upsilon)_I(\mu_{[d_n]})))] = \\
\quad [f^{\mathsf{F}_{k''}}((\eta^{k''_1,k''-k''_1}_\Upsilon)_I(\mu_{[d_1]}),\ldots,(\eta^{k''_n,k''-k''_n}_\Upsilon)_I(\mu_{[d_n]}))] = \\
\quad f^{\mathsf{F}_\omega}([\llbracket t_1\rrbracket^{E^{k'_1}_\Upsilon(I)\rho^{k'_1}}],\ldots,[\llbracket t_n\rrbracket^{E^{k'_n}_\Upsilon(I)\rho^{k'_n}}]) = \\
\quad f^{\mathsf{F}_\omega}(\llbracket t_1\rrbracket^{E^\omega_\Upsilon(I)\rho^\omega},\ldots,\llbracket t_n\rrbracket^{E^\omega_\Upsilon(I)\rho^\omega}) = \\
\quad \llbracket f(t_1,\ldots,t_n)\rrbracket^{E^\omega_\Upsilon(I)\rho^\omega}
\end{array}
$$

where $d_i = \llbracket t_i\rrbracket^{E^{k'_i}_\Upsilon(I)\rho^{k'_i}}$ and $k'' = \max(k''_1,\ldots,k''_n)$, using the induction hypothesis and Lemma 5.3. Observe that $k \geq k'_1,\ldots,k'_n$, $k'_i \leq k''_i$ for $i = 1,\ldots,n$ and that $\rho^k(x_i) = (\eta^{k'_i,k-k'_i}_\Upsilon)_I(\rho^{k'_i}(x_i))$ for $i = 1,\ldots,n$. QED

The following result is a direct consequence of Proposition 2.2 since $(\eta^{k,m}_\Upsilon)_I : E^k_\Upsilon(I) \to E^{k+m}_\Upsilon(I)$ is an embedding in $\mathrm{Mod}(\Upsilon)$ and so we omit its proof.

Lemma 5.6 Let φ be a formula in $\exists^+_1$ with free variables $x_1,\ldots,x_n$. Then, for every natural number m, $E^{k+m}_\Upsilon(I)\rho^{k+m} \Vdash_\Sigma \varphi$ whenever $E^k_\Upsilon(I)\rho^k \Vdash_\Sigma \varphi$, where $\rho^{k+m}(x_i) = (\eta^{k,m}_\Upsilon)_I(\rho^k(x_i))$. ◊

We start by showing that satisfaction of quantifier free formulas is preserved from E^k_Υ to E^ω_Υ. It is shown to be preserved from E^ω_Υ to E^k_Υ in order to deal with negation.

Lemma 5.7 Let φ be a quantifier free formula with free variables $x_1,\ldots,x_n$. Then, for every assignment ρ^ω over $E^\omega_\Upsilon(I)$, natural number k_i such that $\mu_{\rho^\omega(x_i)}$ is in D_{k_i}, and natural number k greater than or equal to $\max(k_1,\ldots,k_n)$,

$$E^\omega_\Upsilon(I)\rho^\omega \Vdash_\Sigma \varphi \quad \text{if and only if} \quad E^k_\Upsilon(I)\rho^k \Vdash_\Sigma \varphi$$

where ρ^k is an assignment such that $\rho^k(x_i) = (\eta^{k_i,k-k_i}_\Upsilon)_I(\mu_{\rho^\omega(x_i)})$.

Proof: The proof follows by induction on φ.

Base: φ is $p(t_1, \ldots, t_m)$. Observe that

$$
\begin{aligned}
&[\![p(t_1, \ldots, t_n)]\!]^{E^\omega_\Upsilon(I)\rho^\omega} = \\
&\quad p^{\mathsf{P}_\omega}([\![t_1]\!]^{E^\omega_\Upsilon(I)\rho^\omega}, \ldots, [\![t_n]\!]^{E^\omega_\Upsilon(I)\rho^\omega}) = \\
&\quad p^{\mathsf{P}_\omega}([[\![t_1]\!]^{E^{k'_1}_\Upsilon(I)\rho^{k'_1}}], \ldots, [[\![t_n]\!]^{E^{k'_n}_\Upsilon(I)\rho^{k'_n}}]) = \\
&\quad p^{\mathsf{P}_{k''}}((\eta^{k''_1,k''-k''_1}_\Upsilon)_I(\mu_{[d_1]}), \ldots, (\eta^{k''_n,k''-k''_n}_\Upsilon)_I(\mu_{[d_n]})) = \\
&\quad p^{\mathsf{P}_k}((\eta^{k'',k-k''}_\Upsilon)_I((\eta^{k''_1,k''-k''_1}_\Upsilon)_I(\mu_{[d_1]})), \ldots, (\eta^{k'',k-k''}_\Upsilon)_I((\eta^{k''_n,k''-k''_n}_\Upsilon)_I(\mu_{[d_n]}))) = \\
&\quad p^{\mathsf{P}_k}((\eta^{k''_1,k-k''_1}_\Upsilon)_I(\mu_{[d_1]}), \ldots, (\eta^{k''_n,k-k''_n}_\Upsilon)_I(\mu_{[d_n]})) = \\
&\quad p^{\mathsf{P}_k}((\eta^{k'_1,k-k'_1}_\Upsilon)_I((\eta^{k''_1,k'_1-k''_1}_\Upsilon)_I(\mu_{[d_1]})), \ldots, (\eta^{k'_n,k-k'_n}_\Upsilon)_I((\eta^{k''_n,k'_n-k''_n}_\Upsilon)_I(\mu_{[d_n]}))) = \\
&\quad p^{\mathsf{P}_k}((\eta^{k'_1,k-k'_1}_\Upsilon)_I([\![t_1]\!]^{E^{k'_1}_\Upsilon(I)\rho^{k'_1}}), \ldots, (\eta^{k'_n,k-k'_n}_\Upsilon)_I([\![t_n]\!]^{E^{k'_n}_\Upsilon(I)\rho^{k'_n}})) = \\
&\quad p^{\mathsf{P}_k}([\![t_1]\!]^{E^k_\Upsilon(I)\rho^k}, \ldots, [\![t_n]\!]^{E^k_\Upsilon(I)\rho^k}) = \\
&\quad [\![p(t_1, \ldots, t_n)]\!]^{E^k_\Upsilon(I)\rho^k}
\end{aligned}
$$

where $d_i = [\![t_i]\!]^{E^{k'_i}_\Upsilon(I)\rho^{k'_i}}$, using also Lemma 5.3 and Lemma 5.5. Then $E^\omega_\Upsilon(I)\rho^\omega \Vdash_\Sigma p(t_1, \ldots, t_m)$ if and only if $E^k_\Upsilon(I)\rho^k \Vdash_\Sigma p(t_1, \ldots, t_m)$;

Step:

(a) φ is $\neg\varphi_1$. Let ρ^ω be an assignment over $E^\omega_\Upsilon(I)$, k_i a natural number such that $\mu_{\rho^\omega(x_i)}$ is in D_{k_i}, and k a natural number greater than or equal to $\max(k_1, \ldots, k_n)$. ($\Rightarrow$) Assume that $E^\omega_\Upsilon(I)\rho^\omega \Vdash_\Sigma \neg\varphi_1$. Then $E^\omega_\Upsilon(I)\rho^\omega \not\Vdash_\Sigma \varphi_1$ and so by induction hypothesis $E^k_\Upsilon(I)\rho^k \not\Vdash_\Sigma \varphi_1$. Therefore $E^k_\Upsilon(I)\rho^k \Vdash_\Sigma \varphi$.

The proof of the remaining cases in step follows straightforwardly and so they are omitted. QED

Preservation of satisfaction is extended in Lemma 5.8 to formulas in $\exists^+_1$ capitalizing on the result of Lemma 5.7 for quantifier free formulas. Observe that it is crucial that negations can only be applied in quantifier free subformulas of formulas in $\exists^+_1$.

Lemma 5.8 Let φ be a formula in $\exists^+_1$ with free variables $x_1, \ldots, x_n$. Then, for every assignment ρ^ω over $E^\omega_\Upsilon(I)$, natural number k_i such that $\mu_{\rho^\omega(x_i)} \in D_{k_i}$, and natural number k greater than or equal to $\max(k_1, \ldots, k_n)$,

$$E^\omega_\Upsilon(I)\rho^\omega \Vdash_\Sigma \varphi \text{ whenever } E^k_\Upsilon(I)\rho^k \Vdash_\Sigma \varphi$$

where ρ^k is an assignment such that $\rho^k(x_i) = (\eta^{k_i,k-k_i}_\Upsilon)_I(\mu_{\rho^\omega(x_i)})$.

