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Independence structures on quantales

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Applied Mathematics and Computation
Diploma Thesis

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May 2002

Acknowledgments

Firstly, I would like to thank my supervisor, Pedro Resende, for his guidance and support before and during the elaboration of this thesis.

I would also like to thank professors Amílcar and Cristina Sernadas for their constant support and care about my work.

I wish to thank my parents and my brother for their love and infinite patience.

Finally, I thank Alexandre for his patience and friendship.

Contents

List of symbols

$Q(L)$	Quantale of sup-lattice endomorphisms of L
$Q(C)$	Quantale of small category C
$\bigwedge$	Infimum
$\bigvee$	Supremum
$f[S]$	Image of a set S by a map f
SL	Category of sup-lattices

Chapter 1

Introduction

A system with many actions that run in parallel, and which can interact with each other, is designated by concurrent system. In computer science, such systems appear in the form of multiprocessor or distributed systems. The study of these systems and their interaction with the environment leads to concurrency theory, which provides mathematical models of concurrent systems.

Quantales were introduced in [Mul86] with the purpose of studying the spectrum of C*-algebras and the foundations of quantum mechanics. From this last point of view, a quantale is a semigroup whose multiplication $a\&b$ can be temporally interpreted as "a and then b". This idea has also appeared in [Yet90], when studying non-commutative versions of the linear logic of Girard [Gir87], and later in [AV93], where a quantale can be understood as an algebra of observations on concurrent systems. This work and [Res01, RVnt] study interleaving concurrency models such as trace equivalence, acceptance-trace equivalence, failure-trace equivalence, ready-trace equivalence, simulation, ready-simulation and bisimulation, (see also [Res00]).

In these works, such models are uniformly studied in an algebraic framework based on quantales. However, this is not done for other models of concurrency, of the so-called "true-concurrency", in particular geometrical models as in [Pra91], in which concurrency is associated with the dimension of a space. This kind of concurrency is preliminarly studied in [Res98, Res99b], using models based on quantales although not directly involving the geometrical ideas.

An interesting idea is to extend notions of "true concurrency" to the quantale based framework, in particular relating them to geometric models. Another would be to study the possibility of relating interleaving models and true concurrency via quantales, in order to treat simultaneously both aspects.

In the present work, we will apply quantales to true concurrency, via a notion of independence structure that we define, and we verify how it arises in two concurrency models. Firstly they will be applied to event structures, which is a well known model of concurrent systems. Then they will be extended to a geometric model. In this last case, which will enable us to study a continuous version of concurrent systems, higher dimensions appear in a way akin to other geometric models, e.g. [Pra91, Gou00].

In Chapter 2, we describe the basic concepts of sup-lattice and quantale as well as stating some of their properties.

The aim of Chapter 3 is to discuss the characterization of the finite observations over a system as forming a quantale. More specifically, if S is a system, finite observations over S form a quantale whose multiplication describes the sequential composition of finite observations. We describe how to represent the dynamics of a system by a relational representation of a quantale of finite observations (as is done in [AV93, Res99b, Res00, Res01]). Also, we motivate a notion of independence between two finite observations over a system and a general definition is given in terms of so-called independence structures on quantales.

Event structures [NPW81] are described in Chapter 4. The aim of this chapter is to illustrate the application of quantales to a well known model of true concurrency, and to assess the applicability of independence structures in this case.

Finally, in Chapter 5, we study the possibility of characterizing systems by a continuous model. In order to do this, we represent each system by a topological space, where states are points of the space, and each action is expressed by a set of (higher dimensional) paths that join two points of the space, the states before and after the action's occurrence. Similarly to what is done in Chapter 3, we will study the ideas discussed in Chapter 2 in this context. We conclude by examining some properties of independence structure in this model. In particular, we remark that, whereas in Chapter 4 the independence structures are sup-lattice bimorphisms, in Chapter 5 they are not, which suggests that the definition of independence structure put forward in Chapter 3, which only requires monotonicity, is the right one.

Chapter 2

Preliminaries

In this chapter we introduce the basic definitions and results that will be used in other chapters. First, we describe sup-lattices because almost everything will be based on these, and then we address quantales.

2.1 Sup-lattices

We begin by recalling some basic definitions about partial orders and sup-lattices, as can be found in [Bir67] or [JT84].

Definition 2.1.1 A *poset* is a set equipped with a binary relation $\leq$, which satisfies the following conditions for all x, y, z:

1. $x \leq x$. (Reflexivity)
2. If $x \leq y$ and $y \leq x$, then $x = y$. (Anti-symmetry)
3. If $x \leq y$ and $y \leq z$, then $x \leq z$. (Transitivity)

A poset P is *bounded* if P has a greatest element, i.e., an element 1 such that $x \leq 1$ for all $x \in P$, and a least element, i.e., an element 0 such that $0 \leq x$ for all $x \in P$.

Definition 2.1.2 Let S be a poset. Then $a \in S$ is an *upper bound* of a subset X of S if $x \leq a$ for all $x \in X$. Similarly, $b \in S$ is a *lower bound* of a subset X of S if $b \leq x$ for all $x \in X$.

Definition 2.1.3 Let S be a poset. Then $a \in S$ is the *supremum*, or *join* of a subset X of S if a is an upper bound of X and, for all upper bounds a' of

X, we have $a \leq a'$. Similarly, $b \in S$ is the *infimum*, or *meet* of a subset X of S if b is a lower bound of X and, for all lower bounds b' of X, we have $b' \leq b$.

The join (resp. meet) of X, if it exists, is unique and we denote it by $\bigvee X$ (resp. $\bigwedge X$), or, for sets of two elements, $x \vee y$ (resp. $x \wedge y$).

Definition 2.1.4 A *lattice* is a poset P such that for all x, $y \in P$ there is $x \vee y$ and $x \wedge y$. A lattice P is *complete* when there is $\bigvee X$ and $\bigwedge X$ for every subset X of L.

Definition 2.1.5 Let L and F be two complete lattices. We say that $f : L \longrightarrow F$ is a *homomorphism of sup-lattices* if it preserves arbitrary joins: $f(\bigvee S) = \bigvee f[S]$.

Definition 2.1.6 The category SL of *sup-lattices* has as its objects complete lattices and as morphisms the homomorphisms of sup-lattices.

Complete lattices are usually called sup-lattices when they are thought of as objects of SL.

Definition 2.1.7 Let P be a poset. An order preserving function $j : P \longrightarrow P$ is called a *closure operator* if it satisfies, for all $a \in P$:

$$a \leq j(a)$$

$$j(j(a)) = j(a)$$

Given a closure operator j on a poset P, P_j is the poset whose elements are j-closed elements, i.e, $P_j = \{a \in P \mid j(a) = a\}$. Clearly, $j : P \longrightarrow P_j$ is a surjective monotone map.

It is also important to note that if j is a closure operator, then $a \leq j(b)$ if and only if $j(a) \leq j(b)$, with a and $b \in Q$. In fact this condition is equivalent to j being a closure operator.

Closure operators are useful for representing quotients of sup-lattices. If j is a closure operator on a sup-lattice L then L_j is also a sup-lattice with joins given by $\bigvee^j(X) = j(\bigvee X)$.

2.2 Nuclei on sup-lattices

Definition 2.2.1 Let L be a sup-lattice and $f : L^n \longrightarrow L$ a n-ary monotone operation on L. An *f-nucleus* on L is a closure operator $j : L \longrightarrow L$ such that, for all $x_1, ..., x_n \in L$,

$$f(j(x_1), ..., j(x_n)) \leq j(f(x_1, ..., x_n)).$$

Proposition 2.2.2 *Let j be an f-nucleus on a sup-lattice L. Then,*

$$j(f(j(x_1), ..., j(x_n))) = j(f(x_1, ..., x_n)).$$

Proof: By the fact that j is an f-nucleus, $f(j(x_1), ..., j(x_n)) \leq j(f(x_1, ..., x_n))$. Then, $j(f(j(x_1), ..., j(x_n))) \leq j(f(x_1, ..., x_n))$, because j is a closure operator. The other inequality is also verified because $f(x_1, ..., x_n) \leq f(j(x_1), ..., j(x_n))$, since f is monotone. ∎

Definition 2.2.3 Let $f : L^n \longrightarrow L$ a monotone operation on a sup-lattice L and j an f-nucleus. We define $f_j : L_j^n \longrightarrow L_j$ as $f_j = j \circ f$.

Proposition 2.2.4 *If $f : L^n \longrightarrow L$ is an operation on a sup-lattice L then $j : L \longrightarrow L_j$ is an f-homomorphism, i.e.,*

$$j(f(x_1, ..., x_n)) = f_j(j(x_1), ..., j(x_n)).$$

Proof: It follows from proposition 2.2.2. ∎

Proposition 2.2.5 *Let $f : L^n \longrightarrow L$ be an operation on a sup-lattice L which preserves joins in all variables, i.e., for all $x_1, x_2, ..., x_n \in L$ and $S \subseteq L$:*

$$\forall_{1 \leq i \leq n}\ f(x_1, ..., x_{i-1}, \bigvee S, x_{i+1}, ..., x_n) = \bigvee_{x \in S} f(x_1, ..., x_{i-1}, x, x_{i+1}, ..., x_n).$$

Then, if j is an f-nucleus, f_j is distributive over $\bigvee^j$.

Proof: Suppose that $x_1, ..., x_n \in L_j$ and $S \subseteq L_j$. Then, for all $1 \leq i \leq n$:

$$\begin{aligned}
&f_j(x_1, ..., x_{i-1}, \bigvee^j S, x_{i+1}, ..., x_n) \\
&\quad = f_j(x_1, ..., x_{i-1}, j(\bigvee S), x_{i+1}, ..., x_n) \\
&\quad = j(f(x_1, ..., x_{i-1}, j(\bigvee S), x_{i+1}, ..., x_n)) \\
&\quad = j(f(j(x_1), ..., j(x_{i-1}), j(\bigvee S), j(x_{i+1}), ..., j(x_n))) \\
&\quad = j(f(x_1, ..., x_{i-1}, \bigvee S, x_{i+1}, ..., x_n)) \\
&\quad = j(\bigvee_{s \in S} f(x_1, ..., x_{i-1}, s, x_{i+1}, ..., x_n)) \\
&\quad = j(\bigvee_{s \in S} f(j(x_1), ..., j(x_{i-1}), j(s), j(x_{i+1}), ..., j(x_n))) \\
&\quad = j(\bigvee_{s \in S} j(f(j(x_1), ..., j(x_{i-1}), j(s), j(x_{i+1}), ..., j(x_n))))
\end{aligned}$$

$$= \bigvee_{s \in S}^{j} j(f(j(x_1), ..., j(x_{i-1}), j(s), j(x_{i+1}), ..., j(x_n)))$$
$$= \bigvee_{s \in S}^{j} f_j(x_1, ..., x_{i-1}, s, x_{i+1}, ..., x_n).$$

$\blacksquare$

This last proposition states that the algebraic law (distributivity of f on $\bigvee$ in each variable) is preserved by j, i.e., is verified in L_j.

Generally, if j is an f_λ-nucleus for a family $\{f_\lambda\}$ of operations of arity n_λ on L, then any algebraic law expressed in terms of f_λ and $\bigvee$ will be preserved.

Example 2.2.6 If f is an associative operation on a sup-lattice L and j an f-nucleus, then f_j is associative. In fact, if $x, y, z \in L_j$,

$$\begin{aligned} f_j(f_j(x,y),z) &= j(f(j(f(x,y)),z)) \\ &= j(f(f(x,y),z)) \\ &= j(f(x,f(y,z))) \\ &= j(f(x,j(f(y,z)))) \\ &= f_j(x,f_j(y,z)). \end{aligned}$$

Proposition 2.2.7 *Let $\{f_\lambda\}$ be a family of operations of arity n_λ on a sup-lattice L. Then, the set of all f_λ-nuclei, $N(L)$, is a complete lattice.*

Proof: Since all f_λ-nuclei are monotone functions and the set of all monotone functions is a complete lattice, we only need to prove that $N(L)$ is closed under meets.

Let $\{j_\alpha\}$ be a family of f_λ-nuclei. If we define j as $j = \bigwedge_\alpha j_\alpha$, j is a closure operator. In fact, if $x, y \in L$:

$$\begin{aligned} x \leq j(y) &\Longleftrightarrow x \leq \bigwedge_\alpha j_\alpha \\ &\Longleftrightarrow \forall_\alpha\ x \leq j_\alpha(y) \\ &\Longleftrightarrow \forall_\alpha\ j_\alpha(x) \leq j_\alpha(y) \\ &\Longleftrightarrow \bigwedge_\alpha j_\alpha(x) \leq \bigwedge_\alpha j_\alpha(y) \\ &\Longleftrightarrow j(x) \leq j(y). \end{aligned}$$

Now, we will prove that j is a nucleus. In fact, for all λ and $x_1, ..., x_{n_\lambda} \in L$,

$$\begin{aligned} \forall_\alpha\ f_\lambda(j(x_1), ..., j(x_{n_\lambda})) &\leq f_\lambda(j_\alpha(x_1), ..., j_\alpha(x_{n_\lambda})) \\ &\leq j_\alpha(f_\lambda(x_1, ..., x_{n_\lambda})) \end{aligned}$$

Then, $f_\lambda(j(x_1), ..., j(x_{n_\lambda})) \leq j(f_\lambda(x_1, ..., x_{n_\lambda})))$, i.e., j is an f_λ-nucleus. $\blacksquare$

Definition 2.2.8 Let f be an n-ary function over a sup-lattice L, j an f-nucleus and $R \subseteq L \times L$ a relation on L. We say that j *respects* R if for all pairs $(x, y) \in R$, $j(x) = j(y)$.

Proposition 2.2.9 *Let $\{f_\lambda\}$ be a family of operations of arity n_λ on a sup-lattice L and $R \subseteq L \times L$ a relation. There exists the least f_λ-nucleus that respects R.*

Proof: Let $\{j_\alpha\}$ be the family of all the f_λ-nuclei that respect R and $j = \bigwedge_\alpha j_\alpha$. Suppose that $(x, y) \in R$. Then, $j(x) = \bigwedge_\alpha j_\alpha(x) = \bigwedge_\alpha j_\alpha(y) = j(y)$, i.e, j respects R. $\blacksquare$

It should be pointed out that if $j, k \in N(L)$, then $j \leq k$ if and only if $L_j \subseteq L_k$.

2.3 Quantales

2.3.1 Basic definitions and examples

We now present the basic definitions and properties of quantales, as well as some examples. For that we follow essentially [Ros90].

Definition 2.3.1 A *quantale* Q is a sup-lattice equipped with an associative binary operation $\cdot$ (the multiplication) that distributes over joins:

$$a \cdot (\bigvee_{i \in I} c_i) = \bigvee_{i \in I} (a \cdot c_i)$$

$$(\bigvee_{i \in I} c_i) \cdot a = \bigvee_{i \in I} (c_i \cdot a)$$

Notice that the multiplication of a quantale is a particular case of a monotone binary operation defined on a sup-lattice.

Definition 2.3.2 Let Q be a quantale.

1. Q is *commutative* if $a \cdot b = b \cdot a$ for all $a, b \in Q$.

2. Q is *idempotent* if every $a \in Q$ is idempotent, i.e., $a \cdot a = a$.

3. Q is *unital* if the multiplication has a unit, which we denote by e.

Definition 2.3.3 Let Q and Q' be quantales. A *homomorphism of quantales* $h : Q \longrightarrow Q'$ is a homomorphism of sup-lattices that preserves the multiplication of the quantale. A homomorphism is *unital* if preserves the unit.

Example 2.3.4 Let M be a semigroup, with operation $\times$. If $A \subseteq M$ and $B \subseteq M$, define $A \cdot B = \{a \times b \mid a \in A \text{ and } b \in B\}$. We have also that if $X \subseteq 2^M$, then $A \cdot (\bigcup X) = \bigcup (A \cdot X)$ and $(\bigcup X) \cdot A = \bigcup (X \cdot A)$, where $A \cdot X = \{A \cdot B \mid B \in X\}$ and $X \cdot A = \{B \cdot A \mid B \in X\}$. So, we can conclude that 2^M is a quantale.

If M is a monoid, this quantale is unital, with unit given by $\{e\}$, with e the unit of the monoid.

Example 2.3.5 Let $\mathcal{C}$ be a small category and $\hom(\mathcal{C}) = \bigcup_{x,y \in Ob(\mathcal{C})} \hom_{\mathcal{C}}(x, y)$ (where the homsets are assumed to be disjoint). Let $Q(\mathcal{C})$ denote the power set of $\hom(\mathcal{C})$. Then, $(Q(\mathcal{C}), \subseteq)$ is a complete lattice. If $X \in Q(\mathcal{C})$ and $Y \in Q(\mathcal{C})$, let $X \cdot Y = \{f : x \longrightarrow y \mid \exists_{g:x \longrightarrow z \in X, h:z \longrightarrow y \in Y} \ f = h \circ g\}$ where $\circ$ is the composition in $\mathcal{C}$. It can be easily seen that when $X \subseteq Q(\mathcal{C})$ then $A \cdot (\bigcup X) = \bigcup (A \cdot X)$, as well as the corresponding equation with A appearing on the right. Hence, $Q(\mathcal{C})$ is a quantale.