Proof: The proof follows by induction on the structure of φ in $\exists^+_1$. Indeed:

(i) φ is a quantifier free formula. The result follows by Lemma 5.7;

(ii) φ is $\exists x_{n+1}\psi$ and ψ is in $\exists_1$. Assume that $E^k_\Upsilon(I)\rho^k \Vdash_\Sigma \exists x_{n+1}\psi$ and let $k \geq \max(k_1, \ldots, k_n)$. Let σ^k be an assignment over $E^k_\Upsilon(I)$, x_{n+1}-equivalent to ρ^k with $E^k_\Upsilon(I)\sigma^k \Vdash_\Sigma \psi$. Take k_{n+1} such that $\mu_{[\sigma^k(x_{n+1})]} \in D_{k_{n+1}}$. Note that $k \geq \max(k_1, \ldots, k_n, k_{n+1})$. Let σ^ω be the assignment over $E^\omega_\Upsilon(I)$, x_{n+1}-equivalent to ρ^ω with $\sigma^\omega(x_{n+1}) = [\sigma^k(x_{n+1})]$. Then

$$\sigma^k(x_i) = (\eta_\Upsilon^{k_i, k-k_i})_I(\mu_{\sigma^\omega(x_i)})$$

for $i = 1, \ldots, n+1$. For $i \neq n+1$, the result follows since the assignments are x_{n+1}-equivalents. For $i = n+1$ we have:

$$\sigma^k(x_{n+1}) = (\eta_\Upsilon^{k_{n+1}, k-k_{n+1}})_I(\mu_{[\sigma^k(x_{n+1}])}) = (\eta_\Upsilon^{k_{n+1}, k-k_{n+1}})_I(\mu_{\sigma^\omega(x_{n+1})})$$

and so, by the induction hypothesis, $E^\omega_\Upsilon(I)\sigma^\omega \Vdash_\Sigma \psi$;

The proof of the other cases follows straightforwardly. QED

The extension of the preservation result to formulas in $\forall_2$ imply a strengthening of the sufficient conditions, as we show in Lemma 5.9. The extension to $\forall_2^+$ brings difficulties when the formula is a disjunction of formulas in $\forall_2$, for instance φ_1 and φ_2, since in this case from the hypothesis that $E^k_\Upsilon(I)\rho^k \Vdash_\Sigma \varphi_1 \vee \varphi_2$ for every $k \geq \max(k_1, \ldots, k_n)$ we can neither conclude that $E^{k'}_\Upsilon(I)\rho^{k'} \Vdash_\Sigma \varphi_1$ for every $k' \geq \max(k'_1, \ldots, k'_n)$ nor $E^{k''}_\Upsilon(I)\rho^{k''} \Vdash_\Sigma \varphi_2$ for every $k'' \geq \max(k''_1, \ldots, k''_n)$.

Lemma 5.9 Let φ be a formula in $\forall_2$ with free variables $x_1, \ldots, x_n$. Then

$$E^\omega_\Upsilon(I)\rho^\omega \Vdash_\Sigma \varphi \text{ if for every } k \geq \max(k_1, \ldots, k_n), E^k_\Upsilon(I)\rho^k \Vdash_\Sigma \varphi$$

where ρ^ω and ρ^k are assignments such that $\rho^k(x_i) = (\eta_\Upsilon^{k_i, k-k_i})_I(\mu_{\rho^\omega(x_i)})$ and k_i is such that $\mu_{\rho^\omega(x_i)} \in D_{k_i}$.

Proof: The proof follows by induction on the structure of φ. Indeed:

(i) φ is a quantifier free formula. The result follows by Lemma 5.7.

(ii) φ is in $\exists_1^+$. The result follows by Lemma 5.8.

(iii) φ is $\forall x_{n+1}\psi$ where ψ is in $\forall_2$. Let ρ^ω and ρ^k be assignments fulfilling the hypothesis. Assume $E^k_\Upsilon(I)\rho^k \Vdash_\Sigma \varphi$ for every natural number $k \geq \max(k_1, \ldots, k_n)$. Let σ^ω be an assignment over $E^\omega_\Upsilon(I)$ x_{n+1} equivalent to ρ^ω. Let k' be a natural number greater than or equal to $\max(k_1, \ldots, k_n, k_{n+1})$ where k_{n+1} is such that $\mu_{\sigma^\omega(x_{n+1})} \in D_{k_{n+1}}$. Moreover, let $\rho^{k'}$ be an assignment over $E^{k'}_\Upsilon(I)$ such that ρ^ω and $\rho^{k'}$ are in the conditions of the lemma. Observe that $k' \geq k$. Let $\sigma^{k'}$ be an assignment over $E^{k'}_\Upsilon(I)$ such that $\sigma^{k'}(x_i) = (\eta_\Upsilon^{k_i, k'-k_i})_I(\mu_{\sigma^\omega(x_i)})$ for $i = 1, \ldots, n+1$. Since $\sigma^\omega(x_i) = \rho^\omega(x_i)$ for $i = 1, \ldots, n$ then $(\eta_\Upsilon^{k_i, k'-k_i})_I(\mu_{\sigma^\omega(x_i)}) = (\eta_\Upsilon^{k_i, k'-k_i})_I(\mu_{\rho^\omega(x_i)})$ and so $\sigma^{k'}(x_i) = \rho^{k'}(x_i)$ for $i = 1, \ldots, n$. Hence, we can conclude that $\sigma^{k'}$ is x_{n+1}-equivalent to $\rho^{k'}$. Since $E^{k'}_\Upsilon(I)\rho^{k'} \Vdash_\Sigma \varphi$ then $E^{k'}_\Upsilon(I)\sigma^{k'} \Vdash_\Sigma \psi$. So, using the induction hypothesis, we conclude that $E^\omega_\Upsilon(I)\sigma^\omega \Vdash_\Sigma \psi$. QED

Capitalizing on Lemma 5.9, we can now finally prove that E^ω_Υ is indeed a functor from Mod(Υ) to Mod(Υ) whenever $\Upsilon \subseteq \forall_2$. As mentioned in paragraph immediately before Lemma 5.9, the result cannot be extended to a theory of sentences in $\forall_2^+$.

Proposition 5.10 Assuming that Υ is contained in $\forall_2$ then E^{ω}_{Υ} is a functor from Mod(Υ) to Mod(Υ).

Proof: By Proposition 5.4, it enough to show that, for every I in Mod(Υ), $E^{\omega}_{\Upsilon}(I)$ is a model of Υ. That is,

$$E^{\omega}_{\Upsilon}(I)\rho^{\omega} \Vdash_{\Sigma} \varphi$$

for every φ in Υ and assignment ρ^{ω} over $E^{\omega}_{\Upsilon}(I)$. This fact follows immediately by Lemma 5.9 since $E^{k}_{\Upsilon}(I)\rho^{k} \Vdash_{\Sigma} \varphi$ for every φ in Υ because $E^{k}_{\Upsilon}(I) \in \text{Mod}(\Upsilon)$ by Lemma 5.1. QED

Let Δ be a set of sentences in $\forall_2$ over the same signature Σ as Υ. We say that functor E_{Υ} and natural transformation η_{Υ} *induce satisfaction for* Δ if

$$E_{\Upsilon}(I)(\eta_{\Upsilon})_I \circ \rho \Vdash_{\Sigma} \delta \text{ whenever } I\rho \not\Vdash_{\Sigma} \delta$$

for every model I of Υ, assignment ρ over I and δ in $\exists^{+}_{1}$ with $\forall\delta$ in Δ.

Proposition 5.11 Let Υ and Δ be sets of sentences in $\forall_2$ over the same signature Σ. Assume that E_{Υ} and η_{Υ} induce satisfaction for Δ. Then, E^{ω}_{Υ} is a functor from Mod(Υ) to Mod($\Upsilon \cup \Delta$).

Proof: Due to Proposition 5.10, we only need to verify that the image by E^{ω}_{Υ} of a model of Υ satisfies the sentences of Δ.
Let $\forall\delta$ be a sentence in Δ. Therefore, δ is in $\exists^{+}_{1}$. Assume that $x_1, \dots, x_n$ are the free variables in δ. Let ρ^{ω} be an assignment over $E^{\omega}_{\Upsilon}(I)$ and $k = \max(k_1, \dots, k_n)$ where k_i is such that $\mu_{\rho^{\omega}(x_i)} \in D_{k_i}$. Consider an assignment ρ^k such that $\rho^k(x_i) = (\eta^{k_i,k-k_i}_{\Upsilon})_I(\mu_{\rho^{\omega}(x_i)})$. There are two cases:

(a) $E^{k}_{\Upsilon}(I)\rho^{k} \Vdash_{\Sigma} \delta$. Then $E^{\omega}_{\Upsilon}(I)\rho^{\omega} \Vdash_{\Sigma} \delta$ by Lemma 5.8.

(b) $E^{k}_{\Upsilon}(I)\rho^{k} \not\Vdash_{\Sigma} \delta$. Then $E^{k+1}_{\Upsilon}(I)(\eta_{\Upsilon})_{E^{k}_{\Upsilon}(I)} \circ \rho^k \Vdash_{\Sigma} \delta$. Observe that $k+1 \geq \max(k_1, \dots, k_n)$ and that $(\eta_{\Upsilon})_{E^{k}_{\Upsilon}(I)}(\rho^k(x_i)) = (\eta_{\Upsilon})_{E^{k}_{\Upsilon}(I)}((\eta^{k_i,k-k_i}_{\Upsilon})_I(\mu_{\rho^{\omega}(x_i)})) = (\eta^{k_i,k+1-k_i}_{\Upsilon})_I(\mu_{\rho^{\omega}(x_i)})$. Therefore $E^{\omega}_{\Upsilon}(I)\rho^{\omega} \Vdash_{\Sigma} \delta$ by Lemma 5.8. QED

Moreover, under suitable conditions, η^{ω}_{Υ} is a natural transformation satisfying the conditions of Lemma 4.4, that is, such that, given any embedding $h : I \to J_{\Upsilon\cup\Delta,\Upsilon}(I')$ in Mod(Υ), there is a morphism $\bar{h} : E^{\omega}_{\Upsilon}(I) \to I'$ in Mod($\Upsilon \cup \Delta$) such that $J_{\Upsilon\cup\Delta,\Upsilon}(\bar{h}) \circ \eta^{\omega}_{\Upsilon I} = h$.

We say that the functor E_{Υ} and the natural transformation η_{Υ} *extend to* Δ whenever for every morphism $h : I \to I'$ in Mod(Υ) with I' in Mod($\Upsilon \cup \Delta$) there is a morphism $\bar{h} : E_{\Upsilon}(I) \to I'$ in Mod(Υ) such that $\bar{h} \circ (\eta_{\Upsilon})_I = h$.

Proposition 5.12 Let Υ and Δ be sets contained in $\forall_2$. Assume that E_{Υ} and η_{Υ} induce satisfaction and extend to Δ. Then, given any embedding $h : I \to J_{\Upsilon\cup\Delta,\Upsilon}(I')$ in Mod(Υ) there is a morphism $\bar{h} : E^{\omega}_{\Upsilon}(I) \to I'$ in Mod($\Upsilon \cup \Delta$) such that $J_{\Upsilon\cup\Delta,\Upsilon}(\bar{h}) \circ \eta^{\omega}_{\Upsilon I} = h$.

Proof: Observe that, under the hypothesis, there is a morphism $\bar{h}^k : E^k_\Upsilon(I) \to I'$ such that $\bar{h}^k \circ (\eta^k_\Upsilon)_I = h$ for each natural number k, and that $\bar{h}^0$ is h. So $\bar{h}^k \circ (\eta^{k',k-k'}_\Upsilon)_I = \bar{h}^{k'}$.

Let $h : I \to I'$ in Mod(Υ) with I' in Mod($\Upsilon \cup \Delta$). Consider $\bar{h}^\omega : E^\omega_\Upsilon(I) \to I'$ such that $\bar{h}^\omega([d]) = \bar{h}^k(\mu_{[d]})$ where k is such that $\mu_{[d]}$ is in D_k.

(1) $\bar{h}^\omega$ is well defined. Let $d' \in [d]$. In fact:

$$\begin{array}{rcl}\bar{h}^\omega([d]) & = & \bar{h}^k(\mu_{[d]}) \\ & = & \bar{h}^k(\mu_{[d']}) \\ & = & \bar{h}^\omega([d']).\end{array}$$

(2) $\bar{h}^\omega$ is injective.

Assume that $\bar{h}^\omega([d_1]) = \bar{h}^\omega([d_2])$. Then $\bar{h}^k(\mu_{[d_1]}) = \bar{h}^k(\mu_{[d_2]})$, hence $\mu_{[d_1]} = \mu_{[d_2]}$ and so $[d_1] = [d_2]$.

(3) $\bar{h}^\omega$ is an embedding from $E^\omega_\Upsilon(I)$ to I'. By (2) and taking into account that:

$$\begin{array}{rcl}\bar{h}^\omega(f^{\mathsf{F}_\omega}([d_1], \ldots, [d_n])) & = & \bar{h}^\omega([f^{\mathsf{F}_k}((\eta^{k_1,k-k_1}_\Upsilon)_I(\mu_{[d_1]}), \ldots, (\eta^{k_n,k-k_n}_\Upsilon)_I(\mu_{[d_n]}))]) \\ & = & \bar{h}^{k_\ell}(\mu_{[f^{\mathsf{F}_k}((\eta^{k_1,k-k_1}_\Upsilon)_I(\mu_{[d_1]}),\ldots,(\eta^{k_n,k-k_n}_\Upsilon)_I(\mu_{[d_n]}))]}) \\ & = & \bar{h}^k((\eta^{k_\ell,k-k_\ell}_\Upsilon)_I(\mu_{[f^{\mathsf{F}_k}((\eta^{k_1,k-k_1}_\Upsilon)_I(\mu_{[d_1]}),\ldots,(\eta^{k_n,k-k_n}_\Upsilon)_I(\mu_{[d_n]}))]})) \\ & = & \bar{h}^k(f^{\mathsf{F}_k}((\eta^{k_1,k-k_1}_\Upsilon)_I(\mu_{[d_1]}), \ldots, (\eta^{k_n,k-k_n}_\Upsilon)_I(\mu_{[d_n]}))) \\ & = & f^{\mathsf{F}'}(\bar{h}^k((\eta^{k_1,k-k_1}_\Upsilon)_I(\mu_{[d_1]})), \ldots, \bar{h}^k((\eta^{k_n,k-k_n}_\Upsilon)_I(\mu_{[d_n]}))) \\ & = & f^{\mathsf{F}'}(\bar{h}^{k_1}(\mu_{[d_1]}), \ldots, \bar{h}^{k_n}(\mu_{[d_n]})) \\ & = & f^{\mathsf{F}'}(\bar{h}^\omega([d_1]), \ldots, \bar{h}^\omega([d_n])).\end{array}$$

and similarly for the predicate symbols.

(4) $\bar{h}^\omega \circ (\eta^\omega_\Upsilon)_I = h$.

$$\begin{array}{rcl}\bar{h}^\omega((\eta^\omega_\Upsilon)_I(d)) & = & \bar{h}^\omega([d]) \\ & = & \bar{h}^0(\mu_{[d]}) \\ & = & h(d)\end{array}$$

as we wanted to show. QED

Putting all these conditions together and using Lemma 4.4 it is possible to provide sufficient conditions for quantifier elimination. At the end of the section we apply those sufficient conditions in order to prove that the theory of algebraically closed fields enjoys quantifier elimination.

Theorem 5.13 Let Θ be a theory and Υ a set such that $\Theta^\forall \subseteq \Upsilon \subseteq \Theta$. Assume that:

- $\Theta \subseteq \forall_2$;
- Θ is $\exists$-adequate;
- E_Υ and η_Υ induce satisfaction for and extend to $\Theta \setminus \Upsilon$;

- the inclusion functor $J_{\Upsilon,\Theta^\forall}$ has a left adjoint $E_{\Theta^\forall,\Upsilon}$.

Then Θ has quantifier elimination.

Proof:
Let η be the unit of the adjunction between $\mathrm{Mod}(\Theta^\forall)$ and $\mathrm{Mod}(\Upsilon)$ by the functors $J_{\Upsilon,\Theta^\forall}$ and $E_{\Theta^\forall,\Upsilon}$, and consider the natural transformation $J_{\Upsilon,\Theta^\forall}(\eta^\omega_\Upsilon)\circ\eta = \{(J_{\Upsilon,\Theta^\forall}(\eta^\omega_\Upsilon)\circ\eta)_I : I \to J_{\Upsilon,\Theta^\forall}(J_{\Theta,\Upsilon}(E^\omega_\Upsilon(E_{\Theta^\forall,\Upsilon}(I))))\}_{I\in\mathrm{Mod}(\Theta^\forall)}$ from $\mathrm{id}_{\mathrm{Mod}(\Theta^\forall)}$ to $(J_{\Upsilon,\Theta^\forall}\circ J_{\Theta,\Upsilon})\circ(E^\omega_\Upsilon\circ E_{\Theta^\forall,\Upsilon})$ such that $(J_{\Upsilon,\Theta^\forall}(\eta^\omega_\Upsilon)\circ\eta)_I = J_{\Upsilon,\Theta^\forall}(\eta^\omega_{\Upsilon\, E_{\Theta^\forall,\Upsilon}(I)})\circ\eta_I$.