Consider $e = \{1_x \mid x \in Ob(\mathcal{C})\}$. Then, $e \cdot A = A \cdot e = A$ for all $A \in Q(\mathcal{C})$, i.e., $Q(\mathcal{C})$ is unital.

We will call $Q(\mathcal{C})$ the quantale generated by a small category . The previous example coincides with this if we see a monoid as a one object category.

Example 2.3.6 Let L be a sup-lattice and $Q(L) = Hom_{SL}(L, L)$. $Q(L)$ is a unital quantale, whose multiplication is the composition of functions and unit is the identity map.

Example 2.3.7 Let X be a set. $2^{X \times X}$ is a quantale whose operation is the composition of relations and unit is the identity relation in X. In fact $2^{X \times X} \cong Q(2^X)$, by the map from $Q(2^X)$ to $2^{X \times X}$ that maps each endomorphism f to the relation $R_f = \{(x, y) \mid y \in f(\{x\})\}$.

Definition 2.3.8 Let Q be a quantale and L a sup-lattice. A *representation* of Q on L is a quantale homomorphism $\rho : Q \longrightarrow Q(L)$, and we say it is *faithful* if ρ is 1-1.

Definition 2.3.9 A *relational representation* of a quantale Q on a set P is a homomorphism $\rho : Q \longrightarrow 2^{P \times P}$.

A relational representation of a quantale Q on a set P is then $\rho : Q \longrightarrow Q(2^P)$, up to isomorphism.

2.3.2 Quotient quantales

Quotients of quantales can be described by nuclei, whose definition will be recalled in this subsection. We define it as a special case of definition 2.2.1.

Definition 2.3.10 A *quantic nucleus* on Q is a closure operator such that $j(a) \cdot j(b) \leq j(a \cdot b)$ for all $a, b \in Q$.

The following corollaries are obtained by section 2.2

Proposition 2.3.11 *If j is a quantic nucleus, then $j(a \cdot b) = j(j(a) \cdot j(b))$, with a and $b \in Q$.*

Theorem 2.3.12 ([Ros90]) *If $j : Q \longrightarrow Q$ is a quantic nucleus, then Q_j is a quantale with $a \cdot_j b = j(a \cdot b)$, and $j : Q \longrightarrow Q_j$ is a surjective quantale homomorphism.*

Proof: This follows essentially from proposition 2.2.6 and example 2.2.7. ∎

Proposition 2.3.13 *The set of all nuclei on the quantale Q, $N(Q)$, is a complete lattice.*

Chapter 3

Concurrent systems via quantales

The most general way to describe a concurrent system is in terms of a set of actions with the property that any one of them can occur. If we have a multiprocessor or a distributed system, we will have concurrency, since in these systems there is more than one CPU executing actions.

The research area that studies concurrent systems is designated by concurrency. Models of concurrent systems, usually called models of concurrency, have appeared such as Petri nets, event structures, Chu spaces, etc.

Models of concurrent systems are usually called models of concurrency, and split into two models: the interleaving models, for which one assumes that actions are interleaved in time, and the so-called "true-concurrency" models, for which actions may have duration or be related by conflict, causality, or independence relations.

In this section we work under the assumption that sets of actions are extended so as to include all the "finite observations" that may be performed on a system. Then, we describe systems as being certain relational representations of quantales. Finally, we define a new operation over the quantale that will represent the notion of independence between finite observations, in order to be able to address models of true-concurrency.

3.1 Finite observations

An observation on a system consists of an exchange of information between the system and the observer, and we call it finite if it only involves finite amounts of information in finite time; see [Res00]. Given a system S, we can provide the set of all observations that we can perform on S with an algebraic structure. In fact, if we have two observations, a and b, on some system, we can obtain another denoted by $a.b$ which consists of performing a and then b. When observations are finite, a finite sequence of finite observations should also be a finite observation, which suggests the idea that finite observations should form a monoid (the unit corresponds to the null observation). Note that an infinite sequence of finite observations may not be a finite observation. We can also define the disjunctive observation $\bigvee U$, with U a finite or infinite set of possible observations, which consists of performing one observation of this set. Clearly, when U is a set of finite observations, $\bigvee U$ is also finite. So, what is suggested is that the set of finite observations on a system can be modeled as being a unital quantale, with disjunctions and "and then" being modeled as joins and as the multiplication of the quantale, respectively.

3.2 Systems

We start with the assumption that any system has a set of states and can be observed by means of finite observations, whose set forms a quantale. Let S be a system and P its set of states. If S is in state $p \in P$, and a non-null finite observation over S is made, S may change its state. However, if the observation made is null, S should stay in the same state. The dynamics of the system can be described by a transition relation $\longrightarrow \subseteq P \times Q \times P$, with Q the quantale of finite observations over S, in the following way: if the system is at state p and an observation a is made, the system may change to state q, and we put $< p, a, q > \in \longrightarrow$.

Notation : $p \stackrel{a}{\longrightarrow} q \stackrel{\text{def}}{\Longleftrightarrow} < p, a, q > \in \longrightarrow$.

In fact, we are just seeing a system as a labeled transition system:

Definition 3.2.1 A *labeled transition system* $(P, L, \longrightarrow)$ consists of a set P (of states), a set L (of labels), and a ternary relation $\longrightarrow \subseteq P \times L \times P$ (the transition relation).

Since the set of labels is a quantale Q, the transition relations should have the following natural properties:

1. $p \stackrel{e}{\longrightarrow} q \Longleftrightarrow p = q$

2. $p \stackrel{a.b}{\longrightarrow} q \Longleftrightarrow \exists_{r \in P}(p \stackrel{a}{\longrightarrow} r \wedge r \stackrel{b}{\longrightarrow} q)$

3. $p \xrightarrow{\vee S} q \iff \exists_{a \in S}(p \xrightarrow{a} q)$

As we have seen in the previous chapter, a relational representation of a quantale Q on a set P is a homomorphism $\rho : Q \longrightarrow 2^{P \times P}$.

Consider that Q is the quantale of finite observations and P the set of possible states of the system. Clearly, the map $\rho : Q \longrightarrow 2^{P \times P}$ defined by $(p, q) \in \rho(a) \iff p \xrightarrow{a} q$ is a quantale homomorphism if and only if the properties (1-3) above are satisfied. This means that the system can be identified with a relational representation of the quantale of finite observations.

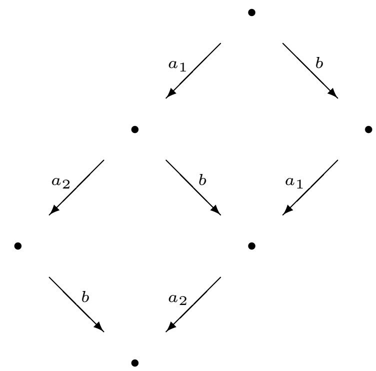
Indeed, the system's dynamics is modeled by this representation. As we can see in [Res99a, Res98], this is useful for instance when working with notions of implementation or refinement that involve changes of "interface".

3.3 Independence structures

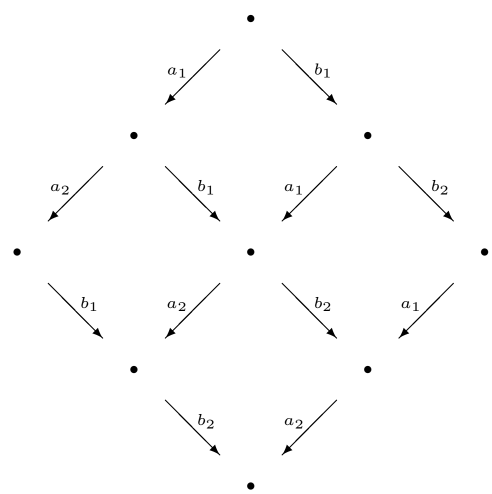
As we have seen before, the multiplication of the quantale of finite observations models the sequentialization of finite observations. However, quantales can also model aspects of true concurrency. In fact, for some observations (not necessary all) a, b, we may have an observation $a \| b$, which means that both a and b were observed and were independent from each other.

Indeed, seeing a system as a labeled transition system, it is expected that if a system can change from state p to q with $a \| b$ then it can also change from p to q with $a.b$ and with $b.a$ since the order of occurrence between two independent observations should not matter. Then, it is expected that the new operation over the quantale that will model this independence verifies $a \| b \leq a.b$ and $a \| b \leq b.a$, thus implying that if $p \xrightarrow{a \| b} q$ then $p \xrightarrow{a.b} q$ and $p \xrightarrow{b.a} q$.