Let $h : I \to (J_{\Upsilon,\Theta^\forall}\circ J_{\Theta,\Upsilon})(I')$ be an embedding in $\mathrm{Mod}(\Theta^\forall)$. Since $E_{\Theta^\forall,\Upsilon}$ is left adjoint to $J_{\Upsilon,\Theta^\forall}$ then there is an embedding $\bar{\bar{h}} : E_{\Theta^\forall,\Upsilon}(I) \to J_{\Theta,\Upsilon}(I')$ in $\mathrm{Mod}(\Upsilon)$ such that $J_{\Upsilon,\Theta^\forall}(\bar{\bar{h}})\circ\eta_I = h$. Moreover, by Proposition 5.12 and the hypothesis, there is $\bar{h} : E^\omega_\Upsilon(E_{\Theta^\forall,\Upsilon}(I)) \to I'$ in $\mathrm{Mod}(\Theta)$ such that $J_{\Theta,\Upsilon}(\bar{h})\circ\eta^\omega_{\Upsilon\, E_{\Theta^\forall,\Upsilon}(I)} = \bar{\bar{h}}$. Hence $J_{\Upsilon,\Theta^\forall}(J_{\Theta,\Upsilon}(\bar{h}))\circ J_{\Upsilon,\Theta^\forall}(\eta^\omega_{\Upsilon\, E_{\Theta^\forall,\Upsilon}(I)})\circ\eta_I = h$.

So, using the fact that Θ is an $\exists$-adequate theory, by Lemma 4.4, we conclude that Θ has quantifier elimination. QED

When the "extension" condition on the functor E_Υ and natural transformation η_Υ is strengthen by imposing a uniqueness requirement it is possible to characterize quantifier elimination through an adjunction between $\mathrm{Mod}(\Theta^\forall)$ and $\mathrm{Mod}(\Theta)$, capitalizing on Theorem 4.5.

We say that E_Υ and natural transformation η_Υ *universally extend to* Δ whenever for every morphism $h : I \to I'$ in $\mathrm{Mod}(\Upsilon)$ with I' is in $\mathrm{Mod}(\Upsilon\cup\Delta)$ there is a unique morphism $\bar{h} : E_\Upsilon(I) \to I'$ in $\mathrm{Mod}(\Upsilon)$ such that $\bar{h}\circ(\eta_\Upsilon)_I = h$.

Proposition 5.14 Let Υ and Δ be sets contained in $\forall_2$. Assume that E_Υ and η_Υ induce satisfaction and universally extend to Δ. Then E^ω_Υ is left adjoint of $J_{\Upsilon\cup\Delta,\Upsilon}$ with η^ω_Υ as the unit.

Proof:
We only show the uniqueness requirement of the adjunction since the proofs of the other conditions are immediately obtained by the proof of Proposition 5.12.

Let $g : E^\omega_\Upsilon(I) \to I'$ be an embedding such that $g\circ(\eta^\omega_\Upsilon)_I = h$. Let $g^k : E^k_\Upsilon(I) \to I'$ be such that $g^k(d) = g([d])$. We show first that g^k is an embedding and $g^k\circ(\eta^k_\Upsilon)_I = h$.

(a) g^k is injective.
Assume that $g^k(d_1) = g^k(d_2)$. Then $g([d_1]) = g([d_2])$, hence $[d_1] = [d_2]$ and so, since $d_1, d_2 \in D_k$, $d_1 = d_2$.

(b) g^k is an embedding. By (a) and taking into account that:

$$
\begin{aligned}
&g^k(f^{\mathsf{F}_k}(d_1,\ldots,d_n)) = \\
&\quad g([f^{\mathsf{F}_k}(d_1,\ldots,d_n)]) = \\
&\quad g([f^{\mathsf{F}_k}((\eta^{k'_1,k-k'_1}_\Upsilon)_I(\mu_{[d_1]}),\ldots,(\eta^{k'_n,k-k'_n}_\Upsilon)_I(\mu_{[d_n]}))]) = \\
&\quad g([f^{\mathsf{F}_k}((\eta^{k',k-k'}_\Upsilon)_I((\eta^{k'_1,k'-k'_1}_\Upsilon)_I(\mu_{[d_1]})),\ldots,(\eta^{k',k-k'}_\Upsilon)_I((\eta^{k'_n,k'-k'_n}_\Upsilon)_I(\mu_{[d_n]})))]) =
\end{aligned}
$$

$$
\begin{aligned}
&g([(\eta_\Upsilon^{k',k-k'})_I(f^{\mathsf{F}_{k'}}((\eta_\Upsilon^{k'_1,k'-k'_1})_I(\mu_{[d_1]}),\ldots,(\eta_\Upsilon^{k'_n,k'-k'_n})_I(\mu_{[d_n]})))]) = \\
&g([f^{\mathsf{F}_{k'}}((\eta_\Upsilon^{k'_1,k'-k'_1})_I(\mu_{[d_1]}),\ldots,(\eta_\Upsilon^{k'_n,k'-k'_n})_I(\mu_{[d_n]}))]) = \\
&g(f^{\mathsf{F}_\omega}([d_1],\ldots,[d_n])) = \\
&f^{\mathsf{F}'}(g([d_1]),\ldots,g([d_n])) = \\
&f^{\mathsf{F}'}(g^k(d_1),\ldots,g^k(d_n))
\end{aligned}
$$

and similarly for predicate symbols.

(c) $g^k \circ (\eta_\Upsilon^k)_I = h$.

$$
\begin{aligned}
g^k((\eta_\Upsilon^k)_I(d)) = &= g([(\eta_\Upsilon^k)_I(d)]) \\
&= g([d]) \\
&= g((\eta_\Upsilon^\omega)_I(d)) \\
&= h(d).
\end{aligned}
$$

Hence $g^k = \bar{h}^k$. Finally, we conclude that g is $\bar{h}^\omega$:

$$
\begin{aligned}
\bar{h}^\omega([d]) &= \bar{h}^k(\mu_{[d]}) \\
&= g^k(\mu_{[d]}) \\
&= g([\mu_{[d]}]) \\
&= g([d]).
\end{aligned}
$$

Hence η_Υ^ω is the unit of the adjunction. QED

So, if the inclusion functor $J_{\Upsilon,\Theta^\forall}$ has a left adjoint $E_{\Theta^\forall,\Upsilon}$ then $E_\Upsilon^\omega \circ E_{\Theta^\forall,\Upsilon}$ is left adjoint to $J_{\Upsilon,\Theta^\forall} \circ J_{\Theta,\Upsilon}$, since the composition of left adjoints is a left adjoint. Hence, if additionally Θ is an $\exists$-adequate theory, by Theorem 4.5, it is possible to conclude that Θ has quantifier elimination. We have just proved the following theorem, Theorem 5.15.

Theorem 5.15 Let Θ be a theory and Υ a set such that $\Theta^\forall \subseteq \Upsilon \subseteq \Theta$. Assume that:

- $\Theta \subseteq \forall_2$;
- Θ is $\exists$-adequate;
- E_Υ and η_Υ induce satisfaction for and universally extend to $\Theta \setminus \Upsilon$;
- the inclusion functor $J_{\Upsilon,\Theta^\forall}$ has a left adjoint $E_{\Theta^\forall,\Upsilon}$.

Then Θ has quantifier elimination. $\diamond$

Algebraically closed fields

An algebraically closed field extending a field is herein obtained by applying ω-times a certain construction originally proposed by Emil Artin, see [3, 22]. So, for algebraically closed fields, Υ is Θ_f and E_Υ is the functor that associates with each field an algebraic extension of it such that every polynomial in one variable of degree at least one with coefficients in the field, has a root in that algebraic extension.

Recall the theory $\Theta_{\rm acf}$ of algebraically closed fields presented in Section 3. Observe that each model of $\Theta^{\forall}_{\rm acf}$ is an integral domain, since

$$\Theta_{\rm acf} \vDash_{\Sigma_{\rm f}} \forall x_1 \forall x_2\, (((\neg(x_1 \cong 0)) \wedge (\neg(x_2 \cong 0))) \Rightarrow (\neg(x_1 \times x_2) \cong 0)).$$

The functor $E_{\Theta^{\forall}_{\rm acf},\Theta_{\rm acf}}$ is defined by composition as follows:

$$\underbrace{\mathrm{Mod}(\Theta^{\forall}_{\rm acf}) \stackrel{\rm FF}{\rightarrow} \mathrm{Mod}(\Theta_{\rm f}) \stackrel{E^{\omega}_{\Theta_{\rm f},\Theta_{\rm acf}}}{\rightarrow}}_{E_{\Theta^{\forall}_{\rm acf},\Theta_{\rm acf}}} \mathrm{Mod}(\Theta_{\rm acf})$$

where functor FF associates to each integral domain in $\mathrm{Mod}(\Theta^{\forall}_{\rm acf})$ its *field of fractions* and functor $E^{\omega}_{\Theta_{\rm f},\Theta_{\rm acf}}$ associates to each field an algebraically closure field extending it, obtained by the Artin construction.

The functor FF is identical to oFF except for the conditions for the order, see Proposition 4.11. So the proof of Proposition 5.16 and of Proposition 5.17 are omitted since they follow almost identically the corresponding proofs for the functor oFF, that is, the proof of the Proposition 4.11 and of the Proposition 4.12, respectively.

Proposition 5.16 FF is a functor from $\mathrm{Mod}(\Theta^{\forall}_{\rm acf})$ to $\mathrm{Mod}(\Theta_{\rm f})$.

Given a model I of $\Theta^{\forall}_{\rm acf}$ the interpretation structure $\mathrm{FF}_0(I)$ is called the *field of fractions* or the field of quotients induced by the integral domain I. Observe that $\mathrm{Mod}(\Theta_{\rm f}) \subseteq \mathrm{Mod}(\Theta^{\forall}_{\rm acf})$ since every field is an integral domain. Let $J_{\Theta_{\rm f},\Theta^{\forall}_{\rm acf}} : \mathrm{Mod}(\Theta_{\rm f}) \to \mathrm{Mod}(\Theta^{\forall}_{\rm acf})$ be the inclusion functor.

Proposition 5.17 Functor FF is left adjoint of the inclusion functor $J_{\Theta_{\rm f},\Theta^{\forall}_{\rm acf}}$.

Now that we concluded that there is an adjunction between $\mathrm{Mod}(\Theta^{\forall}_{\rm acf})$ and $\mathrm{Mod}(\Theta_{\rm f})$ having the inclusion functor as the right adjoint, we concentrate on $E^{\omega}_{\Theta_{\rm f},\Theta_{\rm acf}}$. For that we follow the work presented in the beginning of this section. In this case Υ is the theory $\Theta_{\rm f}$ and the base functor $E_{\Theta_{\rm f},\Theta_{\rm acf}}$ is the composition of the functors PR and QF such that:

$$\mathrm{Mod}(\Theta_{\rm f}) \stackrel{\rm PR}{\rightarrow} \mathrm{pRg} \stackrel{\rm QF}{\rightarrow} \mathrm{Mod}(\Theta_{\rm f}).$$

where pRg is a category of polynomial rings, PR is a functor that associates to each field a polynomial ring and QF is a functor that associates to each polynomial ring a field generated by the quotient with a specific maximal field. First, we need some preliminary notions.

Given a set V, let $\mathbf{M}(V)$ be the free monoid over V. A *polynomial* over I in $\mathrm{Mod}(\Theta_{\rm f})$ and a set V is a map $q : \mathbf{M}(V) \to I$ with non zero values only for a finite number of elements of $\mathbf{M}(V)$. In general when defining a polynomial we only indicate the non zero values. We may use $\mathbf{M}(v)$ for $\mathbf{M}(\{v\})$. A polynomial q is said to be *constant* whenever $q(m) = 0^{\mathsf{F}}$ for every $m \in \mathbf{M}(V)$ different from

ϵ. A polynomial q over a singleton set V and a field I in $\mathrm{Mod}(\Theta_\mathrm{f})$ is said to be *monic* if $q(m) = 1^\mathsf{F}$ and $q(m') = 0^\mathsf{F}$ for every m' such that $|m'| > |m|$.

Let $I[V]$ be the set of all polynomials over V and a field I in $\mathrm{Mod}(\Theta_\mathrm{f})$. The tuple

$$(I[V], +_{I[V]}, \times_{I[V]}, 0_{I[V]}, 1_{I[V]})$$

where

- $q_1 +_{I[V]} q_2 = q$ such that $q(m) = q_1(m) +^\mathsf{F} q_2(m)$;
- $q_1 \times_{I[V]} q_2 = q$ such that $q(m) = \displaystyle\sum_{\substack{m_1,m_2 \\ m=m_1m_2}} q_1(m_1) \times^\mathsf{F} q_2(m_2)$;
- $0_{I[V]}$ is such that $0_{I[V]}(m) = 0^\mathsf{F}$;
- $1_{I[V]}$ is such that $1_{I[V]}(\epsilon) = 1^\mathsf{F}$ and $1_{I[V]}(m) = 0^\mathsf{F}$ otherwise;

is a ring called the *polynomial ring* over V and I. Let V be a singleton set.

Given I in $\mathrm{Mod}(\Theta_\mathrm{f})$ and a map $\mu : V \to D$, we define the map

$$\mathrm{ev}_\mu : I[V] \to D$$

as follows:

$$\mathrm{ev}_\mu(q) = \sum_{m \in \mathbf{M}(V)} \mu^*(m) \times^\mathsf{F} q(m),$$

where μ^* is the obvious extension of μ to sequences where the concatenation is replaced by product in I. For simplification we may omit $\times^\mathsf{F}$ when considering ev_μ. Observe that ev_μ is a ring homomorphism from $I[V]$ to I.

Given a polynomial q over V and I, we denote by $\mathrm{var}(q)$ the set of elements of V in the sequences in $\{m \in \mathbf{M}(V) : q(m) \neq 0^\mathsf{F}\}$. Observe that this set is finite.

Lemma 5.18 Let q be a polynomial over V and I, $\mu_1 : V \to D$ and $\mu_2 : V \to D$ maps such that $\mu_1(v) = \mu_2(v)$ for every v in $\mathrm{var}(q)$. Then

$$\mathrm{ev}_{\mu_1}(q) = \mathrm{ev}_{\mu_2}(q).$$

$\diamond$

The proof of this lemma is omitted since it follows straightforwardly by definition of ev_μ.

According to Lemma 5.18 it is enough to consider a map $\mu : \mathrm{var}(q) \to D$ when evaluating a polynomial q. A map $\mu : \mathrm{var}(q) \to D$ is a *root* of a polynomial q if $\mathrm{ev}_\mu(q) = 0^\mathsf{F}$. It is common to consider that the root of a polynomial is the values given by μ when applied to $\mathrm{var}(q)$.

It is possible to embed a field in an algebraically closed field. Most of the proofs of this fact are non constructive and involve Zorn's lemma. Herein, we were inspired by the construction proposed by Emil Artin [3] for getting an algebraic closure of a field. We start by setting up a category of polynomial rings.

Let pRg be the category whose *objects* are polynomial rings of the form $I[V_I]$ for each I in Mod(Θ_{f}) where

$$V_I = \{v_q : q \in I[x]^+\}$$

and $I[x]^+$ is the set of all non-constant polynomials in $I[x]$. For each element $d \in I$, let p^I_d be the constant polynomial in $I[V_I]$ such that $p^I_d(\epsilon) = d$ and $p^I_d(m) = 0^{\mathsf{F}}$ for $m \neq \epsilon$, and $p^I_{v_q}$ the polynomial in $I[V_I]$ such that $p^I_{v_q}(v_q) = 1^{\mathsf{F}}$ and $p^I_{v_q}(m) = 0^{\mathsf{F}}$ whenever $m \neq v_q$. Let P_I be the set $\{p^I_d : d \in D\}$ and P_{V_I} the set $\{p^I_{v_q} : v_q \in V_I\}$.

A *morphism in pRg* from $I_1[V_{I_1}]$ to $I_2[V_{I_2}]$ is a map from $I_1[V_{I_1}]$ to $I_2[V_{I_2}]$ induced by an embedding $h : I_1 \to I_2$ in Mod(Θ_{f}), denoted by

$$\widehat{h}$$

such that

$$\widehat{h}(q) = \lambda\, m_2\, . \begin{cases} (h \circ q)(v_{q_{11}} \dots v_{q_{1n}}) & \text{if } m_2 = v_{h\circ q_{11}} \dots v_{h\circ q_{1n}} \\ 0^{\mathsf{F}_2} & \text{otherwise.} \end{cases}$$

The *composition in pRg* of morphisms $\widehat{h}_1 : I_1[V_{I_1}] \to I_2[V_{I_2}]$ and $\widehat{h}_2 : I_2[V_{I_2}] \to I_3[V_{I_3}]$, denoted by $\widehat{h}_2 \circ \widehat{h}_1$, is the morphism $\widehat{h_2 \circ h_1}$.