A possibility would be to try to define the operation $\|$ as: given two finite observations a, b, let $a \| b = a.b \wedge b.a$. In fact, with this definition, the independence of two observations would mean that if the system was in state p, it would change to q if $a.b$ or $b.a$ was made. So, although this definition seems to ignore the order of occurrence between two independent observations, this does not happen, for suppose that the observation a is in fact non-atomic and is composed of two subobservations a_1 and a_2, i.e., $a = a_1.a_2$. By the previous definition we would have that $a \| b = a_1.a_2.b \wedge b.a_1.a_2$. This no longer reflects the idea of independence, since the order of occurrence of two independent observations should not matter and the definition misses the sequence $a_1.b.a_2$ which should be possible, as we can see in the following figure:

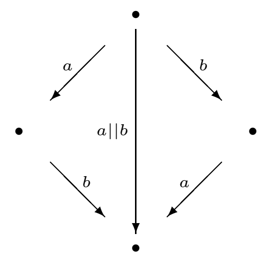


Even if we redefine this operation by $a \| b = a_1.a_2.b \wedge a_1.b.a_2 \wedge b.a_1.a_2$ in order to solve the previous problem, a similar one appears if we suppose that b is a non-atomic observation, $b = b_1.b_2$, with b_1 and b_2 finite observations. This new definition will still miss some paths, as we can see in the next figure:



In fact, if we continue with the refinement of observations a and b, we are led to thinking that they are independent if and only if any transition arisen by alternating some parts of the action a (the refinement of a) with some parts of the action b (the refinement of b) in a random order until both actions are completed leads the system changing to a same state q.

In other words, what really represents the fact of a and b being independent is the following filled-in square:



Since it seems to be difficult to say what are the generators of this new operation, it makes sense to give the axioms that will describe it. The argumentation above leads to defining an operation that will represent the independence between two actions, a structure of independence, as :

Definition 3.3.1 A *structure of independence* on Q is an associative operation $\| : Q \times Q \longmapsto Q$ which is monotone and such that for all $a, b, c, d \in Q$,

(A) $a\|e = a = e\|a$

(B) $(a \cdot b)\|(c \cdot d) \leq (a\|c) \cdot (b\|d)$

Proposition 3.3.2 *Let $||$ be a structure of independence on Q. Then, for all $a, b, c, d \in Q$,*

(*i*) $a\|(c \cdot d) \leq (a\|c) \cdot d$ (*ii*) $(a \cdot c)\|d \leq (a\|d) \cdot c$

(*iii*) $a\|b \leq a \cdot b$ (*iv*) $a\|b \leq b \cdot a$

Proof: For (i) we have

$$a\|(c \cdot d) = (a \cdot e)\|(c \cdot d) \leq (a\|c) \cdot d$$

The others are similar. ∎

Indeed, when a and b are independent, for instance when $a = a_1 \cdot a_2$ and $b = b_1 \cdot b_2$ as in the previous example, these axioms allow us to confirm what is suggested in the motivation, i.e., $a\|b \leq a_1 \cdot a_2 \cdot b_1 \cdot b_2$, $a\|b \leq a_1 \cdot b_1 \cdot a_2 \cdot b_2$, $a\|b \leq a_1 \cdot b_1 \cdot b_2 \cdot a_2$, $a\|b \leq b_1 \cdot a_1 \cdot b_2 \cdot a_2$ and $a\|b \leq b_1 \cdot b_2 \cdot a_1 \cdot a_2$. More generally, if $a = a_1 ... a_n$ and $b = b_1 ... b_m$, let c be any "shuffling" of a and b, i.e., a product of a_i's and b_i's, keeping their relative order in the product, and with exactly a copy of each a_i and b_i. Then it is straightforward to prove $a\|b \leq c$.

Since $\|$ is a new operation on the quantale Q of finite observations over a system, we may wonder if we could define systems to be quantale homomorphisms $Q \longrightarrow 2^{P\times P}$ that also preserve $\|$, with P the set of states of the system in question. In this situation we would have to define $\|$ on $2^{P\times P}$. However, the following example shows that this is in general impossible:

Example 3.3.3 Let $P = \mathbb{N}$ and suppose that a binary operation $\|$ is defined in $2^{\mathbb{N}\times\mathbb{N}}$. Consequently, for all relations $R, S, T, U \subseteq \mathbb{N} \times \mathbb{N}$ we have that:

1. $R\|\triangle_{\mathbb{N}} = R$
2. $\triangle_{\mathbb{N}}\|R = R$
3. $(R; S)\|(T; U) \subseteq (R\|T); (S\|U)$, where ; represents the composition of relations.

Let $R = \{(n, 2n) \mid n \in \mathbb{N}\}$ and $S = \{(n, 2n + 1) \mid n \in \mathbb{N}\}$. So, $R; R^{-1} = \triangle_{\mathbb{N}}$ and $S^{-1}; S = \triangle_{\text{odd}} \neq \emptyset$. But then
$\triangle_{\text{odd}} = S^{-1}; S = (R; R^{-1})\|(S_{-1}; S) \subseteq (R\|S^{-1}); (R^{-1}\|S) \subseteq R; S^{-1}; R^{-1}; S^{-1} = \emptyset$, which is absurd.

However, although the operation is not preserved, representations are monotone, and thus they preserve all the desired inequalities.

Chapter 4

Event structures

Concurrent systems can be modeled by event structures. These include notions of state and state transitions, which means we can define labeled transitions systems from them. As we have seen before, a system can be represented by a relational representation of a quantale of finite observations. Therefore, our main interest is to describe this quantale on event structures and also define the operations $\cdot$ and $\|$ in it, which will express the notions of sequencing of finite observations and independence between finite observations, respectively. But first, we will review the basic concepts about event structures.

4.1 Definitions

An event structure consists just of a set E together with a binary relation of *causality* $\leq$, and a binary relation of *conflict* $\#$, as follows:

Definition 4.1.1 An *event structure* is a triple $(E, \leq, \#)$, where E is a set (of *events*), $\leq$ is a partial order on E, called the *causal dependency* relation, and $\#$ is an irreflexive and symmetric binary relation on E, called the *conflict* relation, such that for all $x, y, z \in E$

- $\downarrow(x) (= \{y \in E | y \leq x\})$ is finite;
- $x \# y \leq z \Rightarrow x \# z$.

Causality between two events e and e', $e \leq e'$, means that the execution of e' can only start when the execution of e has totally finished. So, we admit that the events may not be atomic, i.e., may have internal structure.

4.2 Configurations and transitions

The states of a system will be represented in event structures by configurations, i.e., by sets of events that have occurred at a given time instant.

Definition 4.2.1 Let $\mathcal{E} =< E, \leq, \# >$ be an event structure. A *configuration* is a set $C \subseteq E$ that is lower-closed and conflict-free, i.e., such that for all $x, y \in E$

- $x \leq y \in C \Rightarrow x \in C$
- $x, y \in C \Rightarrow \urcorner(x \# y)$.

Notation 4.2.2 We write cfgs($\mathcal{E}$) for the set of configurations of $\mathcal{E}$.

Example 4.2.3 Let $\mathcal{E}_1$ be an event structure with events $\{e_1, e_2, e_3\}$ such that $e_1 \leq e_2$ and $\mathcal{E}_2$ an event structure with events $\{e_1, e_2, e_3, e_4\}$ such that $e_2 \leq e_3$, $e_1 \# e_2$ and $e_1 \# e_3$. The configurations of $\mathcal{E}_1$ are $\emptyset$, $\{e_1\}$ and $\{e_1, e_2\}$ and of $\mathcal{E}_2$ are $\emptyset$, $\{e_1\}$, $\{e_2\}$ and $\{e_2, e_3\}$.

A transition is a located or situated set of events, that is, a set of events together with conceptual information about the status of other events. It corresponds to changes of state of systems:

Definition 4.2.4 Define a set $X \subseteq E$ to be a *transition* when there exist configurations C and D for which $C \subseteq D$ and $X = D \setminus C$, and let tran($\mathcal{E}$) denote the set of transitions of $\mathcal{E}$. We will write $C \stackrel{X}{\Longrightarrow} D$ to mean $C \subseteq D$ and $D \setminus C = X$.

Proposition 4.2.5 ([Res99b]) *A set $X \subseteq E$ is a transition if and only if it satisfies the following conditions:*

- $\forall_{x,y \in X} \urcorner(x \# y)$ *(i.e, X is conflict-free);*
- $\forall_{x,y \in X}\ \forall_{z \in E}(x \leq z \leq y \Rightarrow z \in X)$ *(i.e, X is convex) .*

Proof: ($\Rightarrow$): Assume that X is a transition. By definition there exist configurations C and D such that $C \stackrel{X}{\Longrightarrow} D$. Then $X \subseteq D$ and thus X is conflict-free because D is. Now assume that $x \leq z \leq y$, where $x, y \in X$. D is lower-closed and thus $z \in D$. Hence, if $z \notin X$ it must be the case that $z \in C$; but then $x \in C$ because C is lower-closed, which implies $x \notin X$, a contradiction. Hence, X must be convex.

($\Leftarrow$): Assume that X is convex and conflict-free, and define $D \stackrel{\text{def}}{=} \downarrow(X)$ and $C \stackrel{\text{def}}{=} D \setminus X$; hence, $C \subset D$ and $X = D \setminus C$. Now we will see that C and D are configurations. First, by the condition of conflict heredity it follows that D is conflict-free because X is conflict-free, and thus C is conflict-free because it is contained in D. Furthermore, D is lower-closed by construction and thus it is a configuration. If C is empty then it is a configuration, so assume $x \in C$. By construction there must exist $z \in X$ such that $x \leq z$. If $y \leq x$ and $y \notin C$ then we must have $y \in X$ because D is lower-closed, and thus by convexity $x \in X$, which contradicts $x \in C$. Hence, C is lower-closed. ∎

Example 4.2.6 In $\mathcal{E}_2$ of example 4.2.3 $\{e_3\}$ is a transition, but is not true that $\{e_1, e_3\}$ is a transition.

Definition 4.2.7 ([Res99b]) Define $<$ and $\lessdot$ to be the binary relations over $\text{tran}(\mathcal{E})$ given by

$$X < Y \stackrel{\text{def}}{\Longleftrightarrow} x < y, \text{ for all } x \in X \text{ and } y \in \text{ Y},$$

$$X \lessdot Y \stackrel{\text{def}}{\Longleftrightarrow} x \not\geq y, \text{ for all } x \in X \text{ and } y \in \text{ Y}.$$

When $X < Y$ we say that X is *causally before* Y, and when $X \lessdot Y$ we say that X *does not depend on* Y.

Proposition 4.2.8 *The relations $<$ and $\lessdot$ satisfy the following properties, for all transitions X and Y:*

1. $X < Y \Rightarrow X \lessdot Y$;
2. $X, Y \neq \emptyset \Rightarrow ((X < Y \Rightarrow \neg(Y \lessdot X))$;
3. $X \lessdot Y \Rightarrow X \cap Y = \emptyset$;
4. $\lessdot$ *is irreflexive over non-empty transitions;*
5. $<$ *is a strict partial order over non-empty transitions;*
6. $\bigcup_{i \in I} X_i < \bigcup_{j \in J} Y_i \Longleftrightarrow \forall_{i \in I} \forall_{j \in J} (X_i < Y_i)$;
7. $\bigcup_{i \in I} X_i \lessdot \bigcup_{j \in J} Y_i \Longleftrightarrow \forall_{i \in I} \forall_{j \in J} (X_i \lessdot Y_i)$;
8. $\emptyset < X$ *and* $X < \emptyset$;
9. $\emptyset \lessdot X$ *and* $X \lessdot \emptyset$.

Proof: 1 - If $x < y$ then $x \not\geq y$. 2 - Let $x \in X$ and $y \in Y$. If $x < y$ then $x \leq y$ and thus $\neg(Y \lessdot X)$. 3 - Assume $x \in X \cap Y$. Then $x \in X$, $x \in Y$ and $x \leq x$, contradicting $X \lessdot Y$. 4 - If $X \lessdot X$ then $X = X \cap X = \emptyset$. 5 - $<$ is clearly transitive, and it is irreflexive over non-empty transitions because $\lessdot$ is and, by 1, $< \subseteq \lessdot$. The remaining properties are equally simple and we omit their proofs. ∎

Theorem 4.2.9 ([Res99b]) *Let X and Y be transitions and let B and D be configurations. The following propositions are equivalent:*

1. *There exists a configuration C such that $B \overset{X}{\Longrightarrow} C$ and $C \overset{Y}{\Longrightarrow} D$.*

2. *$B \overset{X \cup Y}{\Longrightarrow} D$ and $X \lessdot Y$.*

Proof: ($1 \Rightarrow 2$) We clearly have $B \overset{X \cup Y}{\Longrightarrow} D$. If either $X = \emptyset$ or $Y = \emptyset$ the result is immediate, so let $x \in X$ and $y \in Y$. If $y \leq x$ then $y \in C$ because C is lower-closed, and thus $y \notin Y$, a contradiction. Hence, $y \not\leq x$, and we conclude $X \lessdot Y$.

($2 \Rightarrow 1$) The set $C \overset{\text{def}}{=} B \cup X$ is obviously conflict-free. If $C = \emptyset$ the result is immediate, so let $x \in C$. There are two possibilities, namely $x \in B$ or $x \in X$. We deal with the latter first. If $y \leq x$ then $y \in D$, because $x \in D$ and D is lower-closed. Hence, if $x \in X$ and $y \notin C$ then $y \in Y$, which contradicts $X \lessdot Y$, whence it follows that $y \in C$. Now we deal with the case $x \in B$. If $y \leq x$ then $y \in B$ because B is lower-closed, which implies $y \in C$. We have thus showed that C is a configuration. Furthermore, we have $B \overset{X}{\Longrightarrow} C$ and $C \overset{Y}{\Longrightarrow} D$, which concludes the proof. ∎

4.3 The quantale of an event structure

An event structure represents both states and finite observations of a concurrent system; a set of transitions of an event structure should express a finite observation and an union of sets of transitions can be regarded as a disjunction of finite observations.

So, for each event structure $\mathcal{E}$, the quantale of finite observations is $Q_{\mathcal{E}} = (2^{\text{tran}(\mathcal{E})}, \subseteq)$, whose multiplication is given by:

$$A \cdot B = \{T \in \text{tran}(\mathcal{E}) \mid \exists_{X \in A} \exists_{Y \in B}\ (T = X \cup Y \text{ and } X \lessdot Y)\},$$
$$\text{for } A, B \in Q_{\mathcal{E}}. \qquad (4.2)$$

It is simple to see that the operation $\cdot$ is associative and that $A \cdot \{\emptyset\} = A = \{\emptyset\} \cdot A$ for all $A \in Q_{\mathcal{E}}$. It is also true that this operation distributes over joins. Thus, $(Q_{\mathcal{E}}, \{\emptyset\}, \cdot)$ is a unital quantale, more specifically the quantale associated to the event structure $\mathcal{E}$.

A particular relational representation of the quantale is given by the following proposition:

Proposition 4.3.1 ([Res99b]) *The mapping* $t : Q_{\mathcal{E}} \longrightarrow 2^{cfgs(\mathcal{E}) \times cfgs(\mathcal{E})}$ *given by* $A \longmapsto \{(C, D) | C \subseteq D \text{ and } D \setminus C \in A\}$ *is a faithful representation of* $Q_{\mathcal{E}}$.

Proof: We have $t(\{\emptyset\}) = \Delta_{\mathrm{cfgs}(\mathcal{E})}$, i.e., the quantale unit is preserved. Now let $(C_1, C_2) \in t(A \cdot B)$. By definition of t this condition holds if and only if there exists a transition T in $A \cdot B$ such that $C_1 \stackrel{T}{\Longrightarrow} C_2$. The condition $T \in A \cdot B$ holds if and only if there exist transitions $X \in A$ and $Y \in B$ such that $T = X \cup Y$ and $X \lessdot Y$, and thus from theorem 4.2.9 it follows that $(C_1, C_2) \in t(A \cdot B)$ exactly when there exist transitions $X \in A$ and $Y \in B$ and a configuration C such that $C_1 \stackrel{X}{\Longrightarrow} C$ and $C \stackrel{Y}{\Longrightarrow} C_2$, which means that the quantale multiplication is preserved by t. Furthermore, joins (i.e. unions) are clearly preserved, and thus t is a unital homomorphism of quantales. Now we prove that t is injective. Let A and B be sets of transitions such that $A \nsubseteq B$, and let $X \in A \setminus B$. Then there is a pair (C, D) in $t(A)$ such that $C \stackrel{X}{\Longrightarrow} D$. If (C, D) were in $t(B)$ then we would have $D \setminus C = X \in B$, a contradiction. Hence, $t(A) \nsubseteq t(B)$. ∎

The monomorphism t of this proposition defines a system, and it shows that the multiplication of this quantale really has an adequate meaning, i.e., it represents a sequencing of finite observations.