The *identity morphism in pRg* of an object $I[V_I]$, denoted by $\mathrm{id}_{I[V_I]}$ is the morphism $\widehat{\mathrm{id}_I}$.

Observe that $\widehat{h}(p^{I_1}_{v_{q_1}}) = p^{I_2}_{v_{h\circ q_1}}$ and $\widehat{h}(p^{I_1}_{d_1}) = p^{I_2}_{h(d_1)}$. As expected, pRg constitutes a category.

Proposition 5.19 pRg is a category.

The proof of Proposition 5.19 is omitted since it follows by a not so complicated case analysis from the definition of pRg.

Proposition 5.20 Any morphism in pRg is a ring homomorphism.

Proof:
We only show that $\widehat{h} : I_1[V_{I_1}] \to I_2[V_{I_2}]$ satisfies the homomorphism condition for $+_{I_1[V_{I_1}]}$, that is, $\widehat{h}(q_1 +_{I_1[V_{I_1}]} q_1') = \widehat{h}(q_1) +_{I_2[V_{I_2}]} \widehat{h}(q_1')$, since the other conditions follow similarly. Let m_2 be in $\mathbf{M}(V_{I_2})$. Consider two cases:
(a) m_2 is $v_{h\circ q_{11}} \dots v_{h\circ q_{1n}}$. Then:

$$\begin{aligned}
\widehat{h}(q_1 +_{I_1[V_{I_1}]} q_1')(m_2) &= (h \circ (q_1 +_{I_1[V_{I_1}]} q_1'))(v_{q_{11}} \dots v_{q_{1n}}) \\
&= h((q_1 +_{I_1[V_{I_1}]} q_1')(v_{q_{11}} \dots v_{q_{1n}})) \\
&= h(q_1(v_{q_{11}} \dots v_{q_{1n}}) +^{\mathsf{F}_1} q_1'(v_{q_{11}} \dots v_{q_{1n}})) \\
&= h(q_1(v_{q_{11}} \dots v_{q_{1n}})) +^{\mathsf{F}_2} h(q_1'(v_{q_{11}} \dots v_{q_{1n}})) \\
&= (h \circ q_1)(v_{q_{11}} \dots v_{q_{1n}}) +^{\mathsf{F}_2} (h \circ q_1')(v_{q_{11}} \dots v_{q_{1n}}) \\
&= \widehat{h}(q_1)(v_{h\circ q_{11}} \dots v_{h\circ q_{1n}}) +^{\mathsf{F}_2} \widehat{h}(q_1')(v_{h\circ q_{11}} \dots v_{h\circ q_{1n}}) \\
&= (\widehat{h}(q_1) +_{I_2[V_{I_2}]} \widehat{h}(q_1'))(m_2)
\end{aligned}$$

(b) m_2 is not of the form $v_{h\circ q_{11}} \dots v_{h\circ q_{1n}}$. Then:

$$\begin{aligned}
\widehat{h}(q_1 +_{I_1[V_{I_1}]} q_1')(m_2) &= 0^{\mathsf{F}_2} \\
&= 0^{\mathsf{F}_2} +^{\mathsf{F}_2} 0^{\mathsf{F}_2} \\
&= \widehat{h}(q_1)(m_2) +^{\mathsf{F}_2} \widehat{h}(q_1')(m_2) \\
&= (\widehat{h}(q_1) +_{I_2[V_{I_2}]} \widehat{h}(q_1'))(m_2)
\end{aligned}$$

as we wanted to show. QED

We are now ready to define a functor from the category Mod($\Theta_{\rm f}$) to the category pRg of polynomial rings described above. We omit the proof that it is indeed a functor since it follows straightforwardly.

Proposition 5.21 The pair

$$((\text{PR})_0, (\text{PR})_1)$$

such that:

- $(\text{PR})_0(I) = I[V_I]$;
- $(\text{PR})_1(h) = \widehat{h}$;

is a functor from Mod($\Theta_{\rm f}$) to pRg.

Our objective now is to associate with a polynomial ring $I[V_I]$, as defined above, an element of Mod($\Theta_{\rm f}$), that is a field, such that the polynomials in P_{V_I} are roots of the corresponding polynomials in the image field.

Given a polynomial q in $I[x]^+$ denote by $q^x_{v_q}$ the polynomial in $I[V_I]$ obtained from q by replacing x by v_q, that is, such that $q^x_{v_q}(v_q^n) = q(x^n)$ for every natural number n and $q^x_{v_q}(m) = 0^{\mathsf{F}}$ when m is not of the form v_q^n for some natural number n.

Let J_{V_I} be a maximal ideal of $I[V_I]$ containing the ideal generated by the set of polynomials $q^x_{v_q}$ for q in $I[x]^+$ which can be shown to be proper.

Observe that, given an embedding $h : I_1 \to I_2$ in Mod($\Theta_{\rm f}$) and a polynomial q_1 in $I_1[x]^+$, the polynomial $h \circ q_1$ is in $I_2[x]^+$, and the polynomial $\widehat{h}({q_1}^x_{v_{q_1}})$ is $(h \circ q_1)^x_{v_{h\circ q_1}}$ and so is in the generators of the ideal that is encompassed by $J_{V_{I_2}}$. So, we assume without loss of generality that

$$\widehat{h}(J_{V_{I_1}}) \subseteq J_{V_{I_2}}$$

which is important when defining the functor QF from pRg to Mod($\Theta_{\rm f}$) in Proposition 5.23. Furthermore if the image by $\widehat{h}$ of a polynomial is in $J_{V_{I_2}}$ then that polynomial is in $J_{V_{I_1}}$, see Lemma 5.22.

Lemma 5.22 Given an embedding $h : I_1 \to I_2$ in Mod($\Theta_{\rm f}$) and a polynomial q_1 in $I_1[x]^+$,

if $\widehat{h}(q_1)$ is in J_{V_2} then q_1 is in J_{V_1}.

Moreover, $p_{0^{\mathsf{F}}}$ is the only constant polynomial in J_{V_I}, for any model I of $\Theta_{\rm f}$.

Proof:
(1) If $\widehat{h}(q_1)$ is in J_{V_2} then q_1 is in J_{V_1}. Suppose by contradiction that $\widehat{h}(q_1) \in J_{V_2}$ and $q_1 \notin J_{V_1}$. Then $J_{V_{I_1}} + I_1[V_{I_1}] \times q_1$ is an ideal properly including $J_{V_{I_1}}$. Therefore, since $J_{V_{I_1}}$ is maximal, we can conclude that $J_{V_{I_1}} + I_1[V_{I_1}] \times q_1 = I_1[V_{I_1}]$. In particular, $1_{I_1[V_{I_1}]} = q'' +_{I_1[V_{I_1}]} q' \times_{I_1[V_{I_1}]} q_1$ where $q'' \in J_{V_{I_1}}$ and $q' \in I_1[V_{I_1}]$. Hence $1_{I_2[V_{I_2}]} = \widehat{h}(1_{I_2[V_{I_2}]}) = \widehat{h}(q'' +_{I_1[V_{I_1}]} q' \times_{I_1[V_{I_1}]} q_1) = \widehat{h}(q'') +_{I_2[V_{I_2}]} \widehat{h}(q') \times_{I_2[V_{I_2}]} \widehat{h}(q_1) \in J_{V_{I_2}}$ which can not happen since $J_{V_{I_2}}$ is maximal.

(2) $p_{0^{\mathsf{F}}}$ is the only constant polynomial in J_{V_I}. Suppose by contradiction that $p_d \in J_{V_I}$ and $d \neq 0^{\mathsf{F}}$. Then $p_{1^{\mathsf{F}}} = p_d \times_{I[V_I]} p_{d^{-1}} \in J_{V_I}$ and so J_{V_I} would not be proper, which contradicts the fact that J_{V_I} is maximal. QED

We can now present the functor QF from pRg to Mod(Θ_{f}) that, when composed with PR, will constitute the counterpart, in terms of our study of quantifier elimination for algebraically closed fields, of the functor E_Υ considered in the general study of quantifier elimination presented in the beginning of this section.