In order to represent independence between actions, we can define $\|$ in this quantale as:

$$A \| B = \{T \in \mathrm{tran}(\mathcal{E}) \mid \exists_{X \in A} \exists_{Y \in B} \ (T = X \cup Y \text{ and } X \lessdot Y \text{ and } Y \lessdot X)\},$$
$$\text{for} A, B \in Q(\mathcal{E}). \tag{4.4}$$

It follows easily from its definition that this operation is associative, and that $A \| \{\emptyset\} = A = \{\emptyset\} \| A$, and is also monotone in both variables (in fact it distributes over joins), then making $(Q_{\mathcal{E}}, \{\emptyset\}, \|)$ a unital quantale.

In fact, $\|$ is an independence structure on $Q_{\mathcal{E}}$, as we can see in the following proposition:

Proposition 4.3.2 *Let* $\mathcal{E}$ *be an event structure and consider the quantale associated to this event structure,* $Q_{\mathcal{E}}$*. Then,* $\|$ *defined as in (4.4) is an independence structure.*

Proof: We have already seen that $A\|\{\emptyset\} = A = \{\emptyset\}\|A$ and that $\|$ is monotone. It remains to prove the second condition of independence. Suppose that A, B, C and $D \in Q_{\mathcal{E}}$ and that $T \in (A \cdot B)\|(C \cdot D)$. By definition of $\|$, $T \in (A \cdot B)\|(C \cdot D)$ if and only if there exist transition $X \in A \cdot B$ and $Y \in C \cdot D$ such that $T = X \cup Y$, $X \lessdot Y$ and $Y \lessdot X$. Since $X \in A \cdot B$, there are transitions $X_1 \in A$ and $X_2 \in B$ such that $X = X_1 \cup X_2$ and $X_1 \lessdot X_2$. By the fact that $Y \in A \cdot B$, there exist transitions $Y_1 \in C$ and $Y_2 \in D$ such that $Y = Y_1 \cup Y_2$ and $Y_1 \lessdot Y_2$. We have that $X = X_1 \cup X_2$, $Y = Y_1 \cup Y_2$, $X_1 \cup X_2 \lessdot Y_1 \cup Y_2$ and $Y_1 \cup Y_2 \lessdot X_1 \cup X_2$. So, by proposition 4.2.8, $X_1 \lessdot Y_1$, $X_2 \lessdot Y_2$, $Y_1 \lessdot X_1$ and $Y_2 \lessdot X_2$. Consequently, $X_1 \cup X_2 \in A\|C$ and $Y_1 \cup Y_2 \in B\|D$. This same proposition also implies that $X_1 \cup Y_1 \lessdot X_2 \cup Y_2$, since $X_1 \lessdot X_2$ and $Y_1 \lessdot Y_2$ by hypothesis. Then, $T \in (A\|C) \cdot (B\|D)$ ∎

So, as we have noticed in the previous chapter, the fact that $\|$ is an independence structure means that the definition that was presented for this operation really expresses the desired independence notion between two observations.

Chapter 5

Continuous behaviour

Since systems may have a continuous behaviour, we will study the possibility of characterizing such systems. So, we will see a system as a topological space, i.e., the states of a system would be points of a specific topological space. Further, considering that S is a system and X is a topological space which represents S, an action of this system should be represented in X by a path "of higher dimension" that joins x and y, with x and y belonging to X, when x corresponds to the state before the action occurs and y corresponds to the state after the occurrence. These paths of higher dimension, which will be just designated by paths, are like usual paths, i.e., maps $I \longrightarrow X$ with $I = [0, 1]$, but whose domain may have dimension greater than 1. Consequently, a finite observation over a system should be represented in this model by a set of such paths.

Our main interest in this chapter is to apply this idea to the study of concurrent systems, not only introducing the quantale of finite observations for each system in this context, but also defining the operations $\cdot$ and $\|$ in it, with the purpose of representing notions of sequencing of finite observations and independence between finite observations, respectively.

In §5.4 we will study some properties of the operation $\|$, in particular for specific topological spaces.

5.1 Higher dimensional paths

Definition 5.1.1 A *bi-pointed topological space* is a topological space X equipped with two specific points, 0_X and 1_X. Given two such spaces, X and Y, we define their *product* to be $X \times Y$ with $0_{X\times Y} = (0_X, 0_Y)$ and $1_{X\times Y} = (1_X, 1_Y)$, and their *sequential composition*, $X \rhd Y$, to be the quotient space of the disjoint union $X \coprod Y$ in which 1_X and 0_Y are identified.

Definition 5.1.2 Consider $\mathbb{R}^n$ partially ordered by $(x_1, ..., x_n) \leq (y_1, ..., y_n)$ iff $x_i \leq y_i$ for all $i = 1, ..., n$. By *hyperinterval* in $\mathbb{R}^n$ we mean a bi-pointed space $\mathbb{I} \subseteq \mathbb{R}^n$ which, for some $x_0 \in \mathbb{R}^n$, $x_0 \geq 0$, is a connected topological subspace of the interval $\{x \in \mathbb{R}^n \mid 0 \leq x \leq x_0\}$, containing both 0 and x_0. Then $0_{\mathbb{I}} = 0 = \min \mathbb{I}$, and $1_{\mathbb{I}} = x_0 = \max \mathbb{I}$.

The previous definition of the product and the sequential composition of bi-pointed spaces were only up to homeomorphism, but for hyperintervals we shall adopt concrete definitions, as follows.

Definition 5.1.3 Let $\mathbb{I}$ and $\mathbb{J}$ be hyperintervals in $\mathbb{R}^n$. $\mathbb{I} \rhd \mathbb{J}$ is the hyperinterval $\mathbb{I} \cup (\mathbb{J} + 1_{\mathbb{I}})$[1].

Proposition 5.1.4 *The operation $\rhd$ is associative.*

Proof:
Suppose that $\mathbb{I}$, $\mathbb{J}$ and $\mathbb{K}$ are hyperintervals.

$$\begin{aligned}
(\mathbb{I} \rhd \mathbb{J}) \rhd \mathbb{K} &= (\mathbb{I} \cup (\mathbb{J} + 1_{\mathbb{I}})) \cup (\mathbb{K} + 1_{\mathbb{I} \rhd \mathbb{J}}) \\
&= (\mathbb{I} \cup (\mathbb{J} + 1_{\mathbb{I}})) \cup (\mathbb{K} + 1_{\mathbb{I}} + 1_{\mathbb{J}}) \\
&= \mathbb{I} \cup ((\mathbb{J} + 1_{\mathbb{I}}) \cup (\mathbb{K} + 1_{\mathbb{I}} + 1_{\mathbb{J}})) \\
&= \mathbb{I} \cup ((\mathbb{J} \cup (\mathbb{K} + 1_{\mathbb{J}})) + 1_{\mathbb{I}}) \\
&= \mathbb{I} \cup ((\mathbb{J} \rhd \mathbb{K}) + 1_I) \\
&= \mathbb{I} \rhd (\mathbb{J} \rhd \mathbb{K}).
\end{aligned}$$

Clearly,

$$\begin{aligned}
\dim\ ((\mathbb{I} \rhd \mathbb{J}) \rhd \mathbb{K}) &= \dim\ (\mathbb{I} \rhd (\mathbb{J} \rhd \mathbb{K})) \\
&= \max\{\dim \mathbb{I}, \dim \mathbb{J}, \dim \mathbb{K}\}.
\end{aligned}$$

∎

Definition 5.1.5 Let $\mathbb{I}$ and $\mathbb{J}$ be two hyperintervals, in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. $\mathbb{I} \times \mathbb{J}$ is the hyperinterval in $\mathbb{R}^{n+m}$ $\{(x_1, ..., x_n, y_1, ..., y_m) \mid (x_1, ..., x_n) \in \mathbb{I}, (y_1, ..., y_m) \in \mathbb{J}\}$.

[1] Hence, $1_{\mathbb{I} \rhd \mathbb{J}} = 1_{\mathbb{I}} + 1_{\mathbb{J}}$.

Example 5.1.6

- I in $\mathbb{R}$ is the hyperinterval $[0,1]$.

- $\mathbf{1}$ is the hyperinterval $\{0\}$.

From now on, we will consider a fixed set of hyperintervals, $\mathcal{H}$, assuming only that it includes I and $\mathbf{1}$, and that is closed under $\triangleright$ and $\times$.

We have already seen that $\triangleright$ is associative and it is clear that $\mathbf{1} \triangleright \mathbb{I} = \mathbb{I} \triangleright \mathbf{1} = \mathbb{I}$ for any $\mathbb{I}$. So, $(\mathcal{H}, \triangleright, \mathbf{1})$ is a monoid.

Let X be a topological space, which will remain fixed throughout this section. Let $x, y \in X$.

Definition 5.1.7 A *higher dimensional path* in X from x to y is a continuous function $s : \mathbb{I} \longrightarrow X$, where $\mathbb{I} \in \mathcal{H}$, such that $s(0) = x$ and $s(1_{\mathbb{I}}) = y$, and we write $s : x \underset{\mathbb{I}}{\longrightarrow} y$ or simply $s : x \longrightarrow y$.