Proposition 5.23 The pair

$$((\mathrm{QF})_0, (\mathrm{QF})_1)$$

such that:

- $(\mathrm{QF})_0(I[V_I]) = I_{I[V_I]/J_{V_I}}$ where $I_{I[V_I]/J_{V_I}} = (I[V_I]/J_{V_I}, \cdot^{\mathsf{F}\sim}, \cdot^{\mathsf{P}\sim})$ is such that:
  - $[q_1] +^{\mathsf{F}\sim} [q_2] = [q_1 +_{I[V_I]} q_2]$;
  - $-^{\mathsf{F}\sim}[q] = [-_{I[V_I]}q]$;
  - $[q_1] \times^{\mathsf{F}\sim} [q_2] = [q_1 \times_{I[V_I]} q_2]$;
  - $0^{\mathsf{F}\sim} = [0_{I[V_I]}]$;
  - $1^{\mathsf{F}\sim} = [1_{I[V_I]}]$;
- $(\mathrm{QF})_1(\widehat{h} : I_1[V_{I_1}] \to I_2[V_{I_2}])([q_1]) = [\widehat{h}(q_1)]$;

is a functor from pRg to Mod(Θ_{f}).

Proof:
(1) QF($I[V_I]$) is a field. We only show some of the conditions since the others follow similarly:
(a) $0^{\mathsf{F}\sim} \neq 1^{\mathsf{F}\sim}$. It is enough to note that $1_{I[V_I]} \notin 0^{\mathsf{F}\sim}$. Suppose, by contradiction, that $1_{I[V_I]} \in 0^{\mathsf{F}\sim}$. Then $1_{I[V_I]} +_{I[V_I]} (-_{I[V_I]}0_{I[V_I]}) = 1_{I[V_I]} +_{I[V_I]} 0_{I[V_I]} = 1_{I[V_I]} \in J_{V_I}$. Hence $q \in J_{V_I}$ for every $q \in I[V_I]$, which contradicts the fact that J_{V_I} is proper.
(b) For each $q \in I[V_I]$ such that $[q] \neq 0^{\mathsf{F}\sim}$ there is $[q']$ such that $[q] \times^{\mathsf{F}\sim} [q'] = 1^{\mathsf{F}\sim}$.

Let $q \in I[V_I]$ be such that $[q] \neq 0^{\mathsf{F}_\sim}$. Observe that

$$J_{V_I} +_{I[V_I]} I[V_I] \times_{I[V_I]} q$$

is an ideal properly including J_{V_I}. Therefore, since J_{V_I} is maximal, we can conclude that $J_{V_I} +_{I[V_I]} I[V_I] \times_{I[V_I]} q = I[V_I]$. In particular, $1_{I[V_I]} = q'' +_{I[V_I]} q' \times_{I[V_I]} q$ where $q'' \in J_{V_I}$ and $q' \in I[V_I]$. Therefore $1^{\mathsf{F}_\sim} = [q''] +^{\mathsf{F}_\sim} [q'] \times^{\mathsf{F}_\sim} [q]$ and so, since $[q''] = 0^{\mathsf{F}_\sim}$ we conclude that there is a multiplicative inverse for $[q]$.

(2) $\mathrm{QF}(\widehat{h})$ is well-defined for every morphism $\widehat{h} : I_1[V_{I_1}] \to I_2[V_{I_2}]$ in pRg. It is enough to show that if $q_1 +_{I_1[V_{I_1}]} (-_{I_1[V_{I_1}]} q_2) \in J_{V_{I_1}}$ then $\widehat{h}(q_1) +_{I_2[V_{I_2}]} (-_{I_2[V_{I_2}]} \widehat{h}(q_2)) \in J_{V_{I_2}}$. Assuming that

$$q_1 +_{I_1[V_{I_1}]} (-_{I_1[V_{I_1}]} q_2) \in J_{V_{I_1}}$$

then

$$\widehat{h}(q_1 +_{I_1[V_{I_1}]} (-_{I_1[V_{I_1}]} q_2)) \in J_{V_{I_2}}$$

since we are assuming that $\widehat{h}(J_{V_{I_1}}) \subseteq J_{V_{I_2}}$ as justified in the paragraph immediately before this proposition. So the result follows since $\widehat{h}$ is an homomorphism;

(3) $\mathrm{QF}(\widehat{h})$ is injective for every morphism $\widehat{h} : I_1[V_{I_1}] \to I_2[V_{I_2}]$ in pRg. Assuming that $\mathrm{QF}(\widehat{h})([q_1]) = \mathrm{QF}(\widehat{h})([q_2])$, then $[\widehat{h}(q_1)] = [\widehat{h}(q_2)]$ and so $\widehat{h}(q_1) +_{I_2[V_{I_2}]} (-_{I_2[V_{I_2}]} \widehat{h}(q_2)) \in J_{V_{I_2}}$. Therefore $\widehat{h}(q_1 +_{I_1[V_{I_1}]} (-_{I_1[V_{I_1}]} q_2)) \in J_{V_{I_2}}$, and so by Lemma 5.22, $q_1 +_{I_1[V_{I_1}]} (-_{I_1[V_{I_1}]} q_2) \in J_{V_{I_1}}$ as we wanted to show.

(4) $\mathrm{QF}(\widehat{h})$ is an embedding for every morphism $\widehat{h} : I_1[V_{I_1}] \to I_2[V_{I_2}]$ in pRg. By item (3) and taking into account that:

$$\begin{array}{rcl} \mathrm{QF}(\widehat{h})([q_1] +^{\mathsf{F}_{\sim_1}} [q_2]) & = & \mathrm{QF}(\widehat{h})([q_1 +_{I_1[V_{I_1}]} q_2]) \\ & = & [\widehat{h}(q_1 +_{I_1[V_{I_1}]} q_2)] \\ & = & [\widehat{h}(q_1) +_{I_2[V_{I_2}]} \widehat{h}(q_2)] \\ & = & [\widehat{h}(q_1)] +^{\mathsf{F}_{\sim_2}} [\widehat{h}(q_2)] \\ & = & \mathrm{QF}(\widehat{h})([q_1]) +^{\mathsf{F}_{\sim_2}} \mathrm{QF}(\widehat{h})([q_2])) \end{array}$$

and similarly for the other homomorphic conditions. QED

We now consider a natural transformation from $\mathrm{id}_{\mathrm{Mod}(\Theta_\mathrm{f})}$ to $\mathrm{QF} \circ \mathrm{PR}$ that together with the functor $\mathrm{QF} \circ \mathrm{PR}$, induce satisfaction for $\Theta_\mathrm{acf} \setminus \Theta_\mathrm{f}$, see Proposition 5.26, and extend to $\Theta_\mathrm{acf} \setminus \Theta_\mathrm{f}$, see Proposition 5.27.

Proposition 5.24 The family of maps

$$\{\eta_I : I \to I_{I[V_I]/J_{V_I}}\}_{I \in \mathrm{Mod}(\Theta_f)}$$

such that $\eta_I(d) = [p_d^I]$ is a natural transformation from $\mathrm{id}_{\mathrm{Mod}(\Theta_\mathrm{f})}$ to $\mathrm{QF} \circ \mathrm{PR}$.

Proof:
(1) η_I is injective for each I in $\mathrm{Mod}(\Theta_f)$. Let d_1 and d_2 be distinct elements of D and suppose by contradiction that $p^I_{d_1} +_{I[V_I]} (-_{I[V_I]} p^I_{d_2}) \in J_{V_I}$. Since $p^I_{d_1} +_{I[V_I]} (-_{I[V_I]} p^I_{d_2})$ is the constant polynomial $p^I_{d_1 +^{\mathsf{F}} (-^{\mathsf{F}} d_2)}$ then by Lemma 5.22 $d_1 +^{\mathsf{F}} (-^{\mathsf{F}} d_2) = 0^{\mathsf{F}}$ and so $d_1 = d_2$ which contradicts the initial assumption that d_1 and d_2 are distincts.
(2) η_I is an embedding for each model I of Θ_{f}. By (1) and taking into account that:

$$\begin{aligned}
\mathrm{QF}(\widehat{h})([q_1] +^{\mathsf{F}_{\sim_1}} [q_2]) &= \mathrm{QF}(\widehat{h})([q_1 +_{I_1[V_{I_1}]} q_2]) \\
&= [\widehat{h}(q_1 +_{I_1[V_{I_1}]} q_2)] \\
&= [\widehat{h}(q_1) +_{I_2[V_{I_2}]} \widehat{h}(q_2)] \\
&= [\widehat{h}(q_1)] +^{\mathsf{F}_{\sim_2}} [\widehat{h}(q_2)] \\
&= \mathrm{QF}(\widehat{h})([q_1]) +^{\mathsf{F}_{\sim_2}} \mathrm{QF}(\widehat{h})([q_2]))
\end{aligned}$$

and similarly for the other homomorphic conditions.
(3) $(\mathrm{QF} \circ \mathrm{PR})(h) \circ \eta_{I_1} = \eta_{I_2} \circ h$ for every embedding $h : I_1 \to I_2$ in $\mathrm{Mod}(\Theta_{\mathrm{f}})$. In fact

$$\begin{aligned}
((\mathrm{QF} \circ \mathrm{PR})(h) \circ \eta_{I_1})(d_1) &= (\mathrm{QF}(\mathrm{PR}(h))([p_{d_1}]) \\
&= (\mathrm{QF}(\widehat{h}))([p_{d_1}]) \\
&= [\widehat{h}(p_{d_1})] \\
&= [p_{h(d_1)}] \\
&= \eta_{I_2}(h(d_1)) \\
&= (\eta_{I_2} \circ h)(d_1)
\end{aligned}$$

as we wanted to show. QED

Observe that every non-constant polynomial of $I_{I[V_I]/J_{V_I}}[x]$ with coefficients from I has at least a root in $I_{I[V_I]/J_{V_I}}$. Indeed, in Proposition 5.25, we show that given a non-constant polynomial q in $I[x]$, the polynomial $\eta_I \circ q$ in $I_{I[V_I]/J_{V_I}}[x]$ has at least the element $[p^{I[V_I]}_{v_q}]$ as a root in $I_{I[V_I]/J_{V_I}}$.

Proposition 5.25 Given a model I of Θ_{f} and a non-constant polynomial q in $I[x]$, the map $\mu : \{x\} \to I_{I[V_I]/J_{V_I}}$ such that $\mu(x) = [p^{I[V_I]}_{v_q}]$ is a root of $\eta_I \circ q$ in $\mathrm{PR}(\mathrm{QF}(I))$, that is, in $I_{I[V_I]/J_{V_I}}$.

Proof: Indeed:

$$\begin{aligned}
\mathrm{ev}_\mu(\eta_I \circ q) &= \sum_{m \in \mathbf{M}(x)} \mu^*(m) \times^{\mathsf{F}_\sim} \eta_I(q(m)) \\
&= \sum_{m \in \mathbf{M}(x)} [p^{I[V_I]}_{v_q}]^{|m|} \times^{\mathsf{F}_\sim} [p^{I[V_I]}_{q(m)}] \\
&= [\sum_{m \in \mathbf{M}(x)} (p^{I[V_I]}_{v_q})^{|m|} \times_{I[V_I]} p^{I[V_I]}_{q(m)}] \\
&= [q^x_{v_q}] \\
&= 0^{\mathsf{F}_\sim}
\end{aligned}$$

as we wanted to show. QED

We now show that QF $\circ$ PR and η induce satisfaction for $\Theta_{\mathrm{acf}} \setminus \Theta_{\mathrm{f}}$ and so, using Proposition 5.11, that $(\mathrm{QF} \circ \mathrm{PR})^{\omega}$ is a functor from Θ_{f} to Θ_{acf}.

Proposition 5.26 The functor QF $\circ$ PR and the natural transformation η induce satisfaction for $\Theta_{\mathrm{acf}} \setminus \Theta_{\mathrm{f}}$.

Proof: Let I be a model of Θ_{f}, ρ an assignment over I and δ a formula in $\exists_1^+$ such that $\forall\delta$ is in $\Theta_{\mathrm{acf}} \setminus \Theta_{\mathrm{f}}$. Then δ is of the form $\exists y\,(y^n + x_1 y^{n-1} + \cdots + x_n \cong 0)$ for $n > 0$. Observe that $x_1, \ldots, x_n$ are the free variables of δ. Let n be a natural number greater than 0. Assume that $I\rho \not\Vdash_{\Sigma_{\mathrm{f}}} \exists y\,(y^n + x_1 y^{n-1} + \cdots + x_n \cong 0)$. Let $q : \mathbf{M}(\{x\}) \to I$ be the polynomial such that $q(x^i) = \rho(x_{n-i})$ for $i = 1, \ldots, n-1$, $q(x^n) = 1^{\mathsf{F}}$ and $q(x^j) = 0^{\mathsf{F}}$ for $j > n$. Then, by Proposition 5.25, $[p_{v_q}^{I[V_I]}]$ is a root of $\eta_I \circ q$ in PR(QF(I)), that is, in $I_{I[V_I]/J_{V_I}}$. Consider an assignment ρ' over PR(QF(I)), y-equivalent to $\eta_I \circ \rho$, such that $\rho'(y) = [p_{v_q}^{I[V_I]}]$. Since

$$\mathrm{ev}_{\lambda y.[p_{v_q}^{I[V_I]}]}(\eta_I \circ q) = [\![y^n + x_1 y^{n-1} + \cdots + x_n]\!]^{\mathrm{PR}(\mathrm{QF}(I))\rho'}$$

then $\mathrm{PR}(\mathrm{QF}(I))\rho' \Vdash_{\Sigma_{\mathrm{f}}} y^n + x_1 y^{n-1} + \cdots + x_n \cong 0$ and so $\mathrm{PR}(\mathrm{QF}(I))\eta_I \circ \rho \Vdash_{\Sigma_{\mathrm{f}}} \exists y (y^n + x_1 y^{n-1} + \cdots + x_n \cong 0)$ as we wanted to show. QED

Similarly we show that QF $\circ$ PR and η also extend to $\Theta_{\mathrm{acf}} \setminus \Theta_{\mathrm{f}}$ and so they satisfy the conditions of Proposition 5.12.

Proposition 5.27 The functor QF $\circ$ PR and the natural transformation η extend to $\Theta_{\mathrm{acf}} \setminus \Theta_{\mathrm{f}}$.

Proof: Let $h : I \to I'$ be an embedding in $\mathrm{Mod}(\Theta_{\mathrm{f}})$ with I' in $\mathrm{Mod}(\Theta_{\mathrm{acf}})$. Observe that there are no proper algebraic extensions of I' since it is an algebraically closed field, so $\eta_{I'}$ is an isomorphism from I' to PR(QF(I')). Consider the map

$$\eta_{I'}^{-1} \circ \mathrm{PR}(\mathrm{QF}(h))$$

from PR(QF(I)) to I'. Then

$$\begin{aligned}
(\eta_{I'}^{-1} \circ \mathrm{PR}(\mathrm{QF}(h))) \circ \eta_I &= \eta_{I'}^{-1} \circ (\mathrm{PR}(\mathrm{QF}(h)) \circ \eta_I) \\
&= \eta_{I'}^{-1} \circ (\eta_{I'} \circ h) \\
&= h
\end{aligned}$$

as we wanted to show. QED

Hence we can now conclude that Θ_{acf} has quantifier elimination by Theorem 5.13 since Θ_{acf} is contained in $\forall_2$, Θ_{acf} is $\exists$-adequate by Proposition 3.8 and since, by Proposition 5.26 and Proposition 5.27, the functor QF $\circ$ PR and the natural transformation η induce satisfaction for and extend to $\Theta_{\mathrm{acf}} \setminus \Theta_{\mathrm{f}}$, respectively. So, we have just proved the following theorem, Theorem 5.28.

Theorem 5.28 The theory Θ_{acf} has quantifier elimination.

6 Concluding remarks

We introduce a new (categorial) perspective on model theoretic quantifier elimination. A sufficient condition is proved which states that quantifier elimination holds in a theory Θ whenever there is a close relationship between the category of models of Θ and the category of $\Theta^\forall$ as well as reflection of satisfaction for existential formulas. A systematic way for proving quantifier elimination is devised. We illustrated the results obtained by applying them to some first-order theories.

We believe that the sufficient condition can be used in a wide range of situations in order to prove that a certain theory has quantifier elimination capitalizing on the generality of the approach. Moreover, it seems to avoid some intricacies that can be found in some proofs of quantifier elimination.

We intend to explore the techniques presented herein to establish results about the decidability of a theory Θ that comes as a combination of two decidable theories Θ_1 and Θ_2. Assuming that quantifier elimination was proved in Θ_1 and Θ_2 using our sufficient condition, we want to investigate whether or not the sufficient conditions also hold for Θ, capitalizing and adapting some of the work in [36, 37] on how to combine logics. A lot of research [44] has been under way about preservation of decidability when combining theories since the work of [29, 39]. Most of the methods are either syntactic or algorithmic, and not algebraic/categorial as we intend to pursue, based on the results obtained herein.

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