Definition 5.1.8 Consider two higher dimensional paths $s : x \underset{\mathbb{I}}{\longrightarrow} y$, $t : y \underset{\mathbb{J}}{\longrightarrow} z$. We define their *composition* , $s.t : x \underset{\mathbb{I}\triangleright\mathbb{J}}{\longrightarrow} y$, by:

$$s.t(x) = \begin{cases} s(x) & \text{if} \quad x \in \mathbb{I} \\ t(x - 1_{\mathbb{I}}) & \text{if} \quad x \in \mathbb{J} + 1_{\mathbb{I}}. \end{cases}$$

Proposition 5.1.9 *X, equipped with all the higher dimensional paths, forms a small category.*

Proof: X is the set of objects. For each $x, y \in X$, $\hom(x, y)$ is the set of higher dimensional paths from x to y. For each x the identity function 1_x is the higher dimensional path $s : x \underset{\mathbf{1}}{\longrightarrow} x$, and composition is that of definition 5.1.7. Let us verify that it is associative. Assume that $s : x \longrightarrow y$, $t : y \longrightarrow z$, $u : z \longrightarrow w$ are the higher dimensional paths $s : \mathbb{I} \longrightarrow X$, $t : \mathbb{J} \longrightarrow X$ and $u : \mathbb{K} \longrightarrow X$. Then,

$$((s.t).u)(x) = \begin{cases} (s.t)(x) & \text{if} \quad x \in \mathbb{I} \triangleright \mathbb{J} \\ u(x - 1_{\mathbb{I}\triangleright\mathbb{J}}) & \text{if} \quad x \in \mathbb{K} + 1_{\mathbb{I}\triangleright\mathbb{J}} \end{cases}$$

$$= \begin{cases} s(x) & \text{if} \quad x \in \mathbb{I} \\ t(x - 1_{\mathbb{I}}) & \text{if} \quad x \in \mathbb{J} + 1_{\mathbb{I}} \\ u(x - 1_{\mathbb{I}} - 1_{\mathbb{J}}) & \text{if} \quad x \in \mathbb{K} + 1_{\mathbb{I}\triangleright\mathbb{J}} \end{cases}$$

$$(s.(t.u))(x) = \begin{cases} s(x) & \text{if} \quad x \in \mathbb{I} \\ (t.u)(x - 1_{\mathbb{I}}) & \text{if} \quad x \in \mathbb{J} \triangleright \mathbb{K} + 1_{\mathbb{I}} \end{cases}$$

$$= \begin{cases} s(x) & \text{if} \quad x \in \mathbb{I} \\ t(x - 1_{\mathbb{I}}) & \text{if} \quad x \in \mathbb{J} + 1_{\mathbb{I}} \\ u(x - 1_{\mathbb{I}} - 1_{\mathbb{J}}) & \text{if} \quad x \in \mathbb{K} + 1_{\mathbb{I}} + 1_{J}. \end{cases}$$

∎

We denote this category by $\Pi(X)$. According to 2.3.5, $Q(\Pi(X))$ is a quantale.

We conclude this section by defining a new relation that can be established between two higher dimensional paths, according to some conditions.

Definition 5.1.10 The *restriction* of the higher dimensional path $s : x \underset{\mathbb{I}}{\longrightarrow} y$ to $\mathbb{J} \subseteq \mathbb{I}$, with and $1_{\mathbb{J}} = 1_{\mathbb{I}}$, is the higher dimensional path $s|_{\mathbb{J}} : x \underset{\mathbb{J}}{\longrightarrow} y$.

Definition 5.1.11 Consider two higher dimensional paths $s : x \underset{\mathbb{I}}{\longrightarrow} y$, $t : x \underset{\mathbb{J}}{\longrightarrow} y$. We write $s \leq t$ if $\mathbb{I} \subseteq \mathbb{J}$ and $s = t|_{\mathbb{I}}$.

Proposition 5.1.12 $(\Pi(X), \leq)$ *is a partial order.*

Proof: It follows immediately from the definition. ∎

Notice that $\Pi(X)$ is enriched in Poset, i.e., each homset is partially ordered and the composition is monotone on both sides.

5.2 Equivalence of paths

In the category $\Pi(X)$, higher dimensional paths like $s : x \underset{\mathbf{1}}{\longrightarrow} x$ and $s : x \underset{\mathbf{1}\times\mathbf{1}}{\longrightarrow} x$ are distinct. So, instead of considering higher dimensional paths, we can consider equivalence classes of higher dimensional paths in order to avoid distinguishing higher dimensional paths that run through the same points of space and are defined on homeomorphic hyperintervals. Then, another possible way of representing states of systems and their actions is to consider that states of a system lie in a specific topological space as previously, but instead of representing an action of the system by a set of higher dimensional paths, represent it by a set of equivalence classes of higher dimensional paths, as follows:

Definition 5.2.1 Let $s : x \underset{\mathbb{I}}{\longrightarrow} y$ and $t : x \underset{\mathbb{J}}{\longrightarrow} y$ be two higher dimensional paths. We write $s \cong t$ if there is a homeomorphism $i : \mathbb{I} \longrightarrow \mathbb{J}$ of bi-pointed spaces such that $t \circ i = s$.

By $[s]$, equivalence class of higher dimensional path s, we will mean all higher dimensional paths s' such that $s' \cong s$.

Notice that the relation $\cong$ between higher dimensional paths defines a congruence in the category $\Pi(X)$, and $\Omega(X)$ is the quotient. Hence, the composition of two classes of higher dimensional paths, $[s]$ and $[v]$, is given by $[s.v]$. In fact $\cong$ respects $\leq$, so we have a quotient of enriched categories.

5.3 Independence structures

For the purposes described above, we will give a definition of $\|$ in both quantales, $Q(\Pi(X))$ and $Q(\Omega(X))$, and see some properties of this operation.

In $Q(\Pi(X))$ it makes sense to define $\|$ as:

$$S\|T = \{s : \mathbb{I} \times \mathbb{J} \longrightarrow X | \ \forall_{x \in \mathbb{I}}\, s(x, \_) : \mathbb{J} \longrightarrow X \in T, \\ \forall_{y \in \mathbb{J}}\, s(\_, y) : \mathbb{I} \longrightarrow X \in S\}.$$

Example 5.3.1 Let X be the square $I \times I$, $A = \{t_i \mid \forall_{i \in I}\, t_i : I \longrightarrow I \times I, t_i(j) = (i, j)\}$ and $B = \{v_j \mid \forall_{j \in I}\, v_j : I \longrightarrow I \times I, v_j(i) = (i, j)\}$. A and B are sets of higher dimensional paths. The only element of $A\|B$ is the identity $id_{I \times I}$.

It is important to notice that, although in event structures the structure of independence that was defined was a distributive operation over joins, this structure of independence is not. Indeed, the fact of not being distributive is what allow us to describe the geometrical idea and notion of continuity that was intended.

Example 5.3.2 If we consider the previous example and divide the set B into two sets, $B_1 = \{v_j \mid \forall_{j \in I \bigcap \mathbb{Q}}\, v_j : I \longrightarrow I \times I, v_j(i) = (i, j)\}$ and $B_2 = \{v_j \mid \forall_{j \in I \bigcap (\mathbb{R} \setminus \mathbb{Q})}\, v_j : I \longrightarrow I \times I, v_j(i) = (i, j)\}$, then $A\|(\cup_i B_i) \neq \cup_i (A\|B_i) = \emptyset$.

Next we see that how to obtain a structure of independence from $\|$. In fact, $\|$ does not satisfy either condition (A) or (B) of definition 3.3.1. However, it is important to notice that a relation can be established between a higher dimensional path in $(A.B)\|(C.D)$ and a higher dimensional path in $(A\|C).(B\|D)$, with A, B, C and $D \in Q(\Pi(X))$, as we can observe in the following proposition:

Proposition 5.3.3 *For all paths $s \in (A.B)\|(C.D)$ there is a higher dimensional path $t \in (A\|C).(B\|D)$ such that $t \leq s$.*

Proof: Let $s \in (A.B)\|(C.D)$. So $s : (\mathbb{I} \rhd \mathbb{J}) \times (\mathbb{K} \rhd \mathbb{L}) \longrightarrow X$ such that for all $z \in (\mathbb{I} \rhd \mathbb{J})$, $s(z, \_) : \mathbb{K} \rhd \mathbb{L} \longrightarrow X \in C.D$ and for all $w \in (\mathbb{K} \rhd \mathbb{L})$, $s(\_, w) : \mathbb{K} \rhd \mathbb{L} \longrightarrow X \in A.B$.

Notice that $(\mathbb{I} \times \mathbb{J}) \rhd (\mathbb{K} \times \mathbb{L}) \subseteq (\mathbb{I} \rhd \mathbb{J}) \times (\mathbb{K} \rhd \mathbb{L})$. Admit that $v = s|_{(\mathbb{I}\times\mathbb{J})\rhd(\mathbb{K}\times\mathbb{L})}$. Then v is a higher dimensional path and also $v \leq s$.

In more detail, $v = v_1.v_2$, where $v_1 = s|_{\mathbb{I}\times\mathbb{K}}$, and $v_2(j, l) = s(j + 1_{\mathbb{I}}, l + 1_{\mathbb{K}})$, $j \in \mathbb{J}$ and $l \in \mathbb{L}$. So, $v_1 \in A\|C$ and $v_2 \in B\|D$. Consequently, $v \in (A\|C).(B\|D)$. ∎

We remark that $\|$ is not in general an independence structure. For instance, we may fail to have $A\|B \subseteq A \cdot B$, as can easily be concluded from example 5.3.1. In order to remedy this, define a $\|$-*nucleus* on $Q(\Pi(X))$ to be a quantale nucleus j which is also an f-nucleus with respect to the operation $\|$, i.e., satisfying $j(A)\|j(B) \subseteq j(A\|B)$. From proposition 2.2.9 we conclude that there exists the least $\|$-nucleus j on $Q(\Pi(X))$ satisfying, for all $A, B, C, D \in Q(\Pi(X))$,

$$j(e\|A) = j(A\|e) = j(A)$$

$$j((A \cdot B)\|(C \cdot D)) \subseteq j((A\|C) \cdot (B\|D)).$$

[Define $R \subseteq Q(\Pi(X)) \times Q(\Pi(X))$ to consist of the pairs of the form $(A, A\|e)$, $(e\|A, A)$, $((A \cdot B)\|(C \cdot D) \bigcup (A\|C) \cdot (B\|D), (A\|C) \cdot (B\|D))$.] Then $Q(\Pi(X))_j$ is precisely the largest quotient of $Q(\Pi(X))$ for which $\|_j$ is an independence structure, with the quantale operations and $\|_j$ defined by:

$$C \cdot_j B = j(C \cdot B);$$

$$C\|_j B = j(C\|B);$$

$$\bigcup_i^j C_i = j(\bigcup_i C_i)$$

$$e_j = j(e).$$

As we see in the following proposition, if we define $\rho : Q(\Pi(X)) \longrightarrow 2^{X\times X}$ as $\rho(A) = \{(x, y) \mid \exists_{s\in A}\ s : x \longrightarrow y\}$ then ρ is a relational representation of $Q(\Pi(X))$ on X.

Proposition 5.3.4 *Let $\rho : Q(\Pi(X)) \longrightarrow 2^{X\times X}$ be defined as $\rho(A) = \{(x, y) \mid \exists_{s\in A}\ s : x \longrightarrow y\}$. Then ρ is a relational representation of $Q(\Pi(X))$ on X.*

Proof: It follows from the definition of ρ that it preserves the unit of the quantale. It remains to prove that it preserves the operation $\cdot$ and arbitrary joins.

Let $A, B \in Q(\Pi(X))$. Then,

$$\begin{aligned}\rho(A \cdot B) &= \{(x,z) \mid \exists_{s \in A \cdot B}\ s : x \longrightarrow z\} \\ &= \{(x,z) \mid \exists_{s_1 \in A}\ \exists_{s_2 \in B}\ s = s_1 \cdot s_2,\ s_1 : x \longrightarrow y \text{ and } s_2 : y \longrightarrow z\} \\ &= \{(x,y) \mid \exists_{s_1 \in A}\ s_1 : x \longrightarrow y\} \circ \{(y,z) \mid \exists_{s_2 \in B}\ s_2 : y \longrightarrow z\} \\ &= \rho(A) \circ \rho(B),\end{aligned}$$

i.e., ρ preserves the operation $\cdot$.

Admit that $A_i \in Q(\Pi(X))$, with $i \in I$ a set of indexes. So,

$$\begin{aligned}\rho(\bigcup_i A_i) &= \{(x_i, z_i) \mid \exists_{s \in \bigcup_i A_i}\ s : x_i \longrightarrow z_i\} \\ &= \bigcup_i \{(x_i, z_i) \mid \exists_{s \in A_i}\ s : x_i \longrightarrow z_i\} \\ &\bigcup_i \rho(A_i),\end{aligned}$$

i.e., ρ preserves arbitrary joins.

Then, ρ is a homomorphism of quantales. ∎

Proposition 5.3.5 *Let $\rho : Q(\Pi(X)) \longrightarrow 2^{X \times X}$ be defined as $\rho(A) = \{(x,y) \mid \exists_{s \in A}\ s : x \longrightarrow y\}$. Then, for all $A, B, C, D \in Q(\Pi(X))$, $\rho((A \cdot B)\|(C \cdot D)) \subseteq \rho((A\|C) \cdot (B\|D))$, $\rho(A) = \rho(A\|e)$ and $\rho(e\|A) = \rho(A)$.*

Proof: Suppose that $(x,y) \in \rho((A \cdot B)\|(C \cdot D))$. Then, there exists $s : x \longrightarrow y \in (A \cdot B)\|(C \cdot D)$. So, by proposition 5.3.3 there is a higher dimension path $t : x \longrightarrow y \in (A\|C) \cdot (B\|D)$. Thus $(x,y) \in \rho((A\|C) \cdot (B\|D))$. The others are equally simple. ∎

So, by the universal property of $Q(\Pi(X))_j$ there exists one and only one relational representation of $\rho' : Q(\Pi(X))_j \longrightarrow 2^{X \times X}$ such that $\rho' \circ j = \rho$.

Considering now the quantale $Q(\Omega(X))$, what seems to be the most appropriate definition of $\|$ is:

$$S\|T = \{[s] \mid \exists_{s' \in [s]} s' : \mathbb{I} \times \mathbb{J} \longmapsto X \text{such that} \forall_{x \in \mathbb{I}}\ [s'(x, \_)] \in T, \\ \forall_{y \in \mathbb{J}}\ [s'(\_, y)] \in S\}$$

In fact, this definition of $\|$ satisfies the first condition of the definition of structure of independence, i.e., if E denotes the set of equivalence classes of the higher dimensional paths 1_x, for $x \in X$, then we have the following result:

Proposition 5.3.6 *Let E denote the set of the equivalence classes of the higher dimensional paths 1_x, for $x \in X$. Then, for all $S \in Q(\Omega(X))$, $S \| E = E \| S = S$.*

Proof: If $[s : \mathbb{I} \times \mathbf{1} \longrightarrow X] \in S \| E$, then $[s(x, \_) : \mathbf{1} \longrightarrow X] \in E$, for all $x \in \mathbb{I}$, and $[s(\_, y) : \mathbb{I} \longrightarrow X] \in S$, for all $y \in \mathbf{1}$. Since $\mathbf{1}$ is a singular set, we have that $i : \mathbb{I} \longrightarrow \mathbb{I} \times \mathbf{1}$ such that $i(x) = (x, y)$ is a homeomorphism and thus $s \circ i = s(\_, y)$, i.e., $s \cong s(\_, y)$. Therefore, $[s] \in S$.

Assume that $\mathbf{1}$ is a singular set and that $[s : \mathbb{I} \longrightarrow X] \in S$. We have a homeomorphism $i : \mathbb{I} \times \mathbf{1} \longrightarrow \mathbb{I}$ and thus $s' = s \circ i : \mathbb{I} \times \mathbf{1} \longrightarrow X \in [s]$. As $[s'(x, \_) : \mathbf{1} \longrightarrow X] \in E$, for all $x \in \mathbb{I}$, we conclude that $[s'] = [s] \in S \| E$. Therefore, $S = S \| E$. Similarly, it follows that $S = E \| S$. ∎

In fact, although it satisfies the first condition of definition of structure of independence, the second one is not satisfied, because for instance it is not true in general that $S \| T \subseteq S.T$. Hence, in this quantale we still do not have a structure of independence and it still makes sense to factor it by an appropriate nucleus.

To conclude we remark that in general $\|$ does not preserve joins. For instance, in example 5.3.2 we have verified this. Furthermore, in general the canonical representation does not equal $\rho(S \| \bigcup_i T_i)$ and $\bigcup_i \rho(S \| T_i)$, which means it does not make sense to factor out the quantale in order to obtain such distributivity. This is then a justification for independence structures to be only monotone in general, rather than sup-lattice bimorphism as in event structure example of Chapter 4.

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